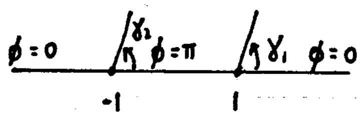
MATH 301

HW # 4 SOLUTIONS

SECTION 7.1 # 3

$\phi_{xx} + \phi_{yy} = 0$



NOW $\phi = C_0 + C_1 \gamma_1 + C_2 \gamma_2$

$\phi = 0 = C_0 \qquad \gamma_1 = \gamma_2 = 0$

IF $\gamma_1 = \pi,\ \gamma_2 = 0$ THEN $\phi = C_1 \pi = \pi \rightarrow C_1 = 1$

IF $\gamma_1 = \pi,\ \gamma_2 = \pi$ THEN $\phi = C_1 \pi + C_2 \pi = 0 \rightarrow C_2 = -1$

HENCE,

$$\phi = \gamma_1 - \gamma_2 \qquad \gamma_1 = \tan^{-1}\left(y/_{x-1} \right) \qquad \gamma_2 = \tan^{-1}\left(y/_{x+1} \right)$$

$$\phi(x,y) = \tan^{-1}\left(y/_{x-1} \right) - \tan^{-1}\left(y/_{x+1} \right)$$

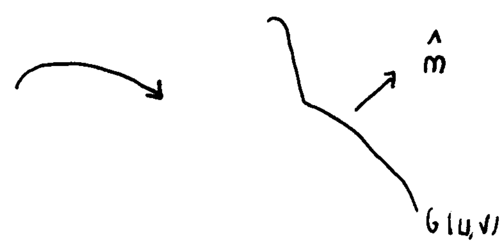
$$\phi(x,y) = \text{ARG}(z-1) - \text{ARG}(z+1)$$

SECTION 7.1 # 6

Z-PLANE

$\hat{n} = \dfrac{\nabla F}{|\nabla F|}$

NOW $\dfrac{\partial U}{\partial n} = \nabla U \cdot \hat{n} = 0$



$\hat{m} = \dfrac{(G_u, G_v)}{|\nabla G|}$

let $\quad W = f(z) \quad$ so $\quad x = x(u,v)$, $\ y = y(u,v)$ $\Longrightarrow$ image curve $\quad G(u,v) = F[x(u,v), y(u,v)] = 0$

MUST SHOW $\quad \dfrac{\partial U}{\partial n} = \dfrac{(U_x F_x + U_y F_y)}{|\nabla F|} = 0 \quad$ IMPLIES $\quad \dfrac{U_u G_u + U_v G_v}{|\nabla G|} = 0.$

NOW
$$U_x = U_u u_x + U_v v_x$$
$$F_x = G_u u_x + G_v v_x$$

$$U_x F_x + U_y F_y = U_u G_u u_x^2 + U_u G_v u_x v_x + U_v G_u u_x v_x + U_v G_v v_x^2 + U_u G_u u_y^2 + U_u G_v u_y v_y + U_v G_u u_y v_y + U_v G_v v_y^2$$

BUT $\quad u_x v_x + u_y v_y = 0 \quad$ AND $\quad |f'(z)|^2 = u_x^2 + u_y^2 = v_x^2 + v_y^2$

HENCE $\quad U_u F_x + U_y F_y = |f'(z)|^2 \left(U_u G_u + U_v G_v \right) \quad$ which yields the result.

SECTION 7.2 #7

$W = z + 1/z \qquad |z| = \rho \quad \text{with} \quad \rho \neq 1$

let $\quad z = \rho e^{i\varphi} \qquad W = \rho e^{i\varphi} + \frac{1}{\rho} e^{-i\varphi}$

NOW $\qquad W = (\rho + 1/\rho)\cos\varphi + i(\rho - 1/\rho)\sin\varphi = u + iv$

HENCE,

$$u = (\rho + 1/\rho)\cos\varphi$$

$$v = (\rho - 1/\rho)\sin\varphi$$

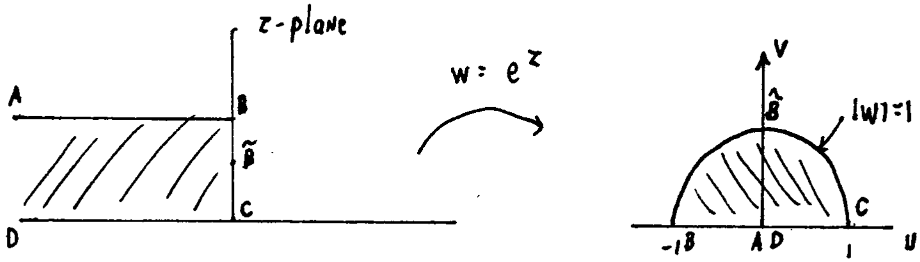
$$\frac{u^2}{[\rho + 1/\rho]^2} + \frac{v^2}{[\rho - 1/\rho]^2} = 1 \quad \text{ellipse}$$

SECTION 7.2 #11c)

$W = e^z$

$\operatorname{Re} z < 0 \qquad 0 < \operatorname{Im}(z) < \pi.$

let $\qquad z = x + iy$



let $\qquad W = u + iv \qquad\qquad u + iv = e^x(\cos y + i\sin y)$

$u = e^x \cos y$

$v = e^x \sin y \qquad\qquad y \in (0, \pi),\ x < 0 \qquad\qquad u^2 + v^2 = e^{2x}$

$v > 0$ circle

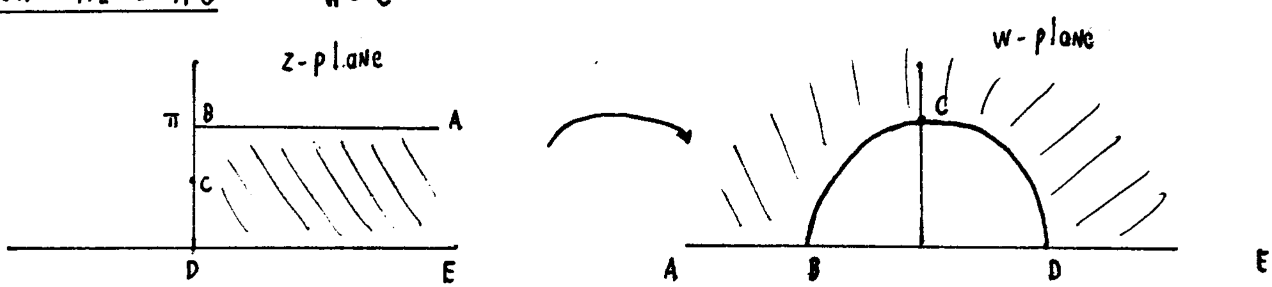
$x = 0 \longrightarrow \quad u = \cos y \quad v = \sin y \quad \text{so} \quad u^2 + v^2 = 1,\ v > 0$

$y = 0 \longrightarrow \quad v = 0,\ u = e^x \in (0, 1) \quad \text{WHEN} \quad x < 0$

$y = \pi \longrightarrow \quad v = 0,\ u = -e^x \in (-1, 0) \quad \text{WHEN} \quad x < 0$

SECTION 7.2 # 11d

$w = e^z$



THE IMAGE IS $|W| > 1$ WITH $\operatorname{Im}(W) > 0$.

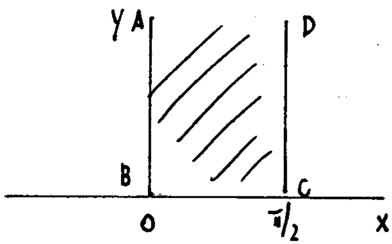
SECTION 7.2 # 13b)

$w = \cos z = \cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$

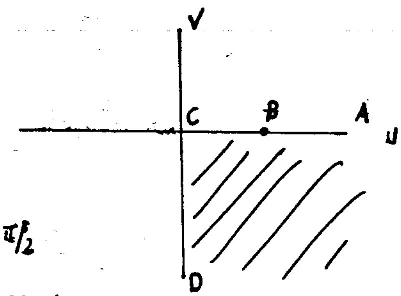
$u = \cos x \cosh y$

$v = -\sin x \sinh y$

let $0 < x < \pi/2, \quad y > 0$



NOW $x = 0,\ y > 0 \rightarrow u = \cosh y > 1 \quad v = 0$



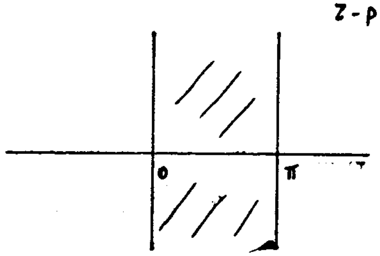
NOW $x = \pi/2,\ y > 0 \longrightarrow u = 0$, $v = -\sinh y < 0$ | $y = 0;\ 0 < x < \pi/2$ $\rightarrow v = 0 \quad u = \cos x$

NOW let $y = y_0 \rightarrow \quad u^2/\cosh^2 y_0 + v^2/\sinh^2 y_0 = 1$

$v < 0,\ u > 0.$

IMAGE is the fourth quadrant $u < 0,\ v > 0.$

SECTION 7.2 # 13 c)

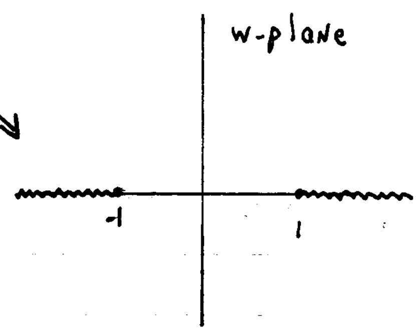


z-plane

$w = \cos z$

$\rightarrow u = \cos x \cosh y$

$v = -\sin x \sinh y$



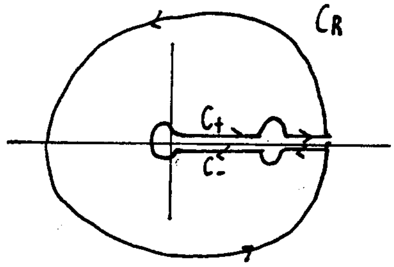
THIS problem was discussed in class.

the image is $\mathbb{C} \setminus \{(-\infty, -1), (1, \infty)\}$.

EXTRA PROBLEM

CALCULATE $\int_0^1 [x(1-x)]^{1/2}\,dx$

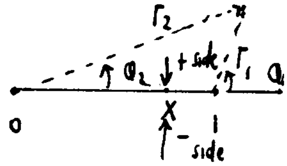
CONSIDER $\int_C [z(1-z)]^{1/2}\,dz$



let $g(z) = [z(1-z)]^{1/2} = i\sqrt{(z)(z-1)}$

NOW CONSTRUCT A BRANCH FOR $g(z)$ between $0 < z < 1$

$(*)$ $g(z) = i\, r_1^{1/2} r_2^{1/2} e^{i(\varphi_1+\varphi_2)/2}$ $\quad r_1 = |z-1| \quad r_2 = |z|$



to put a branch cut between 0 and 1
we take $0 < \varphi_1 < 2\pi$, $0 < \varphi_2 < 2\pi$.

- THEN CONSIDER $z = x$ WITH $0 < x < 1$ evaluated FROM above (+ side).

THEN, $\varphi_1 = \pi$, $\varphi_2 = 0$ $\quad r_1 = 1-x$, $r_2 = x$

$\longrightarrow g(z) = i[x(1-x)]^{1/2} e^{i\pi/2} = -[x(1-x)]^{1/2}$. ↗ FOR C_+

- NOW TAKE $z = x$ WITH $0 < x < 1$ EVALUATED FROM BELOW (− side)

THEN, $\varphi_1 = \pi$, $\varphi_2 = 2\pi$ $\quad r_1 = 1-x$, $r_2 = x$

$g(z) = i[x(1-x)]^{1/2} e^{3\pi i/2} = [x(1-x)]^{1/2}$. ↗ FOR C_-

- FIND BEHAVIOR AS $z \to \infty$

$(+)$ $g(z) = i[z(z-1)]^{1/2} = i\left[z^2(1-1/z)\right]^{1/2} = \pm iz\left(1 - \frac{1}{2z} - \frac{1}{8z^2} + \dots\right)$ $\quad z \to \infty$.

HENCE THE TERM a_{-1}/z IS $a_{-1} = \mp 1/8$.

TO DETERMINE the sign consistent with branch cut, we approach ∞ along positive real axis where $\varphi_1 = \varphi_2 = 0 \longrightarrow g(z) \sim ix$.

FOR this to agree with (+) we need + sign so $g(z) \sim iz\left(1 - \frac{1}{2z} + \frac{1}{8z^2}\right.$

Thus $a_{-1} = -i/8$.

Now in the usual way

$$\int_{C_+} + \int_{C_-} + \lim_{R\to\infty}\int_{C_R} = 0$$

$$-\int_0^1 [x(1-x)]^{1/2}\,dx + \int_1^0 [x(1-x)]^{1/2}\,dx = -\lim_{R\to\infty}\int_{C_R} g(z)\,dz$$

$$\rightarrow\ -2\int_0^1 [x(1-x)]^{1/2}\,dx = -\lim_{R\to\infty} a_{-1}\int_0^{2\pi}\frac{1}{Re^{i\varphi}}\,iRe^{i\varphi}\,d\varphi = -2\pi i\,a_{-1} = +2\pi i$$

$$\rightarrow\ \int_0^1 [x(1-x)]^{1/2}\,dx = \pi/8 \quad \checkmark\checkmark$$