# Vortex rings for the Gross-Pitaevskii equation in $\mathbb{R}^3$

Tai-chia Lin

*Department of Mathematics, National Taiwan University, Taipei, Taiwan 106, China.*
*Email: tclin@math.ntu.edu.tw*

Juncheng Wei

*Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong, China.*
*Email: wei@math.cuhk.edu.hk*

Jun Yang[1]

*College of mathematics and computational sciences, Shenzhen University, Nanhai Ave 3688, Shenzhen, China, 518060.*
*Email: jyang@szu.edu.cn*

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**Abstract**

By rigorous mathematical method, we construct traveling wave solutions with a stationary or traveling vortex ring to the Gross-Pitaevskii equation

$$i\hat{u}_t = \varepsilon^2 \triangle \hat{u} + \Big(\tilde{V} - |\hat{u}|^2\Big)\hat{u}, \quad \hat{u} \in H^1(\mathbb{R}^3),$$

where the unknown function $\hat{u}$ is defined as $\hat{u} : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$, $\varepsilon$ is a small positive parameter and $\tilde{V}$ is a smooth potential.

*Keywords:* Vortex Ring, Bose-Einstein Condensates, Traveling Wave

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## 1. Introduction

In this paper, we consider the existence of traveling wave solutions with vortex rings to the nonlinear schrödinger type problem

$$i\hat{u}_t = \varepsilon^2 \triangle \hat{u} + \Big(\tilde{V} - |\hat{u}|^2\Big)\hat{u}, \quad \hat{u} \in H^1(\mathbb{R}^3), \tag{1.1}$$

where the unknown function $\hat{u}$ is defined as $\hat{u} : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$, $\triangle$ is the Laplace operator in $\mathbb{R}^3$, $\varepsilon$ is a small positive parameter and $\tilde{V}$ is a smooth potential. The equation (1.1) called Gross-Pitaevskii equation[49] is a well-known mathematical model to describe Bose-Einstein condensates.

Vortex flow is one of the fundamental types of fluid and gas motion. The most spectacular form, called concentrated vortices, is characterized by local circulation of fluid around a core[3]. Among all vortical structures, vortex rings with closed-loop cores are perhaps the most familiar to our daily experience such as the well-known smoke rings of cigarettes and the vortex rings observed in the wakes of aircraft. The simplicity of their generation and observation sparked interest of many researchers in mechanics and physics for more than a century[53], [2]. Quantized vortices have gained major interest in the past few years due to the experimental realization of Bose-Einstein

*Corresponding author: Jun Yang, jyang@szu.edu.cn, Tel: 86-755-26534582, Fax: 86-755-26538959

condensates (cf. [5]). Vortices in Bose-Einstein condensates are quantized, and their size, origin, and significance are quite different from those in normal fluids since they exemplify superfluid properties (cf.[20], [6], [7]).

In addition to the simpler two-dimensional point vortices, two types of individual topological defects in three-dimensional Bose-Einstein condensates have focused attention of the scientific community in recent years: vortex lines[59, 56, 26] and vortex rings. Quantized vortex rings with cores have proved to exist when charged particles are accelerated through superfluid helium[50]. The achievements of quantized vortices in a trapped Bose-Einstein condensate [60], [44], [43] have suggested the possibility of producing vortex rings in ultracold atoms. The existence and dynamics of vortex rings in a trapped Bose-Einstein condensates have been studied by several authors [4], [30], [31], [21], [51], [25], [52], [29]. Vortex ring and their two-dimensional analogy(vortex-antivortex pair) have played an important role in the study of complex quantized structures such as superfluid turbulence and so attracted much attention [7], [6], [36], [28]. The reader can refer to the review papers [22], [24], [7] for more details of quantized vortices in physical works.

In this paper, we concern the construction of vortex rings by rigorous mathematical method. We are looking for a traveling wave solution to problem (1.1) in form

$$\hat{u}(\tilde{y}, t) \,=\, e^{i\nu_\varepsilon t}\, \tilde{u}\Big(\tilde{y}_1,\, \tilde{y}_2,\, \tilde{y}_3 - \kappa\, \varepsilon^2 |\log \varepsilon| t\Big),$$

which also has a vortex ring. Here $\kappa$ and $\nu_\varepsilon$ are two constants to be determined latter (c.f. (1.10), (1.15) and (1.19)). Then $\tilde{u}$ is a solution of the nonlinear elliptic problem

$$-\, i\varepsilon^2 |\log \varepsilon| \kappa \,\frac{\partial \tilde{u}}{\partial \tilde{y}_3} \,=\, \varepsilon^2 \triangle \tilde{u} \,+\, \Big(\nu_\varepsilon + \tilde{V}(\tilde{y}) - |\tilde{u}|^2\Big)\tilde{u}, \quad \tilde{u} \in H^1(\mathbb{R}^3). \tag{1.2}$$

Here we have assumed that the trapping potential $\tilde{V}$ is of the form $\tilde{V}(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3 - \kappa\varepsilon^2 |\log \varepsilon| t)$. To prove the existence and describe the profile of a traveling wave solution with vortex ring, we will use the powerful reduction method in partial differential equation theory, other than the formal expansion method in physical works. We believe that our study here also provides a relatively simple and unified approach to more complex vortex structures such as vortex helices and skyrmions.

For the stationary case, i.e. $\kappa = 0$, it becomes the eigenvalue problem

$$\varepsilon^2 \triangle \tilde{u} + \Big(\nu_\varepsilon + \tilde{V}(\tilde{y}) - |\tilde{u}|^2\Big)\tilde{u} = 0, \quad \tilde{u} \in H^1(\mathbb{R}^3), \tag{1.3}$$

where the unknown function $\tilde{u}$ is defined as $\tilde{u} : \mathbb{R}^3 \to \mathbb{C}$, $\varepsilon$ is a small positive parameter and $\tilde{V}$ is a smooth potential. The study of the problem (1.3) in homogeneous case, i.e. $\nu_\varepsilon + \tilde{V} \equiv 1$, on bounded domain with suitable boundary condition started from [8] by F. Bethuel, H. Brezis, F. Helein in 1994, see also the book by K. Hoffmann and Q. Tang[27]. Since then, there is a large pool of literatures on the existence, asymptotic behavior, and dynamical behavior of solutions. We refer to the books [1] and [54] for references and backgrounds. Regarding to the construction of solutions, we mention two papers which are relevant to this paper. F. Pacard and T. Riviere derived a non-variational method to construct solutions with coexisting degrees of +1 and -1 in [48]. The proof is based on a analysis of the linearized operator around an approximation. M. del Pino, M. Kowalczyk and M. Musso [19] derived a reduction method for general existence for vortex solutions under Neumann (or Dirichlet)boundary conditions. The reader can refer to [37]-[39], [40], [57], [61], [16]-[17], [32]-[35], [58] and the references therein.

On the other hand, when $\nu_\varepsilon + \tilde{V} \equiv 1$, there are references on the construction of the traveling wave(i.e. $\kappa \neq 0$) on the whole unbounded domain. In two dimensional plane, F. Bethuel and J. Saut constructed a traveling wave with two vortices of degree $\pm 1$ in [12]. In higher dimension, by minimizing the energy, F. Bethuel, G. Orlandi and D. Smets constructed solutions with a vortex ring[11]. See [15] for another proof by Mountain Pass Lemma and the extension of results in [10]. The reader can refer to the review paper [9] by F. Bethuel, P. Gravejat and J. Saut and the references therein. For a similar existence result of vortex rings for Shrödinger map, F. Lin and J. Wei [41] gave a new proof by using a reduction method.

In this paper, we will prove the existence of solutions possessing vortex rings for the Gross-Pitaevskii equation with inhomogeneous trap potential $\tilde{V}$. Due to the existence of trap potential, we can show the existence of a stationary vortex ring for problem (1.3) by the reduction method in [41]. Then we construct a traveling vortex ring for (1.2).

We first consider the stationary case $\kappa = 0$, i.e. the problem (1.3). We assume that the real function $\tilde{V}$ in (1.3) has the following properties **(A1)-(A3)**.
**(A1):** *$\tilde{V}$ is a symmetric function with the form*

$$\tilde{V}(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3) = \tilde{V}\big(\tilde{r}, \tilde{y}_3\big) = \tilde{V}\big(\tilde{r}, -\tilde{y}_3\big) \quad \textit{with } \tilde{r} = \sqrt{\tilde{y}_1^2 + \tilde{y}_2^2}\,.$$

**(A2):** *There is a point $\tilde{r}_0$ such that the following solvability condition holds*

$$\frac{\partial \tilde{V}}{\partial \tilde{r}}\Big|_{(\tilde{r}_0, 0)} + \frac{d}{\tilde{r}_0} = 0. \tag{1.4}$$

*Here $d$ is a positive constant defined by (c.f. (6.3))*

$$d \equiv \frac{1}{\pi}\int_{\mathbb{R}^2} w(|s|)w'(|s|)\frac{1}{|s|}\,\mathrm{d}s > 0, \tag{1.5}$$

*where $w$ is defined by (2.1). We also assume that $\tilde{r}_0$ is non-degenerate in the sense that*

$$\frac{\partial^2 \tilde{V}}{\partial \tilde{r}^2}\Big|_{(\tilde{r}_0, 0)} - \frac{d}{\tilde{r}_0^2} \neq 0. \tag{1.6}$$

**Remark 1.1.** *There are some works on the dynamics of vortex line with the action of trapped potential, base on formal expansion. In fact, A. Svidzinsky and A. Fetter [59] showed that the vortex velocity has the form*

$$\mathcal{V} \sim -(T \times \bigtriangledown \tilde{V} + kB)A(\varepsilon) + ...$$

*where $T$ and $B$ are tangent vector and binormal of the vortex line. $k$ is the curvature of the vortex line. For more details, the reader can refer to [59] and the references therein. Here we want the stationary vortex ring is trapped by the potential $\tilde{V}$, so we impose the condition (1.4) because of the symmetry.* □

We will construct a solution to (1.3) with a vortex ring, characterized by the curve

$$\sqrt{\tilde{y}_1^2 + \tilde{y}_2^2} = \tilde{r}_0 + \tilde{f} \equiv \tilde{r}_{1\varepsilon}, \quad \tilde{y}_3 = 0, \tag{1.7}$$

where $\tilde{f}$ is a parameter of order $O(\varepsilon)$ to be determined in the reduction procedure.
**(A3):** *There exists a number $\tilde{r}_{2\varepsilon}$ with $\tilde{r}_{2\varepsilon} - \tilde{r}_{1\varepsilon} = \tau_0 + O(\varepsilon)$ such that the following conditions*

$$1 + \Big(\tilde{V}(\tilde{r}, \tilde{y}_3) - \tilde{V}(\tilde{r}_{1\varepsilon}, 0)\Big) = 0, \quad \tilde{V}'(\tilde{r}, \tilde{y}_3) < 0, \quad \tilde{V}''(\tilde{r}, \tilde{y}_3) \leq 0, \tag{1.8}$$

*hold along the circle* $\sqrt{\tilde r^2+\tilde y_3^2}=\tilde r_{2\varepsilon}$*. In (1.8),* $\tau_0$ *is a universal positive constant independent of* $\varepsilon$ *and the derivatives were taken with respect to the out normal of the circle* $\sqrt{\tilde r^2+\tilde y_3^2}=\tilde r_{2\varepsilon}$*. We also assume that*

$$
\begin{aligned}
1\,+\,\Big[\tilde V(\tilde r,\tilde y_3)-\tilde V(\tilde r_{1\varepsilon},0)\Big]&\geq c_1, \quad \text{if } \sqrt{\tilde r^2+\tilde y_3^2}\in\big(0,\,\tilde r_{2\varepsilon}-\tau_1\big),\\
1\,+\,\Big[\tilde V(\tilde r,\tilde y_3)-\tilde V(\tilde r_{1\varepsilon},0)\Big]&\leq -c_2, \quad \text{if } \sqrt{\tilde r^2+\tilde y_3^2}\in(\tilde r_{2\varepsilon}+\tau_2,+\infty),
\end{aligned}
\tag{1.9}
$$

*for some positive constants* $c_1,c_2,\tau_1$ *and* $\tau_2$*.*

**Remark 1.2.** *A typical form of* $\tilde V$ *in physical model is the harmonic type, see [59]*

$$\tilde V(\tilde y_1,\tilde y_2,\tilde y_3)=-\tilde y_1^2-\tilde y_2^2-\tilde y_3^2.$$

*It is obvious that this special* $\tilde V$ *possess the properties in* **(A1)-(A3)**. □

By setting

$$\nu_\varepsilon=1-\tilde V(\tilde r_{1\varepsilon},0), \tag{1.10}$$

to problem (1.3) and then defining $V(\tilde r,\tilde y_3)=\tilde V(\tilde r,\tilde y_3)-\tilde V(\tilde r_{1\varepsilon},0)$, we shall consider the following problem

$$\varepsilon^2\bigtriangleup\tilde u+\Big(1+V(\tilde r,\tilde y_3)-|\tilde u|^2\Big)\tilde u=0, \quad \tilde u\in H^1(\mathbb R^3). \tag{1.11}$$

Here the new potential $V$ possesses the properties:

$$\frac{\partial V}{\partial\tilde y_3}\Big|_{(\tilde r,0)}=0, \qquad \frac{\partial V}{\partial\tilde r}\Big|_{(0,\tilde y_3)}=0, \qquad V(\tilde r_{1\varepsilon},0)=0, \qquad \frac{\partial V}{\partial\tilde r}\Big|_{(\tilde r_0,0)}+\frac{d}{\tilde r_0}=0, \tag{1.12}$$

and also

$$1\,+\,V(\tilde r,\tilde y_3)=0,$$

along the circle $\sqrt{\tilde r^2+\tilde y_3^2}=\tilde r_{2\varepsilon}$.

The main object of this paper is to construct a solution to problem (1.11) with a stationary vortex ring approaching the circle $(\tilde r_0,0)$ in the $(\tilde r,\tilde y_3)$ coordinates. The result reads:

**Theorem 1.3.** *For* $\varepsilon$ *sufficiently small, there exists an axially symmetric solution to problem (1.11) in the form* $u=u(|\tilde y'|,\tilde y_3)\in C^\infty(\mathbb R^3,\mathbb C)$ *with a stationary vortex ring of degree* $+1$ *locating at the circle* $(|\tilde y'|,\tilde y_3)=(a_\varepsilon,0)$*, where* $a_\varepsilon\sim\tilde r_0$*. More precisely, the solution* $u$ *posses the following asymptotic profile*

$$
u(\tilde y_1,\tilde y_2,\tilde y_3)\sim\left\{
\begin{array}{ll}
w\Big(\frac{\tilde\rho}{\varepsilon}\Big)\,e^{i\varphi_0^+}, & \tilde y\in\mathcal D_2=\{\tilde\rho<\tau_0\},\\[2ex]
\sqrt{1+V(\tilde r,\tilde y_3)}\,e^{i\varphi_0^+}, & \tilde y\in\mathcal D_1=\{\tilde\ell<\tilde r_{2\varepsilon}-\tau_1\}\setminus\mathcal D_2,\\[2ex]
\delta_\varepsilon^{1/3}q\Big(\delta_\varepsilon^{1/3}\,\frac{\tilde\ell-\tilde r_{2\varepsilon}}{\varepsilon}\Big)\,e^{i\varphi_0^+}, & \tilde y\in\mathcal D_3=\{\tilde\ell>\tilde r_{2\varepsilon}-\tau_1\},
\end{array}\right.
$$

*where we have denoted*

$$\tilde\rho=\sqrt{\tilde y_1^2+\tilde y_2^2+\tilde y_3^2-\tilde r_{1\varepsilon}^2}, \qquad \tilde\ell=\sqrt{\tilde y_1^2+\tilde y_2^2+\tilde y_3^2}\,, \qquad \delta_\varepsilon=-\varepsilon\frac{\partial V}{\partial\tilde\ell}\Big|_{(\tilde r_{2\varepsilon},0)}>0,$$

*and* $\varphi_0^+(\tilde y_1,\tilde y_2,\tilde y_3)=\varphi_0^+(\tilde r,\tilde y_3)$ *is the angle argument of the vector* $(\tilde r-\tilde r_{1\varepsilon},\tilde y_3)$ *in the* $(\tilde r,\tilde y_3)$ *plane. Here* $q$ *is the function defined by Lemma 2.4.* □

**Remark 1.4.** *Due to the assumption* **(A3)**, *in the region* $\mathcal{D}_1$ *we use the classical Thomas-Fermi approximation to describe the wave function. The reader can refer to then monograph [49] for more discussions. For the asymptotic behavior of* $u$ *in* $\mathcal{D}_3$, *there are also some formal expansions in physical works such as [42] and [23]. Here we use* $q$ *in Lemma 2.4 to describe the profile beyond the Thomas-Fermi approximation.* $\square$

We now consider the problem (1.2) for the case $\kappa \neq 0$. We assume that the real function $\tilde{V}$ in (1.2) has the following two properties **(P2)** and **(P3)**, as well as **(A1)**.
**(P2):** *There is a point* $\hat{r}_0$ *such that*

$$\frac{\partial \tilde{V}}{\partial \tilde{r}}\Big|_{(\hat{r}_0,0)} + \frac{d}{\hat{r}_0} \neq 0 \quad \text{and} \quad \frac{\partial \tilde{V}}{\partial \tilde{r}}\Big|_{(\hat{r}_0,0)} < 0, \tag{1.13}$$

*where* $d$ *is a positive constant defined in (1.5). We also assume that* $\hat{r}_0$ *is non-degenerate in the sense that*

$$\frac{\partial^2 \tilde{V}}{\partial \tilde{r}^2}\Big|_{(\hat{r}_0,0)} - \frac{d}{\hat{r}_0^2} \neq 0. \tag{1.14}$$

Then we set the parameter $\kappa$ by the relation (c.f. (7.35))

$$\frac{\partial \tilde{V}}{\partial \tilde{r}}\Big|_{(\hat{r}_0,0)} + \frac{d}{\hat{r}_0} = \kappa\, d. \tag{1.15}$$

We assume that the vortex ring is characterized by the curve

$$\sqrt{\tilde{y}_1^2 + \tilde{y}_2^2} = \hat{r}_0 + \hat{f} \equiv \hat{r}_{1\varepsilon}, \quad \tilde{y}_3 = 0, \tag{1.16}$$

where $\hat{f}$ is a parameter of order $O(\varepsilon)$ to be determined in the reduction procedure.
**(P3):** *There also exists a number* $\hat{r}_{2\varepsilon}$ *with* $\hat{r}_{2\varepsilon} - \hat{r}_{1\varepsilon} = \hat{\tau}_0 + O(\varepsilon)$ *such that the following conditions*

$$1 + \Big(\tilde{V}(\tilde{r}, \tilde{y}_3) - \tilde{V}(\hat{r}_{1\varepsilon}, 0)\Big) = 0, \quad \tilde{V}'(\tilde{r}, \tilde{y}_3) < 0, \quad \tilde{V}''(\tilde{r}, \tilde{y}_3) \leq 0, \tag{1.17}$$

*hold along the circle* $\sqrt{\tilde{r}^2 + \tilde{y}_3^2} = \hat{r}_{2\varepsilon}$. *In the above,* $\hat{\tau}_0$ *is a universal positive constant independent of* $\varepsilon$ *and the derivatives were taken with respect to the outer normal of the circle* $\sqrt{\tilde{r}^2 + \tilde{y}_3^2} = \hat{r}_{2\varepsilon}$. *We also assume that*

$$\begin{aligned} 1 + \Big[\tilde{V}(\tilde{r}, \tilde{y}_3) - \tilde{V}(\hat{r}_{1\varepsilon}, 0)\Big] &\geq \hat{c}_1, \quad \text{if } \sqrt{\tilde{r}^2 + \tilde{y}_3^2} \in \big(0,\, \hat{r}_{2\varepsilon} - \hat{\tau}_1\big), \\ 1 + \Big[\tilde{V}(\tilde{r}, \tilde{y}_3) - \tilde{V}(\hat{r}_{1\varepsilon}, 0)\Big] &\leq -\hat{c}_2, \quad \text{if } \sqrt{\tilde{r}^2 + \tilde{y}_3^2} \in (\hat{r}_{2\varepsilon} + \hat{\tau}_2, +\infty), \end{aligned} \tag{1.18}$$

*for some positive constants* $\hat{c}_1, \hat{c}_2, \hat{\tau}_1$ *and* $\hat{\tau}_2$.

By setting

$$\nu_\varepsilon = 1 - \tilde{V}(\hat{r}_{1\varepsilon}, 0), \tag{1.19}$$

to problem (1.2) and then defining $\check{V}(\tilde{r}, \tilde{y}_3) = \tilde{V}(\tilde{r}, \tilde{y}_3) - \tilde{V}(\hat{r}_{1\varepsilon}, 0)$, we shall consider the following problem

$$\varepsilon^2 \triangle \tilde{u} + \Big(1 + \check{V}(\tilde{r}, \tilde{y}_3) - |\tilde{u}|^2\Big)\tilde{u} + i\varepsilon^2 |\log \varepsilon| \kappa \frac{\partial \tilde{u}}{\partial \tilde{y}_3} = 0, \quad \tilde{u} \in H^1(\mathbb{R}^3). \tag{1.20}$$

Here the new potential $\check{V}$ possesses the properties:

$$\frac{\partial \check{V}}{\partial \tilde{y}_3}\Big|_{(\tilde{r},0)} = 0, \quad \frac{\partial \check{V}}{\partial \tilde{r}}\Big|_{(0,\tilde{y}_3)} = 0, \quad \check{V}(\hat{r}_{1\varepsilon}, 0) = 0, \quad \frac{\partial \check{V}}{\partial \tilde{r}}\Big|_{(\hat{r}_0,0)} + \frac{d}{\hat{r}_0} = \kappa\, d, \tag{1.21}$$

and also

$$1 + \check{V}(\tilde{r}, \tilde{y}_3) = 0,$$

along the circle $\sqrt{\tilde{r}^2 + \tilde{y}_3^2} = \hat{r}_{2\varepsilon}$.

The main object of the last section is to construct a solution to problem (1.20) with a traveling vortex ring approaching the circle $(\hat{r}_0, 0)$ in the $(\tilde{r}, \tilde{y}_3)$ coordinates.

**Theorem 1.5.** *For $\varepsilon$ sufficiently small, there exists an axially symmetric solution of problem (1.20) with the form $u = u(|\tilde{y}'|, \tilde{y}_3) \in C^\infty(\mathbb{R}^3, \mathbb{C})$ possessing a traveling vortex ring of degree $+1$ locating at $(|\tilde{y}'|, \tilde{y}_3) = (\hat{a}_\varepsilon, 0)$, where $\hat{a}_\varepsilon \sim \hat{r}_0$. The profile of $u$ is the same as the solution in Theorem 1.3.* □

**Remark 1.6.** *In both theorems, the solutions we have constructed satisfy*

$$\int_{\mathbb{R}^3} \left(|\nabla \tilde{u}|^2 + |\tilde{u}|^2\right) < +\infty. \tag{1.22}$$

*Thus the asymptotic behavior of the solutions is quite different from those constructed with constant trapping potential ([11]). The reason for this is clear: because of the trapping potential, there exists a vortexless solution satisfying (1.22). Outside the vortex our solutions behaves like this vortexless solution. A major difficulty (or problem) is the matching of vortex solution with vortexless solution.* □

The remaining part of this paper is devoted to the complete proof of Theorem 1.3 and Theorem 1.5. The organization is as follows: in section 2, we give some preliminary results. Sections 3-6 are devoted to the proof of Theorem 1.3, with arguments on details, while we sketch the similar proof for Theorem 1.5 in Section 7.

## 2. Preliminaries

By $(\ell, \varphi)$ designating the usual polar coordinates $s_1 = \ell \cos\varphi$, $s_2 = \ell \sin\varphi$, we introduce the standard vortex block solution

$$U_0(s_1, s_2) = w(\ell) e^{i\varphi}, \tag{2.1}$$

with degree $+1$ in the whole plane, where $w(\ell)$ is the unique solution of the problem

$$w'' + \frac{1}{\ell} w' - \frac{1}{\ell^2} w + (1 - |w|^2) w = 0 \quad \text{for } \ell \in (0, +\infty), \quad w(0) = 0, \quad w(+\infty) = 1. \tag{2.2}$$

The properties of the function $w$ are stated in the following lemma.

**Lemma 2.1.** *There hold the following properties:*
*(1) $w(0) = 0, \quad w'(0) > 0, \quad 0 < w(\ell) < 1, \quad w'(\ell) > 0$ for all $\ell > 0$,*
*(2) $w(\ell) = 1 - \frac{1}{2\ell^2} + O(\frac{1}{\ell^4})$ for large $\ell$,*
*(3) $w(\ell) = k\ell - \frac{k}{8}\ell^3 + O(\ell^5)$ for $\ell$ close to $0$,*
*(4) Define $T = \frac{\mathrm{d}w}{\mathrm{d}\ell} - \frac{w}{\ell}$, then $T < 0$ in $(0, +\infty)$.*

**Proof.** Partial proof of this lemma can be found in [14] and the references therein. □

We introduce the bilinear form

$$\mathcal{B}(\phi,\phi)=\int_{\mathbb{R}^2}|\bigtriangledown\phi|^2\,-\,\int_{\mathbb{R}^2}(1-w^2)|\phi|^2\,+\,2\int_{\mathbb{R}^2}|\mathrm{Re}(\bar{U}_0\phi)|^2, \tag{2.3}$$

defined in the natural space $\mathcal{H}$ of all locally-$H^1$ functions with

$$||\phi||_{\mathcal{H}}=\int_{\mathbb{R}^2}|\bigtriangledown\phi|^2\,-\,\int_{\mathbb{R}^2}(1-w^2)|\phi|^2\,+\,2\int_{\mathbb{R}^2}|\mathrm{Re}(\bar{U}_0\phi)|^2<+\infty. \tag{2.4}$$

Let us consider, for a given $\phi$, its associated $\psi$ defined by the relation

$$\phi=iU_0\psi. \tag{2.5}$$

Then we decompose $\psi$ by the form

$$\psi=\psi_0(\ell)+\sum_{m\geq1}\Big[\psi_m^1+\psi_m^2\Big], \tag{2.6}$$

where we have denoted

$$\begin{aligned}\psi_0&=\psi_{01}(\ell)+i\psi_{02}(\ell),\\ \psi_m^1&=\psi_{m1}^1(\ell)\cos(m\vartheta)+i\psi_{m2}^1(\ell)\sin(m\vartheta),\\ \psi_m^2&=\psi_{m1}^2(\ell)\sin(m\vartheta)+i\psi_{m2}^2(\ell)\cos(m\vartheta).\end{aligned}$$

This bilinear form is non-negative, as it follows from various results in [8, 13, 45, 46, 55], see also [18, 47]. The nondegeneracy of $U_0$ is contained in the following lemma, whose proof can be found in the appendix of [19].

**Lemma 2.2.** *There exists a constant $C>0$ such that if $\phi\in\mathcal{H}$ decomposes like in (2.5)-(2.6) with $\psi_0\equiv0$, and satisfies the orthogonality conditions*

$$\mathrm{Re}\int_{B(0,1/2)}\bar{\phi}\,\frac{\partial U_0}{\partial s_l}=0,\quad l=1,2,$$

*then there holds*

$$\mathcal{B}(\phi,\phi)\geq C\int_{\mathbb{R}^2}\frac{|\phi|^2}{1+\ell^2}.$$

□

The linear operator $L_0$ corresponding to the bilinear form $\mathcal{B}$ can be defined by

$$L_0(\phi)=\Big(\frac{\partial^2}{\partial s_1^2}+\frac{\partial^2}{\partial s_2^2}\Big)\phi+(1-|w|^2)\phi-2\mathrm{Re}\big(\bar{U}_0\phi\big)U_0.$$

The nondegeneracy of $U_0$ can be also stated as following lemma, whose proof can be found in [18].

**Lemma 2.3.** *Suppose that $L_0[\phi]=0$ with $\phi\in\mathcal{H}$, then*

$$\phi=c_1\frac{\partial U_0}{\partial s_1}+c_2\frac{\partial U_0}{\partial s_2}, \tag{2.7}$$

*for some real constants $c_1,c_2$.* □

To construct approximate solution in Section 3, we also prepare the following lemma.

**Lemma 2.4.** *There exists a unique solution $q$ to the following problem*

$$q'' - q\big(\ell + q^2\big) = 0 \quad on\ \mathbb{R}, \tag{2.8}$$

*such that the properties hold*

$$q(\ell) > 0 \quad for\ all\ \ell \in \mathbb{R}, \qquad q'(\ell) < 0 \quad for\ any\ \ell > 0,$$
$$q(\ell) \sim \exp\left(-\ell^{3/2}\right) \quad as\ \ell \to +\infty, \qquad q(\ell) \sim \sqrt{-\ell} \quad as\ \ell \to -\infty.$$

**Proof.** We first prove the existence by sub-super solution method. To this end, we set

$$q_1(\ell) = \delta \exp(-4\ell - \ell^2/2).$$

By choosing $\delta$ such that $11 - 4\delta^2 e^{16} = 0$, we obtain

$$\begin{aligned} q_1'' - q_1\big(\ell + q_1^2\big) \ &= \ \delta \exp(-4\ell - \ell^2/2)\Big[\ell^2 + 7\ell + 15 - \delta^2 \exp(-8\ell - \ell^2)\Big] \\ &\geq \ \delta \exp(-4\ell - \ell^2/2)\Big[\frac{11}{4} - \delta^2 e^{16}\Big] = 0. \end{aligned}$$

We also let $q_2(\ell) = \exp\left(-\ell/2\right)$. Then it is easy to derive that

$$q_2'' - q_2\big(\ell + q_2^2\big) \ = \ \exp(-\ell/2)\Big[\frac{1}{4} - \ell - \exp(-\ell)\Big] \leq 0.$$

Note that $q_1$ is a subsolution, while $q_2$ is a supersolution with the property $q_1 \leq q_2$. By standard elliptic theory, there exists a positive solution to (2.8) with the property

$$q(\ell) \to 0 \text{ as } \ell \to +\infty, \quad q(\ell) \to \infty \text{ as } \ell \to -\infty.$$

It is easy to check that $q'(\ell) < 0$ as $\ell \to +\infty$. Moreover, there exists a sequence of $\{\ell_n\}_n$ approaching $-\infty$ such that $q'(\ell_n) < 0$. Now, we claim that $q'(\ell) < 0$ for all $\ell \in \mathbb{R}$. Suppose it is not the case, i.e. $q'(\ell_0) > 0$ for some point $\ell_0$. Then there exists an interval $(a, b)$ such that

$$q'(a) = q'(b) = 0, \ q'(\ell) > 0 \text{ for } \ell \in (a, b).$$

We further define the function $\mathfrak{g} = q'/q$ on $(a, b)$. There still holds

$$\mathfrak{g}(a) = \mathfrak{g}(b) = 0, \ \mathfrak{g}(\ell) > 0 \text{ for } \ell \in (a, b).$$

Hence, $\mathfrak{g}$ attains a local positive maximum at $\ell_* \in (a, b)$ with $\mathfrak{g}'(\ell_*) = 0$. By using the inequality

$$q''' - q'\big(\ell + q^2\big) = 2q^2 q' + q > 0 \quad \text{on } (a, b),$$

we have $\mathfrak{g}''(\ell_*) > 0$. It is a contradiction with the fact that $\ell_*$ is a local maximum point of $\mathfrak{g}$.

Let $\bar{q}_1$ and $\bar{q}_2$ be two solutions of (2.8). We claim that $\bar{q}_1 = \bar{q}_2$. In fact, the previous argument shows that either $\bar{q}_1 > \bar{q}_2$ or $\bar{q}_1 < \bar{q}_2$. Without loss of generality, we assume that $\bar{q}_1 > \bar{q}_2$. Let $v = \bar{q}_1 - \bar{q}_2$. Then there holds

$$v'' - v\big(\ell + \bar{q}_1^2 + \bar{q}_1\bar{q}_2 + \bar{q}_2^2\big) = 0.$$

Now we also consider $\bar{v} = \frac{v}{\bar{q}_1}$. There also hold

$$\bar{v}'' + \frac{2\bar{q}_1'}{\bar{q}_1}\bar{v}' - \bar{v}\big(\bar{q}_1\bar{q}_2 + \bar{q}_2^2\big) = 0. \tag{2.9}$$

It is easy to see that $\bar{v}(\ell) \to 0$ as $\ell \to -\infty$. So $\bar{v}' > 0$ and $\bar{v} \to C_0$ as $\ell \to +\infty$. From (2.9), we have $\bar{v}'' > 0$. It contradicts with the fact $\bar{v}'(\ell) \to 0$ as $\ell \to \pm\infty$. The uniqueness is also proved. $\square$

## 3. Outline of the proof and approximate solutions

By using the symmetry, we will first transfer the problem (1.11) to a two dimensional case in form (3.3)-(3.4) and then give an outline of the proof for Theorem 1.3. The main object of this section will focus on the construction a good approximate solution in a suitable form and then estimate its error.

*3.1. The reduction of the problem*

Making rescaling $\tilde{y} = \varepsilon\hat{y}$, problem (1.11) takes the form

$$\triangle u + \Big(1 + V(\varepsilon\hat{y}) - |u|^2\Big)u = 0. \tag{3.1}$$

Introduce a new coordinates $(r, \theta, \hat{y}_3) \in (0, +\infty) \times (0, 2\pi] \times \mathbb{R}$ as the form

$$\hat{y}_1 = r\cos\theta, \quad \hat{y}_2 = r\sin\theta, \quad \hat{y}_3 = \hat{y}_3.$$

Then problem (3.1) takes the form

$$S[u] \,=\, \Big(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial \hat{y}_3^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\Big)u + \Big(1 + V(\varepsilon r, \varepsilon\hat{y}_3) - |u|^2\Big)u = 0. \tag{3.2}$$

In this paper, we want to construct a solution with a vortex ring, which does not depend on the variable $\theta$. Hence, we consider a two-dimensional problem, for $(x_1, x_2) \in \mathbb{R}^2$

$$S[u] \equiv \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)u + \Big(1 + V(\varepsilon|x_1|, \varepsilon x_2) - |u|^2\Big)u = 0, \tag{3.3}$$

with Neumann boundary condition

$$\frac{\partial u}{\partial x_1}(0, x_2) = 0, \quad |u| \to 0 \text{ as } |x| \to +\infty. \tag{3.4}$$

**Notations:** *For further convenience, we have used $x_1, x_2$ to denote $r, \hat{y}_3$ in the above equations, and also $x = (x_1, x_2)$, $\ell = |x|$ in the sequel. In this rescaled coordinates, we write*

$$r_{1\varepsilon} = \tilde{r}_{1\varepsilon}/\varepsilon = \tilde{r}_0/\varepsilon + f \quad \textit{with } f = \tilde{f}/\varepsilon, \qquad r_{2\varepsilon} = \tilde{r}_{2\varepsilon}/\varepsilon, \tag{3.5}$$

*where the constants $\tilde{f}$, $\tilde{r}_{1\varepsilon}$ and $\tilde{r}_{2\varepsilon}$ are defined in (1.7) and (1.8). By setting, $\xi_+ = (r_{1\varepsilon}, 0)$ and $\xi_- = (-r_{1\varepsilon}, 0)$, we also introduce the translated variable*

$$s = x - \xi_+ \quad \textit{or} \quad s = x - \xi_-, \tag{3.6}$$

*in a small neighborhood of the vortices. We will use these notations without any further statement in the sequel.* □

To handle the influence of the potential, we here look for vortex ring solutions vanishing as $|x|$ approaching $+\infty$. As we stated in (1.7), we assume that the vortex ring is characterized by the curve, in the original coordinates $\hat{y} = (\hat{y}_1, \hat{y}_2, \hat{y}_3)$

$$\sqrt{\hat{y}_1^2 + \hat{y}_2^2} = r_0/\varepsilon + f = r_{1\varepsilon}, \quad \hat{y}_3 = 0. \tag{3.7}$$

In other words, in the two dimensional situation with $(x_1, x_2)$ coordinates, we will construct a vortex with degree $+1$ at $(r_{1\varepsilon}, 0)$ and its anti-pair with degree $-1$ at $(-r_{1\varepsilon}, 0)$.

Finally, we decompose the operator as

$$S[u] \equiv S_0[u] + S_1[u], \tag{3.8}$$

with the explicit form

$$S_0[u] \,\equiv\, \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)u, \quad S_1[u] \,\equiv\, \Big(1 + V(\varepsilon|x_1|, \varepsilon x_2) - |u|^2\Big)u. \tag{3.9}$$

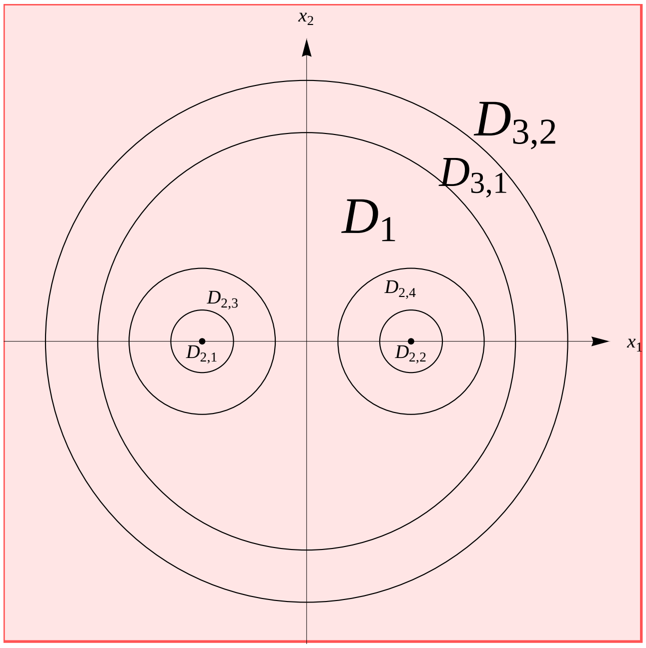


Figure 1: Decomposition of Domain: $D_2 = \cup_{j=1}^4 D_{2,j}$, $D_3 = D_{3,1} \cup D_{3,2}$. The components of $D_2$ center at $\xi_+$ or $\xi_-$.

*3.2. Outline of the Proof*

To construct a solution to (3.3)-(3.4) and prove the result in Theorem 1.3, the first step is to construct an approximate solution, denoted by $u_2$ in (3.34), possessing a pair of vortices with degree $\pm 1$ locating at $\xi_+ = (r_{1\varepsilon}, 0)$ and $\xi_- = (-r_{1\varepsilon}, 0)$. The heuristic method is to find suitable approximations in different regions and then patch them together. So we decompose the plane into different regions $D_1, D_2, D_3$ as in (3.13), see Figure 1. Note that the components of $D_2$ center at $\xi_+$ or $\xi_-$. The first approximation $u_1$ to a solution has a profile of a pair of standard vortices in $D_2$, which possess the degrees $\pm 1$ and centers $\xi_+$ and $\xi_-$, see (3.14). Then in $D_1$ we set $u_1$ by Thomas-Fermi approximation in form (3.15) and make a trivial extension to the region $D_3$.

Now there are two types of singularities caused by the phase term of standard vortices and the Thomas-Fermi approximation, which will be described in subsection 3.3. In fact, to cancel the singularity caused by $\frac{1}{x_1}\frac{\varphi_0}{\partial x_1}$ with the standard phase $\varphi_0$ in (3.14) we here add one more correction term $\varphi_1$ in (3.21) to the pase component as the work [41]. Moreover, by some type of rescaling, in $D_3$ we use $q$ in Lemma 2.4 as a bridge when $|x|$ crossing $r_{2\varepsilon}$ and then reduce the norm of the approximate solution to zero as $|x|$ tends to $\infty$. Finally we get the approximate solution $u_2$ in (3.34), which has the symmetry

$$u_2(x_1, x_2) = \overline{u_2(x_1, -x_2)}, \qquad u_2(x_1, x_2) = u_2(-x_1, x_2). \tag{3.10}$$

These are done in subsections 3.3 and 3.4. The subsection 3.5 is devoted to estimation of the errors in suitable weighted norms. The reader can refer to the papers [19] and [41].

To get explicit information of the linearized problem, we then also divide further $D_2$ and $D_3$ into small parts in (4.8), see Figure 1. In section 4, we then express the error and formulate the problem in suitable local forms in different regions by the method in [19]. More precisely, for the perturbation $\psi = \psi_1 + i\psi_2$ with symmetry (4.5), we take the solution $u$ in form (4.4). The key points that we shall mention are the roles of local forms of the linearized problem for further deriving of the linear resolution theory in section 5. In $D_1$, the linear operators have approximate forms, (c.f. (4.12))

$$\tilde{L}_1(\psi_1) \equiv \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_1 + \frac{2}{\beta_1}\,\nabla\beta_1\cdot\nabla\psi_1,$$
$$\bar{L}_1(\psi_2) \equiv \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_2 - 2|u_2|^2\psi_2 + \frac{2}{\beta_1}\,\nabla\beta_1\cdot\nabla\psi_2.$$

The type of the linear operator $\tilde{L}_1$ was handled in [41], while $\bar{L}_1$ is a good operator since $|u_2|$ stays uniformly away from 0 in $D_1$ by the assumption (**A3**), see (4.10). In the vortex core regions $D_{2,1}$ and $D_{2,2}$, we use a type of symmetry (3.10) to deal with the kernel of the linear operator related to the standard vortex. In $D_{3,1}$, the lowest approximations of the linear operators are, (c.f. (4.34))

$$L_{31*}(\psi_1) = \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_1 - \big(z+q^2(z)\big)\psi_1\,,$$
$$L_{31**}(\psi_2) = \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_2 - \big(z+3q^2(z)\big)\psi_2\,.$$

By Lemma 2.4, the facts that $L_{31*}(q)=0$ and $L_{31**}(-q')=0$ with $-q'>0$ and $q>0$ on $\mathbb{R}$ will give the application of maximum principle. The linear operators in the region $D_{3,2}$ can be approximated by a good linear operator of the form, (c.f.(4.39))

$$L_{32*}[\tilde{\psi}] \equiv \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\tilde{\psi} + \big(1+V\big)\tilde{\psi},$$

with $\big(1+V\big)<-c_2<0$ by the assumption (**A3**). For more details, the reader can refer to proof of Lemma 5.1.

After deriving the linear resolution theory by Lemmas 5.1 and 5.2, and then solving the nonlinear projected problem (4.41) in section 5, as the standard reduction method we adjust the parameter $\tilde{f}$ to get a solution with a vortex ring in Theorem 1.3. It is showed in section 6 that this is equivalent to solve the following algebraic equation, (c.f. (6.5))

$$c(\tilde{f}) = -2\,\varepsilon\pi\left[\frac{\partial V}{\partial\tilde{r}}\Big|_{(\tilde{r}_0+\tilde{f},0)}\log\frac{1}{\varepsilon} + \frac{d}{\tilde{r}_0+\tilde{f}}\log\frac{\tilde{r}_0+\tilde{f}}{\varepsilon}\right] + O(\varepsilon), \tag{3.11}$$

where $O(\varepsilon)$ is a continuous function of the parameter $\tilde{f}$. By the solvability condition (1.4) and the non-degeneracy condition (1.6), we can find a zero of $c(\tilde{f})$ at some small $\tilde{f}$ with the help of the simple mean-value theorem.

However, to prove Theorem 1.5 in section 7, we need to solve the equation, (c.f.(7.35))

$$c(\hat{f}) = -2\,\pi\,\varepsilon\left[\frac{\partial\check{V}}{\partial\tilde{r}}\Big|_{(\hat{r}_0+\hat{f},0)}\log\frac{1}{\varepsilon} + \frac{d}{\hat{r}_0+\hat{f}}\log\frac{\hat{r}_0+\hat{f}}{\varepsilon} - d\,\kappa\log\frac{1}{\varepsilon}\right] + O(\varepsilon),$$

where $O(\varepsilon)$ is a continuous function of the parameter $\hat{f}$. By simple mean-value theorem and the solvability condition (1.13) and the non-degeneracy condition (1.14), we can find a zero of $c(\hat{f})$ at some small $\hat{f}$.

*3.3. First approximate solution*

For any given $(x_1, x_2) \in \mathbb{R}^2$, let $\varphi_0^+(x_1, x_2)$ and $\varphi_0^-(x_1, x_2)$ be respectively the angle arguments of the vectors $(x_1 - r_{1\varepsilon}, x_2)$ and $(x_1 + r_{1\varepsilon}, x_2)$ in the $(x_1, x_2)$ plane. We also let

$$\ell_2(x_1, x_2) = \sqrt{(x_1 - r_{1\varepsilon})^2 + x_2^2}, \quad \ell_1(x_1, x_2) = \sqrt{(x_1 + r_{1\varepsilon})^2 + x_2^2} \tag{3.12}$$

be the distance functions between the point $(x_1, x_2)$ and the pair of vortices of degree $\pm 1$ at the points $\xi_+$ and $\xi_-$. In this subsection, we only consider the case for $x_1 > 0$ because of the symmetry of the problem. We decompose the plane into different regions $D_1, D_2$ and $D_3$ in the following form, see Figure 1

$$\begin{aligned} D_2 \ &\equiv \ \Big\{ (x_1, x_2) : \ell_1 < \frac{\tau_0}{\varepsilon} \text{ or } \ell_2 < \frac{\tau_0}{\varepsilon} \Big\}, \\ D_1 \ &\equiv \ \Big\{ (x_1, x_2) : |x| < r_{2\varepsilon} - \frac{\tau_1}{\varepsilon} \Big\} \setminus D_2, \\ D_3 \ &\equiv \ \Big\{ \ (x_1, x_2) : \ |x| > r_{2\varepsilon} - \frac{\tau_1}{\varepsilon} \ \Big\}. \end{aligned} \tag{3.13}$$

Here $\tau_0$ and $\tau_1$ are given in the assumption (**A3**). Recalling the definition of the standard vortex of degree $+1$ in (2.1), then it can be roughly done as follows:

(1) If $(x_1, x_2) \in D_2$, we choose $u_1$ by

$$u_1(x_1, x_2) \ = \ U_2(x_1, x_2) \ \equiv \ w(\ell_2) w(\ell_1) e^{i\varphi_0}, \tag{3.14}$$

where the phase term $\varphi_0$ is defined by $\varphi_0 = \varphi_0^+ \ - \ \varphi_0^-$.

(2) If $(x_1, x_2) \in D_1$,, we write

$$u_1(x_1, x_2) \ = \ U_1(x_1, x_2) \ \equiv \ \sqrt{1 + V(\varepsilon|x_1|, \varepsilon x_2)}\, e^{i\varphi_0}\,. \tag{3.15}$$

The choice of $u_1$ here is well defined due to the assumption (**A3**). Here we use the standard Thomas-Fermi approximation, see [49].

(3) As we have stated that we look for solutions vanishing at infinity, so we heuristically define $u_1 \ = \ \tilde{U}_3 \ \equiv \ 0$ for $(x_1, x_2) \in D_3$,.

For further improvement of the approximation, it is crucial to evaluate the error of this approximation, which will be carried out as follows. Obviously, there hold the trivial formulas

$$\begin{aligned} \nabla_{x_1,x_2} w(\ell_2) \ &= \ \frac{w'(\ell_2)}{\ell_2}\big(x_1 - r_{1\varepsilon},\, x_2\big), \qquad \nabla_{x_1,x_2} w(\ell_1) \ = \ \frac{w'(\ell_1)}{\ell_1}\big(x_1 + r_{1\varepsilon},\, x_2\big), \\ \nabla_{x_1,x_2} \varphi_0(x_1, x_2) \ &= \ \bigg( \frac{-x_2}{(\ell_2)^2} + \frac{x_2}{(\ell_1)^2},\ \frac{x_1 - r_{1\varepsilon}}{(\ell_2)^2} - \frac{x_1 + r_{1\varepsilon}}{(\ell_1)^2} \bigg). \end{aligned} \tag{3.16}$$

As we have stated, we work directly in the half space $\mathbb{R}^2_+ \ = \ \{(x_1, x_2) \ : \ x_1 \ > \ 0\}$ in the sequel because of the symmetry of the problem.

Firstly, we estimate the error near the vortex ring. Note that for $x_1 > 0$, the error between 1 and $w(\ell_1)$ is $(\ell_1)^2$, which is of order $\varepsilon^2$, we may ignore $w(\ell_1)$ in the computations below. Note that

$$\begin{aligned} S_0[U_2] \ = \ & S_0[w(\ell_1)] w(\ell_2) e^{i\varphi_0} \ + \ S_0[w(\ell_2)] w(\ell_1) e^{i\varphi_0} \ + \ 2e^{i\varphi_0} \nabla w(\ell_2) \cdot \nabla w(\ell_1) \\ & - \ U_2 \big|\nabla\varphi_0\big|^2 \ + \ iS_0[\varphi_0] U_2 \ + \ 2ie^{i\varphi_0} \nabla\Big( w(\ell_2) w(\ell_1) \Big) \cdot \nabla\varphi_0. \end{aligned}$$

Then, there holds

$$
\begin{aligned}
& S_0\big[w(\ell_1)\big]\, w(\ell_2)e^{i\varphi_0} \,+\, S_0\big[w(\ell_2)\big]\, w(\ell_1)e^{i\varphi_0} \,-\, U_2\big|\nabla\varphi_0\big|^2 \\
&= \Big[\, w''(\ell_1) \,+\, \frac{1}{\ell_1}w'(\ell_1) \,-\, \frac{1}{(\ell_1)^2}w(\ell_1)\Big]\, \frac{U_2}{w(\ell_1)} \,+\, \frac{x_1-d_1}{x_1\ell_1}w'(\ell_1)\frac{U_2}{w(\ell_1)} \\
&\quad + \Big[\, w''(\ell_2) \,+\, \frac{1}{\ell_2}w'(\ell_2) \,-\, \frac{1}{(\ell_2)^2}w(\ell_2)\Big]\, \frac{U_2}{w(\ell_2)} \,+\, \frac{x_1-d_2}{x_1\ell_2}w'(\ell_2)\frac{U_2}{w(\ell_2)} \\
&\quad -\, 2U_2\frac{x_2^2+(x_1-r_{1\varepsilon})(x_1+r_{1\varepsilon})}{(\ell_1)^2(\ell_2)^2},
\end{aligned}
$$

and also

$$
2e^{i\varphi_0}\nabla w(\ell_2)\cdot\nabla w(\ell_1) \,=\, 2U_2\frac{x_2^2+(x_1-r_{1\varepsilon})(x_1+r_{1\varepsilon})}{\ell_1\ell_2}\,\frac{w'(\ell_1)}{w(\ell_1)}\frac{w'(\ell_2)}{w(\ell_2)} \,=\, O(\varepsilon^2).
$$

Note that $\nabla w(\ell_2)\cdot\nabla\varphi_0^+ = 0$ and $\nabla w(\ell_1)\cdot\nabla\varphi_0^- = 0$. By the formulas in (3.16), we get

$$
\begin{aligned}
& 2ie^{i\varphi_0}\nabla\Big(w(\ell_2)w(\ell_1)\Big)\cdot\nabla\varphi_0 \\
&= 2ie^{i\varphi_0}\nabla w(\ell_2)\cdot\nabla\varphi_0^- \,+\, 2ie^{i\varphi_0}\nabla w(\ell_1)\cdot\nabla\varphi_0^+ \,+\, O(\varepsilon^2) \\
&= -\,4iU_2\frac{x_2r_{1\varepsilon}}{\ell_1(\ell_2)^2}\,\frac{w'(\ell_1)}{w(\ell_1)} \,-\, 4iU_2\frac{x_2r_{1\varepsilon}}{\ell_2(\ell_1)^2}\,\frac{w'(\ell_2)}{w(\ell_2)} \,+\, O(\varepsilon^2) \\
&= O(\varepsilon^2).
\end{aligned}
$$

Recall that

$$
\frac{\partial V}{\partial\tilde{y}_3}\Big|_{(\tilde{r},0)} = 0.
$$

In a small neighborhood of the point $(\tilde{r}_{1\varepsilon},0) = (\varepsilon r_{1\varepsilon},0)$, by Taylor expansion we also write $V(\varepsilon|x_1|,\varepsilon x_2)$ as the form

$$
V(\varepsilon|x_1|,\varepsilon x_2) \,=\, \varepsilon\,\frac{\partial V}{\partial\tilde{r}}\Big|_{(\varepsilon r_{1\varepsilon},0)}(x_1-r_{1\varepsilon}) \,+\, \varepsilon^2\, O(\ell_2^2),
$$

where we have used the assumption (1.12). It is easy to derive that

$$
S_1[U_2] = \Big(1+V-|U_2|^2\Big)U_2 \,=\, \Big(1-|w(\ell_2)|^2\Big)U_2 \,+\, \varepsilon U_2\frac{\partial V}{\partial\tilde{r}}\Big|_{(\varepsilon r_{1\varepsilon},0)}(x_1-r_{1\varepsilon}) \,+\, \varepsilon^2 O(\ell_2^2)U_2.
$$

By using the equation (2.2), the error, near the vortex ring, takes the form

$$
\begin{aligned}
S[U_2] &= U_2\,\frac{x_1-r_{1\varepsilon}}{x_1\ell_2}\,\frac{w'(\ell_2)}{w(\ell_2)} \,+\, iS_0[\varphi_0]\,U_2 \\
&\quad +\, U_2\Big[\varepsilon\frac{\partial V}{\partial\tilde{r}}\Big|_{(\varepsilon r_{1\varepsilon},0)}(x_1-r_{1\varepsilon})+\varepsilon^2 O(\ell_2^2)\Big] \\
&\equiv F_{21} \,+\, F_{22}.
\end{aligned}
\tag{3.17}
$$

In the above, we have denoted the term $E_{21}$ by

$$
F_{21} \,\equiv\, iS_0[\varphi_0]\,U_2 \,=\, iU_2\frac{1}{x_1}\frac{\partial\varphi_0}{\partial x_1} \,=\, -iU_2\frac{4x_2\,r_{1\varepsilon}}{(\ell_2)^2(\ell_1)^2},
\tag{3.18}
$$

which is a singular term. Whence we need a further correction to improve the approximation.

Secondly, we compute the error for $U_1$. There holds

$$
\begin{aligned}
\frac{\partial}{\partial x_1}\sqrt{1+V(\varepsilon|x_1|,\varepsilon x_2)} \;&=\; \frac{\varepsilon}{2}\big(1+V\big)^{-1/2}\,\frac{\partial V}{\partial \tilde r},\\
\frac{\partial^2}{\partial x_1^2}\sqrt{1+V(\varepsilon|x_1|,\varepsilon x_2)} \;&=\; -\,\frac{\varepsilon^2}{4}\big(1+V\big)^{-3/2}\,\Big|\frac{\partial V}{\partial \tilde r}\Big|^2 \;+\; \frac{\varepsilon^2}{2}\big(1+V\big)^{-1/2}\,\frac{\partial^2 V}{\partial \tilde r^2}.
\end{aligned}
\tag{3.19}
$$

It is easy to check that the error of $U_1$ is

$$
S[U_1] \;=\; S_0[U_1].
$$

Note that

$$
\begin{aligned}
S_0[U_1] \;=\; & S_0\big[\sqrt{1+V}\,\big]e^{i\varphi_0} \;+\; 2ie^{i\varphi_0}\nabla\sqrt{1+V}\cdot\nabla\varphi_0 \;-\; \sqrt{1+V}e^{i\varphi_0}\big|\nabla\varphi_0\big|^2\\
& +\; iS_0[\varphi_0]\sqrt{1+V}e^{i\varphi_0}.
\end{aligned}
$$

Hence the error is

$$
\begin{aligned}
S[U_1] \;=\; & -\,\frac14\,\varepsilon^2\,\big|\tilde\nabla V\big|^2\frac{U_1}{\big(1+V\big)^2} \;+\; \frac12\,\varepsilon^2\,\tilde\triangle V\,\frac{U_1}{1+V} \;-\; i\varepsilon\,\frac{U_1}{1+V}\tilde\nabla V\cdot\nabla\varphi_0 \;+\; U_1\big|\nabla\varphi_0\big|^2\\
& +\; \frac12\,\varepsilon\,U_1\,\frac{1}{x_1}\frac{\partial V}{\partial \tilde r}\,\frac{1}{1+V} \;+\; iU_1S_0[\varphi_0]\\
\equiv\; & F_{11}\,,
\end{aligned}
$$

where we have denoted

$$
\tilde\nabla V \;=\; \Big(\frac{\partial V}{\partial \tilde r},\frac{\partial V}{\partial \tilde y_3}\Big), \quad \tilde\triangle V \;=\; \frac{\partial^2 V}{\partial \tilde r^2}+\frac{\partial^2 V}{\partial \tilde y_3^2}.
$$

Note that the following term in the above formula

$$
iS_0[\varphi_0]\,U_1 \;=\; iU_1\frac{1}{x_1}\frac{\partial\varphi_0}{\partial x_1} \;=\; -iU_1\,\frac{4x_2\,r_{1\varepsilon}}{(\ell_2)^2(\ell_1)^2}.
$$

is not a singular term. The condition $\frac{\partial V}{\partial \tilde r}\Big|_{(0,\tilde y_3)}=0$ implies that

$$
\frac{1}{x_1}\frac{\partial V}{\partial \tilde r} = O(\varepsilon).
$$

Moreover, in the region $D_1$, by the assumption (**A3**) we have $V\geq c_1>0$. Note that the formula in (3.16) implies that $|\nabla\varphi_0|$ is of order $O(\varepsilon)$ in $D_1$. Whence, the error is small in the sense that

$$
S[U_1] = O(\varepsilon^2). \tag{3.20}
$$

On the other hand, if $|x|$ is close to $r_{2\varepsilon}$, then $\big(1+V\big)^{-1}$ brings singularity. So we need another correction term to improve the approximation in a neighborhood of the curve $|x|=r_{2\varepsilon}$.

*3.4. Further improvement of approximation*

To handle the singular term $F_{21}=iS_0[\varphi_0]\,U_2$, as the argument in [41], we here introduce a further correction $\varphi_1(x_1,x_2)$ to the phase term in form

$$
\varphi_1 = \varphi_s + \varphi_r. \tag{3.21}
$$

By setting the smooth cut-off function

$$\eta(s)=\left\{\begin{array}{ll}1, & |s|\leq 1/10,\\ 0, & |s|\geq 1/5,\end{array}\right. \tag{3.22}$$

the singular part $\varphi_s$ is defined as

$$\varphi_s=\frac{x_2}{4r_{1\varepsilon}}\,\eta\big(\varepsilon\ell_2\big)\log\frac{(\ell_2)^2}{(\ell_1)^2}. \tag{3.23}$$

Note that the function $\varphi_s$ is continuous but $\bigtriangledown\varphi_s$ is not. The singularity of $\varphi_s$ comes from its derivatives, which will play an important role in the final reduction procedure.

**Remark 3.1.** The reader can also refer to formula (16) in [59] for the formal derivation of general type improvement of the phase term. $\square$

On the other hand, we choose the regular part $\varphi_r$ by solving the equation

$$\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\varphi_r=-\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\big(\varphi_0+\varphi_s\big). \tag{3.24}$$

It can be done as follows. For further references, we first compute:

$$\frac{\partial\varphi_s}{\partial x_1}=\frac{x_2}{4r_{1\varepsilon}}\,\varepsilon\eta'\big(\varepsilon\ell_2\big)\frac{x-r_{1\varepsilon}}{\ell_2}\log\frac{(\ell_2)^2}{(\ell_1)^2}\,+\,\eta\big(\varepsilon\ell_2\big)\frac{x_2\big[x_1^2-x_2^2-(r_{1\varepsilon})^2\big]}{(\ell_2)^2(\ell_1)^2}, \tag{3.25}$$

$$\begin{aligned}\frac{\partial\varphi_s}{\partial x_2}&=\frac{x_2}{4r_{1\varepsilon}}\,\varepsilon\eta'\big(\varepsilon\ell_2\big)\frac{x_2}{\ell_2}\log\frac{(\ell_2)^2}{(\ell_1)^2}\,+\,\frac{1}{4r_{1\varepsilon}}\,\eta\big(\varepsilon\ell_2\big)\log\frac{(\ell_2)^2}{(\ell_1)^2}\\&\quad+\,\eta\big(\varepsilon\ell_2\big)\frac{2x_1x_2^2}{(\ell_2)^2(\ell_1)^2}.\end{aligned} \tag{3.26}$$

Trivial computation gives that, for $(x_1,x_2)\in B_{1/10}(r_{1\varepsilon}\vec{e}_1)$

$$\begin{aligned}\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)[\varphi_0+\varphi_s]&=\frac{4x_2(x_1-r_{1\varepsilon})}{(\ell_2)^2(\ell_1)^2}\,+\,\frac{x_2[x_1^2-x_2^2-(r_{1\varepsilon})^2]}{x_1(\ell_2)^2(\ell_1)^2}\\&=O({x_1}^{-2})\,=\,O(\varepsilon^2).\end{aligned}$$

For $(x_1,x_2)\in\big(B_{1/5}(r_{1\varepsilon}\vec{e}_1)\big)^c$, the error is also $O(\varepsilon^2)$. In fact, for $(x_1,x_2)\in\big(B_{1/5}(r_{1\varepsilon}\vec{e}_1)\big)^c$, we have $\varphi_s=0$ and then

$$\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)[\varphi_0+\varphi_s]\,=\frac{-4x_2\,r_{1\varepsilon}}{(\ell_2)^2(\ell_1)^2}.$$

Going back to the original variables $(\tilde r,\tilde y_3)$ and setting $\hat\varphi(\tilde r,\tilde y_3)=\varphi_r(\tilde r/\varepsilon,\tilde y_3/\varepsilon)$, we see that

$$\Big(\frac{\partial^2}{\partial\tilde r^2}+\frac{\partial^2}{\partial\tilde y_3^2}+\frac{1}{\tilde r}\frac{\partial}{\partial\tilde r}\Big)\hat\varphi\,\leq\,C\Big(1+\tilde r^2+\tilde y_3^2\Big)^{-3/2}.$$

Whence, by solving problem (3.24), we can choose $\varphi_r$ such that there holds

$$\hat\varphi=O\Big(\frac{1}{\sqrt{1+\tilde r^2+\tilde y_3^2}}\Big).$$

Moreover the term $\varphi_r$ is $C^1$-smooth. As a consequence, we have chosen $\varphi_1$ in way such that the singular term $\frac{4x_2\,r_{1\varepsilon}}{(\ell_2)^2(\ell_1)^2}$ has been canceled and moreover there holds

$$\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\varphi_1=-\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\varphi_0. \tag{3.27}$$

Now we shall deal with the singularity as $x$ approaching the circle $|x| = r_{2\varepsilon}$. By the assumption (**A3**), there exists a small positive $\varepsilon_0$ such that for $0 < \varepsilon < \varepsilon_0$

$$\delta_\varepsilon \;=\; -\varepsilon \frac{\partial V}{\partial \tilde{\ell}}\Big|_{(\tilde{r}_{2\varepsilon},0)} > 0, \tag{3.28}$$

where $\tilde{\ell} = \sqrt{\tilde{r}^2 + \tilde{y}_3^2}$. Then for $(\tilde{r}, \tilde{y}_3)$ with foot point $(\tilde{p}_1, \tilde{p}_2)$ on the circle of radius $\tilde{r}_{2\varepsilon}$, there holds

$$\begin{aligned} 1 + V(\tilde{r}, \tilde{y}_3) \;&=\; 1 + V(\tilde{p}_1, \tilde{p}_2) + \frac{\partial V}{\partial \tilde{\ell}}\Big|_{(\tilde{p}_1,\tilde{p}_2)} \varepsilon(\tilde{\ell} - \tilde{r}_{2\varepsilon}) + O\big(\varepsilon^2(\tilde{\ell} - \tilde{r}_{2\varepsilon})^2\big) \\ &=\; -\delta_\varepsilon(\ell - r_{2\varepsilon}) + O\big(\varepsilon^2(\ell - r_{2\varepsilon})^2\big). \end{aligned} \tag{3.29}$$

Let $q$ be the unique solution given by Lemma 2.4. Now we define $\tilde{q}(z) = \delta_\varepsilon^{1/3} q\big(\delta_\varepsilon^{1/3} z\big)$. Then it is easy to check that

$$\tilde{q}_{zz} \;-\; \tilde{q}\big(\delta_\varepsilon z + \tilde{q}^2\big) \;=\; 0. \tag{3.30}$$

In other words, if we choose $\hat{q}(x_1, x_2) = \delta_\varepsilon^{1/3} q\Big(\delta_\varepsilon^{1/3}(\ell - r_{2\varepsilon})\Big)$ with $\ell = \sqrt{x_1^2 + x_2^2}$, then $\hat{q}$ satisfies

$$\Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)\hat{q} \;+\; \Big(1 + V(\varepsilon|x_1|, \varepsilon x_2)\Big)\hat{q} \;-\; \hat{q}^{\,3} \;=\; O\big(\varepsilon^2\big). \tag{3.31}$$

This implies that we can use

$$U_3(x_1, x_2) = \hat{q}(x_1, x_2) e^{i\varphi_0}, \tag{3.32}$$

as an approximation near $r_{2\varepsilon}$.

By defining smooth cut-off functions as follows

$$\tilde{\eta}_2(s) = \begin{cases} 1, & |s| \le \tau_3, \\ 0, & |s| \ge 2\tau_3, \end{cases} \qquad \tilde{\eta}_3(s) = \begin{cases} 1, & s \ge -\tau_4, \\ 0, & s \le -2\tau_4, \end{cases} \tag{3.33}$$

where the exponents $\tau_3, \tau_4$ are small enough in such a way that $\tau_3, \tau_4 < \min\{\tilde{r}_0, \tau_0, \tau_1, \tau_2\}/10$, we choose the cut-off functions by

$$\begin{aligned} \eta_2\big(\varepsilon x_1, \varepsilon x_2\big) \;&=\; \tilde{\eta}_2\big(\varepsilon \ell_1\big) \;+\; \tilde{\eta}_2\big(\varepsilon \ell_2\big), \\ \eta_3\big(\varepsilon x_1, \varepsilon x_2\big) \;&=\; \tilde{\eta}_3\big(\varepsilon(\ell - r_{2\varepsilon})\big), \\ \eta_1\big(\varepsilon x_1, \varepsilon x_2\big) \;&=\; 1 - \eta_2 - \eta_3. \end{aligned}$$

We then choose the final approximate solution to (3.3) by, for $(x_1, x_2) \in \mathbb{R}^2$,

$$u_2(x_1, x_2) \;= \sqrt{1 + V(\varepsilon|x_1|, \varepsilon x_2)}\, \eta_1\, e^{i\varphi} \;+\; w(\ell_2) w(\ell_1)\, \eta_2\, e^{i\varphi} \;+\; \hat{q}(x_1, x_2)\, \eta_3\, e^{i\varphi}, \tag{3.34}$$

where the new phase term $\varphi = \varphi_0 + \varphi_1$. By recalling the definition of $U_1, U_2, U_3$ in (3.15), (3.14) and (3.32), we also write the approximation as

$$u_2 \;= U_1\, \eta_1\, e^{i\varphi_1} \;+\; U_2\, \eta_2\, e^{i\varphi_1} \;+\; U_3\, \eta_3\, e^{i\varphi_1}. \tag{3.35}$$

It is easy to check that $u_2$ has the symmetry

$$u_2(x_1, x_2) = \overline{u_2(x_1, -x_2)}, \qquad u_2(x_1, x_2) = u_2(-x_1, x_2). \tag{3.36}$$

Moreover, there holds

$$\frac{\partial u_2}{\partial x_1}(0, x_2) = 0. \tag{3.37}$$

*3.5. Estimates of the error*

As we have stated, we work directly in the half space $\mathbb{R}^2_+ = \{(x_1, x_2) : x_1 > 0\}$ in the sequel because of the symmetry of the problem. Recalling the definitions of the operators in (3.9), let us start to compute the error:

$$
\begin{aligned}
\mathbb{E} \,=\, S[u_2] \,=\, & S[U_1]\eta_1 e^{i\varphi_1} \,+\, U_1 S_0[\eta_1 e^{i\varphi_1}] \,+\, 2\nabla U_1 \cdot \nabla\Big(\eta_1 e^{i\varphi_1}\Big) \\
& + S[U_2]\eta_2 e^{i\varphi_1} \,+\, U_2 S_0[\eta_2 e^{i\varphi_1}] \,+\, 2\nabla U_2 \cdot \nabla\Big(\eta_2 e^{i\varphi_1}\Big) \\
& + S[U_3]\eta_3 e^{i\varphi_1} \,+\, U_3 S_0[\eta_3 e^{i\varphi_1}] \,+\, 2\nabla U_3 \cdot \nabla\Big(\eta_3 e^{i\varphi_1}\Big) \,+\, \mathbb{N},
\end{aligned}
\tag{3.38}
$$

where the nonlinear term $\mathbb{N}$ is defined by

$$
\mathbb{N} = \eta_1 |U_1|^2 U_1 e^{i\varphi_1} \,+\, \eta_2 |U_2|^2 U_2 e^{i\varphi_1} \,+\, \eta_3 |U_3|^2 U_3 e^{i\varphi_1} \,-\, |u_2|^2 u_2.
\tag{3.39}
$$

The main components in the above formula can be estimated as follows.

Using the equation (3.27), the singular term $F_{21} = iS_0[\varphi_0]\, U_2$ in $S[U_2]$ in is canceled and we then get

$$
S[U_2]\,\eta_2\, e^{i\varphi_1} \,+\, U_2 S_0[\eta_2 e^{i\varphi_1}] \,=\, F_{22}\,\eta_2\, e^{i\varphi_1} \,+\, U_2\, S_0[\eta_2]\, e^{i\varphi_1} \,+\, 2iU_2 \nabla\eta_2 \cdot \nabla\varphi_1 \,-\, U_2\,\eta_2\, e^{i\varphi_1}\, \big|\nabla\varphi_1\big|^2.
$$

Whence, there holds

$$
S[U_2]\,\eta_2\, e^{i\varphi_1} \,+\, U_2 S_0[\eta_2 e^{i\varphi_1}] \,= F_{22}\eta_2\, e^{i\varphi_1} \,+\, \varepsilon^2 O(|\ell_2|^2).
$$

The formulas in (3.25)-(3.26) imply that

$$
\begin{aligned}
2\nabla U_2 \cdot \nabla\Big(\eta_2 e^{i\varphi_1}\Big) \,=\, & 2\eta_2 U_2 e^{i\varphi_1} \frac{4x_1 x_2\, r_{1\varepsilon}}{(\ell_1\ell_2)^2}\, \frac{x_2\big[x_1^2 - x_2^2 - (r_{1\varepsilon})^2\big]}{(\ell_2)^2(\ell_1)^2} \\
& + 2i\eta_2 U_2 e^{i\varphi_1} \frac{x_1 - r_{1\varepsilon}}{\ell_2}\, \frac{w'(\ell_2)}{w(\ell_2)} \frac{x_2\big[x_1^2 - x_2^2 - (r_{1\varepsilon})^2\big]}{(\ell_2)^2(\ell_1)^2} \\
& + 2i\eta_2 U_2 e^{i\varphi_1} \frac{x_2}{\ell_2}\, \frac{w'(\ell_2)}{w(\ell_2)} \frac{2x_1 x_2^2}{(\ell_2)^2(\ell_1)^2} \\
& - 4\eta_2 U_2 e^{i\varphi_1} \frac{r_{1\varepsilon}\big[x_1^2 - x_2^2 - (r_{1\varepsilon})^2\big]}{(\ell_1\ell_2)^2} \frac{1}{4r_{1\varepsilon}} \log\frac{(\ell_2)^2}{(\ell_1)^2} \\
& + 2i\eta_2 U_2 e^{i\varphi_1} \frac{x_2}{\ell_2}\, \frac{w'(\ell_2)}{w(\ell_2)} \frac{1}{4r_{1\varepsilon}} \log\frac{(\ell_2)^2}{(\ell_1)^2} \,+\, O(\varepsilon) \\
=\, & 2\eta_2 U_2 e^{i\varphi_1} \frac{(x_1 + r_{1\varepsilon})(x_1 - r_{1\varepsilon})}{(\ell_1\ell_2)^2} \log r_{1\varepsilon} \\
& - i\eta_2 U_2 e^{i\varphi_1} \frac{x_2}{\ell_2}\, \frac{w'(\ell_2)}{w(\ell_2)} \frac{1}{r_{1\varepsilon}} \log r_{1\varepsilon} \,+\, O(\varepsilon \log \ell_2).
\end{aligned}
\tag{3.40}
$$

It is worth to mention that, in the vortex-core region

$$
D_2 \,=\, \Big\{(x_1, x_2) : \ell_1 < \frac{\tau_0}{\varepsilon} \text{ or } \ell_2 < \frac{\tau_0}{\varepsilon}\Big\},
$$

we estimate the error by

$$
\begin{aligned}
\mathbb{E} \,=\, & U_2\eta_2 e^{i\varphi_1} \left[ \frac{x_1 - r_{1\varepsilon}}{x_1 \ell_2} \frac{w'(\ell_2)}{w(\ell_2)} \,+\, \varepsilon(x_1 - r_{1\varepsilon}) \frac{\partial V}{\partial \tilde{r}}\Big|_{(\varepsilon r_{1\varepsilon}, 0)} \right] \\
& + \eta_2 U_2 e^{i\varphi_1} \frac{2(x_1 + r_{1\varepsilon})(x_1 - r_{1\varepsilon})}{(\ell_1\ell_2)^2} \log r_{1\varepsilon} \\
& - i\eta_2 U_2 e^{i\varphi_1} \frac{x_2}{\ell_2}\, \frac{w'(\ell_2)}{w(\ell_2)} \frac{1}{r_{1\varepsilon}} \log r_{1\varepsilon} \,+\, O(\varepsilon \log \ell_2).
\end{aligned}
\tag{3.41}
$$

The singularity of the last formula will play an important role in the final reduction step.

We then consider the error in the region

$$D_1 \;=\; \Big\{(x_1,x_2) : |x| < r_{2\varepsilon} - \frac{\tau_1}{\varepsilon}\Big\} \setminus D_2.$$

From the relation (3.27), there holds

$$\eta_1 e^{i\varphi_1} S[U_1] \,+\, U_1 S_0[\eta_1 e^{i\varphi_1}] \;=\; \eta_1\, e^{i\varphi_1}\, E_{11} \,+\, U_1\, S_0[\eta_1]\, e^{i\varphi_1} \,+\, 2iU_1 e^{i\varphi_1}\nabla\eta_1\cdot\nabla\varphi_1 \\ -\, U_1\,\eta_1\, e^{i\varphi_1}\big|\nabla\varphi_1\big|^2.$$

In this region, $|\bigtriangledown\varphi_0| = O(\varepsilon)$ and $|\bigtriangledown\varphi_1| = O(\varepsilon)$. Whence, by using (3.20), we obtain

$$\eta_1 e^{i\varphi_1} S[U_1] \,+\, U_1 S_0[\eta_1 e^{i\varphi_1}] = \eta_1 e^{i\varphi_1} U_1 O(\varepsilon^2).$$

Using the formulas (3.25)-(3.26), we obtain

$$\begin{aligned}
2\nabla U_1\cdot\nabla\Big(\eta_1 e^{i\varphi_1}\Big) \;&=\; 2\eta_1 i U_1 e^{i\varphi_1}\left[\frac{1}{2}\big(1+V\big)^{-1}\varepsilon\,\frac{\partial V}{\partial\tilde r} \,-\, i\frac{4x_1x_2\, r_{1\varepsilon}}{(\ell_2)^2(\ell_1)^2}\right] \\
&\qquad\times\left(\frac{x_2}{4r_{1\varepsilon}}\,\varepsilon\eta'\big(\varepsilon\ell_2\big)\frac{x_1-r_{1\varepsilon}}{\ell_2}\log\frac{(\ell_2)^2}{(\ell_1)^2}\right. \\
&\qquad\qquad\left. +\,\eta\big(\varepsilon\ell_2\big)\frac{x_2\big(x_1^2-x_2^2-(r_{1\varepsilon})^2\big)}{(\ell_2)^2(\ell_1)^2} \,+\, O(\varepsilon)\right) \\
&\quad+\,2\eta_1 U_1 e^{i\varphi_1}\left[\frac{1}{2}\big(1+V\big)^{-1}\varepsilon\,\frac{\partial V}{\partial\tilde y_3} - \frac{2r_{1\varepsilon}\big[{x_1}^2-x_2^2-(r_{1\varepsilon})^2\big]}{(\ell_2)^2(\ell_1)^2}\right] \\
&\qquad\times\left[\frac{x_2}{4r_{1\varepsilon}}\,\varepsilon\eta'\big(\varepsilon\ell_2\big)\frac{x_2}{\ell_2}\log\frac{(\ell_2)^2}{(\ell_1)^2} \,+\, \frac{1}{4r_{1\varepsilon}}\,\eta\big(\varepsilon\ell_2\big)\log\frac{(\ell_2)^2}{(\ell_1)^2}\right. \\
&\qquad\qquad\left. +\,\eta\big(\varepsilon\ell_2\big)\frac{2x_1x_2^2}{(\ell_2)^2(\ell_1)^2} \,+\, O(\varepsilon)\right] \\
&\quad+\,2U_1 e^{i\varphi_1}\left[\frac{1}{2}\big(1+V\big)^{-1}\varepsilon\,\frac{\partial V}{\partial\tilde r} \,-\, i\frac{4x_1x_2\, r_{1\varepsilon}}{(\ell_2)^2(\ell_1)^2}\right]\times\frac{\partial\eta_1}{\partial x_1} \\
&\quad+\,2U_1 e^{i\varphi_1}\left[\frac{1}{2}\big(1+V\big)^{-1}\varepsilon\,\frac{\partial V}{\partial\tilde y_3} - \frac{2r_{1\varepsilon}\big[{x_1}^2-x_2^2-(r_{1\varepsilon})^2\big]}{(\ell_2)^2(\ell_1)^2}\right]\times\frac{\partial\eta_1}{\partial x_2} \\
&=\;\eta_1 U_1 e^{i\varphi_1} O(\varepsilon^2).
\end{aligned}$$

Whence we conclude that, in $D_1$, the error is estimated by

$$\mathbb{E} \;=\; \eta_1 e^{i\varphi_1} U_1 O(\varepsilon^2).$$

In the region

$$D_3 \;=\; \Big\{(x_1,x_2) : |x| > r_{2\varepsilon} - \frac{\tau_1}{\varepsilon}\;\Big\}.$$

we finally compute the error of $U_3$

$$S[U_3] \;=\; S_0[U_3] \,+\, S_1[U_3],$$

where

$$S_0[U_3] \;=\; \delta_\varepsilon\, q''\Big(\delta_\varepsilon^{1/3}(\ell-r_{2\varepsilon})\Big)\, e^{i\varphi_0} \,+\, 2ie^{i\varphi_0}\nabla\hat q\cdot\nabla\varphi_0 \,-\, \hat q\, e^{i\varphi_0}\big|\nabla\varphi_0\big|^2 \,+\, iS_0[\varphi_0]\,\hat q\, e^{i\varphi_0}.$$

We also write $S_1[U_3]$ of the form

$$S_1[U_3] \,=\, \delta_\varepsilon\Big[\delta_\varepsilon^{1/3}(\ell - r_{2\varepsilon}) - q^2\Big(\delta_\varepsilon^{1/3}(\ell - r_{2\varepsilon})\Big)\Big]\, q\Big(\delta_\varepsilon^{1/3}(\ell - r_{2\varepsilon})\Big)e^{i\varphi_0}$$
$$+\,\Big[(1+V) - \delta_\varepsilon(\ell - r_{2\varepsilon})\Big]\hat{q}e^{i\varphi_0}.$$

The equation of $q$ in Lemma 2.4 implies that there holds

$$S[U_3] \,=\, 2ie^{i\varphi_0}\nabla\hat{q}\cdot\nabla\varphi_0 \,-\, \hat{q}\,e^{i\varphi_0}\big|\nabla\varphi_0\big|^2 \,+\, iS_0[\varphi_0]\,\hat{q}\,e^{i\varphi_0} \,+\, \Big[(1+V) - \delta_\varepsilon(\ell - r_{2\varepsilon})\Big]\hat{q}e^{i\varphi_0},$$
$$\equiv F_{31} \,+\, F_{32},$$

where the term $F_{31}$ of the form

$$F_{31} \,\equiv\, iS_0[\varphi_0]\,\hat{q}\,e^{i\varphi_0}.$$

is also not a singular term. Using the equation (3.31) and the similar computations as before, we now obtain

$$\eta_3\, e^{i\varphi_1} S[U_3] \,+\, U_3 S_0[\eta_3 e^{i\varphi_1}] \,=\, \eta_3 e^{i\varphi_1}F_{32} \,+\, \eta_3 e^{i\varphi}O(\varepsilon^2) \,=\, \eta_3 e^{i\varphi}O(\varepsilon^2).$$

In the above, we have use the relation (3.27). Hence, there holds

$$\mathbb{E} \,=\, \eta_3 e^{i\varphi}O(\varepsilon^2).$$

For a function $h = h_1 + ih_2$ with real functions $h_1, h_2$, define a norm of the form

$$||h||_{**} \equiv \sum_{j=1}^{2} ||iu_2 h||_{L^p(\ell_j<3)} \,+\, \sum_{j=1}^{2}\Big[\,||\ell_j^{2+\sigma}h_1||_{L^\infty(\tilde{D})} \,+\, ||\ell_j^{1+\sigma}h_2||_{L^\infty(\tilde{D})}\,\Big]$$
$$+\,\sum_{j=1}^{2}||h_j||_{L^p(D_3)}\,, \tag{3.42}$$

where we have denoted

$$\tilde{D} = D_2 \cup D_1 \setminus \{\ell_1 < 3 \text{ or } \ell_2 < 3\}, \tag{3.43}$$

for $\ell_1$ and $\ell_2$ defined in (3.12). As a conclusion, we have the following lemma.

**Lemma 3.2.** *There holds for $\ell_1 > 2$ and $\ell_2 > 2$*

$$\big|\mathrm{Re}(\mathbb{E})\big| \,\le\, \frac{C\varepsilon^{1-\sigma}}{(1+\ell_1)^3} + \frac{C\varepsilon^{1-\sigma}}{(1+\ell_2)^3},$$
$$\big|\mathrm{Im}(\mathbb{E})\big| \,\le\, \frac{C\varepsilon^{1-\sigma}}{(1+\ell_1)^{1+\sigma}} + \frac{C\varepsilon^{1-\sigma}}{(1+\ell_2)^3},$$
$$\big\|\mathbb{E}\big\|_{L^p(\{\ell_1<3\}\cup\{\ell_2<3\})} \,\le\, C\varepsilon|\log\varepsilon|,$$

*where $\sigma \in (0,1)$ is a constant. As a consequence, there also holds*

$$||\mathbb{E}||_{**} \le C\varepsilon^{1-\sigma}.$$

□

## 4. Local setting-up of the Problem

We look for a solution $u = u(x_1, x_2)$ to problem (3.3)-(3.4) in the form of small perturbation of $u_2$, with additional symmetry:

$$u(x_1, x_2) = \bar{u}(x_1, -x_2). \tag{4.1}$$

Let $\tilde{\chi} : \mathbb{R} \to \mathbb{R}$ be a smooth cut-off function defined by

$$\tilde{\chi}(s) = \begin{cases} 1, & s \leq 1, \\ 0, & s \geq 2. \end{cases} \tag{4.2}$$

Recalling (3.33)-(3.35) and setting the components of the approximation $u_2$ as

$$v_1(x_1, x_2) \,=\, \eta_1\, U_1\, e^{i\varphi_1}, \quad v_2(x_1, x_2) \,=\, \eta_2\, U_2\, e^{i\varphi_1}, \quad v_3(x_1, x_2) \,=\, \eta_3\, U_3\, e^{i\varphi_1}, \tag{4.3}$$

we want to choose the ansatz of the form

$$u = \Big[\chi\big(v_2 + i v_2 \psi\big) + \big(1-\chi\big)(v_1 + v_2)e^{i\psi}\Big] \,+\, \Big[v_3 + i\eta_3 e^{i\varphi}\psi\Big], \tag{4.4}$$

where $\chi(x_1, x_2) = \tilde{\chi}(\ell_2) + \tilde{\chi}(\ell_1)$. The above nonlinear decomposition of the perturbation was first introduced in [19], see also [41].

To find the perturbation terms, the main object of this section is to write the equation for the perturbation as a linear one with a right hand side given by a lower order nonlinear term. The symmetry imposed on $u$ can be transmitted to the symmetry on the perturbation terms

$$\psi(x_1, -x_2) = -\overline{\psi(x_1, x_2)}, \quad \psi(x_1, x_2) = \psi(-x_1, x_2). \tag{4.5}$$

This type of symmetry will play an important role in our further arguments. Let us observe that

$$\begin{aligned} u &= \Big[(v_1 + v_2) + i(v_1 + v_2)\psi + \big(1-\chi\big)(v_1 + v_2)\big(e^{i\psi} - 1 - i\psi\big)\Big] \,+\, \Big(v_3 + \eta_3 e^{i\varphi}\psi\Big) \\ &= u_2 \,+\, i(v_1 + v_2)\psi \,+\, \eta_3 e^{i\varphi}\psi \,+\, \Gamma \end{aligned}$$

where we have denoted

$$\Gamma = \big(1-\chi\big)(v_1 + v_2)\big(e^{i\psi} - 1 - i\psi\big). \tag{4.6}$$

A direct computation shows that $u$ is a solution to problem (3.3)-(3.4) if and only if

$$\begin{aligned} & i(v_1 + v_2)S_0[\psi] \,+\, 2i \bigtriangledown (v_1 + v_2) \cdot \bigtriangledown\psi \,+\, i\Big[\,1 + V - |u_2|^2\,\Big](v_1 + v_2)\psi \\ & +\, iS_0[v_1 + v_2]\psi \,-\, 2\mathrm{Re}\big(\bar{u}_2 i(v_1 + v_2)\psi\big)u_2 \,+\, \eta_3 e^{i\varphi} S_0[\psi] \\ & +\, \eta_3 e^{i\varphi}\Big[\,1 + V - |u_2|^2\,\Big]\psi \,-\, 2\eta_3 \mathrm{Re}\big(\bar{u}_2 e^{i\varphi}\psi\big)u_2 \\ & +\, 2 \bigtriangledown \Big[\,\eta_3 e^{i\varphi}\Big] \cdot \bigtriangledown\psi \,+\, S_0\Big[\,\eta_3 e^{i\varphi}\Big]\psi \,=\, -E + N, \end{aligned} \tag{4.7}$$

where the error term $E$ is defined as $\mathbb{E}$ in (3.38) and $N$ is the nonlinear operator defined by

$$\begin{aligned} N \,=\, & -\, S_0\big[\Gamma\big] \,-\, \big(1 + V - |u_2|^2\big)\Gamma \\ & +\, \Big[2\mathrm{Re}\big(\bar{u}_2 i(v_1 + v_2)\psi\big) + 2\eta_3 \mathrm{Re}(\bar{u}_2 e^{i\varphi}\psi)\Big] \times \Big(i(v_1 + v_2)\psi + \eta_3 e^{i\varphi}\psi + \Gamma\Big) \\ & +\, \Big[2\mathrm{Re}(\bar{u}_2\Gamma) + \big|i(v_1 + v_2)\psi + \eta_3 e^{i\varphi}\psi + \Gamma\big|^2\Big] \times \Big(u_2 + i(v_1 + v_2)\psi + \eta_3 e^{i\varphi}\psi + \Gamma\Big). \end{aligned}$$

We shall explicitly write the equation in suitable local forms and then analyze the property of the corresponding linear operators, which will be done in the following.

Before going further, we first give some notations, see Figure 1. By recalling the notation $\ell_1$ and $\ell_2$ in (3.12), and also $D_1, D_2, D_3$ in (3.13), we set

$$\begin{aligned} D_{2,1} &\equiv \big\{(x_1,x_2): \ell_1<1\big\}, \qquad & D_{2,2} &\equiv \big\{(x_1,x_2): \ell_2<1\big\}, \\ D_{2,3} &\equiv \Big\{(x_1,x_2): \ell_1<\frac{\tau_0}{\varepsilon}\Big\}\setminus D_{2,1}, \qquad & D_{2,4} &\equiv \Big\{(x_1,x_2): \ell_2<\frac{\tau_0}{\varepsilon}\Big\}\setminus D_{2,2}, \\ D_{3,1} &\equiv \Big\{\,(x_1,x_2): r_{2\varepsilon}-\frac{\tau_1}{\varepsilon}<|x|<r_{2\varepsilon}+\frac{\tau_2}{\varepsilon}\,\Big\}, \qquad & D_{3,2} &\equiv \Big\{(x_1,x_2): |x|>r_{2\varepsilon}+\frac{\tau_2}{\varepsilon}\,\Big\}. \end{aligned} \tag{4.8}$$

Here $\tau_0$, $\tau_1$, $\tau_2$, $r_{1\varepsilon}$ and $r_{2\varepsilon}$ are given in the assumption (**A3**).

In the region $D_1$ far from the vortex core region, directly from the form of the ansatz $u=u_2e^{i\psi}$ with the approximation as

$$u_2(x_1,x_2) \;=\; \sqrt{1+V}\,\eta_1\, e^{i(\varphi_0+\varphi_1)} \;+\; w(\ell_2)w(\ell_1)\,\eta_2\, e^{i(\varphi_0+\varphi_1)},$$

we see that the equation takes the simple form

$$\begin{aligned} L_1(\psi) &\equiv \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi+2\frac{\bigtriangledown u_2}{u_2}\cdot\bigtriangledown\psi-2i|u_2|^2\psi_2 \\ &= E_1 \;-\; i(\bigtriangledown\psi)^2 \;+\; i|u_2|^2\big(1-e^{-2\psi_2}+2\psi_2\big), \end{aligned}$$

where $E_1=i\,\mathbb{E}/u_2$. We intend next to describe in more accurate form the equation above. Let us also write

$$u_2=e^{i\varphi}\beta_1 \quad \text{with } \beta_1=\sqrt{1+V}\,\eta_1\;+\;w(\ell_2)w(\ell_1)\,\eta_2.$$

For $|x|<r_{2\varepsilon}-2\tau_1/\varepsilon$, there holds,

$$u_2=\beta_1e^{i\varphi}=\sqrt{1+V}\,e^{i\varphi}, \tag{4.9}$$

and hence, by using the assumption (**A**3), we have

$$|u_2|^2=1+V>1. \tag{4.10}$$

Direct computation also gives that

$$\begin{aligned} 2\frac{\bigtriangledown u_2}{u_2}\cdot\bigtriangledown\psi &= \frac{2}{\beta_1}\bigtriangledown\beta_1\cdot\bigtriangledown\psi_1-2\bigtriangledown\varphi\cdot\bigtriangledown\psi_2+i\frac{2}{\beta_1}\bigtriangledown\beta_1\cdot\bigtriangledown\psi_2+2i\bigtriangledown\varphi\cdot\bigtriangledown\psi_1 \\ &= (A_1,0)\cdot\bigtriangledown\psi_1-(A_2,B_2)\cdot\bigtriangledown\psi_2+i(A_1,0)\cdot\bigtriangledown\psi_2+i(A_2,B_2)\cdot\bigtriangledown\psi_1, \end{aligned}$$

where $A_1=O(\varepsilon|\log\varepsilon|)$, $A_2=O(\varepsilon)$, $B_2=O(\varepsilon)$. For $r_{2\varepsilon}-2\tau_1/\varepsilon<|x|<r_{2\varepsilon}-\tau_1/\varepsilon$, similar estimates hold. The equations become

$$\tilde{L}_1(\psi_1)=\tilde{E}_1\;+\;\tilde{N}_1, \qquad \bar{L}_1(\psi_2)=\bar{E}_1\;+\;\bar{N}_1. \tag{4.11}$$

In the above, we have denoted the linear operators by

$$\begin{aligned} \tilde{L}_1(\psi_1) &\equiv \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_1\;+\;\frac{2}{\beta_1}\bigtriangledown\beta_1\cdot\bigtriangledown\psi_1, \\ \bar{L}_1(\psi_2) &\equiv \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_2\;-\;2|u_2|^2\psi_2\;+\;\frac{2}{\beta_1}\bigtriangledown\beta_1\cdot\bigtriangledown\psi_2. \end{aligned} \tag{4.12}$$

The nonlinear operators are

$$\tilde N_1 \;=\; -\,2 \bigtriangledown \varphi \cdot \bigtriangledown \psi_2 \;+\; 2 \bigtriangledown \psi_1 \cdot \bigtriangledown \psi_2,$$
$$\bar N_1 \;=\; 2 \bigtriangledown \varphi \cdot \bigtriangledown \psi_1 \;+\; |u_2|^2 \big(1 - e^{-2\psi_2} + 2\psi_2\big) \;+\; |\bigtriangledown \psi_1|^2 \;-\; |\bigtriangledown \psi_2|^2.$$

Consider the linearization of the problem on the vortex-core region $D_{2,1} \cup D_{2,2}$. Here we only argue in the region $D_{2,2} = \big\{(x_1, x_2) : \ell_2 < 1\big\}$. It is more convenient to do this in the translated variable $(s_1, s_2) = (x_1 - r_{1\varepsilon}, x_2)$ and then denote $\ell = \ell_2$ for brevity of notation. Now the term $\psi$ is small, however possibly unbounded near the vortex. Whence, in the sequel, by setting

$$\tilde\phi = i v_2 \psi \quad \text{with } \psi = \psi_1 + i\psi_2, \tag{4.13}$$

we shall require that $\tilde\phi$ is bounded (and smooth) near the vortices. We shall write the equation in term of a type of the function $\tilde\phi$ for $\ell < \delta/\varepsilon$. In the region $D_{2,2}$, let us write $u_2$, i.e. $v_2$, as the form

$$v_2 = \beta U_0 \quad \text{ with } \beta = w(\ell_1) e^{-i\varphi_0^- + i\varphi_1}, \tag{4.14}$$

where $U_0$, $\varphi_0^-$ and $\varphi_1$ are defined in (2.1), (3.12) and (3.21). We define the function

$$\phi(s) = iU_0 \psi \quad \text{for } |s| < \delta/\varepsilon, \tag{4.15}$$

namely

$$\tilde\phi = \beta\phi. \tag{4.16}$$

Hence, in the translated variable, the ansatz becomes in this region

$$u_2 \;=\; \beta(s) U_0 \;+\; \beta(s)\phi \;+\; (1-\chi)\beta(s) U_0 \Big( e^{\phi/U_0} - 1 - \frac{\phi}{U_0}\Big). \tag{4.17}$$

We also call $\Gamma_{2,2} = (1-\chi)U_0\Big(e^{\phi/U_0} - 1 - \frac{\phi}{U_0}\Big)$. The support of this function is contained in set $|s| > 1$. In this vortex-core region, the problem, written in $(s_1, s_2)$ coordinates, can be stated as

$$L_{2,2}(\phi) = E_{2,2} + N_{2,2}. \tag{4.18}$$

Let us consider the linear operator defined in the following way: for $\phi$ and $\psi$ linked through formula (4.15) we set

$$\begin{aligned} L_{2,2}(\phi) \;=&\, L_0(\phi) \;+\; \frac{1}{s_1 + r_{1\varepsilon}} \frac{\partial}{\partial s_1}\phi \;+\; 2\big(1 - |\beta|^2\big) \mathrm{Re}\big(\bar U_0 \phi\big) U_0 \\ &+\; \left[ \varepsilon \frac{\partial V}{\partial \tilde r}\Big|_{(r_{1\varepsilon} + \vartheta s_1, 0)} + 1 - |\beta|^2 \right]\phi \;+\; 2\frac{\bigtriangledown\beta}{\beta}\cdot\bigtriangledown\phi \;+\; \chi\frac{E_{2,2}}{U_0}\phi, \end{aligned} \tag{4.19}$$

where $\vartheta$ is a small constant. Here we also have defined $L_0$ as

$$L_0(\phi) = \Big(\frac{\partial^2}{\partial s_1^2} + \frac{\partial^2}{\partial s_2^2}\Big)\phi + (1 - |w|^2)\phi - 2\mathrm{Re}\big(\bar U_0 \phi\big) U_0.$$

Here, by writing the error $\mathbb{E}$ in the translated variable $s$, the error $E_{2,2}$ is given by

$$E_{2,2} = \mathbb{E}/\beta. \tag{4.20}$$

Observe that, in the region $D_{2,2}$, the error $E_{2,2}$ takes the expression

$$
\begin{aligned}
E_{2,2} \;=\; & w(\ell_2)\, e^{i\varphi_0^+} \left[ \frac{x_1 - r_{1\varepsilon}}{x_1 \ell_2} \frac{w'(\ell_2)}{w(\ell_2)} \;+\; \varepsilon \, \frac{\partial V}{\partial \tilde{r}}\Big|_{(\varepsilon r_{1\varepsilon},0)} (x_1 - r_{1\varepsilon}) \right] \\
& +\; w(\ell_2)\, e^{i\varphi_0^+} \, \frac{2(x_1 + r_{1\varepsilon})(x_1 - r_{1\varepsilon})}{(\ell_1 \ell_2)^2} \log r_{1\varepsilon} \\
& -\; i w(\ell_2)\, e^{i\varphi_0^+} \, \frac{x_2}{\ell_2} \, \frac{w'(\ell_2)}{w(\ell_2)} \frac{1}{r_{1\varepsilon}} \log r_{1\varepsilon} \;+\; O(\varepsilon \log \ell_2),
\end{aligned}
\tag{4.21}
$$

while the nonlinear term is given by

$$
\begin{aligned}
N_{2,2}(\phi) = & -\; \frac{\triangle(\beta\, \Gamma_{2,2})}{\beta} \;+\; \Big(1 + V - |U_0|^2\Big)\Gamma_{2,2} \;-\; 2|\beta|^2\, \mathrm{Re}(\bar{U}_0 \phi)\big(\phi + \Gamma_{2,2}\big) \\
& -\; \Big( 2|\beta|^2 \mathrm{Re}\big(\bar{U}_0 \Gamma_{2,2}\big) + |\beta|^2 \big|\phi + \Gamma_{2,2}\big|^2 \Big) \big(U_0 + \phi + \Gamma_{2,2}\big) \;+\; (\chi - 1)\, \frac{E_{2,2}}{U_0} \phi.
\end{aligned}
\tag{4.22}
$$

Taking into account to the explicit form of the function $\beta$ we get

$$
\triangledown \beta = O(\varepsilon), \quad \triangle \beta = O(\varepsilon^2), \quad |\beta| \sim 1 + O(\varepsilon^2), \tag{4.23}
$$

provided that $|s| < \delta/\varepsilon$. With this in mind, we see that the linear operator is a small perturbation of $L_0$.

In the region $D_{2,4}$ far from the vortex core, directly from the form of the ansatz $u = \big(1-\chi\big) u_2 e^{i\psi}$, we see that, for $\ell_2 > 2$, the equation takes the simple form

$$
\begin{aligned}
L_{2,4}(\psi) \;\equiv\; & \Big( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1} \frac{\partial}{\partial x_1} \Big) \psi + 2 \frac{\triangledown u_2}{u_2} \cdot \triangledown \psi - 2i|u_2|^2 \psi_2 \\
\;=\; & E_{2,4} \;-\; i(\triangledown \psi)^2 \;+\; i|u_2|^2 \big(1 - e^{-2\psi_2} + 2\psi_2\big),
\end{aligned}
\tag{4.24}
$$

where $E_{2,4} = i\mathbb{E}/u_2$. We intend next to describe in more accurate form the equation above. As before, let us also write

$$
u_2 = \beta U_0 \quad \text{ with } \beta = \; w(\ell_1) e^{-i\varphi_0^- + i\varphi_1}. \tag{4.25}
$$

where $U_0$, $\varphi_0^+$ and $\varphi_1$ are defined in (2.1), (3.12) and (3.21). For $\ell_2 < \frac{\delta}{\varepsilon}$, there are two real functions $A$ and $B$ such that

$$
\beta = e^{iA+B}, \tag{4.26}
$$

furthermore, a direct computation shows that, in this region, there holds

$$
\triangledown A = O(\varepsilon), \qquad \triangle A = O(\varepsilon^2), \qquad \triangledown B = O(\varepsilon^3), \qquad \triangle B = O(\varepsilon^4). \tag{4.27}
$$

The equations become

$$
\tilde{L}_{2,4}(\psi_1) = \tilde{E}_{2,4} + \tilde{N}_{2,4}, \qquad \bar{L}_{2,4}(\psi_2) = \bar{E}_{2,4} + \bar{N}_{2,4}. \tag{4.28}
$$

In the above, we have denoted the linear operators by

$$
\begin{aligned}
\tilde{L}_{2,4}(\psi_1) \;\equiv\; & \Big( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1} \frac{\partial}{\partial x_1} \Big) \psi_1 \;+\; \Big( \triangledown B + \frac{w'(\ell_2)}{w(\ell_2)} \frac{s}{\ell_2} \Big) \cdot \triangledown \psi_1, \\
\bar{L}_{2,4}(\psi_2) \;\equiv\; & \Big( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1} \frac{\partial}{\partial x_1} \Big) \psi_2 \;-\; 2|u_2|^2 \psi_2 \;+\; 2\Big( \triangledown B + \frac{w'(\ell_2)}{w(\ell_2)} \frac{s}{\ell_2} \Big) \cdot \triangledown \psi_2,
\end{aligned}
$$

where have used $s = (x_1 - r_{1\varepsilon}, x_2)$. The nonlinear operators are

$$\begin{aligned}
\tilde{N}_{2,4} \;&=\; -2(\nabla A + \nabla \varphi_0^+)\cdot \nabla \psi_2 \;+\; 2\nabla\psi_1 \nabla \psi_2,\\
\bar{N}_{2,4} \;&=\; -2(\nabla A + \nabla \varphi_0^+)\cdot \nabla \psi_1 \;+\; |u_2|^2\big(1 - e^{-2\psi_2} + 2\psi_2\big) \;+\; |\nabla\psi_1|^2 \;-\; |\nabla\psi_2|^2.
\end{aligned}$$

In the region

$$D_{3,1} = \Big\{ (x_1, x_2) : \, r_{2\varepsilon} - \tau_1/\varepsilon < |x| < r_{2\varepsilon} + \tau_2/\varepsilon \Big\},$$

the approximation takes the form

$$u_2 \;=\; w(\ell_2) w(\ell_1)\, \eta_2\, e^{i\varphi} \;+\; \hat{q}\, \eta_3\, e^{i\varphi}.$$

We write the ansatz as

$$u = u_2 \;+\; ie^{i\varphi}\psi \;+\; \Gamma_{3,1}, \tag{4.29}$$

where $\Gamma_{3,1}$ is defined as

$$\Gamma_{3,1} = i\eta_2\Big(w(\ell_1)w(\ell_2) - 1\Big)e^{i\varphi}\psi \;+\; \eta_2 w(\ell_1) w(\ell_2) e^{i\varphi}\Big(e^{i\psi} - 1 - i\psi\Big). \tag{4.30}$$

The equation becomes

$$\begin{aligned}
L_{3,1}[\psi] \;\equiv\; & S_0[\psi] \;+\; 2i\nabla\varphi\cdot\nabla\psi \;-\; |\nabla\varphi|^2\psi \;+\; iS_0[\varphi]\psi\\
&+\; \Big(1 + V - |u_2|^2\Big)\psi \;+\; 2ie^{-i\varphi}\mathrm{Re}\big(\bar{u}_2 i e^{i\varphi}\psi\big)u_2\\
=\;& E_{3,1} \;+\; N_{3,1},
\end{aligned}$$

where $E_{3,1} = ie^{-i\varphi}\mathbb{E}$. The nonlinear operator is defined by

$$\begin{aligned}
N_{3,1}(\psi) =\; & ie^{-i\varphi}\Big[\; \triangle \Gamma_4 + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Gamma_4 + \big(1 + V - |u_2|^2\big)\Gamma_4 \Big]\\
&- ie^{-i\varphi}\Big[ 2\mathrm{Re}(\bar{u}_2\Gamma_4) - |ie^{i\varphi}\psi + \Gamma_4|^2 \Big](u_2 + ie^{i\varphi}\psi + \Gamma_4)\\
&- 2ie^{-i\varphi}\mathrm{Re}(\bar{u}_2 ie^{i\varphi}\psi)(ie^{i\varphi}\psi + \Gamma_4).
\end{aligned}$$

More precisely, in the region $D_{3,1}$, the linear operator $L_{3,1}$ is defined as

$$\begin{aligned}
L_{3,1}[\psi] =\; & S_0[\psi] \;-\; \big(\delta_\varepsilon(\ell - r_{2\varepsilon}) + \hat{q}^2\big)\psi \;+\; 2ie^{-i\varphi}\mathrm{Re}\big(\bar{u}_2 ie^{i\varphi}\psi\big)u_2\\
&+\; \Big[1 + V + \delta_\varepsilon(\ell - r_{2\varepsilon})\Big]\psi \;+\; 2i\nabla\varphi\cdot\nabla\psi \;+\; S_0[\varphi]\psi \;-\; |\nabla\varphi|^2\psi.
\end{aligned}$$

where we have used the definition of $\hat{q}$ in (3.31). We shall analyze other terms in the linear operator $L_{3,1}$. For $r_{2\varepsilon} - \tau_1/\varepsilon < |x| < r_{2\varepsilon} + \tau_2/\varepsilon$, there holds $u_2 = \hat{q}\, e^{i\varphi}$. It is obvious that

$$2ie^{-i\varphi}\mathrm{Re}\big(\bar{u}_2 ie^{i\varphi}\psi\big)u_2 \;=\; -2i\,\hat{q}^2\,\psi_2. \tag{4.31}$$

For $r_{2\varepsilon} + \tau_2/\varepsilon < |x| < r_{2\varepsilon} + 2\tau_2/\varepsilon$, there holds

$$u_2 = w(\ell_2)w(\ell_1)\,\eta_2\, e^{i\varphi} \;+\; \hat{q}\,\eta_3\, e^{i\varphi}.$$

Whence we decompose the equation in form

$$\begin{aligned}
\tilde{L}_{3,1}[\psi_1] \;\equiv\; & \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)\psi_1 \;-\; \big(\delta_\varepsilon(\ell - r_{2\varepsilon}) + \hat{q}^2\big)\psi_1 \;+\; \Big[1 + V + \delta_\varepsilon(\ell - r_{2\varepsilon})\Big]\psi_1\\
&+\; \frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_1 \;-\; 2\nabla\varphi\cdot\nabla\psi_2 \;+\; S_0[\varphi]\psi_1 \;-\; |\nabla\varphi|^2\psi_1\\
=\;& \tilde{E}_{3,1} + \tilde{N}_{3,1},
\end{aligned} \tag{4.32}$$

$$\begin{aligned}\bar{L}_{3,1}[\psi_2] &\equiv \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_2 \,-\, \big(\delta_\varepsilon(\ell-r_{2\varepsilon})+3\hat{q}^2\big)\psi_2 \,+\, \Big[1+V+\delta_\varepsilon(\ell-r_{2\varepsilon})\Big]\psi_2\\ &\quad+\,\frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_2 \,+\, 2\bigtriangledown\varphi\cdot\bigtriangledown\psi_1 \,+\, S_0[\varphi]\psi_2 \,-\, |\bigtriangledown\varphi|^2\psi_2\\ &= \bar{E}_{3,1}+\bar{N}_{3,1}.\end{aligned}\tag{4.33}$$

If $r_{2\varepsilon}-\tau_1/\varepsilon<|x|<r_{2\varepsilon}+\tau_2/\varepsilon$, by using (3.28), we then have

$$\Xi_{3,1} \,\equiv\, 1+V+\delta_\varepsilon(\ell-r_{2\varepsilon}) \,=\, \frac{\varepsilon^2}{2}\frac{\partial^2 V}{\partial\tilde{\ell}^2}(\ell-r_{2\varepsilon})^2 \,+\, O\big(\varepsilon^3(\ell-r_{2\varepsilon})^3\big).$$

The other terms with $\varphi_0$ are also lower order terms. Whence the linear operators $\tilde{L}_{3,1}$ and $\bar{L}_{3,1}$ are small perturbations of the following linear operators

$$\begin{aligned}L_{31*}[\psi_1] &\equiv \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_1 \,-\, \big(\delta_\varepsilon(\ell-r_{2\varepsilon})+\hat{q}^2\big)\psi_1,\\ L_{31**}[\psi_2] &\equiv \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_2 \,-\, \big(\delta_\varepsilon(\ell-r_{2\varepsilon})+3\hat{q}^2\big)\psi_2.\end{aligned}\tag{4.34}$$

In the region $D_{3,2}$ the approximation takes the form

$$u_2 \,=\, \hat{q}(x_1,x_2)e^{i\varphi},$$

and the ansatz is

$$u \,=\, u_2 \,+\, ie^{i\varphi}\psi.$$

The equation becomes

$$\begin{aligned}L_{3,2}[\psi] &\equiv S_0[\psi] \,+\, \Big(1+V\Big)\psi \,-\, |u_2|^2\psi \,+\, 2ie^{-i\varphi}\mathrm{Re}\big(\bar{u}_2 ie^{i\varphi}\psi\big)u_2\\ &\quad-\,|\bigtriangledown\varphi|^2\psi \,+\, iS_0[\varphi]\psi \,+\, 2i\bigtriangledown\varphi\cdot\bigtriangledown\psi\\ &= E_{3,2} \,+\, N_{3,2},\end{aligned}\tag{4.35}$$

where $E_{3,2}=ie^{-i\varphi}\mathbb{E}$. The nonlinear operator is defined by

$$N_{3,2}(\psi) \,=\, -\,ie^{-i\varphi}(u_2+ie^{i\varphi}\psi)|\psi|^2 \,+\, 2i\mathrm{Re}(\bar{u}_2 ie^{i\varphi}\psi)\psi.$$

More precisely, for other term, we have

$$-|u_2|^2\psi \,+\, 2ie^{-i\varphi}\mathrm{Re}\big(\bar{u}_2 ie^{i\varphi}\psi\big)u_2 \,=\, -\hat{q}^2\,\psi_1 \,-\, 3i\hat{q}^2\,\psi_2.$$

The equation can be decomposed in the form

$$\begin{aligned}\tilde{L}_{3,2}[\psi_1] &\equiv \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_1+\Big(1+V\Big)\psi_1-\hat{q}\,\psi_1\\ &\quad+\,\frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_2-|\bigtriangledown\varphi|^2\psi_1+iS_0[\varphi]\psi_1-2\bigtriangledown\varphi\cdot\bigtriangledown\psi_2\\ &= \tilde{E}_{3,2} \,+\, \tilde{N}_{3,2},\end{aligned}\tag{4.36}$$

$$\begin{aligned}\bar{L}_{3,2}[\psi_2] &\equiv \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_2+\Big(1+V\Big)\psi_2-\hat{q}\,\psi_2\\ &\quad+\,\frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_2-|\bigtriangledown\varphi|^2\psi_2+iS_0[\varphi]\psi_2+2\bigtriangledown\varphi\cdot\bigtriangledown\psi_1\\ &= \bar{E}_{3,2} \,+\, \bar{N}_{3,2}.\end{aligned}\tag{4.37}$$

The assumption (A3) implies that, for any sufficiently small $\varepsilon$ there holds

$$\Xi_{3,2} \;=\; 1+V < -c_2 \quad \text{for } |x| > r_{2\varepsilon} + \tau_2/\varepsilon. \tag{4.38}$$

The other terms with $\varphi_0$ are lower order terms. From the asymptotic properties of $q$ in Lemma 2.4, $\hat{q}\,\psi_2$ and $\hat{q}\,\psi_1$ are also lower order term. Whence the linear operators $\tilde{L}_{3,2}$ and $\bar{L}_{3,2}$ are small perturbations of the following linear operator

$$L_{32*}[\tilde{\psi}] \equiv \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)\tilde{\psi} \,+\, \big(1+V\big)\tilde{\psi}. \tag{4.39}$$

Let $\chi$ be the cut-off function defined in (4.2). By recalling the definition of $\beta$ in (4.14), we define

$$\Lambda \equiv \frac{\partial u_2}{\partial f}\cdot\frac{\chi\big(|x-\xi_+|/\varepsilon\big)+\chi\big(|x-\xi_-|/\varepsilon\big)}{\beta}\,. \tag{4.40}$$

In summary, for any given $f$ in (3.7), we want to solve the projected equation for $\psi$ satisfying the symmetry (4.5)

$$\mathcal{L}(\psi) = \mathcal{N}(\psi) + \mathcal{E} + c\Lambda, \qquad \mathrm{Re}\int_{\mathbb{R}^2}\bar{\phi}\Lambda \;=\; 0, \tag{4.41}$$

where have denoted

$$\mathcal{L}(\psi) = L_1(\phi) \quad \text{in } D_1, \qquad \mathcal{L}(\psi) = L_{2,j}(\psi) \quad \text{in } D_{2,j} \text{ for } j=1,2,3,4,$$

$$\mathcal{L}(\psi) = L_{3,1}(\psi) \quad \text{in } D_{3,1}, \qquad \mathcal{L}(\psi) = L_{3,2}(\psi) \quad \text{in } D_{3,2},$$

with the relation

$$\phi = iu_2\psi \quad \text{in } D_2. \tag{4.42}$$

As we have stated, the nonlinear operator $\mathcal{N}$ and the error term $\mathcal{E}$ also have suitable local forms in different regions.

## 5. The Resolution of the Projected Nonlinear Problem

### *5.1. The linear resolution theory*

The main object is to consider the resolution of the linear part in previous section, which was stated in Lemma 5.2.

For that purpose, we shall firs get a priori estimates expressed in suitable norms. By recalling the norm $\|\cdot\|_{**}$ defined in (3.42), for fixed small positive numbers $0<\sigma<1$, $0<\gamma<1$, we define

$$\begin{aligned}||\psi||_* \equiv \sum_{i=1}^{2}\Big[\,||\phi||_{W^{2,p}(\ell_i<3)} \,+\, ||\ell_i^{\sigma}\psi_1||_{L^\infty(\tilde{D})} + ||\ell_i^{1+\sigma}\bigtriangledown\psi_1||_{L^\infty(\tilde{D})}\\ +\, ||\ell_i^{1+\sigma}\psi_2||_{L^\infty(\tilde{D})} + ||\ell_i^{2+\sigma}\bigtriangledown\psi_2||_{L^\infty(\tilde{D})}\Big] \,+\, ||\psi||_{W^{2,p}(D_3)},\end{aligned}$$

where we have use the relation $\phi = iu_2\psi$ and the region $\tilde{D}$ is defined in (3.43). We then consider the following problem: finding $\psi$ with the symmetry in (4.5)

$$\mathcal{L}(\psi) = h \quad \text{in } \mathbb{R}^2, \quad \mathrm{Re}\int_{\mathbb{R}^2}\bar{\phi}\Lambda \;=\; 0 \;\; \text{with } \phi = iu_2\psi. \tag{5.1}$$

**Lemma 5.1.** *There exists a constant $C$, depending on $\gamma, \sigma$ only, such that for all $\varepsilon$ sufficiently small, and any solution of (5.1), we have the estimate*

$$||\psi||_* \leq ||h||_{**}.$$

**Proof.** We prove the result by contradiction. Suppose that there is a sequence of $\varepsilon = \varepsilon_n$, functions $\psi^n$, $h_n$ which satisfy (5.1) with

$$||\psi^n||_* = 1, \quad ||h_n||_{**} = o(1).$$

Before any further argument, by the symmetry assumption (4.5) for $\psi = \psi_1 + i\psi_2$, we have

$$\begin{aligned} \psi_1(x_1, -x_2) = -\psi_1(x_1, x_2), \quad \psi_1(-x_1, x_2) = \psi_1(x_1, x_2), \\ \psi_2(x_1, -x_2) = \psi_2(x_1, x_2), \quad \psi_2(-x_1, x_2) = \psi_2(x_1, x_2). \end{aligned} \tag{5.2}$$

We may just need to consider the problem in $\mathbb{R}^2_+ = \{(x_1, x_2) : x_1 > 0\}$. Then we have

$$\mathrm{Re} \int_{\mathbb{R}^2} \bar{\phi}_n \Lambda \, = \, 2\mathrm{Re} \int_{\mathbb{R}^2_+} \bar{\phi}_n \Lambda \, = \, 0, \tag{5.3}$$

for any $\phi_n = iu_2\psi^n$. To get good estimate and then derive a contradiction, we will use suitable forms of the linear operator $\mathcal{L}$ in different regions, which was stated in previous section. Hence we divide the proof into five parts.

**Part 1.** In the outer part $D_1$, we use the following barrier function

$$\mathcal{B}(x) \, = \, \mathcal{B}_1(x) \, + \, \mathcal{B}_2(x),$$

where

$$\mathcal{B}_1(x) = |x - \xi_+|^\varrho \, |x_2|^\gamma \, + \, |x - \xi_-|^\varrho \, |x_2|^\gamma, \qquad \mathcal{B}_2(x) = C_1(1 + |x|^2)^{-\sigma/2},$$

where $\varrho + \gamma = -\sigma$, $0 < \sigma < \gamma < 1$, and $C_1$ is a large number depending on $\sigma, \varrho, \gamma$ only. Trivial computations derive that

$$\begin{aligned} \triangle\mathcal{B}_1 \, &\leq \, -C\big(|x - \xi_+|^2 + |x - \xi_-|^2\big)^{-1-\sigma/2}, \\ \triangle\mathcal{B}_2 + \frac{1}{x_1}\frac{\partial\mathcal{B}_2}{\partial x_1} \, &\leq \, -CC_1(1 + |x|^2)^{-1-\sigma/2}. \end{aligned}$$

On the other hand,

$$\frac{1}{x_1}\frac{\partial\mathcal{B}_1}{\partial x_1} \, \leq \, \frac{|x_2|^\gamma}{x_1}\Big[\, |x - \xi_+|^{\varrho-2}(x_1 - r_{1\varepsilon}) \, + \, |x - \xi_-|^{\varrho-2}(x_1 - r_{1\varepsilon})\Big].$$

Thus for $|x - \xi_+| < c_\sigma r_{1\varepsilon}$, where $c_\sigma$ is small, we have

$$\frac{1}{x_1}\frac{\partial\mathcal{B}_1}{\partial x_1} \, \leq \, Cc_\sigma\Big[\, |x - \xi_+|^2 + |x - \xi_-|^2\,\Big]^{-1-\sigma/2}.$$

For $|x - \xi_+| > c_\sigma r_{1\varepsilon}$, where $c_\sigma$ is small, we have

$$\frac{1}{x_1}\frac{\partial\mathcal{B}_1}{\partial x_1} \leq C(1 + |x|^2)^{-1-\sigma/2}.$$

By choosing $C_1$ large, we have

$$\triangle\mathcal{B} + \frac{1}{x_1}\frac{\partial\mathcal{B}}{\partial x_1} \, \leq \, -C\big(|x - \xi_+|^2 + |x - \xi_-|^2\big)^{-1-\sigma/2}.$$

For the details of the above computations, the reader can refer to the proof of Lemma 7.2 in [41].

In $D_1$, we have

$$\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_1\;+\;\frac{2}{\beta_1}\bigtriangledown\beta_1\cdot\bigtriangledown\psi_1\;=\;h_1.$$

By comparison principle on the set $D_1$, we obtain

$$|\psi_1|\;\leq\;C\mathcal{B}\big(||h||_{**}+o(1)\big),\quad\forall\,x\in D_1.$$

On the other hand, the equation for $\psi_2$ is

$$\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_2\;-\;2|u_2|^2\psi_2\;+\;\frac{2}{\beta_1}\bigtriangledown\beta_1\cdot\psi_2\;=\;h_2.$$

For $x\in D_1$, there holds $|u_2|\sim 1$. By standard elliptic estimates we have

$$||\psi_2||_{L^\infty(\ell_i>4)}\;\leq\;C||\psi_2||_{L^\infty(\ell_i=4)}(1+||\psi||_*)||h||_{**}(1+\ell_1+\ell_2)^{-1-\sigma},$$

$$|\bigtriangledown\psi_2|\;\leq\;C||\psi_2||_{L^\infty(\ell_i=R)}(1+||\psi||_*)||h||_{**}(1+\ell_1+\ell_2)^{-2-\sigma}.$$

**Part 2.** We here only derive the estimates in the vortex-core region $D_{2,2}$ near $\xi_+$. Since $||h||_{**}=o(1),\psi^n\to\psi^0$, which satisfies

$$L_{2,2}(\psi_0)=0,\quad||\psi^0||_*\leq 1.$$

Whence, we get $L_0(\phi_0)=0$. By the nondegeneracy in Lemma 2.3, we have

$$\phi_0=c_1\frac{\partial U_0}{\partial s_1}+c_2\frac{\partial U_0}{\partial s_2}.$$

Observe that $\phi_0$ inherits the symmetries of $\phi$ and hence $\phi_0=\overline{\phi_0(x_1,-x_2)}$, while the other symmetry is not preserved under the translation $s=x-\xi_+$. Obviously, the term $\frac{\partial U_0}{\partial s_2}$ does not enjoy the above symmetry. This implies that $\phi_0=c_1\frac{\partial U_0}{\partial s_1}$. On the other hand, taking a limit of the orthogonality condition $\mathrm{Re}\int_{\mathbb{R}^2_+}\bar{\phi}_n\Lambda=0$, we obtain

$$\mathrm{Re}\int_{\mathbb{R}^2}\bar{\phi}_0\frac{U_0}{\partial s_1}=0,$$

and moreover $c_1=0$ and $\phi_0=0$. Hence, for any fixed $R>0$, there holds

$$||\phi_1||_{L^\infty(\ell<R)}\;+\;||\phi_2||_{L^\infty(\ell<R)}\;+\;||\bigtriangledown\phi_1||_{L^\infty(\ell<R)}\;+\;||\bigtriangledown\phi_2||_{L^\infty(\ell<R)}\;=\;O(1).$$

**Part 3.** In the outer part $D_{2,3}\cup D_{2,4}$, we use the following barrier function

$$\mathcal{B}(x)\;=\;\mathcal{B}_1(x)\;+\;\mathcal{B}_2(x),$$

where

$$\mathcal{B}_1(x)=|x-\xi_+|^\varrho\,|x_2|^\gamma\;+\;|x-\xi_-|^\varrho\,|x_2|^\gamma,\qquad\mathcal{B}_2(x)=C_1(1+|x|^2)^{-\sigma/2},$$

where $\varrho+\gamma=-\sigma$, $0<\sigma<\gamma<1$, and $C_1$ is a large number depending on $\sigma,\varrho,\gamma$ only. Trivial computations derive that

$$\triangle\mathcal{B}_1\;\leq\;-C\big(|x-\xi_+|^2+|x-\xi_-|^2\big)^{-1-\sigma/2},$$

$$\triangle\mathcal{B}_2+\frac{1}{x_1}\frac{\partial\mathcal{B}_2}{\partial x_1}\ \le\ -CC_1(1+|x|^2)^{-1-\sigma/2}.$$

On the other hand,

$$\frac{1}{x_1}\frac{\partial\mathcal{B}_1}{\partial x_1}\ \le\ \frac{|x_2|^\gamma}{x_1}\Big[\,|x-\xi_+|^{\varrho-2}(x_1-r_{1\varepsilon})\ +\ |x-\xi_-|^{\varrho-2}(x_1-r_{1\varepsilon})\Big].$$

Thus for $|x-\xi_+|<c_\sigma r_{1\varepsilon}$, where $c_\sigma$ is small, we have

$$\frac{1}{x_1}\frac{\partial\mathcal{B}_1}{\partial x_1}\ \le\ Cc_\sigma\Big[\,|x-\xi_+|^2+|x-\xi_-|^2\,\Big]^{-1-\sigma/2}.$$

For $|x-\xi_+|>c_\sigma r_{1\varepsilon}$, where $c_\sigma$ is small, we have

$$\frac{1}{x_1}\frac{\partial\mathcal{B}_1}{\partial x_1}\le C(1+|x|^2)^{-1-\sigma/2}.$$

By choosing $C_1$ large, we have

$$\triangle\mathcal{B}+\frac{1}{x_1}\frac{\partial\mathcal{B}}{\partial x_1}\ \le\ -C\big(|x-\xi_+|^2+|x-\xi_-|^2\big)^{-1-\sigma/2}.$$

For the details of the above computations, the reader can refer to the proof of Lemma 7.2 in [41].

In the region $D_{2,3}\cup D_{2,4}$, we have

$$\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_1\ +\ \Big(\bigtriangledown B+\frac{w'(\ell_2)}{w(\ell_2)}\frac{s}{\ell_2}\Big)\cdot\bigtriangledown\psi_1\ =\ h_1,$$

where have used $s=(x_1-r_{1\varepsilon},x_2)$. By comparison principle on the set $D_{2,3}\cup D_{2,4}$, we obtain

$$|\psi_1|\le C\mathcal{B}\big(||h||_{**}+o(1)\big),\quad \forall x\in D_{2,3}\cup D_{2,4}.$$

On the other hand, the equation for $\psi_2$ is

$$\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_2\ -\ 2|u_2|^2\psi_2\ +\ 2\Big(\bigtriangledown B+\frac{w'(\ell_2)}{w(\ell_2)}\frac{s}{\ell_2}\Big)\cdot\bigtriangledown\psi_2\ =\ h_2.$$

For $x\in D_{2,3}\cup D_{2,4}$, there holds $|u_2|\sim 1$. By standard elliptic estimates we have

$$||\psi_2||_{L^\infty(\ell_i>4)}\ \le\ C||\psi_2||_{L^\infty(\ell_i=4)}(1+||\psi||_*)||h||_{**}(1+\ell_1+\ell_2)^{-1-\sigma},$$

$$|\bigtriangledown\psi_2|\ \le\ C||\psi_2||_{L^\infty(\ell_i=R)}(1+||\psi||_*)||h||_{**}(1+\ell_1+\ell_2)^{-2-\sigma}.$$

**Part 4.** In the region $D_{3,1}$, we have

$$\begin{aligned}L_{3,1}[\psi_1]\equiv&\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_1\ -\ \big(\delta_\varepsilon(\ell-r_{2\varepsilon})+\hat q^2\big)\psi_1\ +\ \Big[1+V+\delta_\varepsilon(\ell-r_{2\varepsilon})\Big]\psi_1\\&+\ \frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_1\ -\ 2\bigtriangledown\varphi_0\cdot\bigtriangledown\psi_2\ +\ S_0[\varphi_0]\psi_1\ -\ |\bigtriangledown\varphi_0|^2\psi_1\\=&\,h_1,\end{aligned}$$

$$\begin{aligned}L_{3,1}[\psi_2]\equiv&\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_2\ -\ \big(\delta_\varepsilon(\ell-r_{2\varepsilon})+3\hat q^2\big)\psi_2\ +\ \Big[1+V+\delta_\varepsilon(\ell-r_{2\varepsilon})\Big]\psi_2\\&+\ \frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_2\ +\ 2\bigtriangledown\varphi_0\cdot\bigtriangledown\psi_1\ +\ S_0[\varphi_0]\psi_2\ -\ |\bigtriangledown\varphi_0|^2\psi_2\\=&\,h_2.\end{aligned}$$

By defining a new translated variable $z = \delta_\varepsilon^{1/3}(\ell - r_\varepsilon)$, the linear operators $L_{31*}$ and $L_{31**}$ in (4.34) become

$$
\begin{aligned}
L_{31*}(\psi_{1*}) &= \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)\psi_{1*} - \big(z + q^2(z)\big)\psi_{1*}\,,\\
L_{31**}(\psi_{2**}) &= \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)\psi_{2**} - \big(z + 3q^2(z)\big)\psi_{2**}\,.
\end{aligned}
$$

From Lemma 2.4, $-q'(z) > 0$ for all $z \in \mathbb{R}$, and $L_{31**}(-q') = 0$. We apply the maximum principle to $-\psi_2/q'$ and then obtain

$$
|\psi_2| \le C|q'|(||h||_{**} + o(1)), \quad \forall\, x \in D_{3,1}.
$$

On the other hand, $q(z) > 0$ for all $z \in \mathbb{R}$, and $L_{31*}(q) = 0$. We apply the maximum principle to $\psi_1/q$ and then obtain

$$
|\psi_1| \le Cq(||h||_{**} + o(1)), \quad \forall\, x \in D_{3,1}.
$$

**Part 5.** In $D_{3,2}$, we consider the problem

$$
\begin{aligned}
L_{3,2}[\psi_1] \equiv\ & \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)\psi_1 + \big(1+V\big)\psi_1 - \hat{q}\,\psi_1\\
& + \frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_2 - |\bigtriangledown \varphi_0|^2\psi_1 + iS_0[\varphi_0]\psi_1 - 2\bigtriangledown \varphi_0\cdot \bigtriangledown\psi_2\\
=\ & h_1,
\end{aligned}
$$

$$
\begin{aligned}
L_{3,2}[\psi_2] \equiv\ & \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)\psi_2 + \big(1+V\big)\psi_2 - \hat{q}(|x_1|)\psi_2\\
& + \frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_2 - |\bigtriangledown \varphi_0|^2\psi_2 + iS_0[\varphi_0]\psi_2 + 2\bigtriangledown \varphi_0\cdot \bigtriangledown\psi_1\\
=\ & h_2.
\end{aligned}
$$

By using the properties of $\Xi_{3,2}$ in (4.38), i.e.

$$
\Sigma_{3,2} = \big(1+V\big) < -c_2 \quad \text{in } D_5.
$$

we have

$$
||\psi_2||_{L^\infty(\ell_i>4)} \le C||\psi_2||_{L^\infty(\ell_i=4)}(1+||\psi||_*)||h||_{**}(1+\ell_1+\ell_2)^{-1-\sigma},
$$

$$
|\bigtriangledown \psi_2| \le C||\psi_2||_{L^\infty(\ell_i=R)}(1+||\psi||_*)||h||_{**}(1+\ell_1+\ell_2)^{-2-\sigma}.
$$

Combining all the estimates in the above, we obtain that $||\psi||_* = o(1)$, which is a contradiction. □

We now consider the following linear projected problem: finding $\psi$ with the symmetry in (4.5)

$$
\mathcal{L}[\psi] = h + c\Lambda, \quad \mathrm{Re}\int_{\mathbb{R}^2}\bar{\phi}\Lambda = 0 \quad \text{with } \phi = iu_2\psi. \tag{5.4}
$$

**Lemma 5.2.** *There exists a constant $C$, depending on $\gamma, \sigma$ only, such that for all $\varepsilon$ sufficiently small, the following holds: if $||h||_{**} < +\infty$, there exists a unique solution $(\psi_{\varepsilon,f}, c_{\varepsilon,f}) = \mathcal{T}_{\varepsilon,f}(h)$ to (5.4). Furthermore, there holds*

$$||\psi||_* \le C||h||_{**}.$$

**Proof.** The proof is similar to that of Proposition 4.1 in [19]. Instead of solving (5.4) in $\mathbb{R}^2$, we solve it in a bounded domain first:

$$\mathcal{L}[\psi] \,=\, h \,+\, c\Lambda, \quad \mathrm{Re}\int_{\mathbb{R}^2} \bar{\phi}\Lambda \,=\, 0 \quad \text{ with } \phi = iu_2\psi,$$
$$\phi = 0 \quad \text{ on } \partial B_M(0), \quad \psi \text{ satisfies the symmetry (4.5).}$$

where $M > 10r_{1\varepsilon}$. By the standard proof of a priori estimates, we also obtain the following estimates for any solution $\psi_M$ of above problem

$$||\psi||_* \le C||h||_{**}.$$

By working with the Sobole space $H_0^1(B_M(0))$, the existence will follow by Fredholm alternatives. Now letting $M \to +\infty$, we obtain a solution with the required properties. □

*5.2. Solving the Projected Nonlinear Problem*

We then consider the following problem: finding $\psi$ with the symmetry in (4.5)

$$\mathcal{L}[\psi] \,+\, \mathcal{N}[\psi] \,=\, \mathcal{E} \,+\, c\Lambda, \quad \mathrm{Re}\int_{\mathbb{R}^2} \bar{\phi}\Lambda \,=\, 0 \quad \text{ with } \phi = iu_2\psi. \tag{5.5}$$

**Proposition 5.3.** *There exists a constant $C$, depending on $\gamma, \sigma$ only, such that for all $\varepsilon$ sufficiently small, there exists a unique solution $\psi_{\varepsilon,f}, c_{\varepsilon,f}$ to (5.5), and*

$$||\psi||_* \le C||h||_{**}.$$

*Furthermore, $\psi$ is continuous in the parameter $f$.*

**Proof.** Using of the operator defined by Lemma 5.2, we can write problem (5.5) as

$$\psi = \mathcal{T}_{\varepsilon,f}\big(-\mathcal{N}[\psi] + \mathcal{E}\big) \equiv \mathcal{G}_\varepsilon(\psi).$$

Using Lemma 3.2, we see that

$$||\mathcal{E}||_{**} \le C\varepsilon^{1-\sigma}.$$

Let

$$\psi \in \mathbb{B} = \big\{\, ||\psi||_* < C\varepsilon^{1-\sigma} \,\big\},$$

then we have, using the explicit form of $\mathcal{N}(\psi)$ in section 4

$$||\mathcal{N}(\psi)||_{**} \le C\varepsilon.$$

Whence, there holds

$$||\mathcal{G}_\varepsilon(\psi)||_{**} \le C\Big(||\mathcal{N}(\psi)||_{**} + ||\mathcal{E}||_{**}\Big) \le C\varepsilon^{1-\sigma}.$$

Similarly, we can also show that, for any $\check{\psi}, \hat{\psi} \in \mathbb{B}$

$$||\mathcal{G}_\varepsilon(\check{\psi}) - \mathcal{G}_\varepsilon(\hat{\psi})||_{**} \le o(1)||\check{\psi} - \hat{\psi}||_{**}\,.$$

By contraction mapping theorem, we confirm the result of the Lemma. □

## 6. Reduction procedure

To find a real solution to problem (3.3)-(3.4), in this section, we solve the reduced problem by finding a suitable $f$ such that the constant $c$ in (4.41) is identical zero for any sufficiently small $\varepsilon$.

In previous section, for any given $f$ in (3.7), we have deduced the existence of $\psi$ with the symmetry (4.5) to the projected problem

$$\mathcal{L}(\psi) = \mathcal{N}(\psi) + \mathcal{E} + c\Lambda, \quad \mathrm{Re}\int_{\mathbb{R}^2} \bar{\phi}\Lambda \,=\, 0, \tag{6.1}$$

with the relation

$$\phi = iu_2\psi \quad \text{in } D_2.$$

Multiplying (6.1) by $\bar{\Lambda}$ and integrating, we obtain

$$c\,\mathrm{Re}\int_{\mathbb{R}^2} \bar{\Lambda}\Lambda \,=\, \mathrm{Re}\int_{\mathbb{R}^2} \bar{\Lambda}\mathcal{L}(\psi) \,-\, \mathrm{Re}\int_{\mathbb{R}^2} \bar{\Lambda}\mathcal{N}(\psi) \,-\, \mathrm{Re}\int_{\mathbb{R}^2} \bar{\Lambda}\mathcal{E}. \tag{6.2}$$

Hence we can derive the estimate for $c$ by computing the integrals of the right hand side.

We begin with the computation of $\mathrm{Re}\int_{\mathbb{R}^2} \bar{\Lambda}\mathcal{E}$. The term $\Lambda$ has its support contained in the region $\{(x_1, x_2) : \ell_1 < \tau_0/\varepsilon \text{ or } \ell_2 < \tau_0/\varepsilon\}$. It is convenient to compute $\mathrm{Re}\int_{\mathbb{R}^2} \bar{\Lambda}\mathcal{E}$ on the variables $(s_1, s_2)$. Note that, in the vortex-core region, there holds

$$\frac{\partial u_2}{\partial f} \,=\, \Big[-\frac{w'(\ell_2)}{w(\ell_2)}\,\frac{x_1 - r_{1\varepsilon}}{\ell_2} \,+\, i\,\frac{x_2}{(\ell_2)^2}\Big]u_2 \,+\, O(\varepsilon^2)u_2,$$

which implies that

$$\Lambda \,=\, \chi\big(|x - \xi_+|/\varepsilon\big)\Big[\,-\,\frac{w'(\ell_2)}{w(\ell_2)}\,\frac{x_1 - r_{1\varepsilon}}{\ell_2} \,+\, i\,\frac{x_2}{(\ell_2)^2}\,\Big]w(\ell_2)e^{i\varphi_0^+} \,+\, O(\varepsilon^2).$$

By using of local form of $\mathcal{E}$ in the formula (4.21), we obtain

$$\begin{aligned}
\mathrm{Re}\int_{\mathbb{R}^2} \bar{\Lambda}\mathcal{E}\mathrm{d}x \,=&\, 2\mathrm{Re}\int_{\mathbb{R}^2_+} \bar{\Lambda}\mathcal{E}\mathrm{d}x\\
=&\, -2\int_{\mathbb{R}^2_+} \chi\big(\ell_2/\varepsilon\big)\Big[w'(\ell_2)\Big]^2\frac{(x_1 - r_{1\varepsilon})^2}{x_1(\ell_2)^2}\,\mathrm{d}x\\
&-\, 2\varepsilon\frac{\partial V}{\partial \tilde{r}}\Big|_{(\tilde{r}_{1\varepsilon},0)}\int_{\mathbb{R}^2_+} \chi\big(\ell_2/\varepsilon\big)w(\ell_2)w'(\ell_2)\,\frac{(x_1 - r_{1\varepsilon})(x_1 - r_{1\varepsilon})}{\ell_2}\,\mathrm{d}x\\
&-\, 2\log r_{1\varepsilon}\int_{\mathbb{R}^2_+} \chi\big(\ell_2/\varepsilon\big)w(\ell_2)w'(\ell_2)\,\frac{2(x_1 + r_{1\varepsilon})(x_1 - r_{1\varepsilon})^2}{(\ell_1)^2(\ell_2)^3}\,\mathrm{d}x\\
&-\, 2\frac{1}{r_{1\varepsilon}}\log r_{1\varepsilon}\int_{\mathbb{R}^2_+} \chi\big(\ell_2/\varepsilon\big)w(\ell_2)w'(\ell_2)\,\frac{x_2^2}{(\ell_2)^3}\,\mathrm{d}x \,+\, O(\varepsilon)
\end{aligned}$$

By the translation in (3.6), we further derive that

$$\begin{aligned}
\mathrm{Re}\int_{\mathbb{R}^2} \bar{\Lambda}\mathcal{E}\mathrm{d}x \,=&\, -2\int_{\mathbb{R}^2} \chi\big(|s|/\varepsilon\big)\Big[w'(|s|)\Big]^2\,\frac{s_1^2}{(s_1 + r_{1\varepsilon})|s|^2}\,\mathrm{d}s\\
&-\, 2\varepsilon\frac{\partial V}{\partial \tilde{r}}\Big|_{(\tilde{r}_{1\varepsilon},0)}\int_{\mathbb{R}^2} \chi\big(|s|/\varepsilon\big)w(|s|)w'(|s|)\,\frac{s_1^2}{|s|}\,\mathrm{d}s\\
&-\, 2\log r_{1\varepsilon}\int_{\mathbb{R}^2} \chi\big(|s|/\varepsilon\big)w(|s|)w'(|s|)\,\frac{2(s_1 + 2r_{1\varepsilon} + 2f)s_1^2}{\Big[(s_1 + 2r_{1\varepsilon} + 2f)^2 + s_2^2\Big]|s|^3}\,\mathrm{d}s\\
&-\, 2\frac{1}{r_{1\varepsilon}}\log r_{1\varepsilon}\int_{\mathbb{R}^2} \chi\big(|s|/\varepsilon\big)w(|s|)w'(|s|)\,\frac{s_2^2}{|s|^3}\,\mathrm{d}s \,+\, O(\varepsilon)\,.
\end{aligned}$$

We compute the first two terms in above formula

$$- 2\int_{\mathbb{R}^2} \chi\big(|s|/\varepsilon\big)\Big[w'(|s|)\Big]^2 \frac{s_1^2}{(s_1+r_{1\varepsilon})|s|^2}\,\mathrm{d}s \;=\; O(\varepsilon),$$

and

$$\begin{aligned}
&- 2\varepsilon\frac{\partial V}{\partial \tilde r}\Big|_{(\tilde r_{1\varepsilon},0)}\int_{\mathbb{R}^2} \chi\big(|s|/\varepsilon\big)w(|s|)w'(|s|)\frac{s_1^2}{|s|}\,\mathrm{d}s\\
&\quad = -2\varepsilon\frac{\partial V}{\partial \tilde r}\Big|_{(\tilde r_{1\varepsilon},0)}\int_{\mathbb{R}^2} w(|s|)w'(|s|)\frac{s_1^2}{|s|}\,\mathrm{d}s \;+\; O(\varepsilon)\\
&\quad = -2\pi\varepsilon\,|\log\varepsilon|\,\frac{\partial V}{\partial \tilde r}\Big|_{(\tilde r_{1\varepsilon},0)} \;+\; O(\varepsilon).
\end{aligned}$$

On the other hand, the last two terms can be estimated by

$$\begin{aligned}
&- 2\log r_{1\varepsilon}\int_{\mathbb{R}^2} \chi\big(|s|/\varepsilon\big)w(|s|)w'(|s|)\frac{2(s_1+2r_{1\varepsilon}+2f)s_1^2}{\Big[(s_1+2r_{1\varepsilon}+2f)^2+s_2^2\Big]|s|^3}\,\mathrm{d}s\\
&- 2\frac{1}{r_{1\varepsilon}}\log r_{1\varepsilon}\int_{\mathbb{R}^2} \chi\big(|s|/\varepsilon\big)w(|s|)w'(|s|)\frac{s_2^2}{|s|^3}\,\mathrm{d}s\\
&\quad = -\frac{2}{r_{1\varepsilon}}\log r_{1\varepsilon}\int_{\mathbb{R}^2} w(|s|)w'(|s|)\frac{1}{|s|}\,\mathrm{d}s \;+\; O(\varepsilon)\\
&\quad = -2\frac{d\,\pi}{r_{1\varepsilon}}\log r_{1\varepsilon} \;+\; O(\varepsilon),
\end{aligned}$$

where

$$d = \frac{1}{\pi}\int_{\mathbb{R}^2} w(|s|)w'(|s|)\frac{1}{|s|}\,\mathrm{d}s > 0. \tag{6.3}$$

Hence, there holds

$$I_1 \;\equiv\; \mathrm{Re}\int_{\mathbb{R}^2} \bar\Lambda\mathcal{E}\mathrm{d}x \;=\; -2\pi\varepsilon\,\frac{\partial V}{\partial \tilde r}\Big|_{(\tilde r_{1\varepsilon},0)}\log\frac{1}{\varepsilon} \;-2\,d\,\pi\frac{1}{r_{1\varepsilon}}\log r_{1\varepsilon} \;+\; O(\varepsilon). \tag{6.4}$$

Using Proposition 5.3, and the expression in (4.22), we deduce that

$$\mathrm{Re}\int_{\mathbb{R}^2} \bar\Lambda\mathcal{N}(\psi) \;=\; \mathrm{Re}\int_{\mathbb{R}^2} \bar\Lambda N_2(\psi) \;=\; O(\varepsilon).$$

On the other hand, integration by parts, we have

$$\mathrm{Re}\int_{\mathbb{R}^2} \bar\Lambda\mathcal{L}(\psi) \;=\; \mathrm{Re}\int_{\mathbb{R}^2} \bar\psi\mathcal{L}(\bar\Lambda) \;=\; O(\varepsilon).$$

Combining all estimates together and recalling $\tilde r_{1\varepsilon} = \tilde r_0 + \tilde f$, we obtain the following equation

$$c(\tilde f) = -2\,\varepsilon\pi\left[\,\frac{\partial V}{\partial \tilde r}\Big|_{(\tilde r_0+\tilde f,0)}\log\frac{1}{\varepsilon} \;+\; \frac{d}{\tilde r_0+\tilde f}\log\frac{\tilde r_0+\tilde f}{\varepsilon}\,\right] \;+\; O(\varepsilon), \tag{6.5}$$

where $O(\varepsilon)$ is a continuous function of the parameter $\tilde f$. By the solvability condition (1.4) and the non-degeneracy condition(1.6), we can find a zero of $c\big(\tilde f\big)$ at some small $\tilde f$ with the help of the simple mean-value theorem.

## 7. Traveling wave

The main object of this section to prove Theorem 1.5 by using the same method in previous sections.

*7.1. The formulation of problem*

Making rescaling $\tilde{y} = \varepsilon\check{y}$, problem (1.20) takes the form

$$\triangle u + \Big(1 + \check{V}(\varepsilon\check{y}) - |u|^2\Big)u + i\varepsilon|\log\varepsilon|\kappa\,\frac{\partial u}{\partial \check{y}_3} = 0. \tag{7.1}$$

Introduce a new coordinates $(r, \theta, \check{y}_3) \in (0, +\infty) \times (0, 2\pi] \times \mathbb{R}$ as the form

$$\check{y}_1 = r\cos\theta, \quad \check{y}_2 = r\sin\theta, \quad \check{y}_3 = \check{y}_3.$$

Then problem (7.1) takes the form

$$S[u] \equiv \Big(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial \check{y}_3^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\Big)u + \Big(1 + \check{V}(\varepsilon r, \varepsilon\check{y}_3) - |u|^2\Big)u + i\varepsilon|\log\varepsilon|\kappa\,\frac{\partial u}{\partial \check{y}_3} = 0.$$

In this paper, we want to construct a solution with a vortex ring, which does not depend on the variable $\theta$. Hence, we consider a two-dimensional problem, for $(x_1, x_2) \in \mathbb{R}^2$

$$S[u] = \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)u + \Big(1 + \check{V}(\varepsilon|x_1|, \varepsilon x_2) - |u|^2\Big)u + i\varepsilon|\log\varepsilon|\kappa\,\frac{\partial u}{\partial x_2} = 0, \tag{7.2}$$

with Neumann boundary condition

$$\frac{\partial u}{\partial x_1}(0, x_2) = 0, \quad |u| \to 0 \text{ as } |x| \to +\infty. \tag{7.3}$$

**Notations:** *As before, we have used $x = (x_1, x_2) = (r, \check{y}_3)$ and also write $\ell = |x|$. In this rescaled coordinates, we write*

$$\check{r}_{1\varepsilon} \equiv \hat{r}_0/\varepsilon + \check{f} \equiv \hat{r}_{1\varepsilon}/\varepsilon \quad \textit{with } \check{f} = \hat{f}/\varepsilon, \qquad \check{r}_{2\varepsilon} \equiv \hat{r}_{2\varepsilon}/\varepsilon, \tag{7.4}$$

*where the constants $\hat{f}$, $\hat{r}_{1\varepsilon}$ and $\hat{r}_{2\varepsilon}$ are defined in (1.16) and (1.17). By setting, $\xi_+ = (\check{r}_{1\varepsilon}, 0)$ and $\xi_- = (-\check{r}_{1\varepsilon}, 0)$, we introduce the translated variable*

$$s = x - \xi_+ \quad \textit{or} \quad s = x - \xi_-, \tag{7.5}$$

*in a small neighborhood of the vortices. We will use these notations without any further statement in the sequel.* □

To handle the influence of the potential, we here look for vortex ring solutions vanishing as $|x|$ approaching $+\infty$. As we stated in (1.16), we assume that the vortex ring is characterized by the curve, in the original coordinates $\check{y} = (\check{y}_1, \check{y}_2, \check{y}_3)$

$$\sqrt{\check{y}_1^2 + \check{y}_2^2} = \check{r}_{1\varepsilon}, \quad \check{y}_3 = 0.$$

In other words, in the two dimensional situation with $(x_1, x_2)$ coordinates, we will construct a vortex with degree $+1$ at $(\check{r}_{1\varepsilon}, 0)$ and its anti-pair with degree $-1$ at $(-\check{r}_{1\varepsilon}, 0)$.

Finally, we decompose the operator as

$$S[u] = S_0[u] + S_1[u] + S_2[u],$$

with the explicit form

$$\begin{gathered} S_0[u] = \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)u, \qquad S_1[u] = \Big(1 + \check{V}(\varepsilon|x_1|, \varepsilon x_2) - |u|^2\Big)u, \\ S_2[u] = i\varepsilon\,\kappa\,|\log\varepsilon|\,\frac{\partial u}{\partial x_2}. \end{gathered} \tag{7.6}$$

*7.2. The approximation and its error*

As we have done in (3.13) and (4.8), we decompose the plane into different regions $D_1, D_2$ and $D_3$ in form

$$
\begin{aligned}
D_2 &\equiv \Big\{(x_1,x_2): \ell_1 < \frac{\hat{\tau}_0}{\varepsilon} \text{ or } \ell_2 < \frac{\hat{\tau}_0}{\varepsilon}\Big\},\\
D_1 &\equiv \Big\{(x_1,x_2): |x| < \hat{r}_{2\varepsilon} - \frac{\hat{\tau}_1}{\varepsilon}\Big\} \setminus D_2,\\
D_3 &= \Big\{ (x_1,x_2):\ |x| > \hat{r}_{2\varepsilon} - \frac{\hat{\tau}_1}{\varepsilon} \Big\}.
\end{aligned}
\tag{7.7}
$$

and then we set

$$
\begin{aligned}
D_{2,1} &\equiv \big\{(x_1,x_2): \ell_1 < 1\big\}, \qquad & D_{2,2} &\equiv \big\{(x_1,x_2): \ell_2 < 1\big\},\\
D_{2,3} &\equiv \Big\{(x_1,x_2): \ell_1 < \frac{\hat{\tau}_0}{\varepsilon}\Big\} \setminus D_{2,1}, \qquad & D_{2,4} &\equiv \Big\{(x_1,x_2): \ell_2 < \frac{\hat{\tau}_0}{\varepsilon}\Big\} \setminus D_{2,2},\\
D_{3,1} &\equiv \Big\{ (x_1,x_2): \hat{r}_{2\varepsilon} - \frac{\hat{\tau}_1}{\varepsilon} < |x| < \hat{r}_{2\varepsilon} + \frac{\hat{\tau}_2}{\varepsilon} \Big\}, \qquad & D_{3,2} &\equiv \Big\{(x_1,x_2): |x| > \hat{r}_{2\varepsilon} + \frac{\hat{\tau}_2}{\varepsilon} \Big\}.
\end{aligned}
\tag{7.8}
$$

where $\hat{\tau}_0$, $\hat{\tau}_1$, $\hat{\tau}_2$, $\hat{r}_{1\varepsilon}$ and $\hat{r}_{2\varepsilon}$ are given in the assumption (**P3**). Here we still use the same notations to denote the regions. The reader can refer to Figure 1.

By defining smooth cut-off functions as follows

$$
\tilde{\eta}_2(s) = \begin{cases} 1, & |s| \le \check{\tau}_3,\\ 0, & |s| \ge 2\check{\tau}_3; \end{cases} \qquad \tilde{\eta}_3(s) = \begin{cases} 1, & s \ge -\check{\tau}_4,\\ 0, & s \le -2\check{\tau}_4; \end{cases}
\tag{7.9}
$$

where $\check{\tau}_3, \check{\tau}_4 < \min\{\hat{r}_0, \hat{\tau}_0, \hat{\tau}_1, \hat{\tau}_2\}/10$, we then choose the approximate solution to (7.2)-(7.3) by, for $(x_1,x_2) \in \mathbb{R}^2$,

$$
u_2(x_1,x_2) \ = \sqrt{1+\check{V}(\varepsilon|x_1|,\varepsilon x_2)}\, \eta_1\, e^{i\varphi} \ +\ w(\ell_2) w(\ell_1)\, \eta_2\, e^{i\varphi} \ +\ \hat{q}(x_1,x_2)\, \eta_3\, e^{i\varphi},
\tag{7.10}
$$

where the new phase term $\varphi = \varphi_0 + \varphi_1$ with the functions $\varphi_0$ and $\varphi_1$ defined in (3.14) and (3.27). The cut-off functions are defined by

$$
\begin{aligned}
\eta_1\big(\varepsilon x_1, \varepsilon x_2\big) &= 1 - \eta_2 - \eta_3,\\
\eta_2\big(\varepsilon x_1, \varepsilon x_2\big) &= \tilde{\eta}_2\big(\varepsilon \ell_1\big) + \tilde{\eta}_2\big(\varepsilon \ell_2\big),\\
\eta_3\big(\varepsilon x_1, \varepsilon x_2\big) &= \tilde{\eta}_3\big(\varepsilon(\ell - \check{r}_{2\varepsilon})\big).
\end{aligned}
$$

By recalling the functions $U_1, U_2, U_3$ in (3.15), (3.14), (3.32), we also write the approximation as

$$
u_2 \ = U_1\, \eta_1\, e^{i\varphi_1} \ +\ U_2\, \eta_2\, e^{i\varphi_1} \ +\ U_3\, \eta_3\, e^{i\varphi_1}.
\tag{7.11}
$$

It is easy to check that $u_2$ has the symmetry

$$
u_2(x_1,x_2) = \overline{u_2(x_1,-x_2)}, \qquad u_2(x_1,x_2) = u_2(-x_1,x_2).
\tag{7.12}
$$

Moreover, there holds

$$
\frac{\partial u_2}{\partial x_1}\Big|_{(0,x_2)} = 0.
\tag{7.13}
$$

As we have stated, we work directly in the half space $\mathbb{R}^2_+ = \{(x_1,x_2):\ x_1 > 0\}$ in the sequel because of the symmetry of the problem. Recalling the definitions of the operators in (7.6), let us start to compute the error:

$$
\begin{aligned}
\mathbb{E} \,&=\, S[u_2]\\
&=\, S[U_1]\eta_1 e^{i\varphi_1} \,+\, U_1 S_0[\eta_1 e^{i\varphi_1}] \,+\, 2ie^{i\varphi_1}\nabla U_1\cdot\nabla\varphi_1\\
&\quad+\, S[U_2]\eta_2 e^{i\varphi_1} \,+\, U_2 S_0[\eta_2 e^{i\varphi_1}] \,+\, 2ie^{i\varphi_2}\nabla U_1\cdot\nabla\varphi_1\\
&\quad+\, S[U_3]\eta_3 e^{i\varphi_1} \,+\, U_3 S_0[\eta_3 e^{i\varphi_1}] \,+\, 2ie^{i\varphi_1}\nabla U_3\cdot\nabla\varphi_1 \,+\, \mathbb{N},
\end{aligned}
\tag{7.14}
$$

where the nonlinear term $\mathbb{N}$ is defined by

$$
\mathbb{N} = \eta_1|U_1|^2U_1e^{i\varphi_1} \,+\, \eta_2|U_2|^2U_2e^{i\varphi_1} \,+\, \eta_3|U_3|^2U_3e^{i\varphi_1} \,-\, |u_2|^2u_2.
$$

The main components in the above formula can be estimated as before.

It is worth to mention that, in the vortex-core region $\big\{(x_1,x_2) : \ell_2 < 1 \text{ or } \ell_1 < 1\big\}$, we estimate the error by

$$
\begin{aligned}
\mathbb{E} \,=\, & U_2\eta_2 e^{i\varphi_1}\left[\frac{x_1-\check r_{1\varepsilon}}{x_1\ell_2}\frac{w'(\ell_2)}{w(\ell_2)} \,+\, \varepsilon\,\frac{\partial\check V}{\partial\tilde r}\Big|_{(\varepsilon\check r_{1\varepsilon},0)}(x_1-\check r_{1\varepsilon})\right]\\
&+\,\eta_2U_2e^{i\varphi_1}\frac{2(x_1+\check r_{1\varepsilon})(x_1-\check r_{1\varepsilon})}{(\ell_1\ell_2)^2}\log\check r_{1\varepsilon} \,-\, i\eta_2U_2e^{i\varphi_1}\frac{x_2}{\ell_2}\,\frac{w'(\ell_2)}{w(\ell_2)}\,\frac{1}{\check r_{1\varepsilon}}\log\check r_{1\varepsilon}\\
&+\,\kappa\,\varepsilon|\log\varepsilon|\,\eta_2U_2e^{i\varphi_1}\left[i\frac{x_2}{\ell_2}\,\frac{w'(\ell_2)}{w(\ell_2)} \,-\, \frac{2\big(x_1^2-x_2^2-(\check r_{1\varepsilon})^2\big)\check r_{1\varepsilon}}{(\ell_1\ell_2)^2}\right] \,+\, O(\varepsilon\log\ell_2).
\end{aligned}
\tag{7.15}
$$

The singularity of the last two terms in the above formula will play an important role in the final reduction step.

For a function $h = h_1 + ih_2$ with real functions $h_1, h_2$, define a norm of the form

$$
\begin{aligned}
||h||_{**} \equiv\, & \sum_{j=1}^{2}||iu_2h||_{L^p(\ell_j<3)} \,+\, \sum_{j=1}^{2}\Big[\,||\ell_j^{2+\sigma}h_1||_{L^\infty(\tilde D)} \,+\, ||\ell_j^{1+\sigma}h_2||_{L^\infty(\tilde D)}\,\Big]\\
&+\,\sum_{j=1}^{2}||h_j||_{L^p(D_3)}\,.
\end{aligned}
$$

where we have denoted $\tilde D$ by

$$
\tilde D = D_1\cup D_2\setminus\{\ell_1<3 \text{ or } \ell_2<3\}. \tag{7.16}
$$

As a conclusion, we have the following lemma.

**Lemma 7.1.** *There holds for $\ell_1>2$ and $\ell_2>2$*

$$
\begin{aligned}
\big|\mathrm{Re}(\mathbb{E})\big| \,&\le\, \frac{C\varepsilon^{1-\sigma}}{(1+\ell_1)^3}+\frac{C\varepsilon^{1-\sigma}}{(1+\ell_2)^3},\\
\big|\mathrm{Im}(\mathbb{E})\big| \,&\le\, \frac{C\varepsilon^{1-\sigma}}{(1+\ell_1)^{1+\sigma}}+\frac{C\varepsilon^{1-\sigma}}{(1+\ell_2)^3},\\
\|\mathbb{E}\|_{L^p(\{\ell_1<3\}\cup\{\ell_2<3\})} \,&\le\, C\varepsilon|\log\varepsilon|,
\end{aligned}
$$

*where $\sigma\in(0,1)$ is a constant. As a consequence, there also holds*

$$
||\mathbb{E}||_{**}\le C\varepsilon^{1-\sigma}.
$$

□

*7.3. Local setting-up of the problem*

We look for a solution $u = u(x_1, x_2)$ to problem (7.2)-(7.3) in the form of small perturbation of $u_2$, with additional symmetry:

$$u(x_1, x_2) = \bar{u}(x_1, -x_2). \tag{7.17}$$

Let $\tilde{\chi} : \mathbb{R} \to \mathbb{R}$ be a smooth cut-off function defined by

$$\tilde{\chi}(s) = \begin{cases} 1, & s \leq 1, \\ 0, & s \geq 2. \end{cases} \tag{7.18}$$

Recalling (7.9)-(7.11) and setting the components of the approximation $u_2$ as

$$v_1(x_1, x_2) \;=\; \eta_1\, U_1\, e^{i\varphi_1}, \quad v_2(x_1, x_2) \;=\; \eta_2\, U_2\, e^{i\varphi_1}, \quad v_3(x_1, x_2) \;=\; \eta_3\, U_3\, e^{i\varphi_1}, \tag{7.19}$$

we want to choose the ansatz of the form

$$u = \Big[\chi\big(v_2 + iv_2\psi\big) + \big(1 - \chi\big)(v_1 + v_2)e^{i\psi}\Big] \;+\; \Big[v_3 + i\, \eta_3\, e^{i\varphi}\psi\Big], \tag{7.20}$$

where $\chi(x_1, x_2) = \tilde{\chi}(\ell_2) \;+\; \tilde{\chi}(\ell_1)$.

To find the perturbation terms, the main object of this subsection is to write the equation for the perturbation as a linear one with a right hand side given by a lower order nonlinear term. The symmetry imposed on $u$ can be transmitted to the symmetry on the perturbation terms

$$\psi(x_1, -x_2) = -\overline{\psi(x_1, x_2)}, \quad \psi(x_1, x_2) = \psi(-x_1, x_2). \tag{7.21}$$

Let us observe that

$$\begin{aligned} u \;&=\; \Big[(v_1 + v_2) + i(v_1 + v_2)\psi + \big(1 - \chi\big)(v_1 + v_2)\big(e^{i\psi} - 1 - i\psi\big)\Big] \;+\; \Big(v_3 + \eta_3\, e^{i\varphi}\psi\Big) \\ &= u_2 \;+\; i(v_1 + v_2)\psi \;+\; \eta_3\, e^{i\varphi}\psi \;+\; \Gamma, \end{aligned}$$

where we have denoted

$$\Gamma = \big(1 - \chi\big)(v_1 + v_2)\big(e^{i\psi} - 1 - i\psi\big). \tag{7.22}$$

A direct computation shows that $u$ is a solution to problem (7.2)-(7.3) if and only if

$$\begin{aligned} & i(v_1 + v_2)S_0[\psi] \;+\; 2i \bigtriangledown (v_1 + v_2) \cdot \bigtriangledown\psi \;+\; i(v_1 + v_2)S_2[\psi] \;+\; \eta_3\, e^{i\varphi} S_0[\psi] \\ & \;+\; \eta_3\, e^{i\varphi} S_2[\psi] \;+\; i\Big[1 + \check{V} - |u_2|^2\Big](v_1 + v_2)\psi \;-\; 2\mathrm{Re}\big(\bar{u}_2 i(v_1 + v_2)\psi\big)u_2 \\ & \;+\; iS_0[v_1 + v_2]\psi \;+\; iS_2[v_1 + v_2]\psi \;+\; 2 \bigtriangledown \big(\eta_3\, e^{i\varphi}\big) \cdot \bigtriangledown\psi \;+\; S_0\big[\eta_3\, e^{i\varphi}\big]\psi \\ & \;+\; S_2\Big[\eta_3\, e^{i\varphi}\Big]\psi \;+\; \eta_3\, e^{i\varphi}\big(1 + \check{V} - |u_2|^2\big)\psi \\ & \;-\; 2\eta_3\, \mathrm{Re}\big(\bar{u}_2 e^{i\varphi}\psi\big)u_2 \;=\; -E \;+\; N, \end{aligned} \tag{7.23}$$

where the error term $E$ is defined as $\mathbb{E}$ in (7.14) and $N$ is the nonlinear operator defined by

$$\begin{aligned} N \;=\; & -S_0[\Gamma] \;-\; S_2[\Gamma] \;-\; \big(1 + \check{V} - |u_2|^2\big)\Gamma \\ & \;+\; \Big[2\mathrm{Re}\big(\bar{u}_2 i(v_1 + v_2)\psi\big) + 2\eta_3\, \mathrm{Re}\big(\bar{u}_2 e^{i\varphi}\psi\big)\Big] \times \Big(i(v_1 + v_2)\psi + \eta_3\, e^{i\varphi}\psi + \Gamma\Big) \\ & \;+\; \Big[2\mathrm{Re}(\bar{u}_2\Gamma) + \big|i(v_1 + v_2)\psi + \eta_3\, e^{i\varphi}\psi + \Gamma\big|^2\Big] \times \Big(u_2 + i(v_1 + v_2)\psi + \eta_3\, e^{i\varphi}\psi + \Gamma\Big). \end{aligned}$$

We shall explicitly write the equation in suitable local forms and then analyze the property of the corresponding linear operators, which will be done in the following.

In the region
$$D_1 = \Big\{ (x_1, x_2) : |x| < \check{r}_{2\varepsilon} - \hat{\tau}_1/\varepsilon \Big\} \setminus D_2,$$
far from the vortex core region, directly from the form of the ansatz $u = u_2 e^{i\psi}$ with the approximation as
$$u_2(x_1, x_2) \,=\, \sqrt{1+\check{V}}\,\eta_1\, e^{i(\varphi_0+\varphi_1)} \,+\, w(\ell_2)w(\ell_1)\,\eta_2\, e^{i(\varphi_0+\varphi_1)},$$
we see that the equation (7.23) takes the simple form
$$\begin{aligned} L_1(\psi) \,\equiv\, & \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi + 2\frac{\nabla u_2}{u_2}\cdot\nabla\psi - 2i|u_2|^2\psi_2 + i\varepsilon|\log\varepsilon|\kappa\,\frac{\partial\psi}{\partial x_2} \\ =\, & E_1 \,-\, i(\nabla\psi)^2 \,+\, i|u_2|^2\big(1-e^{-2\psi_2}+2\psi_2\big), \end{aligned}$$
where $E_1 = i\mathbb{E}/u_2$. We intend next to describe in more accurate form the equation above. Let us also write
$$u_2 \,=\, e^{i\varphi}\beta_1 \quad \text{with } \beta_1 \,=\, \sqrt{1+\check{V}}\,\eta_1 \,+\, w(\ell_2)w(\ell_1)\,\eta_2.$$
For $|x| < \check{r}_{2\varepsilon} - 2\hat{\tau}_0/\varepsilon$, there holds,
$$u_2 \,=\, \beta_1 e^{i\varphi} = \sqrt{1+\check{V}}\,e^{i\varphi},$$
and hence, by using the assumption ($\mathbf{P}3$), we have
$$|u_2|^2 \,=\, 1+\check{V} > \hat{c}_1.$$
Direct computation also gives that
$$\begin{aligned} 2\frac{\nabla u_2}{u_2}\cdot\nabla\psi \,=\, & \frac{2}{\beta_1}\nabla\beta_1\cdot\nabla\psi_1 \,-\, 2\nabla\varphi\cdot\nabla\psi_2 \,+\, i\frac{2}{\beta_1}\nabla\beta_1\cdot\nabla\psi_2 \,+\, 2i\nabla\varphi\cdot\nabla\psi_1 \\ =\, & (A_1,0)\cdot\nabla\psi_1 \,-\, (A_2,B_2)\cdot\nabla\psi_2 \,+\, i(A_1,0)\cdot\nabla\psi_2 \,+\, i(A_2,B_2)\cdot\nabla\psi_1, \end{aligned}$$
where $A_1 = O(\varepsilon)$, $A_2 = O(\varepsilon)$, $B_2 = O(\varepsilon)$. For $\check{r}_{1\varepsilon} - 2\hat{\tau}_0/\varepsilon < |x| < \check{r}_{1\varepsilon} - \hat{\tau}_0/\varepsilon$, similar estimates hold. The equations become
$$\tilde{L}_1(\psi_1) \,=\, \tilde{E}_1 \,+\, \tilde{N}_1, \qquad \bar{L}_1(\psi_2) \,=\, \bar{E}_1 \,+\, \bar{N}_1.$$
In the above, we have denoted the linear operators by
$$\begin{aligned} \tilde{L}_1(\psi_1) \,\equiv\, & \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_1 \,-\, \varepsilon|\log\varepsilon|\,\kappa\,\frac{\partial\psi_2}{\partial x_2} \,+\, \frac{2}{\beta_1}\nabla\beta_1\cdot\nabla\psi_1, \\ \bar{L}_1(\psi_2) \,\equiv\, & \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_2 \,-\, 2|u_2|^2\psi_2 \,+\, \varepsilon|\log\varepsilon|\,\kappa\,\frac{\partial\psi_1}{\partial x_2} \,+\, \frac{2}{\beta_1}\nabla\beta_1\cdot\nabla\psi_2. \end{aligned}$$
The nonlinear operators are
$$\begin{aligned} \tilde{N}_1 \,=\, & -2\nabla\varphi\cdot\nabla\psi_2 \,+\, 2\nabla\psi_1\cdot\nabla\psi_2, \\ \bar{N}_1 \,=\, & 2\nabla\varphi\cdot\nabla\psi_1 \,+\, |u_2|^2\big(1-e^{-2\psi_2}+2\psi_2\big) \,+\, |\nabla\psi_1|^2 \,-\, |\nabla\psi_2|^2. \end{aligned}$$

Consider the vortex-core region $D_{21}$ and $D_{22}$. Here we only argue in the region $D_{22}$. It is more convenient to do this in the translated variable $(s_1, s_2) = (x_1 - \check{r}_{1\varepsilon}, x_2)$ and then denote $\ell = \ell_2$ for brevity of notation. Now the term $\psi$ is small, however possibly unbounded near the vortex. Whence, in the sequel, by setting

$$\tilde{\phi} \,=\, iv_2\psi \quad \text{with } \psi = \psi_1 + i\psi_2, \tag{7.24}$$

we shall require that $\tilde{\phi}$ is bounded (and smooth) near the vortices. We shall write the equation in term of a type of the function $\tilde{\phi}$ for $\ell < \delta/\varepsilon$. In the region $D_{22}$, let us write $u_2$, i.e. $v_2$, as the form

$$v_2 \,=\, \beta U_0 \quad \text{ with } \beta \,=\, w(\ell_1)e^{-i\varphi_0^- + i\varphi_1}, \tag{7.25}$$

where $U_0$ is defined in (2.1). We define the function

$$\phi(s) \,=\, iU_0\psi \quad \text{for } |s| < \delta/\varepsilon, \tag{7.26}$$

namely

$$\tilde{\phi} = \beta\phi.$$

Hence, in the translated variable, the ansatz becomes in this region

$$u_2 \,=\, \beta(s)U_0 \,+\, \beta(s)\phi \,+\, (1-\chi)\beta(s)U_0\Big(e^{\phi/U_0} - 1 - \frac{\phi}{U_0}\Big).$$

We also call $\Gamma_{2,2} = (1-\chi)U_0\Big(e^{\phi/U_0} - 1 - \frac{\phi}{U_0}\Big)$. The support of this function is contained in set $|s| > 1$. In this vortex-core region, the problem, written in $(s_1, s_2)$ coordinates, can be stated as

$$L_{2,2}(\phi) \,=\, E_{2,2} \,+\, N_{2,2}.$$

Let us consider the linear operator defined in the following way: for $\phi$ and $\psi$ linked through formula (7.26) we set

$$\begin{aligned} L_{2,2}(\phi) \,=\, & L_0(\phi) \,+\, \frac{1}{s_1 + \check{r}_{1\varepsilon}}\frac{\partial}{\partial s_1}\phi \,+\, 2\big(1-|\beta|^2\big)\mathrm{Re}\big(\bar{U}_0\phi\big)U_0 \,+\, i\varepsilon|\log\varepsilon|\,\kappa\,\frac{\partial\phi}{\partial x_2} \\ & +\, \left[\varepsilon\frac{\partial\check{V}}{\partial\tilde{r}}\Big|_{\left(\check{r}_{1\varepsilon}+\vartheta s_1, 0\right)} + 1 - |\beta|^2\right]\phi \,+\, 2\frac{\nabla\beta}{\beta}\cdot\nabla\phi \,+\, \chi\frac{E_{2,2}}{U_0}\phi, \end{aligned}$$

where $\vartheta$ is a small constant. Here we also have defined $L_0$ as

$$L_0(\phi) = \Big(\frac{\partial^2}{\partial s_1^2} + \frac{\partial^2}{\partial s_2^2}\Big)\phi + (1-|w|^2)\phi - 2\mathrm{Re}\big(\bar{U}_0\phi\big)U_0.$$

Here, by writing the error $\mathbb{E}$ in the translated variable $s$, the error $E_{2,2}$ is given by

$$E_{2,2} \,=\, \mathbb{E}/\beta.$$

Observe that, in the region $D_{22}$, the error $E_{2,2}$ takes the expression

$$\begin{aligned} E_{2,2} \,=\, & w(\ell_2)e^{i\varphi_0^+}\left[\frac{x_1 - \check{r}_{1\varepsilon}}{x_1\ell_2}\frac{w'(\ell_2)}{w(\ell_2)} \,+\, \varepsilon\,\frac{\partial\check{V}}{\partial\tilde{r}}\Big|_{(\varepsilon\check{r}_{1\varepsilon},0)}(x_1 - \check{r}_{1\varepsilon})\right] \\ & +\, w(\ell_2)e^{i\varphi_0^+}\,\frac{2(x_1 + \check{r}_{1\varepsilon})(x_1 - \check{r}_{1\varepsilon})}{(\ell_1\ell_2)^2}\log\check{r}_{1\varepsilon} \,-\, iw(\ell_2)e^{i\varphi_0^+}\,\frac{x_2}{\ell_2}\,\frac{w'(\ell_2)}{w(\ell_2)}\,\frac{1}{\check{r}_{1\varepsilon}}\log\check{r}_{1\varepsilon} \\ & +\, \varepsilon|\log\varepsilon|\kappa\,w(\ell_2)e^{i\varphi_0^+}\left[i\,\frac{x_2}{\ell_2}\,\frac{w'(\ell_2)}{w(\ell_2)} \,-\, \frac{2(x_1^2 - x_2^2 - (\check{r}_{1\varepsilon})^2)\check{r}_{1\varepsilon}}{(\ell_1\ell_2)^2}\right] \,+\, O(\varepsilon\log\ell_2). \end{aligned} \tag{7.27}$$

while the nonlinear term is given by

$$
\begin{aligned}
N_{2,2}(\phi) = & - \frac{\triangle(\beta\Gamma_{2,2})}{\beta} + \Big(1+\check{V}-|U_0|^2\Big)\Gamma_{2,2} - 2|\beta|^2\mathrm{Re}(\bar{U}_0\phi)\big(\phi+\Gamma_{2,2}\big) \\
& - S_2[\Gamma] - \Big(2|\beta|^2\mathrm{Re}(\bar{U}_0\Gamma_{2,2}) + |\beta|^2\big|\phi+\Gamma_{2,2}\big|^2\Big)\big(U_0+\phi+\Gamma_{2,2}\big) \\
& + (\chi-1)\frac{E_{2,2}}{U_0}\phi.
\end{aligned}
\tag{7.28}
$$

Taking into account to the explicit form of the function $\beta$ we get

$$
\bigtriangledown\beta = O(\varepsilon), \quad \triangle\beta = O(\varepsilon^2), \quad |\beta| \sim 1 + O(\varepsilon^2),
$$

provided that $|s| < \delta/\varepsilon$. With this in mind, we see that the linear operator is a small perturbation of $L_0$.

In the region

$$
D_{2,4} = \big\{(x_1,x_2) : \ell_2 < \hat{\tau}_0/\varepsilon\big\} \setminus D_2,
$$

far from the vortex core, directly from the form of the ansatz $u = \big(1-\chi\big)u_2 e^{i\psi}$, we see that, for $\ell_2 > 2$, the equation takes the simple form

$$
\begin{aligned}
L_{2,4}(\psi) \equiv & \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi + 2\frac{\bigtriangledown u_2}{u_2}\cdot\bigtriangledown\psi - 2i|u_2|^2\psi_2 + i\varepsilon|\log\varepsilon|\kappa\frac{\partial\psi}{\partial x_2} \\
= & \ E_{2,4} - i(\bigtriangledown\psi)^2 + i|u_2|^2\big(1-e^{-2\psi_2}+2\psi_2\big),
\end{aligned}
$$

where $E_{2,4} = i\mathbb{E}/u_2$. We intend next to describe in more accurate form the equation above. As before, let us also write

$$
u_2 = \beta U_0 \quad \text{with } \beta = w(\ell_1)e^{-i\varphi_0^- + i\varphi_1}.
$$

For $\ell_2 < \hat{\tau}_0/\varepsilon$, there are two real functions $A$ and $B$ such that

$$
\beta = e^{iA+B},
$$

furthermore, a direct computation shows that, in this region, there holds

$$
\bigtriangledown A = O(\varepsilon), \qquad \triangle A = O(\varepsilon^2), \qquad \bigtriangledown B = O(\varepsilon^3), \qquad \triangle B = O(\varepsilon^4).
$$

The equations become

$$
\tilde{L}_{2,4}(\psi_1) = \tilde{E}_{2,4} + \tilde{N}_{2,4}, \qquad \bar{L}_{2,4}(\psi_2) = \bar{E}_{2,4} + \bar{N}_{2,4}.
$$

In the above, we have denoted the linear operators by

$$
\begin{aligned}
\tilde{L}_{2,4}(\psi_1) \equiv & \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_1 + \Big(\bigtriangledown B + \frac{w'(\ell_2)}{w(\ell_2)}\frac{s}{\ell_2}\Big)\cdot\bigtriangledown\psi_1 - \varepsilon|\log\varepsilon|\kappa\frac{\partial\psi_2}{\partial x_2}, \\
\bar{L}_{2,4}(\psi_2) \equiv & \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_2 - 2|u_2|^2\psi_2 \\
& + 2\Big(\bigtriangledown B + \frac{w'(\ell_2)}{w(\ell_2)}\frac{s}{\ell_2}\Big)\cdot\bigtriangledown\psi_2 + \varepsilon|\log\varepsilon|\kappa\frac{\partial\psi_1}{\partial x_2},
\end{aligned}
$$

where we have used $s = (x_1 - \check{r}_{1\varepsilon}, x_2)$. The nonlinear operators are

$$
\tilde{N}_{2,4} = -2(\bigtriangledown A + \bigtriangledown\varphi_0^+)\cdot\bigtriangledown\psi_2 + 2\bigtriangledown\psi_1\bigtriangledown\psi_2,
$$

$$\bar{N}_{2,4} \;=\; -2(\triangledown A + \triangledown \varphi_0^+)\cdot \triangledown \psi_1 \;+\; |u_2|^2\big(1 - e^{-2\psi_2} + 2\psi_2\big) \;+\; |\triangledown \psi_1|^2 \;-\; |\triangledown \psi_2|^2.$$

In the region

$$D_{3,1} = \big\{(x_1,x_2) : \check{r}_{2\varepsilon} - \hat{\tau}_1/\varepsilon < |x| < \check{r}_{2\varepsilon} + \hat{\tau}_2/\varepsilon\big\},$$

the approximation takes the form

$$u_2 \;=\; w(\ell_2)w(\ell_1)\,\eta_2\, e^{i\varphi} \;+\; \hat{q}(x_1,x_2)\,\eta_3\, e^{i\varphi}.$$

We write the ansatz as

$$u \;=\; u_2 \;+\; ie^{i\varphi}\psi \;+\; \Gamma_{3,1},$$

where $\Gamma_{3,1}$ is defined as

$$\Gamma_{3,1} \;=\; i\eta_2\Big(w(\ell_1)w(\ell_2) - 1\Big)e^{i\varphi}\psi \;+\; \eta_2 w(\ell_1)w(\ell_2)e^{i\varphi}\Big(e^{i\psi} - 1 - i\psi\Big).$$

The equation becomes

$$\begin{aligned}
L_{3,1}[\psi] \;\equiv\;& S_0[\psi] \;+\; 2i\triangledown\varphi\cdot\triangledown\psi \;-\; |\triangledown\varphi|^2\psi \;+\; iS_0[\varphi]\psi \;+\; i\varepsilon|\log\varepsilon|\kappa\,\frac{\partial\psi}{\partial x_2}\\
&+\;\Big(1 + \check{V} - |u_2|^2\Big)\psi \;+\; 2ie^{-i\varphi}\mathrm{Re}\big(\bar{u}_2 ie^{i\varphi}\psi\big)u_2\\
=\;& E_{3,1} \;+\; N_{3,1},
\end{aligned}$$

where $E_{3,1} = ie^{-i\varphi}\mathbb{E}$. The nonlinear operator is defined by

$$\begin{aligned}
N_{3,1}(\psi) \;=\;& ie^{-i\varphi}\Big[\,\triangle\,\Gamma_{3,1} + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Gamma_{3,1} + \big(1 + \check{V} - |u_2|^2\big)\Gamma_{3,1}\Big]\\
&-\; ie^{-i\varphi}\Big[2\mathrm{Re}(\bar{u}_2\Gamma_{3,1}) - \big|ie^{i\varphi}\psi + \Gamma_{3,1}\big|^2\Big](u_2 + ie^{i\varphi}\psi + \Gamma_{3,1})\\
&-\; 2ie^{-i\varphi}\mathrm{Re}(\bar{u}_2 ie^{i\varphi}\psi)(ie^{i\varphi}\psi + \Gamma_{3,1}) \;+\; i\varepsilon|\log\varepsilon|\kappa\,\frac{\partial\Gamma_{3,1}}{\partial x_2}.
\end{aligned}$$

More precisely, in the region $D_{3,1}$, the linear operator $L_{3,1}$ is defined as

$$\begin{aligned}
L_{3,1}[\psi] \;=\;& S_0[\psi] \;-\; \big(\delta_\varepsilon(\ell - \check{r}_{2\varepsilon}) + \hat{q}^2\big)\psi \;+\; 2ie^{-i\varphi}\mathrm{Re}\big(\bar{u}_2 ie^{i\varphi}\psi\big)u_2\\
&+\;\Big[1 + \check{V} + \delta_\varepsilon(\ell - \check{r}_{2\varepsilon})\Big]\psi\\
&+\; i\varepsilon|\log\varepsilon|\kappa\,\frac{\partial\psi}{\partial x_2} \;+\; 2i\triangledown\varphi\cdot\triangledown\psi \;+\; S_0[\varphi]\psi \;-\; |\triangledown\varphi|^2\psi.
\end{aligned}$$

where we have used the definition of $\hat{q}$ in (3.31). We shall analyze other terms in the linear operator $L_{3,1}$. For $\check{r}_{2\varepsilon} - \hat{\tau}_1/\varepsilon < |x| < \check{r}_{2\varepsilon} + \hat{\tau}_2/\varepsilon$, there holds $u_2 = \hat{q}e^{i\varphi}$. It is obvious that

$$2ie^{-i\varphi}\mathrm{Re}\big(\bar{u}_2 ie^{i\varphi}\psi\big)u_2 \;=\; -2i\hat{q}^2\psi_2.$$

For $\check{r}_{2\varepsilon} - 2\hat{\tau}_1/\varepsilon < |x| < \check{r}_{2\varepsilon} - \hat{\tau}_1/\varepsilon$, there holds

$$u_2 \;=\; w(\ell_2)w(\ell_1)\,\eta_2\, e^{i\varphi} \;+\; \hat{q}\,\eta_3\, e^{i\varphi}.$$

The equations become

$$\begin{aligned}
\tilde{L}_{3,1}[\psi_1] \;\equiv\;& \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)\psi_1 \;-\; \big(\delta_\varepsilon(\ell - \check{r}_{2\varepsilon}) + \hat{q}^2\big)\psi_1 \;-\; \varepsilon|\log\varepsilon|\kappa\,\frac{\partial\psi_2}{\partial x_2}\\
&+\;\Big[1 + \check{V} + \delta_\varepsilon(\ell - \check{r}_{2\varepsilon})\Big]\psi_1\\
&+\;\frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_1 \;-\; 2\triangledown\varphi\cdot\triangledown\psi_2 \;+\; S_0[\varphi]\psi_1 \;-\; |\triangledown\varphi|^2\psi_1\\
=\;& \tilde{E}_{3,1} \;+\; \tilde{N}_{3,1},
\end{aligned}$$

$$\begin{aligned}\bar{L}_{3,1}[\psi_2] \,\equiv\, & \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_2 \,-\, \big(\delta_\varepsilon(\ell-\check{r}_{2\varepsilon})+3\hat{q}^2\big)\psi_2 \,+\, \varepsilon|\log\varepsilon|\kappa\,\frac{\partial\psi_1}{\partial x_2}\\ &+\,\Big[1+\check{V}+\delta_\varepsilon(\ell-\check{r}_{2\varepsilon})\Big]\psi_2\\ &+\,\frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_2 \,+\, 2\bigtriangledown\varphi\cdot\bigtriangledown\psi_1 \,+\, S_0[\varphi]\psi_2 \,-\, |\bigtriangledown\varphi|^2\psi_2\\ =\, & \bar{E}_{3,1} \,+\, \bar{N}_{3,1}.\end{aligned}$$

Now, there hold

$$\Xi_{3,1} \,\equiv\, 1+\check{V}+\delta_\varepsilon(\ell-\check{r}_{2\varepsilon}) \,=\, \frac{1}{2}\varepsilon^2\check{V}''(\check{r}_{2\varepsilon})\big(\ell-\check{r}_{2\varepsilon}\big)^2 \,+\, O\Big(\big(\ell-\check{r}_{2\varepsilon}\big)^3\Big).$$

The other terms with $\varphi_0$ are also lower order terms. Whence the linear operators $\tilde{L}_{3,1}$ and $\bar{L}_{3,1}$ are small perturbations of the following linear operators

$$\begin{aligned}L_{31*}[\psi_1] \,&\equiv\, \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_1 \,-\, \big(\delta_\varepsilon(\ell-\check{r}_{2\varepsilon})+\hat{q}^2\big)\psi_1,\\ L_{31**}[\psi_2] \,&\equiv\, \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_2 \,-\, \big(\delta_\varepsilon(\ell-\check{r}_{2\varepsilon})+3\hat{q}^2\big)\psi_2.\end{aligned}$$

In the region $D_{3,2}=\big\{(x_1,x_2):|x|>\check{r}_{2\varepsilon}+\hat{\tau}_2/\varepsilon\big\}$, the approximation takes the form

$$u_2 \,=\, \hat{q}e^{i\varphi},$$

and the ansatz is

$$u \,=\, u_2 \,+\, ie^{i\varphi}\psi.$$

The equation becomes

$$\begin{aligned}L_{3,2}[\psi] \,\equiv\, & S_0[\psi] \,+\, \big(1+\check{V}\big)\psi \,-\, |u_2|^2\psi \,+\, 2ie^{-i\varphi}\mathrm{Re}\big(\bar{u}_2ie^{i\varphi}\psi\big)u_2\\ &+\, i\varepsilon|\log\varepsilon|\kappa\,\frac{\partial\psi}{\partial x_2} \,-\, |\bigtriangledown\varphi|^2\psi \,+\, iS_0[\varphi]\psi \,+\, 2i\bigtriangledown\varphi\cdot\bigtriangledown\psi\\ =\, & E_{3,2} \,+\, N_{3,2},\end{aligned}$$

where $E_{3,2}=ie^{-i\varphi}\mathbb{E}$. The nonlinear operator is defined by

$$N_{3,2}(\psi) \,=\, -\,ie^{-i\varphi}(u_2+ie^{i\varphi}\psi)|\psi|^2 \,+\, 2i\mathrm{Re}(\bar{u}_2ie^{i\varphi}\psi)\psi.$$

More precisely, for other term, we have

$$-|u_2|^2\psi+2ie^{-i\varphi}\mathrm{Re}\big(\bar{u}_2ie^{i\varphi}\psi\big)u_2 \,=\, -\hat{q}^2\psi_1-3i\hat{q}^2\psi_2.$$

The equations are

$$\begin{aligned}\tilde{L}_{3,2}[\psi_1] \,\equiv\, & \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_1 \,+\, \big(1+\check{V}\big)\psi_1 \,-\, \varepsilon|\log\varepsilon|\kappa\,\frac{\partial\psi_2}{\partial x_2}\\ &-\,\hat{q}\,\psi_1 \,+\, \frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_2 \,-\, |\bigtriangledown\varphi|^2\psi_1 \,+\, iS_0[\varphi]\psi_1 \,-\, 2\bigtriangledown\varphi\cdot\bigtriangledown\psi_2\\ =\, & \tilde{E}_{3,2} \,+\, \tilde{N}_{3,2},\end{aligned}$$

$$
\begin{aligned}
\bar{L}_{3,2}[\psi_2] \,\equiv\, & \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_2 \,+\, \big(1+\check{V}\big)\psi_2 \,+\, \varepsilon|\log\varepsilon|\kappa\,\frac{\partial\psi_1}{\partial x_2}\\
& -\,\hat{q}\,\psi_2 \,+\, \frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_2 \,-\, |\bigtriangledown\varphi|^2\psi_2 \,+\, iS_0[\varphi]\psi_2 \,+\, 2\bigtriangledown\varphi\cdot\bigtriangledown\psi_1\\
=\, & \bar{E}_{3,2} \,+\, \bar{N}_{3,2}.
\end{aligned}
$$

The assumption (**P3**) implies that, for any sufficiently small $\varepsilon$ there holds

$$
\Xi_{3,2} \,=\, 1+\check{V} < -\hat{c}_2 \quad \text{for } |x| > \check{r}_{2\varepsilon}+\hat{\tau}_2/\varepsilon.
$$

The other terms with $\varphi_0$ are lower order terms. From the asymptotic properties of $q$ in Lemma 2.4, $\hat{q}\,\psi_2$ and $\hat{q}\,\psi_1$ are also lower order term. Whence the linear operators $L_{51}$ and $L_{52}$ are small perturbations of the following linear operator

$$
L_{32*}[\tilde{\psi}] \equiv \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\tilde{\psi} \,+\, \big(1+\check{V}\big)\tilde{\psi}.
$$

Let $\chi$ be the cut-off function defined in (7.18). By recalling the definition of $\beta$ in (7.25), we define

$$
\Lambda \equiv \frac{\partial u_2}{\partial \check{f}} \cdot \frac{\chi\big(|x-\xi_+|/\varepsilon\big) \,+\, \chi\big(|x-\xi_-|/\varepsilon\big)}{\beta}\,. \tag{7.29}
$$

In summary, for any given $\check{f}$ in (7.4), we want to solve the projected equation for $\psi$ satisfying the symmetry (7.21)

$$
\mathcal{L}(\psi) \,=\, \mathcal{N}(\psi)+\mathcal{E}+c\Lambda, \quad \mathrm{Re}\int_{\mathbb{R}^2}\bar{\phi}\Lambda \,=\, 0, \tag{7.30}
$$

where have denoted

$$
\mathcal{L}(\psi) \,=\, L_1(\phi) \quad \text{in } D_1, \quad \mathcal{L}(\psi) \,=\, L_{2,j}(\psi) \quad \text{in } D_{2,j} \text{ for } j=1,2,3,4,
$$

$$
\mathcal{L}(\psi) \,=\, L_{3,1}(\psi) \quad \text{in } D_{3,1}, \quad \mathcal{L}(\psi) \,=\, L_{3,2}(\psi) \quad \text{in } D_{3,2},
$$

with the relation

$$
\phi \,=\, iu_2\psi \quad \text{in } D_2. \tag{7.31}
$$

As we have stated, the nonlinear operator $\mathcal{N}$ and the error term $\mathcal{E}$ also have suitable local forms in different regions.

*7.4. The resolution of the projected nonlinear problem*

For fixed small positive numbers $0<\sigma<1$, $0<\gamma<1$, we define

$$
\begin{aligned}
||\psi||_* \equiv \sum_{i=1}^{2}\Big[\, & ||\phi||_{W^{2,p}(\ell_i<3)} \,+\, ||\ell_i^{\sigma}\psi_1||_{L^\infty(\tilde{D})} + ||\ell_i^{1+\sigma}\bigtriangledown\psi_1||_{L^\infty(\tilde{D})}\\
& +\, ||\ell_i^{1+\sigma}\psi_2||_{L^\infty(\tilde{D})} + ||\ell_i^{2+\sigma}\bigtriangledown\psi_2||_{L^\infty(\tilde{D})}\Big] \,+\, ||\psi||_{W^{2,p}(D_3)},
\end{aligned}
$$

where we have used the relation $\phi = iu_2\psi$ and $\tilde{D}$ is defined in (7.16).

We now consider the following linear projected problem: finding $\psi$ with the symmetry in (7.21)

$$
\mathcal{L}[\psi] \,=\, h \,+\, c\Lambda, \quad \mathrm{Re}\int_{\mathbb{R}^2}\bar{\phi}\Lambda \,=\, 0 \quad \text{ with } \phi = iu_2\psi. \tag{7.32}
$$

**Lemma 7.2.** *There exists a constant $C$, depending on $\gamma, \sigma$ only, such that for all $\varepsilon$ sufficiently small, the following holds: if $||h||_{**} < +\infty$, there exists a unique solution $(\psi_{\varepsilon,\check f}, c_{\varepsilon,\check f}) = \mathcal{T}_{\varepsilon,\check f}(h)$ to (7.32). Furthermore, there holds*

$$||\psi||_* \leq C||h||_{**}.$$

**Proof.** The proof is similar as that in Lemma 5.2. □

We then consider the following problem: finding $\psi$ with the symmetry in (7.21)

$$\mathcal{L}[\psi] + \mathcal{N}[\psi] = \mathcal{E} + c\Lambda, \quad \mathrm{Re}\int_{\mathbb{R}^2} \bar\phi\Lambda = 0 \quad \text{ with } \phi = iu_2\psi. \tag{7.33}$$

**Proposition 7.3.** *There exists a constant $C$, depending on $\gamma, \sigma$ only, such that for all $\varepsilon$ sufficiently small, there exists a unique solution $(\psi_{\varepsilon,\check f}, c_{\varepsilon,\check f})$ to (7.33), and*

$$||\psi||_* \leq C||h||_{**}.$$

*Furthermore, $\psi$ is continuous in the parameter $\check f$.*

**Proof.** The proof is similar as that in Proposition 5.3. □

*7.5. Reduction procedure*

To find a real solution to problem (7.2)-(7.3), in this subsection, we solve the reduced problem by finding a suitable $\check f$ such that the constant $c$ in (7.30) is identical zero for any sufficiently small $\varepsilon$.

In previous subsection, for any given $\check f$ in (7.4), we have deduced the existence of $\psi$ with the symmetry (7.21) to the projected problem

$$\mathcal{L}(\psi) = \mathcal{N}(\psi) + \mathcal{E} + c\Lambda, \quad \mathrm{Re}\int_{\mathbb{R}^2} \bar\phi\Lambda = 0, \tag{7.34}$$

with the relation

$$\phi = iu_2\psi \quad \text{in } D_2.$$

Multiplying (7.34) by $\bar\Lambda$ and integrating, we obtain

$$c\,\mathrm{Re}\int_{\mathbb{R}^2} \bar\Lambda\Lambda = \mathrm{Re}\int_{\mathbb{R}^2} \bar\Lambda\mathcal{L}(\psi) - \mathrm{Re}\int_{\mathbb{R}^2} \bar\Lambda\mathcal{N}(\psi) - \mathrm{Re}\int_{\mathbb{R}^2} \bar\Lambda\mathcal{E}.$$

Hence we can derive the estimate for $c$ by computing the integrals of the right hand side.

We begin with the computation of $\mathrm{Re}\int_{\mathbb{R}^2} \bar\Lambda\mathcal{E}$. The term $\Lambda$ has its support contained in the region $\{(x_1, x_2) : \ell_1 < 1/\varepsilon \text{ or } \ell_2 < 1/\varepsilon\}$. It is convenient to compute $\mathrm{Re}\int_{\mathbb{R}^2} \bar\Lambda\mathcal{E}$ on the variables $(s_1, s_2)$. Note that, in the vortex-core region, there holds

$$\frac{\partial u_2}{\partial \check f} = \Big[ -\frac{w'(\ell_2)}{w(\ell_2)} \frac{x_1 - \check r_{1\varepsilon}}{\ell_2} + i\,\frac{x_2}{(\ell_2)^2}\Big] u_2 + O(\varepsilon^2)u_2,$$

which implies that

$$\Lambda = \chi\big(|x - \xi_+|/\varepsilon\big)\Big[ -\frac{w'(\ell_2)}{w(\ell_2)} \frac{x_1 - \check r_{1\varepsilon}}{\ell_2} + i\,\frac{x_2}{(\ell_2)^2}\Big] w(\ell_2)e^{i\varphi_0^+} + O(\varepsilon^2).$$

By using of the formula (7.27), we obtain

$$\mathrm{Re}\int_{\mathbb{R}^2}\bar{\Lambda}\mathcal{E}\mathrm{d}x\ \equiv\ I_1\ +\ I_2,$$

where $I_1$ is defined in (6.4). In the above, we have denoted

$$\begin{aligned}I_2\ =\ &2\,\kappa\,\varepsilon\log\frac{1}{\varepsilon}\int_{\mathbb{R}^2_+}\chi\big(\ell_2/\varepsilon\big)w(\ell_2)w'(\ell_2)\frac{2\big[x_1^2-x_2^2-(\check{r}_{1\varepsilon})^2\big]\check{r}_{1\varepsilon}(x_1-\check{r}_{1\varepsilon})}{(\ell_1)^2(\ell_2)^3}\,\mathrm{d}x\\&+\ 2\,\kappa\,\varepsilon\log\frac{1}{\varepsilon}\int_{\mathbb{R}^2_+}\chi\big(\ell_2/\varepsilon\big)w(\ell_2)w'(\ell_2)\frac{x_2^2}{(\ell_2)^3}\,\mathrm{d}x\ +\ O(\varepsilon).\end{aligned}$$

As we have done in section 6, we get $I_2\ =\ 2d\,\pi\,\kappa\,\varepsilon|\log\varepsilon|+O(\varepsilon)$. Hence, there holds

$$\mathrm{Re}\int_{\mathbb{R}^2}\bar{\Lambda}\mathcal{E}\mathrm{d}x\ =\ -2\pi\varepsilon\,\frac{\partial\check{V}}{\partial\tilde{r}}\Big|_{(\varepsilon\check{r}_{1\varepsilon},0)}\ \log\frac{1}{\varepsilon}\ -\ 2d\,\pi\,\frac{1}{\check{r}_{1\varepsilon}}\log\check{r}_{1\varepsilon}\ +\ 2d\,\pi\,\kappa\,\varepsilon|\log\varepsilon|\ +\ O(\varepsilon).$$

Using Proposition 7.3, and the expression in (7.28), we deduce that

$$\mathrm{Re}\int_{\mathbb{R}^2}\bar{\Lambda}\mathcal{N}(\psi)\ =\ \mathrm{Re}\int_{\mathbb{R}^2}\bar{\Lambda}N_2(\psi)\ =\ O(\varepsilon).$$

On the other hand, integration by parts, we have

$$\mathrm{Re}\int_{\mathbb{R}^2}\bar{\Lambda}\mathcal{L}(\psi)\ =\ \mathrm{Re}\int_{\mathbb{R}^2}\bar{\psi}\mathcal{L}(\bar{\Lambda})\ =\ O(\varepsilon).$$

Combining all estimates together and recalling $\varepsilon\,\check{r}_{1\varepsilon}\,=\,\hat{r}_{1\varepsilon}\,=\,\hat{r}_0+\hat{f}$, we obtain the following equation

$$c(\hat{f})\ =\ -2\,\pi\,\varepsilon\left[\frac{\partial\check{V}}{\partial\tilde{r}}\Big|_{(\hat{r}_0+\hat{f},0)}\ \log\frac{1}{\varepsilon}\ +\ \frac{d}{\hat{r}_0+\hat{f}}\ \log\frac{\hat{r}_0+\hat{f}}{\varepsilon}\ -\ d\,\kappa\ \log\frac{1}{\varepsilon}\right]\ +\ O(\varepsilon),\tag{7.35}$$

where $O(\varepsilon)$ is a continuous function of the parameter $\hat{f}$. By simple mean-value theorem and the solvability condition (1.13) and the non-degeneracy condition (1.14), we can find a zero of $c(\hat{f})$ at some small $\hat{f}$.

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## References

[1] A. Aftalion, *Vortices in Bose-Einstein Condensates*, Progress in Nonlinear Differential Equations and their Applications, 67. Birkhauser Boston, Inc. Boston Ma, 2000.

[2] D. G. Akhmetov, *Vortex Rings*, Springer-Verlag, Berlin Heidelberg, 2009.

[3] S. V. Alekseenko, P. A. Kuibin, V. L. Okulov, *Theory of Concentrated Vortices: An Introduction*, Springer, Heidelberg, 2007.

[4] B. Anderson, P. Haljan, C. Regal, D. Feder, L. Collins, C. Clark and E. Cornell, Watching Dark Solitons Decay into Vortex Rings in a Bose-Einstein Condensate, *Physical Review Letters* 86 (2001), no. 14, 2926-2929.

[5] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman and E. A. Cornell, Observation of Bose-Einstein Condensation in a Dilute Atomic Vapor, *Science* 269 (1995), 198-201.

[6] C. Barenghi, Is the Reynolds number infinite in superfluid turbulence? *Physica D* 237 (2008), 2195-2202.

[7] C. Barenghi and R. Donnell, Vortex rings in classical and quantum systems, *Fluid Dyn. Res.* 41 (2009), 051401.

[8] F. Bethuel, H. Brezis and F. Helein, *Ginzburg-Landau vortices*, Progress in Nonlinear Differential Equations and their Applications, 13. Birkhauser Boston, Inc. Boston Ma, 1994.

[9] F. Bethuel, P. Gravejat and J. Saut, Existence and properties of travelling waves for the Gross-Pitaevskii equation, *Stationary and time dependent Gross-Pitaevskii equations*, 55-103, Contemp. Math., 473, Amer. Math. Soc., Providence, RI, 2008.

[10] F. Bethuel, P. Gravejat and J.-G. Saut, Travelling waves for the Gross-Pitaevskii equation, II, *Comm. Math. Phys.* 285 (2009), no. 2, 567-651.

[11] F. Bethuel, G. Orlandi and D. Smets, Vortex rings for the Gross-Pitaevskii equation, *J. Eur. Math. Soc.* 6 (2004), no.1, 17-94.

[12] F. Bethuel and J.-C. Saut, Travelling waves for the Gross-Pitaevskii equation I, *Ann. Inst. Henri Poincar, Physique Thorique.* 70 (1999), no. 2, 147-238.

[13] H. Brezis, F. Merle and T. Rieviere, Quantization effects for $-\triangle u = u(1-|u|^2)$ in $\mathbb{R}^2$, *Arch. Ration. Mech. Anal.* 126 (1994), no. 1, 35-58.

[14] X. Chen, C. M. Elliott and Q. Tang, Shooting method for vortex solutions of a complex-valued Ginzburg-Landau equation, *Proc. Roy. Soc. Edinburgh Sect. A* 124 (1994), no. 6, 1075-1088.

[15] D. Chiron, Travelling waves for the Gross-Pitaevskii equation in dimension larger than two, *Nonlinear Anal.* 58 (2004), no. 1-2, 175-204.

[16] M. del Pino and P. Felmer, Locally energy-minimizing solution of the Ginzburg-Landau equation, *C. R. Acad. Sci. Paris Sr. I Math.* 321 (1995), no. 9, 1207-1211.

[17] M. del Pino and P. Felmer, Local minimizers for the Ginzburg-Landau energy, *Math. Z.* 225 (1997), no. 4, 671-684.

[18] M. del Pino, P. Felmer and M. Kowalczyk, Minimlity and nondegeneracy of degree-one Ginzburg-Landau vortex as a Hardy's type inequality, *Int. Math. Res. Not.* (2004), no. 30, 1511-1627.

[19] M. del Pino, M. Kowalczyk and M. Musso, Variational reduction for Ginzburg-Landau vortices, *J. Funcational Analysis* 239 (2006), 497-541.

[20] R. Donnelly, *Quantized vortices in Helium II*, Cambridge University Press, Cambredge, U. K. 1991.

[21] D. L. Feder, M. S. Pindzola, L. A. Collins, B. I. Schneider and C. W. Clark, Dark-soliton states of Bose-Einstein condensates in anisotropic traps, *Physical Review A* 62 (2000), no. 5, 053606.

[22] A. L. Fetter, Rotating trapped Bose-Einstein condensates, *Laser Physics* 18 (2008), no. 1, 1-11.

[23] A. Fetter and D. Feder, Beyond the Thomas-Fermi approximation for a trapped condensed Bose-Einstein gas, *Phycical Review A* 58 (1998), no. 4, 3185-3194.

[24] A. L. Fetter and A. A Svidzinsky, Vortices in a trapped dilute Bose-Einstein condensate, *J. Phys.: Condens. Matter* 13 (2001), 135-194.

[25] M. Guilleumas, D. M. Jezek, R. Mayol, M. Pi and M. Barranco, Generating vortex rings in Bose-Einstein condensates in the line-source approximation, *Physical Review A* 65 (2002), no. 5, 053609.

[26] M. Guilleumas and R. Graham Off-axis vortices in trapped Bose-condensed gases: Angular momentum and frequency splitting, *Physical Review A* 64 (2001), no. 3, 033607.

[27] K.-H. Hoffmann and Q. Tang, *Ginburg-Landau phase transition theory and superconductivity*, Birkhauser Verlag, Basel, 2001.

[28] T. Horng, C. Hsueh and S. Gou, Transition to quantum turbulence in a Bose-Einstein condensate through the bending-wave instability of a single-vortex ring, *Physical Review A* 77 (2008), 063625.

[29] T. Horng, S. Gou, and T. Lin, Bending-wave instability of a vortex ring in a trapped Bose-Einstein condensate, *Physical Review A* 74 (2006), 041603(R).

[30] B. Jackson, J. F. McCann and C. S. Adams, Vortex rings and mutual drag in trapped Bose-Einstein condensates, *Physical Review A* 60 (1999), no. 6, 4882-4885.

[31] B. Jackson, J. F. McCann and C. S. Adams, Vortex line and ring dynamics in trapped Bose-Einstein condensates, *Physical Review A* 61 (1999), no. 1, 013604.

[32] S. Jimbo and Y. Morita, Stability of nonconstant steady-state solutions to a Ginzburg-Landau equation in higher space dimensions, *Nonlinear Anal.* 22 (1994), no. 6, 753-770.

[33] S. Jimbo and Y. Morita, Vortex dynamics for the Ginzburg-Landau equation with Neumann condition, *Methods Appl. Anal.* 8 (2001), no. 3, 451-477.

[34] S. Jimbo and Y. Morita, Notes on the limit equation of vortex motion for the Ginzburg-Landau equation with Neumann condition, in: Recent Topics in Mathematics Moving toward Science and Engineering, *Japan J. Indust. Appl. Math.* 18 (2001), no. 2, 483-501.

[35] S. Jimbo, Y. Morita and J. Zhai, Ginzburg-Landau equation and stable steady state solutions in a non-trivial domain, *Comm. Partial Differential Equations* 20 (1995), no. 11-12, 2093-2112.

[36] R. Kerr Numerical generation of a vortex ring cascade in quantum turbulence, arXiv:1006.3911v2.

[37] F. Lin, Solutions of Ginzburg-Landau equations and critical points of the renormalized energy, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 12 (1995), no. 5, 599-622.

[38] F. Lin, Mixed vortex-antivortex solutions of Ginzburg-Landau equations, *Arch. Ration. Mech. Anal.* 133 (1995), no. 2, 103-127.

[39] F. Lin, Some dynamical properties of Ginzburg-Landau vortices, *Comm. Pure Appl. Math.* 49 (1996), no. 4, 323-359.

[40] F. Lin and T. Lin, Minimax solutions of the Ginzburg-Landau equations, *Selecta Math. (N.S.)* 3 (1997), no. 1, 99-113.

[41] F. Lin and J. Wei, Travelling wave solutions of Schrödinger map equation, *Comm. Pure Appl. Math.* 63 (2010), no. 12, 1585-1621.

[42] E. Lundh, C. Pethick and H. Smith, Zero-temperature properties of a trapped Bose-condensed gas: beyond the Thomas-Fermi approximation, *Physical Review A* 55 (1997), no. 3, 2126-2131.

[43] K. Madison, F. Chevy, W. Wohlleben and J. Dalibard, Vortex Formation in a Stirred Bose-Einstein Condensate, *Physical Review Letters* 84 (2000), no. 5, 806-809.

[44] M. R. Matthews, B. P. Anderson, P. C. Haljan, D. S. Hall, C. E. Wieman and E. A. Cornell, Vortices in a Bose-Einstein Condensate, *Physical Review Letters* 83 (1999), no. 13, 2498-2501.

[45] P. Mironescu, On the stability of radial solutions of the Ginzburg-Landau equation, *J. Funct. Anal.* 130 (1995), no. 2, 334-344.

[46] P. Mironescu, Les minimiseurs locaux pour l'quation de Ginzburg-Landau sont symtrie radiale [Local minimizers for the Ginzburg-Landau equation are radially symmetric], *C. R. Acad. Sci. Paris Sr. I Math.* 323 (1996), no. 6, 593-598.

[47] Y. N. Ovchinnikov and I. M. Sigal, *Ginzburg-Landau equation. I. Static vortices.* Partial differential equations and their applications (Toronto, ON, 1995), 199-220, CRM Proc. Lecture Notes, 12, Amer. Math. Soc., Providence, RI, 1997.

[48] F. Pacard and T. Riviere, *Linear and nonlinear aspects of vortices.* Nonlinear Differential Equations Appl., vol. 39, Birkhauser Boston, Boston, MA, 2000.

[49] C. Pethick and H. Smith, *Bose-Einstein condensation in dilute gases.* Cambridge University Press, Cambridge, 2002.

[50] G. Rayfield and F. Reif, Evidence for The Creation and Motion of Quantized Vortex Rings in Superfluid Helium, *Physical Review Letters* 11 (1963), no. 1, 305-308.

[51] J. Ruostekoski and J. Anglin, Creating Vortex Rings and Three-Dimensional Skyrmions in Bose-Einstein Condensates, *Physical Review Letters* 86 (2001), no. 18, 3934-3937.

[52] J. Ruostekoski and Z. Dutton, Engineering vortex rings and systems for controlled studies of vortex interactions in Bose-Einstein condensates, *Physical Review A* 72 (2005), no. 6, 063626.

[53] P. Saffman, *Vortex dynamics*, Cambridge University Press, Cambridge, U. K. 1992.

[54] E. Sandier and S. Serfaty, *Vortices in the Magnetic Ginzburg-Landau Model*, Progress in Nonlinear Differential Equations and Their Applications, Vol. 70, Birkhauser, Boston, 2007.

[55] I. Shafrir, Remarks on solutions of $-\Delta u = (1 - |u|^2)u$ in $\mathbb{R}^2$, *C. R. Acad. Sci. Paris Sr. I Math.* 318 (1994), no. 4, 327-331.

[56] F. Sols, Vortex matter in atomic BoseCEinstein condensates *Physica C* 369 (2002), no. 125-134.

[57] M. Struwe, On the asymptotic behavior of minimizers of the Ginzburg-Landau model in 2 dimensions, *Differential and Integral Equations* 7 (1994), no. 5-6, 1613-1624.

[58] S. Serfaty, Stability in 2D Ginzburg-Landau passes to the limit, *Indiana Univ. Math. J.* 54 (2005), no. 1, 199-221.

[59] A. Svidzinsky and A. Fetter, Dynamics of a vortex in trapped Bose-Einstein condensate, *Physical Review A* 62 (2000), 063617: 1-14.

[60] J. E. Williams and M. J. Holland, Preparing topological states of a Bose-Einstein condensate, *Nature(London)* 401 (1999), no. 6753, 568-572.

[61] F. Zhou and Q. Zhou, A remark on multiplicity of solutions for the Ginzburg-Landau equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 16 (1999), no. 2, 255-267.