TRAVELING WAVE SOLUTIONS OF SCHRÖDINGER MAP EQUATION

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ABSTRACT. We first construct traveling wave solutions for the Schrödinger map in $\mathbb{R}^2$

$$\frac{\partial m}{\partial t} = m \times (\Delta m - m_3 \vec{e}_3) \text{ in } \mathbb{R}^2 \times \mathbb{R}$$

of the form $m(x_1, x_2 - \epsilon t)$, where m has exactly two vortices at roughly $(\pm \frac{1}{2\epsilon}, 0) \in \mathbb{R}^2$ of degree ± 1. We use a perturbative approach which gives a complete characterization of the asymptotic behavior of the solutions. With a few modifications, the similar construction yields traveling wave solutions of the Schrödinger map equations in higher dimensions.

1. INTRODUCTION

The aim of this paper is to construct traveling wave solutions for a class of Landau-Lifshitz equations. We shall first concentrate on the two-dimensional traveling wave solutions of the Schrödinger map equation

$$\frac{\partial m}{\partial t} = m \times (\Delta m - m_3 \vec{e}_3) \text{ in } \mathbb{R}^2 \times \mathbb{R} \tag{1.1}$$

or equivalently the equation

$$-m \times \frac{\partial m}{\partial t} = \Delta m - m_3 \vec{e}_3 + (|\nabla m|^2 + m_3^2) m. \tag{1.2}$$

Here $m : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{S}^2$ so that $|m(x,t)| \equiv 1$ and where $\vec{e}_3 = (0,0,1) \in \mathbb{R}^3$.

The equation (1.1) (or equivalently (1.2)) is, in fact, the Landau-Lifshitz equation describing the planar ferromagnets, that is, ferromagnets with an easy-plane anisotropy ([32], [36]). The unit normal to the easy-plane is assumed to be $\vec{e}_3$ in the equations , see for example,[36]. Despite some serious efforts (see e.g. [14], [15], [21], [20], [22] ,[9, 25, 37], [23, 24], [10, 11, 12, 13]) and the references therein), some basic mathematical issues such as local and global well-posedness and global in time asymptotics for the equation (1.1) remain unknown. If one is interested in one-dimensional wave (plane-wave) solutions of (1.1), that is, $m : \mathbb{R} \times \mathbb{R} \to S^2$ (or S^1), a lot were known as (1.1) becomes basically an integrable system (see [20] and [8]). The problem in 2-D or higher dimensions are much more subtle. Even though it is possible to obtain weak solutions of (1.1) (see [8]-[21], [34], [39]), one does not know if such weak solutions are classical (smooth) or unique.

From the physical side of (1.1), one expects topological solitons, which are half magnetic bubbles, exist in solutions of (1.1) (see [30] and [36]). Indeed, in [28]-[29], we have established the corresponding static theory for such magnetic vortices.

Date: October 29, 2009.

1991 *Mathematics Subject Classification.* 35J25, 35J20, 35B33, 35B40.

Key words and phrases. Traveling Waves, Schrödinger Maps, Vortices, Perturbation.

They are very much like vortices in the superconductor described by the Ginzburg-Landau equation, see [6] and references therein.

Let us recall the essential features of vortex dynamics in a classical fluid (or a superfluid modeled by the Gross-Pitaevskii equation, see [35]). A single vortex or antivortex is always spontaneously pinned and hence can move only together with the background fluid. However vortex motion relative to the fluid is possible in the presence of other vortices and it displays characteristics similar to the 2-D Hall motion of interacting electric charges in a uniform magnetic field. In particular, two like vortices orbit around each other while a vortex-antivortex pair undergoes Kelvin motion along parallel trajectories that are perpendicular to the line connecting the vortex and the antivortex. This latter fact had been obtained in a special case by Jones and Robert [31]. They also derived a three dimensional (3-D) solitary wave that describe a vortex ring moving steadily along its symmetric axis. For more precise mathematical proofs, we refer to [3], [5] and [4] for the case of Gross-Pitaevskii equation.

The aim of this paper is to obtain similar result as those of [3] for the Landau-Lifshitz equation (1.1). (In fact our proofs are completely different from [3]. This also gives a new approach to [3].) We note that formal arguments as well as numerical evidences were already presented in the work [36]. We should also note that in the case of the initial date of (1.1) contains only one vortex (one magnetic half-bubble), with its structure as described in the work of [28]-[29] very precisely, the above discussions imply that the vortex will simply stay at its center of mass, and a meaningful mathematical issue to examine would be its global stability. It is, however, unknown to authors that whether such stability result is true or not, see [23], [24],[25] for relevant discussions. On the other hand, it is relatively easy to generalize the work of [33] to the equation (1.1) for the planar ferromagnets. One may obtain the same Kirchhoff vortex dynamical law for these widely separated and slowly moving magnetic half-bubbles, see [26] (as formally derived in [36] and also [35] for the Gross-Pitaevskii equation) of solutions of (1.1).

In order to explain our main result, we consider two magnetic half-bubbles of different orientations (i.e, a pair of vortex and antivortex) in an initial data for the equation (1.1). When they travel (since we are looking traveling wave solutions) in the same direction at the same speed, the "Lorentz force" will try to pull them apart as the sign of charges are different. This force is proportional to the speed of the motion (we shall fix a unit positive and negative charges at these two magnetic vortices). Since the speed of the sound in the equation (1.1) is normalized to be 1 and, since we assume these magnetic vortices move rather slowly, we assume the speed of motion of these vortices is $\epsilon, 0 < \epsilon << 1$. Thus the size of the repelling Lorentz-force between these two magnetic half-bubble is $\approx C\epsilon$. This force must be balanced out by the attraction force (since they carried different signs of charges). The potential of this attraction force (later we will call it the renormalized energy as that in [6] and [28]) is simply the $\log|p-q|$, where p, q are locations of vortices. Thus the size of the attraction force world be $\approx \frac{1}{d}$, where d is the distance between two vortices. From this discussion, we conclude that $d \approx \frac{1}{\epsilon}$ (hence vortices are widely separated and move rather slowly. It is therefore in the so-called particle plus field regime of [35].)

Thus we are looking for a solution of the form $m(x_1, x_2 - \epsilon t)$ (i.e, travel in the x_2-direction with the speed $\epsilon > 0$) of the equation (1.1). Then m must be a solution of

$$-\epsilon\frac{\partial m}{\partial x_2} = m \times (\Delta m - m_3\vec{e}_3). \tag{1.3}$$

After a proper scaling in the space, (1.3) becomes

$$-\frac{\partial m}{\partial x_2} = m \times (\Delta m - \frac{m_3}{\epsilon^2}\vec{e}_3), x \in \mathbb{R}^2 \tag{1.4}$$

or

$$m \times \frac{\partial m}{\partial x_2} = \Delta m - \frac{m_3\vec{e}_3}{\epsilon^2} + (|\nabla m|^2 + \frac{m_3^2}{\epsilon^2})m. \tag{1.5}$$

Note that the distance between the two vortices of solutions of (1.4) or (1.5) is of a unity size. The main result of the paper is the following

Theorem 1.1. *For ϵ sufficiently small there is a solution $m \in C^\infty(\mathbb{R}^2, \mathbb{S}^2)$ of (1.4) such that*

$$E_\epsilon(m) = \int_{\mathbb{R}^2} \frac{1}{2}(|\nabla m|^2 + \frac{m_3}{\epsilon^2})dx < \infty \tag{1.6}$$

and that m has exactly two vortices at $(\pm a_\epsilon, 0) \in \mathbb{R}^2$ of degree ± 1, where $a_\epsilon \approx \frac{1}{2}$.

For more precise description of m, we refer to the details in the proofs, in particular, the construction of approximate solutions in Section 2 and 3 below. Naturally such solution m gives rise to a nontrivial (two-dimensional) traveling wave solution of (1.1) with a pair of vortex and antivortex which undergoes the Kelvin Motion as described above.

A similar result for the traveling vortex ring solutions are obtained in higher dimensions. More precisely, we consider the $N-$dimensional Schrödinger map equation

$$\frac{\partial m}{\partial t} = m \times (\Delta m - m_3\vec{e}_3) \text{ in } \mathbb{R}^N \times \mathbb{R} \tag{1.7}$$

and we look for traveling wave solutions of the type $m(x', x_N - ct)$ to (1.7). Then m must be a solution of

$$-c\frac{\partial m}{\partial x_N} = m \times (\Delta m - m_3\vec{e}_3). \tag{1.8}$$

Our second result concerns (1.8).

Theorem 1.2. *Let $N \geq 3$ and $c = (N-2)\epsilon|\log\frac{1}{\epsilon}|$. Then for ϵ sufficiently small there is an axially symmetric solution $m = m(|x'|, x_N) \in C^\infty(\mathbb{R}^N, \mathbb{S}^2)$ of (1.8) such that*

$$E_\epsilon(m) = \int_{\mathbb{R}^N} \frac{1}{2}(|\nabla m|^2 + \frac{m_3}{\epsilon^2})dx < \infty \tag{1.9}$$

and that m has exactly one vortex at $(|x'|, x_N) = (a_\epsilon, 0)$ of degree $+1$, where $a_\epsilon \approx \frac{1}{2}$.

Solutions constructed in Theorem 1.2 are called *traveling vortex rings.* For more precise asymptotic behavior of the solutions, we refer to Theorem 7.1 of the last section.

We end the introduction with some discussions. In the papers [3] and [4], Bethuel and Saut (when $N = 2$), Bethuel, Orlandi and Smets (when $N \geq 3$) constructed traveling wave solution for the Gross-Pitaevskii equation

$$\frac{\partial u}{\partial t} = \Delta u + u - |u|^2 u, \text{ in } \mathbb{R}^N \times \mathbb{R}. \tag{1.10}$$

Their method is variational. First they constructed a mountain-pass value for the energy functional (in bounded domains) and used variational method and fine estimates to prove the existence of traveling wave solutions to (1.10) with small speed. Here we use a completely different (and more direct) approach to prove Theorems 1.1-1.2. Our method gives more precise asymptotic behavior of solutions as the speed approaches zero. This may yield more information on the spectrum, uniqueness and dynamical properties of the traveling wave solutions. This method can be easily adopted to give a different proof of the existence of traveling wave solutions to (1.10).

2. Outline of the Proof

Before we go to the detailed proofs, we sketch our approach and some key points involved in each steps.

Setting $u = \frac{m_1 + i m_2}{1 + m_3}$. Then (1.3) becomes

$$i\epsilon \frac{\partial u}{\partial x_2} + \Delta u + \frac{1 - |u|^2}{1 + |u|^2} u = \frac{2\bar{u}}{1 + |u|^2} \nabla u \cdot \nabla u \tag{2.1}$$

where we look for solutions u satisfying

$$u(z) \to 1 \quad \text{as } |z| \to +\infty. \tag{2.2}$$

Here and throughout the paper, we use $\bar{u}$ to denote the conjugate of u and use $z = x_1 + i x_2$ to denote a point in $\mathbb{R}^2$.

When $\epsilon = 0$, (2.1) admits solutions of vortex of degree $+1$, i.e. $u = w^+ := \rho(r) e^{i\theta}$, where ρ satisfies

$$\rho^{''} + \frac{\rho^{'}}{r} - \frac{2\rho(\rho^{'})^2}{1 + \rho^2} + (1 - \frac{1}{r^2}) \frac{1 - \rho^2}{1 + \rho^2} \rho = 0. \tag{2.3}$$

Another solution $w^- := \rho(r) e^{-i\theta}$ will be of vortex of degree -1.

Theorem 1.1 follows from the following theorem.

Theorem 2.1. *For ϵ sufficiently small, problem (2.1)-(2.2) has a traveling wave solution u_ϵ with the following asymptotic behavior*

$$u_\epsilon = w^+(z - d_\epsilon \vec{e}_1) w^-(z + d_\epsilon \vec{e}_1) + \phi_\epsilon \tag{2.4}$$

where

$$d_\epsilon = \frac{1 + o(1)}{2\epsilon}, \quad \|\phi_\epsilon\|_{L^p(\mathbb{R}^2)} = o(1) \tag{2.5}$$

for any $p > 2$.

Theorem 2.1 is proved by a *finite dimensional reduction method*. The basic idea is as follows: we look for solutions of (2.1) in the form

$$u = w^+(z - d\vec{e}_1) w^-(z + d\vec{e}_1) + \phi \tag{2.6}$$

where ϕ is small in suitable norms, and $d \sim \frac{1}{\epsilon}$ is suitably chosen. We achieve this in two steps:

Step 1: For d large and ϵ small, we look for a $\phi = \phi_{\epsilon,d}$ and a Lagrange multiplier $c = c_\epsilon(d)$ such that
(2.7)
$$\begin{cases} i\epsilon\frac{\partial u}{\partial x_2} + \Delta u - \frac{2\bar{u}}{1+|u|^2}\nabla u\cdot\nabla u + \frac{1-|u|^2}{1+|u|^2}u = c\frac{\partial}{\partial d}(w^+(z-d\vec{e}_1)w^-(z+d\vec{e}_1)), \\ u = w^+(z-d\vec{e}_1)w^-(z+d\vec{e}_1)+\phi, \\ Re(\int_{\mathbb{R}^2}\bar{\phi}\frac{\partial}{\partial d}(w(x-d\vec{e}_1)w(x+d\vec{e}_1))) = 0. \end{cases}$$

This step is done through *a priori estimates* and contraction mapping theorem. The key element in this step is the **nondegeneracy** of w^+, which will be proven in the appendix. This seems to be **new** and may be of independent interest.

Step 2: We solve the reduced equation for the Lagrange multiplier

$$c_\epsilon(d) = 0. \tag{2.8}$$

By choosing suitable $d = d_\epsilon \approx \frac{1}{2\epsilon}$, we find a zero of the function $c_\epsilon(d)$. This step is done through a balancing condition of between the Lorentz force and the interaction of vortices.

This finite dimensional reduction procedure has been used in many other problems. See [16], [19], [27] and the references therein. M.del Pino, Kowalczyk and Musso [17] were the first to use this procedure to study Ginzburg-Landau equation in a bounded domain. We adopt this approach to the Schrödinger map equation.

The organization of the paper is as follows: In Section 3, we introduce some basic notations and estimates. We consider the set-up of the problem in Section 4 and we solve the projected linear and nonlinear problem (2.7) in Section 5. In Section 6, the reduced problem (2.8) will be solved. In Section 7, we prove Theorem 1.2. The nondegeneracy of w^+ is proved in appendix.

3. Some preliminaries

In this section, we collect some important facts which will be used later. First we introduce the solution operator and the symmetry space. Then we study the nondegeneracy of degree one vortex. Using degree one vortex, we introduce the approximate solutions. Finally we give a detailed estimate of the approximate solution, including its composition and decaying properties.

3.1. **Symmetries and Solution Operator.**

First, we write the equation (2.1) in operator form. Let us define two solution operators

$$\mathbb{S}_0[u] = \Delta u - \frac{2\bar{u}}{1+|u|^2}\nabla u\nabla u + f(u), \tag{3.1}$$

$$\mathbb{S}[u] = \mathbb{S}_0[u] + i\epsilon\frac{\partial u}{\partial x_2}, \tag{3.2}$$

where $f(u) = \frac{1-|u|^2}{1+|u|^2}u$.

It is easy to see the operator $\mathbb{S}$ is invariant under the following two transformations

$$u(z) \to \overline{u(\bar{z})}, \quad u(z) \to u(-\bar{z}). \tag{3.3}$$

Thus we impose the following symmetry on the solution u

$$\Sigma := \{u(z) = \overline{u(\bar{z})},\ u(z) = u(-\bar{z})\}. \tag{3.4}$$

This symmetry will play an important role in our analysis.

3.2. **Nondegeneracy of degree one vortex.** Next, we need to study the non-degeneracy of the degree one vortex w^+. For simplicity, from now on, we use $w = \rho(r)e^{i\theta}$ to denote the degree $+1$ vortex. We also assume that $\sigma \in (0,1)$ is a fixed and small constant.

The following properties of ρ are proved in [28].

Lemma 3.1. *(1) $\rho(0) = 0, 0 < \rho(r) < 1, \rho' > 0$ for $r > 0$,*
(2) $\rho(r) = 1 - c_0\frac{e^{-r}}{\sqrt{r}} + O(r^{-3/2}e^{-r})$ as $r \to +\infty$, where $c_0 > 0$.

Setting $w = w_1 + iw_2$, then we need to study the following linearized operator of $\mathbb{S}_0$ around w:

$$\begin{aligned}\mathbb{L}_0(\phi) = \Delta\phi &- \frac{4(w_1\nabla w_1 + w_2\nabla w_2)}{1+|w|^2}\nabla\phi \\ &- \frac{4\nabla\langle w,\phi\rangle}{1+|w|^2}\nabla w + \frac{8\langle w,\phi\rangle(w_1\nabla w_1 + w_2\nabla w_2)}{(1+|w|^2)^2}\nabla w \\ &+ \frac{4\langle\nabla w,\nabla\phi\rangle}{1+|w|^2}w - \frac{4(1+|\nabla w|^2)\langle w,\phi\rangle}{(1+|w|^2)^2}w \\ &+ \frac{2|\nabla w|^2}{1+|w|^2}\phi + \frac{1-|w|^2}{1+|w|^2}\phi.\end{aligned}$$

The nondegeneracy of w is contained in the following theorem, whose proof will be delayed to the appendix.

Theorem 3.1. *Suppose that*

$$\mathbb{L}_0[\phi] = 0, \tag{3.5}$$

where $\phi = iw\psi$, and $\psi = \psi_1 + i\psi_2$ satisfies the following decaying estimates

$$|\psi_1| + |z||\nabla\psi_1| \le C(1+|z|)^{-\sigma}, |\psi_2| + |z||\nabla\psi_2| \le C(1+|z|)^{-1-\sigma}, \tag{3.6}$$

for some $0 < \sigma < 1$. Then

$$\phi = c_1\frac{\partial w}{\partial x_1} + c_2\frac{\partial w}{\partial x_2}$$

for certain real constants c_1, c_2.

3.3. **Approximate Solution.** Using the degree one vortex, we introduce the *approximate solutions.*

Throughout the paper, we assume that the distance d satisfies

$$d = \frac{\hat{d}}{\epsilon}, \quad \text{where } \hat{d} \in [\frac{1}{100}, 100]. \tag{3.7}$$

The approximate function is then defined by

$$V_d(z) := w^+(z - d\vec{e}_1)w^-(z + d\vec{e}_1). \tag{3.8}$$

It is easy to see that $V_d(z) \in \Sigma$.

We also define the approximate co-kernel

$$\tilde{Z}_d = \frac{\partial}{\partial d} V_d(z). \tag{3.9}$$

Then $\tilde{Z}_d \in \Sigma$. Furthermore, a simple computation shows that

$$V_d(z) \to 1 \text{ as } |z| \to +\infty. \tag{3.10}$$

In fact, for $|z| >> d$, we have

$$\begin{aligned} V_d &\approx e^{i\theta_{d\vec{e}_1} - i\theta_{-d\vec{e}_1}} \\ &\approx \frac{(x_1^2 - d^2 + x_2^2 + 2ix_2 d)}{\sqrt{(x_1-d)^2 + x_2^2}\sqrt{(x_1+d)^2+x_2^2}} \\ &\approx 1 + O(\frac{d^2}{|z|^2} + \frac{d}{|z|}). \end{aligned}$$

Here θ_ξ denotes the angle argument around ξ and $r_\xi = |z-\xi|$. It is easy to see that

$$\nabla r_\xi = \frac{1}{|z-\xi|}(z-\xi), \; \nabla\theta_\xi = \frac{1}{|z-\xi|^2}(z-\xi)^\perp, |\nabla\theta_\xi| = \frac{1}{|z-\xi|} \tag{3.11}$$

where we denote $z^\perp = (-x_2, x_1)$.

3.4. **Error Estimates.** We plug in the approximate function (3.8) into the solution operator $\mathbb{S}$ and obtain the error

$$\mathbb{E}_d := \mathbb{S}[V_d] = \mathbb{S}_0[V_d] + i\epsilon \frac{\partial V_d}{\partial x_2}. \tag{3.12}$$

Our purpose in this subsection is to estimate this error $\mathbb{E}_d$.

By our construction and the properties of ρ, we have

$$V_d = e^{i(\theta_{\vec{d}e_1} - \theta_{-\vec{d}e_1})}(1 + O(e^{-\min(|z-d\vec{e}_1|, |z+d\vec{e}_1|)})). \tag{3.13}$$

We divide $\mathbb{R}^2$ into two regions: $\mathbb{R}^2_+ = \{x_1 > 0\}$, $\mathbb{R}^2_- = \{x_1 < 0\}$. By our symmetry assumption, we just need to consider the region $\mathbb{R}^2_+$.

In the region $\mathbb{R}^2_+$, we consider two cases. First, if $|z + d\vec{e}_1| > \frac{d}{2} + \frac{|z-d\vec{e}_1|}{2}$, then we have

$$\begin{aligned} V_d &= w(z - d\vec{e}_1) e^{-i\theta_{-d\vec{e}_1}}(1 + O(e^{-d/2 - |z-d\vec{e}_1|/2})) \\ |V_d| &= \rho(|z-d\vec{e}_1|)(1 + O(e^{-d/2-|z-d\vec{e}_1|/2})). \end{aligned}$$

Secondly, if $|z + d\vec{e}_1| < \frac{d}{2} + \frac{|z-d\vec{e}_1|}{2}$, we rescale the variable as follows

$$z = d\vec{e}_1 + y.$$

Then we obtain the following estimates

$$\begin{aligned} \frac{1-|V_d|^2}{1+|V_d|^2} V_d &= \frac{1-\rho^2}{1+\rho^2} w(1 + O(e^{-d/2-|y|/2}))e^{-i\theta_{-d\vec{e}_1}}, \\ \Delta V_d &= \left(\Delta w - 2i\nabla w \cdot \nabla\theta_{-d\vec{e}_1} - \frac{w}{|y+2d\vec{e}_1|^2} + O(e^{-d/2-|y|/2})\right) e^{-i\theta_{-d\vec{e}_1}}, \\ \frac{2\bar{V_d}}{1+|V_d|^2}\nabla V_d \cdot \nabla V_d &= \frac{2\bar{w}}{1+\rho^2} e^{-i\theta_{-d\vec{e}_1}} \\ &\quad \times \left[\nabla w \cdot \nabla w - 2iw\nabla w \cdot \nabla\theta_{-d\vec{e}_1} - w^2 \nabla\theta_{-d\vec{e}_1} \cdot \nabla\theta_{-d\vec{e}_1}\right] + O(e^{-d/2-|y|/2}). \end{aligned}$$

Combining the estimates above, we have for $z \in \mathbb{R}^2_+$,

(3.14)
$$\mathbb{S}_0[V_d] = e^{-i\theta_{-d\vec{e}_1}} \left[\frac{2(\rho^2-1)}{\rho^2+1} i\nabla w \cdot \nabla \theta_{-d\vec{e}_1} + \frac{\rho^2-1}{\rho^2+1} \frac{w}{|y+2d\vec{e}_1|^2} + O(e^{-d/2-|y|/2}) \right].$$

On the other hand, we can estimate the derivative term $\frac{\partial V_d}{\partial x_2}$ as follows

$$\begin{aligned} i\epsilon \frac{\partial V_d}{\partial x_2} &= i\epsilon \frac{\partial}{\partial y_2} \left[w(y) e^{-i\theta_{-d\vec{e}_1}} + O(e^{-d/2-|y|/2}) \right] \\ &= i\epsilon \left[\frac{\partial w}{\partial y_2} - iw \frac{\partial \theta_{-d\vec{e}_1}}{\partial y_2} \right] e^{-i\theta_{-d\vec{e}_1}} + O(\epsilon e^{-d/2-|y|/2}) \end{aligned}$$

where

$$\frac{\partial}{\partial y_2} \theta_{-d\vec{e}_1} = \frac{y_1+2d}{(y_1+2d)^2+y_2^2} = O(\frac{1}{d}) = O(\epsilon). \tag{3.15}$$

In summary, we have obtained for $z \in \mathbb{R}^2_+, z = d\vec{e}_1 + y$

$$\mathbb{S}[V_d] = \mathbb{S}_0[V_d] + i\epsilon \frac{\partial V_d}{\partial x_2}$$

(3.16)
$$= e^{-i\theta_{-d\vec{e}_1}} \left[\frac{2(\rho^2-1)}{\rho^2+1} i\nabla w \cdot \nabla \theta_{-d\vec{e}_1} + \frac{\rho^2-1}{\rho^2+1} \frac{w}{|y+2d\vec{e}_1|^2} + i\epsilon \frac{\partial w}{\partial y_2} + \epsilon w \frac{\partial \theta_{-d\vec{e}_1}}{\partial y_2} + O(e^{-d/2-|y|/2}) \right].$$

A similar (and almost identical) estimate also holds in the region $z \in \mathbb{R}^2_-$.

4. Set-up of the Problem

Now we introduce the set-up of the reduction procedure.

We look for solutions of (2.1)-(2.2) in the form

$$u(y) = \eta(V_d + iV_d\psi) + (1-\eta)V_d e^{i\psi} \tag{4.1}$$

where η is a function such that

$$\eta = \tilde{\eta}(|z - d\vec{e}_1|) + \tilde{\eta}(|z + d\vec{e}_1|) \tag{4.2}$$

and $\tilde{\eta}(s) = 1$ for $s \leq 1$ and $\tilde{\eta}(s) = 0$ for $s \geq 2$. This nonlinear decomposition (4.1) was introduced first in [17] for Ginzburg-Landau equation.

The symmetry imposed on u (see (3.4)) can be transmitted to the symmetry on ψ

$$\psi(\bar{z}) = -\overline{\psi(z)}, \quad \psi(z) = \psi(-\bar{z}). \tag{4.3}$$

This symmetry will be important in solving the linear problems. It excludes all but one kernel.

We may write $\psi = \psi_1 + i\psi_2$ with ψ_1, ψ_2 real-valued. Setting

$$v = V_d + \phi, \qquad \phi = \eta i V_d \psi + (1-\eta) V_d (e^{i\psi} - 1). \tag{4.4}$$

In the inner region $\{z \in B_1(d\vec{e}_1) \cup B_1(-d\vec{e}_1)\}$, we have

$$u = V_d + \phi \tag{4.5}$$

and the equation for ϕ becomes

$$\mathbb{L}_d[\phi] + \mathbb{N}_d[\phi] = \mathbb{E}_d \tag{4.6}$$

where

(4.7)

$$\mathbb{L}_d[\phi]=\Delta\phi-\frac{4\bar{V}_d}{1+|V_d|^2}\nabla V_d\cdot\nabla\phi-\frac{2\bar{\phi}}{1+|V_d|^2}\nabla V_d\cdot\nabla V_d+\frac{2\bar{V}_d(V_d\bar{\phi}+\bar{V}_d\phi)}{(1+|V_d|^2)^2}\nabla V_d\cdot\nabla V_d+f'(V_d)\phi$$

$$\mathbb{N}_d[\phi]=f(V_d+\phi)-f(V_d)-f'(V_d)\phi+O((1+|\phi|)|\nabla\phi|^2)+i\epsilon\frac{\partial\phi}{\partial x_2} \tag{4.8}$$

$$\mathbb{E}_d=-\mathbb{S}[V_d]. \tag{4.9}$$

In the outer region $\{z\in(B_2(d\vec{e}_1)\cup B_2(-d\vec{e}_1))^c\}$, we have $u=V_de^{i\psi}$. By simple computations we obtain

$$\begin{aligned}\frac{\mathbb{S}[V_de^{i\psi}]}{iV_de^{i\psi}} &= \Delta\psi+2\frac{1-|V_d|^2+|V_d|^2(e^{-2\psi_2}-1)}{V_d(1+|V_d|^2e^{-2\psi_2})}\nabla V_d\cdot\nabla\psi\\ &+\frac{1}{iV_d}\frac{2|V_d|^2\bar{V}_d(e^{-2\psi_2}-1)}{(1+|V_d|^2)(1+|V_d|^2e^{-2\psi_2})}\nabla V_d\cdot\nabla V_d-i2\frac{|V_d|^2(1-e^{-2\psi_2})}{(1+|V_d|^2e^{-2\psi_2})(1+|V_d|^2)}\\ &-i(\frac{2|V_d|^2}{1+|V_d|^2e^{-2\psi_2}}-1)\nabla\psi\cdot\nabla\psi\\ &+i\epsilon\frac{\partial\psi}{\partial x_2}+\frac{\mathbb{E}_d}{iV_d}.\end{aligned}$$

We can also write it as

$$\tilde{L}_0[\psi]+\tilde{N}_1[\psi]+\tilde{N}_2[\psi]=\tilde{E}_d \tag{4.10}$$

where

$$\begin{aligned}\tilde{L}_0[\psi] &= \Delta\psi+\frac{2(1-|V_d|^2)}{1+|V_d|^2}\nabla V_d\cdot\nabla\psi+\frac{4i\bar{V}_d^{\,2}\psi_2}{(1+|V_d|^2)^2}\nabla V_d\cdot\nabla V_d-i\frac{4|V_d|^2\psi_2}{(1+|V_d|^2)^2}\\ \tilde{N}_1[\psi] &= \frac{1}{V_d}\nabla\psi\cdot\nabla V_dO(\psi_2)+O(||V_d|^2-1|+|\psi_2|)|\nabla\psi\cdot\nabla\psi|+iO(|e^{-\psi_2}-1+\psi_2|)\\ \tilde{N}_2[\psi] &= i\epsilon\frac{\partial\psi}{\partial x_2}\\ \tilde{E}_d &= -\frac{\mathbb{E}_d}{iV_d}.\end{aligned}$$

Recall that $\psi=\psi_1+i\psi_2$. Then setting $z=d\vec{e}_1+y$, we have for $z\in\mathbb{R}^2_+$

$$\tilde{L}_0[\psi]=\begin{pmatrix}\Delta\psi_1+O(e^{-|y|})|\nabla\psi|\\ \Delta\psi_2-\frac{4|V_d|^2}{(1+|V_d|^2)^2}\psi_2+O(e^{-|y|})\nabla\psi_2\end{pmatrix} \tag{4.11}$$

$$\tilde{N}_1[\psi]=\begin{pmatrix}O(e^{-|y|}|\nabla\psi\cdot\nabla\psi|+|\psi_2|^2\frac{1}{(1+|y|)^2}+|\psi_2|\frac{1}{1+|y|}|\nabla\psi|)\\ O(e^{-|y|}|\nabla\psi\cdot\nabla\psi|+|\psi_2||\nabla\psi\cdot\nabla\psi|+|\psi_2|^2),\end{pmatrix} \tag{4.12}$$

$$\tilde{N}_2[\psi]=\begin{pmatrix}O(\epsilon|\frac{\partial\psi_2}{\partial y_2}|\\ O(\epsilon|\frac{\partial\psi_1}{\partial y_2}|\end{pmatrix} \tag{4.13}$$

Let us remark that the explicit form of all the linear and nonlinear terms will be very useful for later analysis.

A direct application of (3.14) and (3.16) yields the following decay estimates for the error $\tilde{E}_d$:

Lemma 4.1. *It holds that for $z \in (B_2(d\vec{e}_1) \cup B_2(d\vec{e}_1))^c$*

$$|Re(\tilde{E}_d)| \leq \frac{C\epsilon^{1-\sigma}}{(1+|z-d\vec{e}_1|)^3} + \frac{C\epsilon^{1-\sigma}}{(1+|z+d\vec{e}_1|)^3}, \tag{4.14}$$

$$|Im(\tilde{E}_d)| \leq \frac{C\epsilon^{1-\sigma}}{(1+|z-d\vec{e}_1|)^{1+\sigma}} + \frac{C\epsilon^{1-\sigma}}{(1+|z+d\vec{e}_1|)^{1+\sigma}}, \tag{4.15}$$

where $\sigma \in (0,1)$ is a constant.

Proof: Since

$$\tilde{E}_d = -\frac{\mathbb{S}_0[V_d]}{iV_d} - \epsilon\frac{\frac{\partial V_d}{\partial x_2}}{V_d},$$

the estimates for the first term follows from (3.14). We just need to estimate the second term.

Let us compute for $z \in \mathbb{R}^2_+$

$$\begin{aligned}
\frac{\epsilon i\frac{\partial V_d}{\partial x_2}}{iV_d} &= \frac{i\epsilon\frac{\partial \rho(z-d\vec{e}_1)e^{i\theta_{d\vec{e}_1}-\theta_{-d\vec{e}_1}}}{\partial x_2}}{iV_d} + O(\epsilon e^{-|z-d\vec{e}_1|} + \epsilon e^{-|z+d\vec{e}_1|}) \\
&= \epsilon\frac{\frac{\partial\rho(|z-d\vec{e}_1|)}{\partial x_2}}{\rho(|z-d\vec{e}_1|)} + i\epsilon\frac{\partial(\theta_{d\vec{e}_1}-\theta_{-d\vec{e}_1})}{\partial x_2} + O(\epsilon e^{-|z-d\vec{e}_1|} + \epsilon e^{-|z+d\vec{e}_1|}) \\
&= O(\epsilon e^{-|z-d\vec{e}_1|} + \epsilon e^{-|z+d\vec{e}_1|}) + i\epsilon\left[\frac{x_1-d}{(x_1-d)^2+x_2^2} - \frac{x_1+d}{(x_1+d)^2+x_2^2}\right].
\end{aligned}$$

(4.14) then follows.

Let us notice that for $z \in \mathbb{R}^2_+,\ |z-d\vec{e}_1| < d$,

$$\epsilon\left|\frac{x_1-d}{(x_1-d)^2+x_2^2} - \frac{x_1+d}{(x_1+d)^2+x_2^2}\right| \leq C\epsilon\frac{1}{(1+|z-d\vec{e}_1|)} \leq C\epsilon^{1-\sigma}\frac{1}{(1+|z-d\vec{e}_1|)^{1+\sigma}}. \tag{4.16}$$

On the other hand, for $z \in \mathbb{R}^2_+, |z-d\vec{e}_1| > d$, we then have

$$\epsilon\left|\frac{x_1-d}{(x_1-d)^2+x_2^2} - \frac{x_1+d}{(x_1+d)^2+x_2^2}\right| \leq C\frac{\epsilon}{(1+|z-d\vec{e}_1|)^2} \leq C\frac{\epsilon^{1-\sigma}}{(1+|z-d\vec{e}_1|)^{1+\sigma}}. \tag{4.17}$$

Thus (4.15) is proved.

□

Finally, we will need the following lemma on decay estimates of a linear problem in $\mathbb{R}^2$.

Lemma 4.2. *Let h satisfy*

$$\Delta h + f(z) = 0, h(\bar{z}) = -h(z), |h| \leq C \tag{4.18}$$

where f satisfies

$$|f(z)| \leq \frac{C}{(1+|z|)^{2+\sigma}}, 0 < \sigma < 1. \tag{4.19}$$

Then

$$|h(z)| \leq \frac{C}{(1+|z|)^{\sigma}}. \tag{4.20}$$

Proof: By Poisson's formula,

$$h(z) = \frac{1}{2\pi}\int_{\{y_2>0\}} \log\frac{|\bar{z}-y|}{|z-y|} f(y)dy. \tag{4.21}$$

Because of (4.19), $h(z) \to 0$ as $z \to +\infty$.

We construct suitable super-solutions on $\{x_2 > 0\}$. Then the result will follow from Maximum Principle. In fact, let

$$h_0(z) = r^\beta x_2^\gamma$$

where $r = |z|$ and the parameters are chosen so that

$$\beta + \gamma = -\sigma, 0 < \sigma < \gamma < 1. \tag{4.22}$$

Then simple computations show that

$$\begin{aligned} \Delta h_0 &= r^\beta x_2^\gamma((\beta^2 + 2\beta\gamma)r^{-2} + \gamma(\gamma-1)x_2^{-2}) \\ &\leq -Cr^\beta x_2^\gamma(r^{-2} + x_2^{-2}) \leq -Cr^{\beta-1}x_2^{\gamma-1} \\ &\leq -Cr^{\beta+\gamma-2} \leq -C(1+|z|)^{\beta+\gamma-2}. \end{aligned}$$

where we have used (4.22).

□

5. Projected Linear and Nonlinear Problem

In this section, we solve a *projected linear* and *nonlinear* problem.

First, we introduce some weighted Sobolev norms. Let us fix two small positive numbers $0 < \gamma < 1, 0 < \sigma < 1$. Recall that $\phi = iV_d\psi, \psi = \psi_1 + i\psi_2$. Denote $r_j = |z - P_j|$ where $P_1 = d\vec{e}_1, P_2 = -d\vec{e}_1$, and define

$$\begin{aligned} \|\psi\|_* &= \sum_{j=1}^2 \|\phi\|_{C^{2,\gamma}(r_j<2)} + \sum_{j=1}^2 \|\phi_j\|_{C^{1,\gamma}(r_j<3)} \\ &+ \sum_{j=1}^2 \left[\|r_j^\sigma \psi_1\|_{L^\infty(r_j>2)} + \|r_j^{1+\sigma}\nabla\psi_1\|_{L^\infty(r_j>2)}\right] \\ &+ \sum_{j=1}^2 \left[\|r_j^{1+\sigma} \psi_2\|_{L^\infty(r_j>2)} + \|r_j^{2+\sigma}\nabla\psi_2\|_{L^\infty(r_j>2)}\right] \end{aligned} \tag{5.1}$$

$$\|h\|_{**} = \sum_{j=1}^2 \|iV_d h\|_{C^{0,\gamma}(r_j<3)} + \sum_{j=1}^2 [\|r_j^{2+\sigma}h_1\|_{L^\infty(r_j>2)} + \|r_j^{1+\sigma}h_2\|_{L^\infty(r_j>2)}]. \tag{5.2}$$

We remark that the choices of these norms are motivated by the expressions of (4.12)-(4.13).

Let $\tilde{\eta}$ be defined as in (4.2) and $R > 0$ be a fixed large positive number. We define

$$Z_d := \frac{\partial V_d}{\partial d}(\tilde{\eta}(\frac{|z-d\vec{e}_1|}{R}) + \tilde{\eta}(\frac{|z+d\vec{e}_1|}{R})). \tag{5.3}$$

In this section, our aim is to solve the following *projected problems*:

$$\begin{cases} \mathbb{S}[V_d + \phi] = cZ_d \\ Re(\int_{\mathbb{R}^2} \bar{\phi} Z_d) = 0. \end{cases} \tag{5.4}$$

5.1. **Projected Linear Problem.** First, we need to consider the following linear problem

$$\left\{ \begin{array}{l} \tilde{L}_0(\psi) = h \quad \text{in } \mathbb{R}^2, \\ \mathrm{Re}(\int_{\mathbb{R}^2} \bar{\phi} Z_d) = 0 \\ \phi = iV_d\psi, \quad \psi \text{ satisfies the symmetry (4.3).} \end{array} \right. \tag{5.5}$$

We have the following *a priori estimates*.

Lemma 5.1. *There exists a constant C, depending on γ, σ only such that for all ϵ sufficiently small, $d \sim \frac{1}{\epsilon}$, and any solution of (5.5), it holds*

$$\|\psi\|_* \leq C\|h\|_{**}. \tag{5.6}$$

Proof: We prove it by contradiction. Suppose that there exists a sequence of $\epsilon = \epsilon_n \to 0$, functions ψ^n, h_n which satisfy (5.5) with

$$\|\psi^n\|_* = 1, \|h_n\|_{**} = o(1). \tag{5.7}$$

By the symmetry assumption (4.3), we have $\psi_1(x_1, -x_2) = -\psi_1(x_1, x_2), \psi_1(-x_1, x_2) = \psi_1(x_1, x_2), \psi_2(x_1, -x_2) = \psi_2(x_1, x_2), \psi_2(-x_1, x_2) = \psi_2(x_1, x_2)$. We may just need to consider the region

$$\Sigma = \{x_1 > 0\}.$$

Then we have

$$\mathrm{Re}(\int_{\mathbb{R}^2} \bar{\phi}_n Z_d) = 2\mathrm{Re}(\int_\Sigma \bar{\phi}_n Z_d) = 0. \tag{5.8}$$

We derive *inner estimates* first. Let $z \in \Sigma, z = d\vec{e}_1 + y$ and $\tilde{\phi}_n(y) = \phi_n(z)$. Then as $n \to +\infty$,

$$V_d = w^+(y)e^{-i\theta_{-d\vec{e}_1}}(1 + O(e^{-d/2})) = -w^+(y) + o(1). \tag{5.9}$$

Since $\|\psi^n\|_* = 1$, we may take a limit so that $\tilde{\phi}_n \to \phi_0$ in $\mathbb{R}^2_{loc}$, where ϕ_0 satisfies

$$\mathbb{L}_0[\phi_0] = 0 \tag{5.10}$$

where $\mathbb{L}_0$ is defined by (3.5). Observe that ϕ_0 satisfies the decay estimate (3.6) because of our assumption on ψ^n.

By Theorem 3.1, we have

$$\phi_0 = c_1 \frac{\partial w}{\partial y_1} + c_2 \frac{\partial w}{\partial y_2}.$$

Observe that ϕ_0 inherits the symmetries of ϕ and hence $\phi_0 = \overline{\phi_0(\bar{z})}$. (The other symmetry is not preserved under the transformation $z = d\vec{e}_1 + y$.) But certainly $\frac{\partial w}{\partial y_2}$ does not enjoy the above symmetry. Hence $\phi_0 = c_1 \frac{\partial w}{\partial y_1}$.

On the other hand, taking a limit of the orthogonality condition $\mathrm{Re}(\int_\Sigma \bar{\phi} Z_d)$, we obtain $Re(\int_{\mathbb{R}^2} \bar{\phi}_0 \frac{\partial w}{\partial y_1}) = 0$. This implies that $c_1 = 0$ and hence we have

$$\phi_n \to 0 \quad \text{in } \mathbb{R}^2_{loc} \tag{5.11}$$

which implies that for any fixed $R > 0$,

(5.12)

$$\sum_{j=1}^2 (\|\phi_1\|_{L^\infty(r_j<R)} + \|\phi_2\|_{L^\infty(r_j<R)}) + \|\nabla\phi_1\|_{L^\infty(r_j<R)} + \|\nabla\phi_2\|_{L^\infty(r_j<R)}) = o(1).$$

Next we shall derive *outer estimates*: let $\tilde{\eta}$ be a cut-off function such that $\tilde{\eta}(s) = 1$ for $s \leq 1$ and $\tilde{\eta}(s) = 0$ for $s > 2$. We consider the new function

$$\tilde{\psi} = \psi\chi(z), \quad \text{where } \chi(z) = 1 - \tilde{\eta}(\frac{|z - d\vec{e}_1|}{4}) - \tilde{\eta}(\frac{|z + d\vec{e}_1|}{4}).$$

Using the explicit forms of $\tilde{L}_0$ in (4.11), the first equation becomes

$$\Delta\tilde{\psi}_1 = O(e^{-|y|})|\nabla\psi| + O(\nabla\chi\nabla\psi_1) + O(\psi_1\Delta\chi) + h_1\chi. \tag{5.13}$$

On the region $\mathbb{R}^2_+ \setminus (B_4(d\vec{e}_1) \cup B_4(-d\vec{e}_1))$, we have $\psi_1 = 0$ on $\partial(\mathbb{R}^2_+ \setminus (B_4(d\vec{e}_1) \cup B_4(-d\vec{e}_1)))$. Moreover we have

$$|\nabla\chi \cdot \nabla\psi| = o(1)(|z - d\vec{e}_1|^2 + |z + d\vec{e}_1|^2)^{-\frac{\sigma+2}{2}}$$

and hence

$$|\Delta\tilde{\psi}_1| \leq C(\|h\|_{**} + o(1))(|z - d\vec{e}_1|^2 + |z + d\vec{e}_1|^2)^{-\frac{\sigma+2}{2}} \tag{5.14}$$

Now we consider the following barrier function

$$B(z) := |z - d\vec{e}_1|^\beta x_2^\gamma + |z + d\vec{e}_1|^\beta x_2^{-\gamma} \tag{5.15}$$

where $\beta + \gamma = -\sigma, 0 < \sigma < \gamma < 1$.

Similar to the computations in Lemma 4.2, we obtain that

$$\Delta B \leq -C(|z - d\vec{e}_1|^2 + |z + d\vec{e}_1|^2)^{-\frac{\sigma+2}{2}}. \tag{5.16}$$

By comparison principle on the set $\mathbb{R}^2_+ \setminus (B_4(d\vec{e}_1) \cup B_4(-d\vec{e}_1))$, we get that

$$|\tilde{\psi}_1| \leq CB(\|h\|_{**} + o(1)), \ \forall\ z \in \mathbb{R}^2_+ \setminus (B_4(d\vec{e}_1) \cup B_4(-d\vec{e}_1)) \tag{5.17}$$

and for any $\sigma < 1$. Elliptic estimates then give

$$\sum_{j=1}^{2} \|r_j^{1+\sigma}|\nabla\tilde{\psi}_1|\|_{L^\infty(r_j>4)} \leq C(\|h\|_{**} + o(1)). \tag{5.18}$$

To estimate ψ_2, we perform the same cut-off and now the second equation becomes

(5.19)

$$\Delta\tilde{\psi}_2 - \frac{4|V_d|^2}{(1+|V_d|^2)^2}\tilde{\psi}_2 + O(\frac{1}{1+|y|})\nabla\psi_1 + O(e^{-|y|})\nabla\psi_2 + +O(\nabla\chi\nabla\psi_2) + O(\Delta\psi_2) = h_2\chi\,.$$

Since for $z \in \mathbb{R}^2_+ \setminus (B_4(d\vec{e}_1) \cup B_4(-d\vec{e}_1))$, $\frac{4|V_d|^2}{(1+|V_d|^2)} \geq \frac{1}{4}$, by standard elliptic estimates we have

(5.20)

$$\|\psi_2\|_{L^\infty(r_j>4)} \leq C(\|\psi_2\|_{L^\infty(r_j=4)})(1 + \|\psi\|_*)\|h\|_{**})(1 + |z - d\vec{e}_1| + |z - d\vec{e}_1|)^{-1-\sigma}$$

$$|\nabla\psi_2| \leq C(\|\psi_2\|_{L^\infty(r_j=R)})(1 + \|\psi\|_*)\|h\|_{**})(1 + |z - d\vec{e}_1| + |z - d\vec{e}_1|)^{-2-\sigma} \tag{5.21}$$

Combining both inner and outer estimates in (5.11) and (5.17)-(5.21), we obtain that $\|\psi\|_* = o(1)$, which is a contradiction.

□

We consider now the following projected linear problem

$$\begin{cases} \tilde{L}_0(\psi) = h + c\frac{Z_d}{iV_d} \ \text{ in } \mathbb{R}^2, \\ \mathrm{Re}(\int_{\mathbb{R}^2} \bar{\phi}\frac{\partial V_d}{\partial d}) = 0 \\ \psi \text{ satisfies the symmetry (4.3).} \end{cases} \tag{5.22}$$

We state the following existence result for the projected linear problem.

Proposition 5.1. *There exists a constant C, depending on γ, σ only such that for all ϵ sufficiently small, the following holds: if $\|h\|_{**} < +\infty$, then there exists a unique solution $\psi = T_\epsilon(h)$ to (5.22). Furthermore it holds that*

$$\|T_\epsilon(h)\|_* \leq C\|h\|_{**} \tag{5.23}$$

Proof: The proof is similar to that of [Prop. 4.1,[17]]. Instead of solving (5.22) in $\mathbb{R}^2$, we solve it in a bounded domain first:

$$\begin{cases} \tilde{L}_0(\psi) = h + c\frac{Z_d}{iV_d} \text{ in } B_M(0), \\ \mathrm{Re}(\int_{B_M} \bar{\phi}\frac{\partial V_d}{\partial d}) = 0 \\ \phi = iV_d\psi, \ \psi \text{ satisfies the symmetry (4.3)} \\ \phi = 0 \text{ on } \partial B_M(0) \end{cases} \tag{5.24}$$

where $M > 10d$. By the same proof of a priori estimates, we also obtain the following estimates for any solution ψ_M of (5.24):

$$\|\psi_M\|_* \leq C\|h\|_{**}. \tag{5.25}$$

By working with the Sobolev space $H_0^1(B_M(0))$, the existence will follow by Fredholm alternatives. Now letting $M \to +\infty$, we obtain a solution to (5.22) with the required properties.

□

5.2. **Projected Nonlinear Problem.** Finally, we consider the full nonlinear *projected problem*

$$\begin{cases} \tilde{L}_0[\psi] + \tilde{N}_1[\psi] + \tilde{N}_2[\psi] = \tilde{E}_d + c\frac{1}{iV_d}Z_d \\ \mathrm{Re}(\int_{\mathbb{R}^2} \bar{\phi}Z_d) = 0 \\ \psi \text{ satisfies the symmetry (4.3)} \end{cases} \tag{5.26}$$

Using the operator T_ϵ defined by Proposition (5.1), we can write (5.26) as

$$\psi = T_\epsilon \circ (-\tilde{N}_1[\psi] - \tilde{N}_2[\psi] + \tilde{E}_d) \tag{5.27}$$

which is equivalent to

$$\psi = G_\epsilon[\psi] \tag{5.28}$$

where G_ϵ is the nonlinear operator at the right hand side of (5.27).

Using Lemma 4.1, we see that

$$\|\tilde{E}_d\|_{**} \leq C\epsilon^{1-\sigma}. \tag{5.29}$$

Let

$$\psi \in \mathbb{B} = \{\|\psi\|_* < C\epsilon^{1-\sigma}\}; \tag{5.30}$$

then we have, using the explicit form of $\tilde{N}_1[\psi]$ at (4.12):

$$\|\tilde{N}_1[\psi]\|_{**} \leq C\epsilon. \tag{5.31}$$

On the other hand, we note that

$$\|N_2[\psi]\|_{**} = \|r_j^{2+\sigma}N_{2,1}[\psi]\|_{L^\infty(r_j>2)} + \|r_j^{1+\sigma}N_{2,2}[\psi]\|_{L^\infty(r_j>2)}$$

where $N_2 = N_{2,1} + iN_{2,2}$. Now $i\frac{\partial\psi}{\partial x_2} = -\frac{\partial\psi_2}{\partial x_2} + i\frac{\partial\psi_2}{\partial x_1}$ and

$$\|i\frac{\partial\psi}{\partial x_2}\|_{**} \leq \|r_j^{2+\sigma}\frac{\partial\psi_1}{\partial x_2}\|_{L^\infty(r_j>2)} + \|r_j^{1+\sigma}\frac{\partial\psi_1}{\partial x_2}\|_{L^\infty(r_j>2)} \leq C\|\psi\|_*$$

Therefore, we obtain

$$\|G_\epsilon[\psi]\|_* \leq C(\|\tilde{N}_1[\psi]\|_{**} + \|\tilde{N}_2[\psi]\|_{**} + \|\tilde{E}_d[\psi]\|_{**}) \leq C\epsilon^{1-\sigma} \tag{5.32}$$

Similarly, we can also show that

$$\|G_\epsilon[\psi^{'}] - G_\epsilon[\psi^{''}]\|_* \leq o(1)\|\psi^{'} - \psi^{''}\|_* \tag{5.33}$$

for all $\psi^{'}, \psi^{''} \in \mathbb{B}$.

By contraction mapping theorem, we conclude

Proposition 5.2. *There exists a constant C, depending on γ, σ only such that for all ϵ sufficiently small, d large, the following holds: there exists a unique solution ψ_ϵ to (5.26) and ψ_ϵ satisfies*

$$\|\psi_\epsilon\|_* \leq C\epsilon^{1-\sigma}. \tag{5.34}$$

Furthermore, $\psi_{\epsilon,d}$ is continuous in d.

6. Reduced Problem and the Proof of Theorem 2.1

We now solve the reduced problem. From Proposition 5.2, we deduce the existence of a solution $(\phi, c) = (\phi_{\epsilon,d}, c_\epsilon(d))$ to

$$\mathbb{S}[V_d + \phi_{\epsilon,d}] = \mathbb{L}_d[\phi_{\epsilon,d}] + \mathbb{N}_d[\phi_{\epsilon,d}] + \mathbb{S}[V_d] = c_\epsilon(d)Z_d. \tag{6.1}$$

Multiplying (6.1) by $\frac{1}{(1+|V_d|^2)^2}\tilde{Z}_d$ and integrating, we obtain

$$\begin{aligned} c_\epsilon \mathrm{Re}(\int_{\mathbb{R}^2} \frac{1}{(1+|V_d|^2)^2}\overline{Z_d}\tilde{Z}_d) \quad &= \quad \mathrm{Re}(\int_{\mathbb{R}^2} \frac{1}{(1+|V_d|^2)^2}\overline{\tilde{Z}_d}\mathbb{S}[V_d]) \\ &\quad + \mathrm{Re}(\int_{\mathbb{R}^2} \frac{1}{(1+|V_d|^2)^2}\overline{\tilde{Z}_d}\mathbb{L}_d[\phi_{\epsilon,d}]) + \mathrm{Re}(\int_{\mathbb{R}^2} \frac{1}{(1+|V_d|^2)^2}\overline{\tilde{Z}_d}\mathbb{N}_d[\phi_{\epsilon,d}]). \end{aligned}$$

Using Proposition 5.2 and the expression in (4.8), we deduce that

$$\mathrm{Re}(\int_{\mathbb{R}^2} \overline{\tilde{Z}_d}\mathbb{N}_d[\phi_{\epsilon,d}]) = o(\epsilon). \tag{6.2}$$

On the other hand, integration by parts, we have

$$\mathrm{Re}(\int_{\mathbb{R}^2} \frac{1}{(1+|V_d|^2)^2}\overline{\tilde{Z}_d}\mathbb{L}_d[\phi_{\epsilon,d}]) = \mathrm{Re}(\int_{\mathbb{R}^2} \frac{1}{(1+|V_d|^2)^2}\overline{\phi_{\epsilon,d}}\mathbb{L}_d[\tilde{Z}_d]).$$

Let us observe that

$$\frac{\partial}{\partial d}\mathbb{S}_0[V_d] = \mathbb{L}_d[\frac{\partial V_d}{\partial d}] = \mathbb{L}_d[\tilde{Z}_d] = O(\epsilon) \tag{6.3}$$

and thus by Proposition 5.2

$$\mathrm{Re}(\int_{\mathbb{R}^2} \frac{1}{(1+|V_d|^2)^2}\overline{\phi_{\epsilon,d}}\mathbb{L}_d[\tilde{Z}_d]) = o(\epsilon). \tag{6.4}$$

It remains to estimate

$$\begin{aligned} &\mathrm{Re}(\int_{\mathbb{R}^2} \frac{1}{(1+|V_d|^2)^2}\overline{\tilde{Z}_d}\mathbb{S}[V_d]) \\ &= 2\mathrm{Re}(\int_{\mathbb{R}^2_+} \frac{1}{(1+|V_d|^2)^2}\overline{\tilde{Z}_d}\mathbb{S}[V_d]) \\ &= 2\mathrm{Re}(\int_{\mathbb{R}^2_+} \frac{1}{(1+|V_d|^2)^2}\overline{\tilde{Z}_d}\mathbb{S}_0[V_d]) + 2\epsilon\mathrm{Re}(i\int_{\mathbb{R}^2_+} \frac{1}{(1+|V_d|^2)^2}\overline{\tilde{Z}_d}\frac{\partial V_d}{\partial x_2}) \end{aligned}$$

On $\mathbb{R}^2_+$, recall that $z = d\vec{e}_1 + y$. Then $V_d = w(y)e^{-i\theta_{-d\vec{e}_1}}(1+O(e^{-d/2-|y|/2}))$ and

$$\begin{aligned}\tilde{Z}_d &= \frac{\partial V_d}{\partial d} = -\frac{\partial w}{\partial y_1}e^{-i\theta_{-d\vec{e}_1}}(1+O(e^{-d/2-|y|/2})) - iw\frac{\partial \theta_{-d\vec{e}_1}}{\partial d}e^{-i\theta_{-d\vec{e}_1}}(1+O(e^{-d/2-|y|/2}))\\ \frac{\partial V_d}{\partial y_2} &= \frac{\partial w}{\partial y_2}e^{-i\theta_{-d\vec{e}_1}}(1+O(e^{-d/2-|y|/2})) - iw\frac{\partial \theta_{-d\vec{e}_1}}{\partial y_2}e^{-i\theta_{-d\vec{e}_1}}(1+O(e^{-d/2-|y|/2}))\end{aligned}$$

where

$$\frac{\partial \theta_{-d\vec{e}_1}}{\partial d} = \frac{\partial \theta_{-d\vec{e}_1}}{\partial x_1} = O(\frac{1}{d}) = O(\epsilon). \tag{6.5}$$

We note that for $w = \rho(r)e^{i\theta}$

$$\frac{\partial w}{\partial y_1} = (\rho^{'}\cos\theta - i\frac{\rho}{r}\sin\theta)e^{i\theta}, \frac{\partial w}{\partial y_2} = (\rho^{'}\sin\theta + i\frac{\rho}{r}\cos\theta)e^{i\theta}.$$

Thus

$$\begin{aligned}-\mathrm{Re}(i\int_{\mathbb{R}^2}\frac{1}{(1+|V_d|^2)^2}\frac{\partial V_d}{\partial y_2}\overline{\tilde{Z}_d}) &= Re(i\int_{\mathbb{R}^2}\frac{1}{(1+|w|^2)^2}\frac{\partial w}{\partial y_2}\overline{\frac{\partial w}{\partial y_1}}) + o(1)\\ &= -\int_{\mathbb{R}^2}\frac{\rho\rho^{'}}{r(1+\rho^2)^2} = -\frac{1}{2}\pi + o(1),\end{aligned}$$

On the other hand, using the estimate (3.14), we have

$$\begin{aligned}&-\mathrm{Re}(\int_{\mathbb{R}^2_+}\frac{1}{(1+|V_d|^2)^2}\mathbb{S}_0[V_d]\overline{\tilde{Z}_d})\\ =\;& Re(\int_{\mathbb{R}^2_+}\frac{1}{(1+|V_d|^2)^2}\mathbb{S}_0[V_d]\overline{\frac{\partial w}{\partial y_1}}e^{i\theta_{-d\vec{e}_1}}) + o(\epsilon)\\ =\;& Re(\int_{\mathbb{R}^2_+}\frac{1}{(1+|w|^2)^2}\left[\frac{2(|w|^2-1)}{|w|^2+1}i\nabla w\cdot\nabla\theta_{-d\vec{e}_1} - \frac{1-|w|^2}{1+|w|^2}\frac{w}{|x+d\vec{e}_1|^2}\right]e^{-i\theta}(\rho^{'}\cos\theta + i\frac{\rho}{r}\sin\theta)) + o(\epsilon).\end{aligned}$$

Observe that in $\mathbb{R}^2_+$

$$\nabla\theta_{-d\vec{e}_1} \approx \frac{\vec{e}_2}{2d}, \quad \nabla w\cdot\nabla\theta_{-d\vec{e}_1} \approx \frac{e^{i\theta}}{2d}[\rho^{'}\sin\theta + i\frac{\rho}{r}\cos\theta]. \tag{6.6}$$

Hence

$$\begin{aligned}&-Re(\int_{\mathbb{R}^2}\frac{1}{(1+|w|^2)^2}\mathbb{S}_0[V_d]\overline{\tilde{Z}_d})\\ \approx\;& Re(\int_{\mathbb{R}^2}\frac{(|w|^2-1)}{(|w|^2+1)^3}\frac{i}{d}[\rho^{'}\sin\theta + i\frac{\rho}{r}\cos\theta][\rho^{'}\cos\theta + i\frac{\rho}{r}\sin\theta])\\ \approx\;& -\frac{1}{d}\int_{\mathbb{R}^2}\frac{(\rho^2-1)}{(\rho^2+1)^3}\frac{\rho^{'}\rho}{r}\\ \approx\;& -\frac{\pi}{d}\int_0^\infty\frac{\rho^2-1}{(\rho^2+1)^3}(\rho^2)^{'} = \frac{\pi}{4d}.\end{aligned}$$

Combining all estimates together, we obtain the following equation

$$c_\epsilon(d) = c_0[\frac{\pi}{4d} - \frac{\epsilon\pi}{2}] + o(\epsilon) \tag{6.7}$$

where $o(\epsilon)$ is continuous function of d (which is a consequence of continuity of $\phi_{\epsilon,d}$ in d) and $c_0 \neq 0$. By simple mean-value theorem, we can find a zero of $c_\epsilon(d)$ in the interval $d \in [\frac{1-\delta}{2\epsilon}, \frac{1+\delta}{2\epsilon}]$ where δ is very small.

From (6.1), we see that $u_\epsilon = V_{d_\epsilon} + \phi_{\epsilon,d_\epsilon}$ is a solution to (2.1). Since $\|\psi_{\epsilon,d_\epsilon}\|_* = O(\epsilon^{1-\sigma})$, we see that
(6.8)
$$|\phi_{\epsilon,d_\epsilon}(z)| = |\eta i V_{d_\epsilon}\psi_{\epsilon,d} + (1-\eta)V_d(e^{i\psi_{\epsilon,d}}-1)| \leq C|V_{d_\epsilon}||\psi_{\epsilon,d_\epsilon}| \leq \frac{C\epsilon^{1-\sigma}}{(1+|z-d_\epsilon\vec{e}_1|+|z+d_\epsilon\vec{e}_1|)^\sigma}.$$
Now choosing σ such that $p\sigma > 2$, this completes the proof of Theorem 2.1. □

Finally Theorem 1.1 follows from Theorem 2.1.

Completion of Proof of Theorem 1.1. To complete the proof of Theorem 1.1, we just need to show that $E_\epsilon[m^\epsilon] < +\infty$, where m^ϵ is determined by $u_\epsilon = \frac{m_1^\epsilon + i m_2^\epsilon}{1+m_3^\epsilon}$. In fact,

$$E_\epsilon(m^\epsilon) = \int_{\mathbb{R}^2} \frac{2}{(1+|u_\epsilon|^2)^2}(|\nabla u_\epsilon|^2 + \frac{(1-|u_\epsilon|^2)^2}{4}). \tag{6.9}$$

By our definition, for $z \in (B_2(d_\epsilon\vec{e}_1) \cup B_2(-d_\epsilon\vec{e}_1))^c$, $u_\epsilon = V_d e^{i\psi}$. Hence $(1-|u_\epsilon|^2)^2 \leq C(e^{-\frac{1}{2}|z-d_\epsilon\vec{e}_1|} + e^{-\frac{1}{2}|z+d_\epsilon\vec{e}_1|} + |\psi_2|^2)$. Since $|\psi_2| \leq \frac{C}{(1+|z-d_\epsilon\vec{e}_1|+|z+d_\epsilon\vec{e}_1|)^{1+\sigma}}$, we deduce that $\int_{\mathbb{R}^2}(1-|u_\epsilon|^2)^2 < +\infty$. Similarly we can prove that $\int_{\mathbb{R}^2}|\nabla u_\epsilon|^2 < +\infty$. □

7. Traveling Vortex Rings Solutions

Using the same ideas in the proofs of Theorem 2.1, we can also construct traveling vortex ring solutions for the Schrödinger map equation.

Same as before, we look for a solution of the Schrödinger map equation of the form $m(x', x_N - ct)$, where $(x', x_N) \in \mathbb{R}^N, N \geq 3$. Then m must satisfy

$$-c\frac{\partial m}{\partial x_N} = m \times (\Delta m - \frac{1}{\epsilon^2} m_3 \vec{e}_3) \tag{7.1}$$

Setting $u = \frac{m_1 - i m_2}{1+m_3}$ and

$$c = (N-2)|\log\epsilon|, \tag{7.2}$$

then (7.1) becomes

$$i(N-2)\epsilon^2|\log\epsilon|\frac{\partial u}{\partial x_N} + \epsilon^2\Delta u + \frac{1-|u|^2}{1+|u|^2}u = \epsilon^2\frac{2\bar{u}}{1+|u|^2}\nabla u \cdot \nabla u \tag{7.3}$$

where we look for solutions u satisfying

$$u(x) \to 1 \quad \text{as } |x| \to +\infty. \tag{7.4}$$

Let us rescale

$$(r', x_N) = (\epsilon x_1, \epsilon x_2),\ z = x_1 + i x_2. \tag{7.5}$$

Thus (7.3) becomes

$$i(N-2)\epsilon|\log\epsilon|\frac{\partial u}{\partial x_2} + \Delta u + \frac{N-2}{x_1}\frac{\partial u}{\partial x_1} + \frac{1-|u|^2}{1+|u|^2}u = \frac{2\bar{u}}{1+|u|^2}\nabla u \cdot \nabla u \tag{7.6}$$

and (7.4) becomes

$$\frac{\partial u}{\partial x_1}(0, x_2) = 0, \quad u(z) \to 1 \quad \text{as } |z| \to +\infty. \tag{7.7}$$

Problem (7.6) becomes a two-dimensional problem with Neumann boundary condition $\frac{\partial u}{\partial x_1}(0, x_2) = 0$. So we can use the method in previous sections. Comparing with equation (2.1), there are two main different terms: the constant c becomes $-\epsilon|\log\epsilon|$, and there is an extra derivative term $\frac{(N-2)}{x_1}\frac{\partial u}{\partial x_1}$. In the following we shall mainly focus on how to deal with these two terms.

Now we state our results on vortex rings.

Theorem 7.1. *For ϵ sufficiently small, problem (7.6)-(7.7) has a traveling wave solution u_ϵ*

$$u_\epsilon = w^+(z - d_\epsilon \vec{e}_1) w^-(z + d_\epsilon \vec{e}_1) e^{i\varphi_\epsilon} + \phi_\epsilon \tag{7.8}$$

where

$$d_\epsilon = \frac{1+o(1)}{2\epsilon}, \quad \|\phi_\epsilon\|_{L^\infty} = O(\sqrt{\epsilon}). \tag{7.9}$$

Here $\varphi_\epsilon(r', x_N) \to \varphi_*(r, x_N)$ which is the unique $L^\infty-$solution of

$$\begin{cases} \Delta_{r', x_N}\varphi_* + \frac{N-2}{r'}\frac{\partial \varphi_*}{\partial r'} - \frac{4(N-2)x_N}{((r'-1)^2+x_N^2)((r'+1)^2+x_N^2)} = 0, \\ \frac{\partial \varphi_*}{\partial r'}(0, x_N) = 0, \ \varphi_* \to 0 \ \textit{as} \ |(r', x_N)| \to +\infty. \end{cases} \tag{7.10}$$

Remark: In the original variable (r', x_N), the solution we constructed has a vortex ring at the pont $(1,0)$. This result has been obtained by Bethuel, Orlandi and Smets [4] in the context of Gross-Pitaevskii equation. The function $w^+(z - d_\epsilon \vec{e}_1)w^-(z + d_\epsilon \vec{e}_1)e^{i\varphi_\epsilon}$ converges to the function ω_*, as defined in [4].

The proof of Theorem 7.1 follows from main steps of the proof of Theorem 2.1. Below we sketch the main points.

7.1. **Approximate Functions.** For each fixed $d := \frac{\hat{d}}{\epsilon}$ with $\hat{d} \in [\frac{1}{100}, \ 100]$, we define an approximate function

$$V_d(z) := w^+(z - d\vec{e}_1) w^-(z + d\vec{e}_1) e^{i\varphi_d}. \tag{7.11}$$

Observe that unlike two-dimensional case, we have to include a new phase function φ_d. This is mainly due to the extra derivative term $\frac{(N-2)}{x_1}\frac{\partial u}{\partial x_1}$.

Let $\chi(z) = 1$ for $z \in B_{\frac{d}{10}}(d\vec{e}_1)$ and $\chi = 0$ for $z \in (B_{\frac{d}{5}}(d\vec{e}_1))^c$.

The phase function φ_d will be decomposed into two parts: singular part and regular part. Let $\varphi_d(z) = \varphi_s(z) + \varphi_r(z)$, where the singular part

$$\varphi_s(z) := \frac{N-2}{4d} x_2 \log \frac{|z - d\vec{e}_1|^2}{|z + d\vec{e}_1|^2} \chi(z) \tag{7.12}$$

and the regular part $\varphi_r(z)$ satisfies

$$\Delta \varphi_r(z) + \frac{N-2}{x_1}\frac{\partial \varphi_r}{\partial x_1} = -[\Delta + \frac{N-2}{x_1}\frac{\partial}{\partial x_1}](\theta_{d\vec{e}_1} - \theta_{-d\vec{e}_1} + \varphi_s(z)). \tag{7.13}$$

Note that the function φ_s is continuous but $\nabla\varphi_s$ is not. The singularity of φ_s comes from its derivatives.

By simple computations, we see that for $z \in B_{\frac{d}{10}}(d\vec{e}_1)$,

$$\begin{aligned}
& [\Delta + \frac{N-2}{x_1}\frac{\partial}{\partial x_1}](\theta_{d\vec{e}_1} - \theta_{-d\vec{e}_1} + \varphi_s(z)) \\
= & \frac{4(N-2)x_2(x_1-d)}{((x_1-d)^2+x_2^2)((x_1+d)^2+x_2^2)} \\
+ & \frac{(N-2)^2}{x_1}\frac{x_2(x_1^2-x_2^2-d^2)}{((x_1-d)^2+x_2^2)((x_1+d)^2+x_2^2)} \\
= & O(d^{-2}) = O(\epsilon^2).
\end{aligned}$$

For $z \in (B_{\frac{d}{10}}(d\vec{e}_1))^c$, it is easy to see that we also get $O(\epsilon^2)$. In fact for $z \in (B_{\frac{d}{5}}(d\vec{e}_1))^c$, $\varphi_s = 0$ and

$$\begin{aligned}
& [\Delta + \frac{N-2}{x_1}\frac{\partial}{\partial x_1}](\theta_{d\vec{e}_1} - \theta_{-d\vec{e}_1} + \varphi_s(z)) \\
= & \frac{-4(N-2)x_2}{((x_1-d)^2+x_2^2)((x_1+d)^2+x_2^2)}
\end{aligned}$$

Going back to the original variable (r', x_N) and letting $\hat{\varphi}(r', x_N) = \varphi_r(z)$ we see that

$$|\Delta_{r', x_N}\hat{\varphi} + \frac{(N-2)}{r'}\frac{\partial \hat{\varphi}}{\partial r'}| \leq \frac{C}{(\sqrt{1+(r')^2+x_N^2})^3}. \tag{7.14}$$

Thus we can choose φ_r such that $\hat{\varphi} = O(\frac{1}{\sqrt{1+(r')^2+|x_N|^2}})$. This term φ_r is C^1 in the original variable (r', x_N).

We observe also that by our definition, the function

$$\tilde{\varphi} := \theta_{d\vec{e}_1} - \theta_{-d\vec{e}_1} + \varphi_s + \varphi_r \tag{7.15}$$

satisfies

$$\Delta\tilde{\varphi} + \frac{N-2}{x_1}\frac{\partial \tilde{\varphi}}{\partial x_1} = 0. \tag{7.16}$$

From the decomposition of φ_d, we see that the singular term contains $x_2 \log|z - d\vec{e}_1|$ which becomes dominant when we calculate the speed.

7.2. **Symmetry Class.** We construct solutions to (7.6) under the following additional symmetry

$$u = u(x_1, x_2), u(x_1, x_2) = \bar{u}(x_1, -x_2). \tag{7.17}$$

This symmetry condition is the same as before, and is important, as in the proof of Theorem 2.1.

We denote, as before

$$\mathbb{S}_0[u] := \Delta u + \frac{N-2}{x_1}\frac{\partial u}{\partial x_1} + \frac{1-|u|^2}{1+|u|^2}u - \frac{2\bar{u}}{1+|u|^2}\nabla u \cdot \nabla u, \ \mathbb{S}[u] := \mathbb{S}_0[u] + i(N-2)\epsilon|\log\epsilon|\frac{\partial u}{\partial x_2} \tag{7.18}$$

Then $\mathbb{S}[u]$ is invariant under (7.17). Because of the symmetry (7.17), we work directly in the half space $\mathbb{R}^2_+ = \{(x_1, x_2) | x_1 > 0\}$.

7.3. **Error Estimates.** Let

$$\tilde{\rho} = \rho(|z - d\vec{e}_1|)\rho(|z + d\vec{e}_1|), \quad \tilde{\varphi} = \theta_{d\vec{e}_1} - \theta_{-d\vec{e}_1} + \varphi_d.$$

So $V_d = \tilde{\rho} e^{i\tilde{\varphi}}$. Note that for $x_1 > 0$, we have

$$\rho(|z + d\vec{e}_1|) = 1 + O(e^{-\frac{d}{2} - \frac{1}{2}|z - d\vec{e}_1|}) \tag{7.19}$$

Since the error between 1 and $\rho(|z + d\vec{e}_1|)$ is exponentially small, we may ignore $\rho(|z + d\vec{e}_1|)$ in the computations below.

Let us start to compute the errors:

$$\Delta V_d = \left[\Delta\tilde{\rho} - |\nabla\tilde{\varphi}|^2\tilde{\rho} + 2i\nabla\tilde{\rho}\cdot\nabla\tilde{\varphi} - i\tilde{\rho}\frac{N-2}{x_1}\frac{\partial\tilde{\varphi}}{\partial x_1}) \right] e^{i\tilde{\varphi}}. \tag{7.20}$$

Here we have used the fact (7.16). We continue to compute other terms:

$$\nabla V_d \cdot \nabla V_d = \left[|\nabla\tilde{\rho}|^2 - \tilde{\rho}^2|\nabla\tilde{\varphi}|^2 + 2i\tilde{\rho}\nabla\tilde{\rho}\cdot\nabla\tilde{\varphi} \right] e^{2i\tilde{\varphi}}$$

We then obtain that

$$\mathbb{S}_0[V_d] = \Delta V_d + \frac{N-2}{x_1}\frac{\partial V_d}{\partial x_1} + \frac{1 - |V_d|^2}{1 + |V_d|^2}V_d - \frac{2\bar{V}_d}{1 + |V_d|^2}\nabla V_d \cdot \nabla V_d$$

(7.21)

$$= e^{i\tilde{\varphi}}\left[\frac{(N-2)}{x_1}\frac{\partial\tilde{\rho}}{\partial x_1} + \frac{\tilde{\rho}(\tilde{\rho}^2 - 1)}{\tilde{\rho}^2 + 1}(|\nabla\tilde{\varphi}|^2 - |\nabla\theta|^2) + 2i\frac{1 - \tilde{\rho}^2}{1 + \tilde{\rho}^2}\nabla\tilde{\rho}\cdot\nabla\tilde{\varphi} + O(e^{-\frac{d}{2} - \frac{|z - d\vec{e}_1|}{2}}) \right].$$

Setting $z = d\vec{e}_1 + y$, we then have

$$\nabla\varphi_s = -\frac{(N-2)}{2d}\log d\nabla y_2 + O(\epsilon\log|y|), \ \nabla\varphi_r = O(\epsilon). \tag{7.22}$$

Thus

$$\begin{aligned} |\nabla\tilde{\varphi}|^2 - |\nabla\theta|^2 &= O(|\nabla\varphi_s\cdot\nabla\theta|) + |\nabla\varphi_s|^2 + O(\epsilon|\nabla\theta|) \\ \nabla\tilde{\rho}\cdot\nabla\tilde{\varphi} &= O(\epsilon|\log\epsilon|\rho') + O(\epsilon\rho'\log|y|). \end{aligned}$$

The estimate for the term $i\epsilon|\log\epsilon|\frac{\partial V_d}{\partial x_2}$ is same as before.

7.4. **Norms.** Let

$$\mathbb{E}_d = \mathbb{S}_0[V_d] + i(N-2)|\epsilon\log\epsilon|\frac{\partial V_d}{\partial x_2}, \quad \tilde{\mathbb{E}_d} = \frac{\mathbb{E}_d}{iV_d}. \tag{7.23}$$

Based on the form of the errors, we need to use suitable norms. Let us fix two positive numbers $p > N + 10, 0 < \sigma < 1$. Recall that $\phi = iV_d\psi, \psi = \psi_1 + i\psi_2$. Denote $r = |z - d\vec{e}_1|$, and define

$$\|h\|_{**} = \|iV_d h\|_{L^p(r<3)} + [\|r^{2+\sigma}h_1\|_{L^\infty(r>2)} + \|r^{1+\sigma}h_2\|_{L^\infty(r>2)}] \tag{7.24}$$

$$\begin{aligned} \|\psi\|_* = \|\phi\|_{W^{2,p}(r<2)} &+ \left[\|r^\sigma\psi_1\|_{L^\infty(r>2)} + \|r^{1+\sigma}\nabla\psi_1\|_{L^\infty(r>2)} \right] \\ &+ \left[\|r^{1+\sigma}\psi_2\|_{L^\infty(r>2)} + \|r^{2+\sigma}\nabla\psi_2\|_{L^\infty(r>2)} \right]. \end{aligned} \tag{7.25}$$

We remark that the main difference between this norm and the norms in the two-dimensional case is that we use L^p_{loc} (or $W^{2,p}_{loc}$) in the inner part. This is mainly due to the fact that the error term contains terms like $\epsilon \log|z-d\vec{e}_1|$ which is not L^∞-bounded.

Using the norms defined above, we can have the following error estimates, similar to Lemma 4.1.

Lemma 7.1. *It holds that for $z \in (B_2(d\vec{e}_1))^c$*

$$|Re(\tilde{\mathbb{E}}_d)| \le \frac{C\epsilon^{1-\sigma}}{(1+|z-d\vec{e}_1|)^3} \tag{7.26}$$

$$|Im(\tilde{\mathbb{E}}_d)| \le \frac{C\epsilon^{1-\sigma}}{(1+|z-d\vec{e}_1|)^{1+\sigma}} \tag{7.27}$$

$$\|\mathbb{E}_d\|_{L^p(\{|z-d\vec{e}_1|<3\})} \le C\epsilon|\log\epsilon| \tag{7.28}$$

where $\sigma \in (0,1)$ is a constant. As a consequence, there holds

$$\|\tilde{\mathbb{E}}_d\|_{**} \le C\epsilon^{1-\sigma}. \tag{7.29}$$

7.5. **Projected Linear and Nonlinear Problems.** Let

$$\begin{aligned}\tilde{L}_0[\psi] \quad &= \quad \Delta\psi + \frac{(N-2)}{x_1}\frac{\partial\psi}{\partial x_1} \\ &\quad + \frac{2(1-|V_d|^2)}{1+|V_d|^2}\nabla V_d\cdot\nabla\psi + \frac{4i\bar{V_d}^2\psi_2}{(1+|V_d|^2)^2}\nabla V_d\cdot\nabla V_d - i\frac{4|V_d|^2\psi_2}{(1+|V_d|^2)^2}\end{aligned}$$

and

$$Z_d := \frac{\partial V_d}{\partial d}(\tilde{\eta}(\frac{|z-d\vec{e}_1|}{R}) + \tilde{\eta}(\frac{|z+d\vec{e}_1|}{R})) \tag{7.30}$$

where $\tilde{\eta}$ is defined at (4.2) before.

As the first step of finite dimensional reduction, we need to consider the following projected linear problem

$$\begin{cases} \tilde{L}_0(\psi) = h \ \text{ in } \mathbb{R}^2_+, \\ \mathrm{Re}(\int_{\mathbb{R}^2_+}\bar{\phi}Z_d) = 0 \\ \phi = iV_d\psi, \quad \psi \text{ satisfies the symmetry (4.3)} \end{cases} \tag{7.31}$$

We have the following *a priori estimates*, similar to lemma 5.1.

Lemma 7.2. *There exists a constant C, depending on γ, σ only such that for all ϵ sufficiently small, $d \sim \frac{1}{\epsilon}$, and any solution of (7.31), it holds*

$$\|\psi\|_* \le C\|h\|_{**}. \tag{7.32}$$

Proof: The proof is exactly as in Lemma 5.1 except two main different points. The first point is the inner part estimates. We use instead the L^p-estimates in the inner part $\{|z-d\vec{e}_1|<1\}$. By choosing $p>N$ large we obtain the embedding $W^{2,p}_{loc}$ into $C^{1,\alpha}_{loc}$ for any $\alpha \in (0,1)$. For the outer part estimates, we use the following new barrier function

$$B(z) := B_1(z) + B_2(z) \tag{7.33}$$

where

$$B_1(z) = |z-d\vec{e}_1|^\beta x_2^\gamma + |z+d\vec{e}_1|^\beta x_2^{-\gamma}, \ B_2(z) = C_1(1+|z|^2)^{-\frac{\sigma}{2}} \tag{7.34}$$

where $\beta+\gamma=-\sigma, 0<\sigma<\gamma<1$, and C_1 is a large number depending on σ,β,γ only.

Similar to the computations in Lemma 4.2, we obtain that

$$\Delta B_1 \leq -C(|z-d\vec{e}_1|^2+|z+d\vec{e}_1|^2)^{-\frac{\sigma+2}{2}} \tag{7.35}$$

$$\Delta B_2 + \frac{(N-2)}{x_1} B_2 \leq -CC_1 \frac{1}{(1+|z|^2)^{1+\frac{\sigma}{2}}} \tag{7.36}$$

On the other hand,

$$\frac{(N-2)}{x_1}\frac{\partial B_2}{\partial x_1} \leq C\frac{x_2^\gamma}{x_1}[|z-d\vec{e}_1|^{\beta-2}(x_1-d)+|z+d\vec{e}_1|^{\beta-2}(x_1-d)] \tag{7.37}$$

Thus for $|z-d\vec{e}_1|<c_\sigma d$, where c_σ is small, we have

$$\frac{(N-2)}{x_1}\frac{\partial B_2}{\partial x_1} \leq Cc_\sigma(|z-d\vec{e}_1|^2+|z+d\vec{e}_1|^2)^{-\frac{\sigma+2}{2}} \tag{7.38}$$

For $|z-d\vec{e}_1|>c_\sigma d$,

$$\frac{(N-2)}{x_1}\frac{\partial B_2}{\partial x_1} \leq C\frac{1}{(1+|z|^2)^{1+\frac{\sigma}{2}}} \tag{7.39}$$

By choosing C_1 large, we have

$$\Delta B + \frac{(N-2)}{x_1}\frac{\partial B}{\partial x_1} \leq -C(|z-d\vec{e}_1|^2+|z+d\vec{e}_1|^2)^{-\frac{\sigma+2}{2}} \tag{7.40}$$

The rest of the proof follows the same line of those in Lemma 5.1. We omit the details.

□

Similar to the proof of Proposition 5.2, by contraction mapping theorem, we conclude

Proposition 7.1. *There exists a constant C, depending on γ,σ only such that for all ϵ sufficiently small, d large, the following holds: there exists a unique solution $(\phi_{\epsilon,d}, c_\epsilon(d))$ to*

$$\mathbb{S}[V_d+\phi_{\epsilon,d}] = c_\epsilon(d) Z_d \tag{7.41}$$

and $\phi_{\epsilon,d}$ satisfies

$$\|\phi_{\epsilon,d}\|_* \leq C\epsilon^{1-\sigma}. \tag{7.42}$$

Furthermore, $\phi_{\epsilon,d}$ is continuous in d.

7.6. **Reduced Problem and the Proof of Theorem 7.1.** Similar to the proof of Theorem 2.1, we are left to estimate the following integral

$$\begin{aligned}
&\mathrm{Re}\Big(\int_{\mathbb{R}^2_+}\frac{1}{(1+|V_d|^2)^2}\overline{\tilde{Z}_d}\mathbb{S}[V_d]\Big)\\
&= \mathrm{Re}\Big(\int_{\mathbb{R}^2_+}\frac{1}{(1+|V_d|^2)^2}\overline{\tilde{Z}_d}\mathbb{S}_0[V_d] + (N-2)\epsilon|\log\epsilon|\mathrm{Re}\Big(i\int_{\mathbb{R}^2_+}\frac{1}{(1+|V_d|^2)^2}\overline{\tilde{Z}_d}\frac{\partial V_d}{\partial x_2}\Big).
\end{aligned}$$

On $\mathbb{R}^2_+$, recall that $z = d\vec{e}_1 + y$. Then

$$-Re\Big(\int_{\mathbb{R}^2_+}\frac{1}{(1+|V_d|^2)^2}\overline{\tilde{Z}_d}\mathbb{S}_0[V_d]\Big) \tag{7.43}$$

$$= \int_{\mathbb{R}^2_+}\frac{1}{(1+\rho^2)^2}\left[\Big(\frac{N-2}{d+y_1}\frac{\partial\tilde\rho}{\partial y_1}+\frac{\tilde\rho(\tilde\rho^2-1)}{\tilde\rho^2+1}(2\nabla\varphi_s\nabla\theta+|\nabla\varphi_s|^2)\Big)\frac{\partial\tilde\rho}{\partial y_1}+\frac{2\tilde\rho(1-\tilde\rho^2)}{1+\tilde\rho^2}\nabla\tilde\rho\cdot\nabla\varphi_s\frac{\partial\theta}{\partial y_1}\right]+O(\epsilon)$$

Let us notice that

$$\nabla \varphi_s = -\frac{(N-2)}{2d} \log d \nabla y_2 + O(\epsilon \log |y|).$$

Hence

$$\int_{\mathbb{R}^2_+} \frac{1}{(1+\rho^2)^2} \frac{N-2}{d+y_1} (\frac{\partial \tilde{\rho}}{\partial y_1})^2 = O(\epsilon)$$

$$\int_{\mathbb{R}^2_+} \frac{1}{(1+\rho^2)^2} \left[(\frac{\tilde{\rho}(\tilde{\rho}^2-1)}{\tilde{\rho}^2+1} (2\nabla\varphi_s \cdot \nabla\theta + |\nabla\varphi_s|^2)) \frac{\partial \tilde{\rho}}{\partial y_1} + \frac{2\tilde{\rho}(1-\tilde{\rho}^2)}{1+\tilde{\rho}^2} \nabla\tilde{\rho} \cdot \nabla\varphi_s \frac{\partial \theta}{\partial y_1} \right] + O(\epsilon)$$

$$= \frac{N-2}{d} \log d \int_{\mathbb{R}^2_+} \frac{1-\rho^2}{(\rho^2+1)^3} \frac{\rho\rho'}{|y|} dy + O(\epsilon)$$

$$= \frac{(N-2)\pi}{8d} \log d + O(\epsilon).$$

On the other hand, by the estimates in Section 6, we have

$$\operatorname{Re}(i \int_{\mathbb{R}^2_+} \frac{1}{(1+|V_d|^2)^2} \frac{\partial V_d}{\partial y_2} \overline{\tilde{Z}_d}) = \frac{\pi}{4} + o(1). \tag{7.44}$$

Thus we obtain

$$c_\epsilon(d) = c_0 [\frac{(N-2)\pi}{8d} \log d - \frac{(N-2)\pi}{4} \epsilon |\log \epsilon| + O(\epsilon)] \tag{7.45}$$

where $c_0 \neq 0$. Therefore, we obtain a solution to $c_\epsilon(d) = 0$ with the following asymptotic behavior:

$$d_\epsilon \sim \frac{1+o(1)}{2\epsilon}. \tag{7.46}$$

This proves Theorem 7.1.

8. Appendix: Nondegeneracy

In this appendix, we shall prove the nondegeneracy result–Theorem 3.1. The proof mainly follows that of [18, Theorem 1] which proved the nondegeneracy of degree one vortex for the Ginzburg-Landau equation.

Let ϕ satisfy (3.5) and $\phi = iw\psi$ satisfy (3.6). We now consider the quadratic form:

$$\begin{aligned} B(\phi,\phi) = 4 \int_{\mathbb{R}^2} \Bigg[& \frac{|\nabla\phi|^2}{(1+|w|^2)^2} - \frac{8\langle w,\phi\rangle\langle\nabla w,\nabla\phi\rangle}{(1+|w|^2)^3} - \frac{2|\nabla w|^2|\phi|^2}{(1+|w|^2)^3} \\ & - \frac{(1-|w|^2)|\phi|^2}{(1+|w|^2)^3} + \frac{12|\nabla w|^2\langle w,\phi\rangle^2}{(1+|w|^2)^4} + \frac{4(2-|w|^2)\langle w,\phi\rangle^2}{(1+|w|^2)^4} \Bigg] \mathrm{d}x \end{aligned} \tag{8.1}$$

Using (3.6), a simple computation shows that

$$B(\phi,\phi) = 0. \tag{8.2}$$

Let us first assume that ϕ is a smooth and compactly supported function with support away from $r = 0$. (In fact, we may replace ϕ by $\phi\eta$, where $\eta(r) = 1$ for

$\delta < r < \frac{1}{\delta}$ and $\eta = 0$ for $r < \frac{\delta}{2}$ or $r > \frac{2}{\delta}$.) We decompose ϕ into the form

$$\phi = \phi^0 + \sum_{j=1}^{\infty} \phi_j^1 + \sum_{j=1}^{\infty} \phi_j^2, \tag{8.3}$$

where

$$\phi^0 = e^{i\theta}[\phi_1^0(r) + i\phi_2^0(r)], \tag{8.4}$$

$$\phi_j^1 = e^{i\theta}[\phi_{j1}^1(r)\sin j\theta + i\phi_{j2}^1(r)\cos j\theta], \tag{8.5}$$

$$\phi_j^2 = e^{i\theta}[\phi_{j1}^2(r)\cos j\theta + i\phi_{j2}^2(r)\sin j\theta]. \tag{8.6}$$

Then we get

$$B(\phi,\phi) = B(\phi^0,\phi^0) + \sum_{j=1}^{\infty} B(\phi_j^1,\phi_j^1) + \sum_{j=1}^{\infty} B(\phi_j^2,\phi_j^2). \tag{8.7}$$

Since $\phi = iw\psi$, we introduce the bilinear form for ψ

$$\mathbb{B}(\psi,\psi) = B(iw\psi, iw\psi).$$

Then we can obtain, writing $\psi = \psi_1 + i\psi_2$ for real ψ_1 and ψ_2,

$$\begin{aligned}\mathbb{B}(\psi,\psi) = &\int_{\mathbb{R}^2} \frac{\rho^2}{(1+\rho^2)^2}|\nabla\psi|^2 + \int_{\mathbb{R}^2} \frac{4\rho^2(1-\rho^2)}{r^2(1+\rho^2)^3}\psi_1\frac{\partial\psi_2}{\partial\theta}\\ &+ \int_{\mathbb{R}^2} \frac{4\rho^2}{r^2(1+\rho^2)^4}[r^2\rho'^2 + (r^2-1)\rho^2]\psi_2^2.\end{aligned} \tag{8.8}$$

Indeed, using

$$\mathrm{Re}(w\bar\psi\nabla\bar w\nabla\psi) = \rho\rho'\mathrm{Re}(\frac{\partial\psi}{\partial r}\bar\psi) - \frac{\rho^2}{r^2}\mathrm{Re}(i\frac{\partial\psi}{\partial\theta}\bar\psi)$$

and integrating by parts, we first get

$$\begin{aligned}\int_{\mathbb{R}^2}\frac{|\nabla\phi|^2}{(1+|w|^2)^2} = &\int_{\mathbb{R}^2}\frac{\rho^2}{(1+\rho^2)^2}|\nabla\psi|^2 + \int_{\mathbb{R}^2}\frac{(\rho'^2+\frac{\rho^2}{r^2})}{(1+\rho^2)^2}|\psi|^2\\ &-\int_{\mathbb{R}^2}\frac{1}{r}\left(\frac{r\rho\rho'}{(1+\rho^2)^2}\right)'|\psi|^2 + \int_{\mathbb{R}^2}\frac{4\rho^2}{r^2(1+\rho^2)^2}\psi_1\frac{\partial\psi_2}{\partial\theta}\\ = &\int_{\mathbb{R}^2}\frac{\rho^2}{(1+\rho^2)^2}|\nabla\psi|^2 + \int_{\mathbb{R}^2}\left(\frac{\rho'^2}{(1+\rho^2)^2} + \frac{\rho^2}{r^2(1+\rho^2)^2}\right)|\psi|^2\\ &+\int_{\mathbb{R}^2}\left[\frac{2\rho^2\rho'^2}{(1+\rho^2)^3} - \frac{\rho'^2}{(1+\rho^2)^2} - (\frac{1}{r^2}-1)\frac{1-\rho^2}{(1+\rho^2)^3}\rho^2\right]|\psi|^2\\ &+\int_{\mathbb{R}^2}\frac{4\rho^2}{r^2(1+\rho^2)^2}\psi_1\frac{\partial\psi_2}{\partial\theta},\end{aligned} \tag{8.9}$$

where (2.3) is used in the last equality. Next we have similarly that

$$\begin{aligned}-\int_{\mathbb{R}^2}\frac{8\langle w,\phi\rangle\langle\nabla w,\nabla\phi\rangle}{(1+|w|^2)^3} = &\int_{\mathbb{R}^2}\frac{4}{r}\left(\frac{r\rho^3\rho'}{(1+\rho^2)^3}\right)'\psi_2^2\\ &-\int_{\mathbb{R}^2}8\left(\frac{\rho^2\rho'^2}{(1+\rho^2)^3} + \frac{\rho^4}{r^2(1+\rho^2)^3}\right)\psi_2^2\\ &-\int_{\mathbb{R}^2}\frac{8\rho^4}{r^2(1+\rho^2)^3}\psi_1\frac{\partial\psi_2}{\partial\theta}.\end{aligned} \tag{8.10}$$

By direct calculation, we easily know that

$$-\int_{\mathbb{R}^2}\frac{2|\nabla w|^2|\phi|^2}{(1+|w|^2)^3}=-\int_{\mathbb{R}^2}2\left(\frac{\rho^2\rho'^2}{(1+\rho^2)^3}+\frac{\rho^4}{r^2(1+\rho^2)^3}\right)|\psi|^2, \tag{8.11}$$

$$-\int_{\mathbb{R}^2}\frac{(1-|w|^2)|\phi|^2}{(1+|w|^2)^3}=-\int_{\mathbb{R}^2}\frac{(1-\rho^2)\rho^2}{(1+\rho^2)^3}|\psi|^2, \tag{8.12}$$

$$\int_{\mathbb{R}^2}\frac{12|\nabla w|^2\langle w,\phi\rangle^2}{(1+|w|^2)^4}=\int_{\mathbb{R}^2}\frac{12\rho^2}{1+\rho^2}\left(\frac{\rho^2\rho'^2}{(1+\rho^2)^3}+\frac{\rho^4}{r^2(1+\rho^2)^3}\right)\psi_2^2, \tag{8.13}$$

$$\int_{\mathbb{R}^2}\frac{4(2-|w|^2)\langle w,\phi\rangle^2}{(1+|w|^2)^4}=\int_{\mathbb{R}^2}\frac{4(2-\rho^2)\rho^4}{(1+\rho^2)^4}\psi_2^2. \tag{8.14}$$

Adding (8.9)-(8.14) together and using (2.3), we finally get (8.8).

On the other hand, defining ψ^0 and ψ_j^ℓ such that $\phi^0=iw\psi^0$ and $\phi_j^\ell=iw\psi_j^\ell$ for $j\in\mathbb{N}^*$, $\ell=1,2$, and using (8.1), we get

$$\mathbb{B}(\psi,\psi)=\mathbb{B}(\psi^0,\psi^0)+\sum_{j=1}^{\infty}\mathbb{B}(\psi_j^1,\psi_j^1)+\sum_{j=1}^{\infty}\mathbb{B}(\psi_j^2,\psi_j^2).$$

Let us set

$$\begin{aligned}\psi^0&=\psi_1^0(r)+i\psi_2^0(r),\\ \psi_j^1&=\psi_{j1}^1(r)\cos j\theta+i\psi_{j2}^1(r)\sin j\theta,\\ \psi_j^2&=\psi_{j1}^2(r)\sin j\theta+i\psi_{j2}^2(r)\cos j\theta.\end{aligned}$$

Consider the bilinear forms, for $j\in\mathbb{N}^*$ and $\ell=1,2$,

$$\mathcal{B}_j^\ell(\varphi,\varphi)=\int_0^\infty\frac{r\rho^2}{(1+\rho^2)^2}|\varphi'|^2+\int_0^\infty\frac{r\rho^2}{(1+\rho^2)^2}B_j^\ell\varphi\cdot\varphi, \tag{8.15}$$

where the function φ: $[0,\infty)\to\mathbb{R}^2$ and

$$B_j^\ell=\frac{1}{r^2}\begin{pmatrix} j^2 & (-1)^{\ell+1}2j\frac{1-\rho^2}{1+\rho^2}\\ (-1)^{\ell+1}2j\frac{1-\rho^2}{1+\rho^2} & j^2+\frac{4}{(1+\rho^2)^2}(r^2\rho'^2+(r^2-1)\rho^2)\end{pmatrix}. \tag{8.16}$$

With these definitions it is easy to check that the following fact holds:

$$\mathbb{B}(\psi_j^\ell,\psi_j^\ell)=\pi\mathcal{B}_j^\ell(\varphi_j^\ell,\varphi_j^\ell),$$

where $\varphi_j^\ell(r)=(\psi_{j1}^\ell,\psi_{j2}^\ell)$.

We first analyze the mode one case.

Proposition 8.1. *For any $\mathbb{R}^2-$ valued smooth function φ, with compact support away from the origin we have*

$$\mathcal{B}_1^1(\varphi,\varphi)=\int_0^\infty\frac{r\rho^2}{(1+\rho^2)^2}|\varphi'-A_1(r)\varphi|^2\mathrm{d}r, \tag{8.17}$$

where $A_1(r)$ is a 2×2 symmetric matrix of smooth functions in $(0,\infty)$ with the property that all the solutions of the system $\varphi'=A_1(r)\varphi$ satisfying $\int_0^1\frac{r\rho^2}{(1+\rho^2)^2}|\varphi|^2<\infty$ are just given by constant multiples of $\varphi_0=(\frac{1}{r},-\frac{\rho'}{\rho})$.

Proof. First we expand, by integrating by parts,

$$\int_0^\infty \frac{r\rho^2}{(1+\rho^2)^2}|\varphi' - A_1(r)\varphi|^2 = \int_0^\infty \frac{r\rho^2}{(1+\rho^2)^2}(|\varphi'|^2 - 2A_1\varphi'\cdot\varphi + A_1^2\varphi\cdot\varphi)$$
$$= \int_0^\infty \frac{r\rho^2}{(1+\rho^2)^2}|\varphi'|^2 + \int_0^\infty \left[\left(\frac{r\rho^2}{(1+\rho^2)^2}A_1\right)' + \frac{r\rho^2}{(1+\rho^2)^2}A_1^2\right]\varphi\cdot\varphi.$$

Thus (8.17) is equivalent to

$$\left(\frac{r\rho^2}{(1+\rho^2)^2}A_1\right)' + \frac{r\rho^2}{(1+\rho^2)^2}A_1^2 = \frac{r\rho^2}{(1+\rho^2)^2}B_1^1 \tag{8.18}$$

where B_1^1 is given by (8.16) for j, $\ell = 1$.

Suppose

$$A_1(r) = \begin{pmatrix} a(r) & c(r) \\ c(r) & b(r) \end{pmatrix}.$$

Let us denote $\varphi_0 = (\sigma_1, \sigma_2) := (\frac{1}{r}, -\frac{\rho'}{\rho})$. Since $\mathcal{B}_1^1(\varphi_0, \varphi_0) = 0$, the matrix A_1 should satisfy

$$\varphi_0' = A_1\varphi_0.$$

This yields the relations

$$a(r) = \frac{\sigma_1' - c\sigma_2}{\sigma_1}, \qquad b(r) = \frac{\sigma_2' - c\sigma_1}{\sigma_2}.$$

Now, substituting these relations into (8.18), direct inspection leads to the fact that (8.18) holds if and only if

$$\left(\frac{r\rho^2}{(1+\rho^2)^2}c\right)' = \frac{r\rho^2}{(1+\rho^2)^2}c^2\left(\frac{\sigma_2}{\sigma_1} + \frac{\sigma_1}{\sigma_2}\right) - \frac{r\rho^2}{(1+\rho^2)^2}c\left(\frac{\sigma_1'}{\sigma_1} + \frac{\sigma_2'}{\sigma_2}\right) + \frac{2\rho^2(1-\rho^2)}{r(1+\rho^2)^3}.$$

Recall $\sigma_1 = \frac{1}{r}$, $\sigma_2 = -\frac{\rho'}{\rho}$, and we can get

$$c' = -c^2\left(\frac{\rho^2 + \rho'^2 r^2}{r\rho\rho'}\right) - c\left(\frac{\rho''}{\rho'} + \frac{\rho'}{\rho} - \frac{4\rho\rho'}{1+\rho^2}\right) + \frac{2(1-\rho^2)}{r(1+\rho^2)}. \tag{8.19}$$

Let us observe that if u satisfies the equation

$$(p(r)u')' + p(r)z(r)u' + q(r)u = 0, \tag{8.20}$$

then $c = pu'/u$ satisfies

$$c' = -\frac{c^2}{p} - zc - q.$$

Set

$$p = \frac{r\rho\rho'}{\rho^2 + \rho'^2 r^2}, \quad q = -\frac{2(1-\rho^2)}{r(1+\rho^2)}, \quad z = \frac{\rho''}{\rho'} + \frac{\rho'}{\rho} - \frac{4\rho\rho'}{1+\rho^2}.$$

Then

$$c = pu'/u$$

satisfies Equation (8.19) on $(0, \infty)$ if u is a solution to (8.20) with the above p, q, z. Calculation tells us

$$u'' - \left(\frac{1}{r} + \frac{2(1-\rho^2)(r^2\rho^4 - \rho^4 - r^4\rho'^4)}{r^2\rho\rho'(1+\rho^2)(\rho^2 + r^2\rho'^2)}\right)u' - \frac{2(1-\rho^2)(\rho^2 + \rho'^2 r^2)}{r^3\rho\rho'(1+\rho^2)}u = 0. \tag{8.21}$$

As $r \to \infty$, we easily have

$$\frac{1}{r} + \frac{2(1-\rho^2)(r^2\rho^4 - \rho^4 - r^4\rho'^4)}{r^2\rho\rho'(1+\rho^2)(\rho^2 + r^2\rho'^2)} \sim \frac{1}{r} + \frac{2(1-\rho)}{\rho'},$$
$$\frac{2(1-\rho^2)(\rho^2 + \rho'^2 r^2)}{r^3\rho\rho'(1+\rho^2)} \sim \frac{2(1-\rho)}{r^3\rho'}.$$

Since $(1-\rho)/\rho' \to \alpha$ ($\alpha > 0$ is a constant), this equation resembles

$$u'' - 2\alpha u' - \frac{2\alpha}{r^3}u = 0.$$

Obviously, any positive constant and $e^{1/2r^2}$ are respectively a super-solution and a sub-solution. Therefore, a positive solution of the above equation exists. Similarly as $r \to 0$, (8.21) tends to

$$u'' + \frac{1}{r}u' - \frac{4}{r^2}u = 0.$$

A solution to this equation is $u(r) = \frac{r^2 + r^{-2}}{2}$. Summarizing, we conclude that Equation (8.19) has a globally defined solution $c(r) = pu'/u$, $r \in (0, \infty)$. The matrix $A_1(r)$ as desired has thus been built.

$\varphi_0 = (\frac{1}{r}, \frac{\rho'}{\rho})$ is already demanded to satisfy $\varphi' = A_1\varphi$. Let $\varphi_1(r)$ be another linearly independent solution. Since, from its definition $p(0^+) = 1/2$, we get $c(r) = -\frac{1}{r} + o(\frac{1}{r})$ as $r \to 0$. So the following property for $A_1(r)$ is automatically checked:

$$A_1(r) = \begin{pmatrix} -\frac{2}{r} & -\frac{1}{r} \\ -\frac{1}{r} & -\frac{2}{r} \end{pmatrix} + o(\frac{1}{r})$$

as $r \to 0$. Then Liouville's formula for the Wronskian gives us that

$$W(\varphi_0, \varphi_1) = Ce^{\int_{r_0}^{r}(a+b)\mathrm{d}r} \sim \frac{C}{r^4}.$$

It follows that $|\varphi_1| \geq \frac{C}{r^3}$ for all small $r > 0$. The conclusion is that the unique solutions of $\varphi' = A_1(r)\varphi$ for which $\int_0^1 \frac{r\rho^2}{(1+\rho^2)^2}|\varphi|^2 < \infty$ are scalar multiples φ_0. The proof is complete. $\square$

Similarly we get

Proposition 8.2. *For any $\mathbb{R}^2$ − valued smooth function φ, with compact support away from the origin we have*

$$\mathcal{B}_1^2(\varphi, \varphi) = \int_0^\infty \frac{r\rho^2}{(1+\rho^2)^2}|\varphi' - A_2(r)\varphi|^2 \mathrm{d}r,$$

where $A_2(r)$ is a 2×2 symmetric matrix of smooth functions in $(0, \infty)$ with the property that all the solutions of the system $\varphi' = A_2(r)\varphi$ satisfying $\int_0^1 \frac{r\rho^2}{(1+\rho^2)^2}|\varphi|^2 < \infty$ are just given by constant multiples of $\bar{\varphi}_0 = (\frac{1}{r}, \frac{\rho'}{\rho})$.

Proof of Theorem 3.1. For $j \geq 2$, we note that

$$(B_j^\ell - B_1^\ell)\varphi \cdot \varphi = \frac{j-1}{r^2}\begin{pmatrix} j+1 & (-1)^{\ell+1}2\frac{1-\rho^2}{1+\rho^2} \\ (-1)^{\ell+1}2\frac{1-\rho^2}{1+\rho^2} & j+1 \end{pmatrix} \geq \frac{(j-1)^2}{r^2}|\varphi|^2,$$

which implies

$$\mathcal{B}_j^\ell(\varphi,\varphi) \geq (j-1)^2 \int_0^\infty \frac{\rho^2}{r(1+\rho^2)^2}|\varphi|^2. \tag{8.22}$$

For $j=0$, we use the following fact from [28, Lemma 5.3]

$$r^2\rho'^2 + (r^2-1)\rho^2 \geq 0 \tag{8.23}$$

and obtain

$$\mathcal{B}_0^\ell(\varphi,\varphi) \geq \int_0^\infty \frac{\rho^2}{r(1+\rho^2)^2}|\varphi|^2. \tag{8.24}$$

In conclusion, we have obtained that for functions ϕ which is smooth and compactly away from $r=0$ and $\phi = iw\psi$, it holds
(8.25)

$$\mathbb{B}(\psi,\psi) = \sum_{l=1}^2 \pi \int_0^\infty \frac{r\rho^2}{(1+\rho^2)^2}|\varphi_1^{l'} - A_l(r)\varphi_1^l|^2 dr + \pi \sum_{l=1}^2 \sum_{j=0}^\infty (j-1)^2 \int_0^\infty \frac{\rho^2}{r(1+\rho^2)^2}|\varphi_j^l|^2$$

Inequality (8.25) also holds for any smooth function ϕ satisfying (3.6). The proof is by a cut-off function. The rest of the proof is similar to that of [18, Theorem 1]. □

Acknowledgments: The research of the first author is partially supported by the NSF grant DMS-0700571. The research of the second author is partially supported by an Earmarked Grant from RGC of Hong Kong and a Focused Research Scheme of CUHK.

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