# The Existence and Stability of Spike Equilibria in the One-Dimensional Gray-Scott Model

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**Abstract**

Results for the existence, stability, and pulse-splitting behavior of spike patterns in the one-dimensional Gray-Scott model on a finite domain in the semi-strong spike-interaction regime are summarized. Conditions on the parameters for the existence of competition instabilities, synchronous oscillatory instabilities, or pulse-splitting behavior of spike patterns are given.

Keywords: spike patterns, nonlocal eigenvalue problem, oscillatory instability, pulse-splitting.

## 1 Introduction

In this letter we summarize some results of [1] and [2] for the existence and stability of spike patterns in the one-dimensional Gray-Scott reaction-diffusion model. In dimensionless variables, this model, first introduced in [3] and studied numerically in [4], can be written as

$$V_T = D_v V_{XX} - (F+k)V + UV^2\,, \qquad 0 < X < L\,, \quad T > 0\,; \qquad V_X = 0\,, \quad X = 0, L\,, \tag{1.1a}$$

$$U_T = D_u U_{XX} + F(1-U) - UV^2\,, \qquad 0 < X < L\,, \quad T > 0\,; \qquad U_X = 0\,, \quad X = 0, L\,. \tag{1.1b}$$

Here $D_u > 0$, $D_v > 0$ are the constant diffusivities, $F > 0$ is the feed rate, and $k > 0$ is a reaction-time constant. In [1] and [2], (1.1) is analyzed in the semi-strong interaction regime where $D_v/D_u$ is asymptotically small, but where the diffusive length scale of $U$ is comparable to the length $L$ of the domain. The change of variables, $v = V/\sqrt{F}$, $x = -1 + 2X/L$, and $t = (F+k)T$, which is similar to that introduced in [5], leads to the dimensionless system

$$v_t = \varepsilon^2 v_{xx} - v + Auv^2\,, \qquad -1 < x < 1\,, \quad t > 0\,; \qquad v_x = 0\,, \quad x = -1, 1\,, \tag{1.2a}$$

$$\tau u_t = D u_{xx} + (1-u) - uv^2 \qquad -1 < x < 1\,, \quad t > 0\,; \qquad u_x = 0\,, \quad x = -1, 1\,. \tag{1.2b}$$

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Here $A > 0$, $D > 0$, $\tau > 1$, and $\varepsilon \ll 1$, are defined in terms of $D_u$, $D_v$, $L$, $F$, and $k$, by $D \equiv 4D_u/(FL^2)$, $\varepsilon^2 \equiv 4D_v/[L^2(F+k)]$, $\tau \equiv (F+k)/F$, and $A \equiv \sqrt{F}/(F+k)$. For the infinite domain problem, the existence and stability of a one-spike solution was studied analytically in [6], [7], [8], and [5]. Our results extend these analyses by determining the existence and stability of multi-spike solutions on a finite domain. The behavior of solutions in the semi-strong interaction regime is significantly different than that for the weak interaction regime, where $D = O(\varepsilon^2)$, studied in [9] and [10]. In this latter regime, where $u$ and $v$ are both local variables, the spike interactions are exponentially weak, and self-replication of spikes together with spatio-temporal chaos are possible.

In the semi-strong regime where $D = O(1)$, there are two main parameter regimes for $A$ where different behaviors occur. In §2, we give some results of [1] for the *low-feed rate* regime where $A = O(\varepsilon^{1/2})$. Some results of [2] for the pulse-splitting regime where $A = O(1)$ are given in §3.

## 2 The Low-Feed Rate Regime: $\mathcal{A} = O(\varepsilon^{1/2})$

In the low feed-rate regime, we define a new $O(1)$ variable $\mathcal{A}$ by $A = \varepsilon^{1/2}\mathcal{A}$. The first result of [1] for this regime characterizes symmetric $k$-spike equilibrium patterns, characterized by spikes in $v$ that have a common amplitude. For each $k = 1, 2\ldots$, and for $\mathcal{A} > \mathcal{A}_{ke}$, there are two branches of such solutions for (1.2) that meet at the saddle-node bifurcation value $\mathcal{A}_{ke}$ given by

$$\mathcal{A}_{ke} \equiv \sqrt{\frac{12\theta_0}{\tanh(\theta_0/k)}}, \qquad \theta_0 \equiv D^{-1/2}. \tag{2.1}$$

These solution branches are conveniently parameterized in terms of $s$, with $0 < s < \infty$, where

$$s \equiv \frac{1-U_\pm}{U_\pm}, \qquad U_\pm = \frac{1}{2}\left[1 \pm \sqrt{1 - \frac{\mathcal{A}_{ke}^2}{\mathcal{A}^2}}\right]. \tag{2.2}$$

We refer to the range $0 < s < 1$ as the *large solution branch*, while the range $s > 1$ is called the *small solution branch.* The precise result is as follows (see Proposition 2.1 of [1]):

**Proposition 1**: *Let $\varepsilon \to 0$, with $\mathcal{A} = O(1)$ and $D = O(1)$. Then, when $\mathcal{A} > \mathcal{A}_{ke}$, the small and large $k$-spike symmetric equilibrium solutions for (1.2), denoted by $u_-$, $v_-$ and $u_+$, $v_+$, respectively, are given asymptotically by*

$$v_\pm(x) \sim \frac{1}{\sqrt{\varepsilon}\mathcal{A}U_\pm}\sum_{j=1}^{k} w\left[\varepsilon^{-1}(x-x_j)\right], \qquad u_\pm(x) \sim 1 - \frac{(1-U_\pm)}{a_g}\sum_{j=1}^{k} G(x;x_j). \tag{2.3}$$

*Here* $x_j = -1 + \frac{(2j-1)}{k}$, $a_g \equiv \sum_{i=1}^{k} G(x_j; x_i)$, $w(y) = \frac{3}{2} sech^2(y/2)$ *satisfies* $w'' - w + w^2 = 0$, *and* $G(x; x_j)$ *is the Green's function satisfying* $DG_{xx} - G = -\delta(x - x_j)$ *with* $G_x(\pm 1; x_j) = 0$.

To analyze the stability of these equilibrium solutions on an $O(1)$ time-scale, we let $u(x,t) = u_{\pm}(x) + e^{\lambda t}\eta(x)$, and $v(x,t) = v_{\pm}(x) + e^{\lambda t}\phi(x)$, where $\phi$ is a localized eigenfunction of the form

$$\phi(x) \sim \sum_{j=1}^{k} c_j \Phi\left[\varepsilon^{-1}(x - x_j)\right] , \tag{2.4}$$

for some coefficients $c_j$ to be found. We consider eigenfunctions for which $\int_{-\infty}^{\infty} w(y)\Phi(y)\, dy \neq 0$. A lengthy derivation as given in §3.1 of [1] leads to the following spectral problem for $\Phi(y)$ (see Proposition 3.2 of [1]).

**Proposition 2**: *Assume that* $0 < \varepsilon \ll 1$. *Then, with* $\Phi = \Phi(y)$, *the* $O(1)$ *eigenvalues determining the stability of* $k$*-spike equilibria satisfy the nonlocal eigenvalue problem*

$$L_0\Phi - \chi_{gs} w^2 \left( \frac{\int_{-\infty}^{\infty} w\Phi\, dy}{\int_{-\infty}^{\infty} w\, dy} \right) = \lambda\Phi\,, \qquad -\infty < y < \infty\,; \qquad \Phi \to 0\,, \quad \text{as} \quad |y| \to \infty\,. \tag{2.5}$$

*Here the operator* $L_0$ *is* $L_0\Phi \equiv \Phi'' - \Phi + 2w\Phi$, *and the multiplier* $\chi_{gs} = \chi_{gs}(z; j)$ *is defined by*

$$\chi_{gs} \equiv 2s \left( s + \frac{\sqrt{1+z}}{\tanh(\theta_0/k)} \left[ \tanh(\theta_\lambda/k) + \frac{(1 - \cos[\pi(j-1)/k])}{\sinh(2\theta_\lambda/k)} \right] \right)^{-1} , \tag{2.6}$$

*where* $z \equiv \tau\lambda$, $\theta_\lambda \equiv \theta_0\sqrt{1+z}$, *and* $\theta_0 \equiv D^{-1/2}$. *Here* $s$ *is given in (2.2). The coefficients* $\boldsymbol{c}^t = (c_1, \ldots, c_k)$ *in (2.4) are given by*

$$\boldsymbol{c}_1^t = \frac{1}{\sqrt{k}} (1, \ldots, 1)\ ; \quad c_{l,j} = \sqrt{\frac{2}{k}} \cos\left( \frac{\pi(j-1)}{k} (l - 1/2) \right) , \quad j = 2, \ldots, k\,. \tag{2.7}$$

There is an equivalent formulation of (2.5) which states that the eigenvalues of (2.5) with $\int_{-\infty}^{\infty} w\Phi\, dy \neq 0$ are the union of the zeros of the functions $g_j(\lambda) = 0$ for $j = 1, .., k$, where

$$g_j(\lambda) \equiv C_j(\lambda) - f(\lambda)\,, \qquad f(\lambda) \equiv \frac{\int_{-\infty}^{\infty} w\,(L_0 - \lambda)^{-1} w^2\, dy}{\int_{-\infty}^{\infty} w^2\, dy}\,, \qquad C_j(\lambda) \equiv [\chi_{gs}]^{-1}\,. \tag{2.8}$$

By analyzing the zeroes of $g_j(\lambda)$ in the right half-plane $\text{Re}(\lambda) \geq 0$, the main stability result for multi-spike solutions is as follows (cf. Proposition 3.10 and 3.13 of [1]):

**Proposition 3**: *The large solution* $u_+$, $v_+$ *is unstable for any* $0 < s < 1$, $k \geq 1$, *and* $D > 0$. *Next, let* $k > 1$, *and consider the multi-spike small solution* $u_-$, $v_-$, *where* $s > 1$. *For* $\mathcal{A} > \mathcal{A}_{kL}$, *the solution* *will be stable on an $O(1)$ time-scale when $0 < \tau < \tau_{hL}$. Alternatively, on the range $\mathcal{A}_{ke} < \mathcal{A} < \mathcal{A}_{kL}$, the small solution is unstable for any $\tau > 0$. The threshold $\mathcal{A}_{kL}$ is given analytically by*

$$\mathcal{A}_{kL} = \frac{\left((\gamma_k/2) + 2\sinh^2(\theta_0/k)\right)}{\left(\left[(\gamma_k/2) + 2\sinh^2(\theta_0/k)\right]^2 - (\gamma_k/2)^2\right)^{1/2}}, \qquad \gamma_k \equiv 1 + \cos\left(\frac{\pi}{k}\right). \tag{2.9}$$

Let $\mathcal{A}$ satisfy $\mathcal{A} > \mathcal{A}_{kL}$. Then, as $\tau$ increases beyond $\tau_{hL}$, a Hopf bifurcation in the spike amplitudes was computed numerically in [1]. The threshold $\tau_{hL}$ is given by the *minimum value* of the set $\tau_j$, $j = 1,..,k$, for which $g_j(\lambda) = 0$, $j = 1,..,k$, has complex conjugate roots on the imaginary axis. Let $\lambda = \pm i\lambda_h$ be the corresponding value of $\lambda$. Then, as was shown in [1], the unstable eigenfunction generically has the form of a *synchronous oscillatory instability* with

$$v = v_- + \delta e^{i\lambda_h t}\phi + \text{c.c}, \quad \phi(x) = \sum_{n=1}^{k} c_n \Phi\left[\varepsilon^{-1}(x - x_n)\right], \quad c_n = 1, \quad n = 1,..,k. \tag{2.10}$$

Here c.c denotes complex conjugate and $\delta << 1$. Alternatively, suppose that $\mathcal{A}_{ke} < \mathcal{A} < \mathcal{A}_{kL}$. Then, for any $\tau > 0$, the dominant initial instability was shown in [1] to have the form

$$v = v_- + \delta e^{\lambda_{Rk} t}\phi, \quad \phi(x) = \sum_{n=1}^{k} c_n \Phi\left[\varepsilon^{-1}(x - x_n)\right], \quad c_n = \cos\left(\frac{\pi(k-1)}{k}(n - 1/2)\right), \quad n = 1,..,k. \tag{2.11a}$$

Here $\delta \ll 1$, and $\lambda_{Rk} > 0$ is the unique root of $g_k(\lambda_R) = 0$. Since $\sum_{n=1}^{k} c_n = 0$, this instability locally conserves the sum of the heights of the spikes. Hence, it is referred to as a *competition instability*. The numerical experiments in §3.3 of [1] show that this instability leads to a spike competition process whereby certain spikes in a spike sequence are ultimately annihilated.

Finally, in [1] it was shown there there is a spectral equivalence principle between (2.5) and a corresponding nonlocal eigenvalue problem derived in Proposition 2.3 of [12] for the following Gierer-Meinhardt (GM) model (cf. [11]) on $-1 < x < 1$ with exponent set $(p, q, m, s)$:

$$a_t = \varepsilon^2 a_{xx} - a + \frac{a^p}{h^q}; \qquad \tau h_t = D h_{xx} - h + \varepsilon^{-1}\frac{a^m}{h^s}; \qquad a_x = h_x = 0, \quad x = \pm 1. \tag{2.12}$$

Equilibrium spike solutions for (2.12) and their stability were analyzed in [12]. In Proposition 3.3 of [1] it was proved that the nonlocal eigenvalue problem for the stability of spike patterns in the GM model with exponent set $(2, s, 2, s)$ has exactly the same form as in (2.5).

## 3 The Pulse-Splitting Regime: $A = O(1)$

In this regime, where $A = O(1)$, it was shown in [2] that equilibrium $k$-spike solutions can be constructed in terms of the solutions $V(y) > 0$ and $U(y) > 0$ to a certain *core problem* defined by

$$V'' - V + V^2 U = 0\,, \qquad U'' = UV^2\,, \qquad 0 < y < \infty\,, \tag{3.1a}$$

$$V'(0) = U'(0) = 0\,; \quad V \to 0\,, \quad U \sim By\,, \quad \text{as} \quad y \to \infty\,; \quad B \equiv A \tanh(\theta_0/k)\,. \tag{3.1b}$$

Here $\theta_0 \equiv D^{-1/2}$. The precise result, as given in Proposition 3.1 of [2], is as follows:

**Proposition 4**: *Let $\varepsilon \to 0$, $A = O(1)$, $\varepsilon A/\sqrt{D} \ll 1$, and suppose that (3.1) has a solution. Then, the $v$-component for a $k$-spike equilibrium solution to (1.2) is given by*

$$v \sim \frac{\sqrt{D}}{\varepsilon} \sum_{j=1}^{k} \left( V\left[\varepsilon^{-1}(x - x_j)\right] + O\left(\frac{\varepsilon A}{\sqrt{D}}\right)\right)\,. \tag{3.2}$$

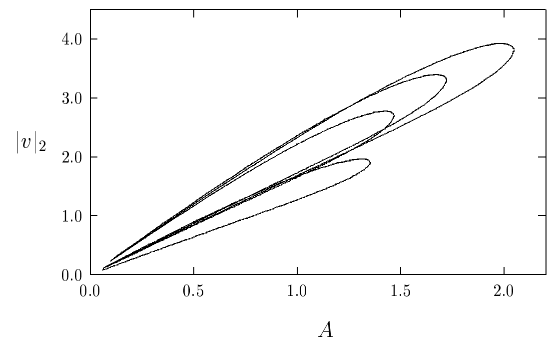
*In the $j^{th}$ inner region, where $|x - x_j| = O(\varepsilon)$, $u$ satisfies $u \sim \frac{\varepsilon}{A\sqrt{D}} U\left[\varepsilon^{-1}(x - x_j)\right]$. The outer solution for $u$, valid for $|x - x_j| \gg O(\varepsilon)$, and $j = 1, \ldots, k$, has the form $u \sim 1 - \frac{1}{a_g}\sum_{j=1}^{k} G(x; x_j)$, where $a_g$, $x_j$, and $G(x; x_j)$ are as defined in Proposition 1.*

The core problem (3.1) was studied qualitatively and numerically in [2] in terms of the parameter $B$. In [2] it was shown that $0 < \gamma < \frac{3}{2}$, where $\gamma \equiv U(0)V(0)$. Numerical solutions to (3.1) were computed for which $V$ has a single maximum at $y = 0$ as $\gamma \to 3/2$ from below, and a resulting curve $B = B(\gamma)$ was computed. This limiting solution for $V$ asymptotically matches onto the solution constructed in the low feed-rate regime in §2. As shown in [2], the curve $B = B(\gamma)$ is double-valued with $B \to 0$ as $\gamma \to 0$ and as $\gamma \to 3/2$, and it has a saddle-node bifurcation point at the maximum value $B_c$ of $B$ given by $B_c = 1.347$ (see also [5]), where $\gamma = \gamma_c = 1.02$. We refer to the range $\gamma_c < \gamma < 3/2$ and $0 < \gamma < \gamma_c$ as the *primary* and *secondary* branches of the $B = B(\gamma)$ bifurcation diagram. Since $B \equiv A\tanh(\theta_0/k)$ from (3.1b), these results of [2] show that equilibrium $k$-spike solutions exist only when $A$ is small enough. In particular, we have (cf. Proposition 3.2 of [2]):

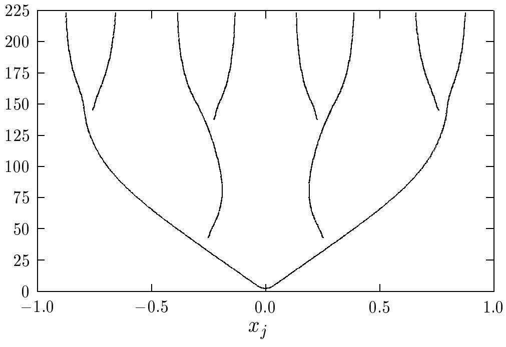
**Proposition 5**: *Let $\varepsilon \ll 1$, $A = O(1)$, and $\varepsilon A/\sqrt{D} \ll 1$. Then, there will be no $k$-spike equilibrium solution to (1.2) that merges onto the low feed-rate regime solution when*

$$A > A_{pk} \equiv 1.347 \coth\left(\frac{1}{k\sqrt{D}}\right)\,. \tag{3.3}$$

Next, it was shown in [2] that the three pulse-splitting criteria of [13] are satisfied when $\tau \ll O(\varepsilon^{-1})$. These criteria are as follows: 1) each $k$-spike equilibrium branch must have a saddle-node




(a) $|v|_2$ versus $A$




(b) $t$ versus $x_j$

Figure 1: Left figure: bifurcation diagram when $A = O(1)$ for $k$-spike solutions with $\varepsilon = 0.02$, $D = 0.1$, and $k = 1, \ldots, 4$. The saddle-node values $A_{pk}$ increase with $k$. Right figure: trajectories of the maxima of $v$ showing pulse-splitting behavior when $\varepsilon = 0.01$, $D = 0.1$, $A = 2.4$, and $\tau = 2.0$.

bifurcation that occurs at approximately the same bifurcation value (i.e. the *lining-up property* of equilibria): 2) for each $k$, one branch of equilibria is unstable while the other is stable: 3) the linearization around the equilibrium solution at each saddle-node value has a *dimple-shaped* eigenfunction associated with a zero eigenvalue. When these criteria of [13] are satisfied, and when the bifurcation parameter is taken to have a value slightly beyond the saddle-node value, pulse-splitting should occur from a single localized initial pulse due to the *ghost* of the dimple eigenfunction. In Fig. 1(a) we plot the norm $|v|_2$, defined by $|v|_2^2 \equiv \int_{-1}^{1} v^2\, dx \sim \varepsilon^{-1} kD \int_{-\infty}^{\infty} V^2\, dy$, versus $A$ when $D = 0.1$, to show that the approximate lining-up condition 1) holds. This condition holds since $\tanh\left(1/k\sqrt{D}\right) \approx 1$ for $D \ll 1$. For $\tau \ll O(\varepsilon^{-1})$ the stability condition 2) of [13] is verified in §5 and §6 of [2] by showing that the primary and secondary branches are stable and unstable, respectively. Finally, the dimple eigenfunction property is verified in §6 of [2]. Therefore, we conjecture:

**Conjecture 1**: *Let $\varepsilon \ll 1$, $\tau \ll O(\varepsilon^{-1})$, and $\varepsilon A/\sqrt{D} \ll 1$. Suppose that we have even one-spike initial data centered at the origin. Then, the final equilibrium state is stable, and it has $2^m$ spikes where, for some integer $m \geq 0$, $A$ is related to $A_{pk}$ by*

$$A_{p2^{m-1}} < A < A_{p2^m} \,. \tag{3.4}$$

To verify this conjecture, we computed numerical solutions to (1.2) starting from an initial pulse in $v$ centered at $x = 0$. In Fig. 1(b) we plot trajectories of the maxima of $v$ versus $t$, showing pulse-splitting behavior when $\varepsilon = 0.01$, $D = 0.1$, $A = 2.4$, and $\tau = 2.0$. For these values we calculate $A_{p4} = 2.045 < A < A_{p8} = 3.583$. Thus, (3.4) correctly predicts the eight-spike final state in Fig. 1(b). Further numerical experiments supporting the conjecture are given in [2].

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