ON BOUND STATES CONCENTRATING ON SPHERES FOR THE MAXWELL-SCHRÖDINGER EQUATION

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Abstract. We study the semiclassical limit for the following system of Maxwell-Schrödinger equations:

$$-\frac{\hbar^2}{2m}\Delta v+v+\omega\phi v-\gamma v^p=0,\quad -\Delta\phi=4\pi\omega v^2$$

where $\hbar,\ m,\ \omega,\ \gamma>0,\ v,\ \phi:\mathbb{R}^3\to\mathbb{R},\ 1<p<\frac{11}{7}$. This system describes standing waves for the nonlinear Schrödinger equation interacting with the electrostatic field: the unknowns v and ϕ represent the *wave function* associated to the particle and the electric potential respectively. By using localized energy method, we construct a family of positive radially symmetric bound states $(v_\hbar,\phi_\hbar)$ such that $v_\hbar$ concentrates around a sphere $\{|x|=s_0\}$ when $\hbar\to 0$.

1. Introduction

In this paper we consider the following system of Maxwell-Schrödinger equations in the electrostatic case:

$$\begin{cases} -\hbar^2\Delta v+v+\omega\phi v-\gamma v^p=0 \text{ in } \mathbb{R}^3, \\ -\Delta\phi=4\pi\omega v^2 \text{ in } \mathbb{R}^3, \\ v,\ \phi>0,\quad v,\ \phi\to 0 \text{ as } |x|\to+\infty, \end{cases} \tag{1.1}$$

where $\hbar,\ \omega,\ \gamma>0$ and $p>1$. This system has been first introduced in the paper [6]: it provides a physical model which describes the interaction of a charged particle with the electrostatic field. In (1.1) we have assumed, for sake of simplicity, $2m=1$, being m the mass of the particle. ω denotes the electric charge of the particle, while $\hbar$ is a constant which is known under the name of *Planck's constant.* The unknowns of the system are the wave function v associated to the particle and the electric potential ϕ. The presence of the nonlinear term in (1.1) simulates the interaction effect among many particles.

The eigenvalue problem for the system (1.1) with $\gamma=0$ has been studied in [6] (in the case in which the charged particle lies in a bounded space region Ω) and in [8] (under the action of an external nonzero potential). Existence results for (1.1) have been established in [11].

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In all the above mentioned results the size of $\hbar$ is not relevant, hence one may assume, without loss of generality, $\hbar = 1$. This paper deals with the *semiclassical limit* of the system (1.1), i.e. it is concerned with the problem of finding nontrivial solutions and studying their asymptotic behaviour when $\hbar \to 0^+$; hence such solutions are usually referred to as *semiclassical* ones. Letting $\hbar$ go to zero has great physical interest since it formally describes the transition from Quantum Mechanics to Classical Mechanics; as one expects, the solutions exhibit some kind of notable behaviour in the semiclassical limit, a concentration behaviour: their form consists of very sharp peaks which become highly concentrated when $\hbar$ is small.

The study of semiclassical phenomena for nonlinear Schrödinger equations has attracted considerable attention in recent years; in particular a large number of papers have been devoted in studying single and multiple spike solutions for the following stationary Schrödinger equation in the presence of an external potential V:

$$-\hbar^2\Delta v + V(x)v - \gamma|v|^{p-1}v = 0, \quad x \in \Omega, \quad V : \Omega \to \mathbb{R}, \tag{1.2}$$

where Ω is an open subset of $\mathbb{R}^N$ $(N \geq 1)$. In the case $p \in (1, \frac{N+2}{N-2})$ for $N \geq 3$ and $p > 1$ for $N = 1, 2$, equation (1.2) admits solutions, called *spike layers*, concentrating at one or multiple points of $\overline{\Omega}$, in the sense that their shape consists of sharp peaks near those points, while it vanishes everywhere else as $\hbar \to 0^+$.

Concerning equation (1.2) in bounded domains Ω with $V \equiv 1$ and Neumann boundary conditions, in [16], [20], [22], [23], [26], [31], [32], [41] the authors look for solutions exhibiting a concentration at one or more points of the boundary which are proved to be critical for the mean curvature of $\partial\Omega$. As regards interior spike solutions, the distance function from the boundary $\partial\Omega$ plays a role similar to that of the mean curvature: in [19], [21], [33], [35], [42] single or multiple spikes are constructed for the Neumann or the Dirichlet problem associated to equation (1.2) with Ω bounded and $V \equiv 1$, and concentration occurs, roughly speaking, at critical points of the distance function d from the boundary $\partial\Omega$.

When $\Omega \equiv \mathbb{R}^N$ and $V \not\equiv$ *const*, spike layer solutions are constructed around the critical points of the potential V. The first result in this line seems due to Floer and Weinstein ([17]). These authors considered the one-dimensional case and constructed for small $\hbar > 0$ such a concentrating family via a Lyapunov-Schmidt reduction around any non-degenerate critical point of the potential V, under the condition that V is bounded and $p = 3$. This line of research has been extensively pursued in a set of recent papers (we recall, among many others, [1], [12], [14], [13], [15], [18], [24], [25], [36], [37], [38], [40]) and semiclassical states are produced using minimax methods or, under suitable non-degeneracy conditions, by finite-dimensional reductions.

In contrast to pointwise concentration, only very recently concentration at higher dimensional sets has been proved. The first progress was the recent paper [29], in which, assuming $V \equiv const$, the authors prove the existence of positive solutions for (1.2) in a smooth bounded subset $\Omega \subset \mathbb{R}^2$ with the Neumann boundary conditions; such solutions concentrate at the whole boundary $\partial\Omega$ or at some of its components. This result has been extended to higher dimensions in [28]. Under symmetry assumptions, higher dimensional spike-layers are proved to exist. In [2] and [4], assuming that V is radial, it is established the existence of radially symmetric solutions for (1.2) in $\mathbb{R}^N$ concentrating on a sphere centered in zero and it is shown that the location of the concentration arises as a balance between the effect of the potential energy due to V and the volume energy of the solutions. In [5] and [9] concentration on circles in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively is produced for a special class of solutions of (1.2). Concerning equation (1.2) in a ball with Dirichlet or Neumann boundary conditions and V radial, in [3] the effect of the boundary of the domain is analyzed; more precisely, it is proved that there exists a family of radial solutions concentrating on the boundary. Finally we recall the recent papers [27] and [30], in which other phenomena of concentration on manifolds are studied.

Motivated by this rich literature, in this paper we want to study which kind of phenomena occurs for the system of Maxwell-Schrödinger equations (1.1) in the semiclassical limit.

More precisely, for small values of the parameter $\hbar$, we prove the existence of a positive radially symmetric wave $v_\hbar$ and a potential $\phi_\hbar$ satisfying (1.1). Furthermore, in the limit when $\hbar \to 0^+$, $v_\hbar$ exhibits a concentration behaviour around a sphere $\{|x| = s_0\}$: the analysis reveals that "most" of the support of $v_\hbar$ is concentrated in a neighbourhood of such sphere whose size depend on $\hbar$ and shrinks to zero as $\hbar \to 0^+$. The radius s_0 is determined by the exponent p and the charge ω. In order to state the exact result, denote by $w(y)$ the unique solution for the following ODE:

$$\begin{cases} w'' - w + w^p = 0 \text{ in } \mathbb{R}, \\ w > 0, \ \ w(0) = \max\limits_{y \in \mathbb{R}} w(y), \\ w(y) \to 0 \text{ as } |y| \to +\infty. \end{cases}$$

It is well known that $w(y)$ can be written explicitly as

$$w(y) = \Big(\frac{p+1}{2}\Big)^{\frac{1}{p-1}} \Big(\cosh\Big(\frac{p-1}{2} y\Big)\Big)^{-\frac{2}{p-1}}. \tag{1.3}$$

Then, roughly speaking, the limit profile of our solutions $v_\hbar$ resembles, after a suitable rescaling in the coordinates, the function w.

More precisely the main result of this paper is the following.

Theorem 1.1. *Assume that* $1 < p < \frac{11}{7}$. *Then there exists* $\hbar_0 > 0$ *such that for every* $\hbar \in (0, \hbar_0)$ *the system (1.1) has a solution* $(v_\hbar, \phi_\hbar)$ *with* $\gamma = \gamma_\hbar = \hbar^{(p-1)/2}$ *and:*

(1) $v_\hbar$, $\phi_\hbar$ *are radially symmetric,* $v_\hbar$, $|\nabla v_\hbar|$, $|\nabla \phi_\hbar| \in L^2(\mathbb{R}^3)$,

(2) $v_\hbar(r) = \hbar^{-\frac{1}{2}} \alpha_\hbar^{\frac{1}{p-1}} w\big(\sqrt{\alpha_\hbar}\frac{r-s_\hbar}{\hbar}\big) + o(1)\exp\big(-\mu\frac{|r-s_\hbar|}{\hbar}\big)$ *uniformly in* $\mathbb{R}^3$ *for a suitable* $\mu \in (0,1)$,

(3) $\phi_\hbar(r) = 4\pi\omega(1+o(1))\alpha_\hbar^{\frac{5-p}{2(p-1)}}(\int_{\mathbb{R}} w^2 dy) G(r; s_\hbar)$, *where* $G(r;\rho)$ *is defined as follows*

$$G(r;\rho) = \frac{\rho}{r}\min(r,\rho). \tag{1.4}$$

Furthermore, as $\hbar \to 0^+$,

$$\alpha_\hbar \to \alpha_0 := \frac{p+3}{11-7p}, \tag{1.5}$$

and

$$s_\hbar \to s_0 := (\alpha_0 - 1)\alpha_0^{\frac{p-5}{2(p-1)}}\Big(4\pi\omega^2\int_{\mathbb{R}} w^2 dy\Big)^{-1}. \tag{1.6}$$

Remark 1.1. *The case* $1 < p < \frac{3}{2}$ *has been proved in* [10] *by variational method. Here we use localized energy method to cover the whole range for* $p \in (1, \frac{11}{7})$. *Notice that for* $p \geq \frac{11}{7}$ *it results* $\alpha_0 - 1 \leq 0$, *hence formula (1.6) does not makes sense; this suggests that the range* $(1, \frac{11}{7})$ *is also necessary to get concentration around a sphere.*

Remark 1.2. *Notice that by (2) of theorem 1.1 it immediately follows that*

$$\int_{\mathbb{R}^3} |v_\hbar|^2 dx = 4\pi\alpha_\hbar^{\frac{5-p}{2(p-1)}} \int_{-\sqrt{\alpha_\hbar}\frac{s_\hbar}{\hbar}}^{+\infty} \Big(\frac{\hbar y}{\sqrt{\alpha_\hbar}} + s_\hbar\Big)^2 |w|^2 dy + o(1) = 4\pi\alpha_0^{\frac{5-p}{2(p-1)}} s_0^2 \int_{\mathbb{R}} |w|^2 dy + o(1),$$

i.e. the L^2*-norm of the waves* $v_\hbar$ *becomes bounded from below and from above. Furthermore, for every* $\delta > 0$ *and for small* $\hbar$:

$$v_\hbar(r) \leq C\exp\Big(-\mu\frac{|r-s_\hbar|}{\hbar}\Big) \text{ for } |r - s_\hbar| \geq \delta$$

where C *is independent on* $\hbar$ *and* δ. *Hence the solutions* $v_\hbar$ *can be viewed as small perturbations of the function* w *rescaled in such a way to be very concentrated near a suitable neighbourhood of* s_0; *then their profile has the form of a solitary elevation which becomes a very sharp peak when* $\hbar$ *is small.*

The proof of theorem 1.1 relies on a local approach and is related to the arguments employed in [34], in which a phenomenon of concentration around a sphere has been obtained for the stationary Gierer-Meinhardt system in $\mathbb{R}^N$. This approach is based on a finite dimensional reduction by using the classical Liapunov-Schmidt reduction method. We first construct some approximate solution (obtained as a small perturbation of rescalation of w) and we solve (1.1) in its normal direction, and then we study the remaining finite dimensional equation. After this reduction process, by using the implicit function theorem we solve (1.1) in a suitable neighbourhood of the approximate solution.

The paper consists of four more sections. In section 2 we introduce some notations and prove some preliminary estimates that play a key role in the rest of the arguments. In section 3 and 4 we construct the approximate solutions and we carry out the Lyapunov-Schmidt procedure that allows us to reduce the problem to the study of a finite dimensional functional. Finally the proof of theorem 1.1 is given in section 5.

NOTATIONS

- If $u : \mathbb{R}^N \to \mathbb{R}$ is a radially symmetric function, we will continue to denote by u the real function $r > 0 \mapsto u(r)$ with $|x| = r$.
- If u is a real valued function, then u_+ is its positive part and u_- its negative part.
- We will often use the symbol C for denoting a positive constant independent on ε. The value of C is allowed to vary from line to line (and also in the same formula).
- Given $A_\varepsilon \subset \mathbb{R}^N$, $f_\varepsilon, g_\varepsilon : A_\varepsilon \to \mathbb{R}$ two families of functions on A_ε and $k \in \mathbb{R}$, we write $f_\varepsilon = o(\varepsilon^k) g_\varepsilon$ *on* A_ε (resp. $f_\varepsilon = O(\epsilon^k) g_\varepsilon$ *on* A_ε) to mean that $\frac{f_\varepsilon}{g_\varepsilon} \to 0$ (resp. $|f_\varepsilon| \le C\varepsilon^k |g|$) uniformly on A_ε as $\varepsilon \to 0$.

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2. Preliminaries

In order to obtain solutions of (1.1) we choose a suitable functional setting. Let $H^1_r(\mathbb{R}^3)$ and $L^2_r(\mathbb{R}^3)$ denote the subspace of the Sobolev space $H^1(\mathbb{R}^3)$ and $L^2(\mathbb{R}^3)$ respectively formed by the radially symmetric functions endowed with the norms

$$\|u\|^2_{H^1(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \left(|\nabla u|^2 + |u|^2\right) dx, \quad \|u\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |u|^2 \, dx.$$

It is well known that $H^1_r(\mathbb{R}^3)$ is compactly imbedded in $L^p(\mathbb{R}^3)$ for $2 < p < 6$. (see [7] Theorem A.I', pg. 341, or [39]). Furthermore if $u \in H^1_r(\mathbb{R}^3)$ then

$$|u(x)| \leq (4\pi)^{-1/2}|x|^{-1}\|u\|_{H^1} \text{ a.e. in } \mathbb{R}^3$$

(see [39]). Define $D^1(\mathbb{R}^3)$ as the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|^2_{D^1(\mathbb{R}^3)} \equiv \int_{\mathbb{R}^3} |\nabla u|^2\, dx,$$

and let $D^1_r(\mathbb{R}^3)$ denote the closed subspace consisting of the radially symmetric functions. The following proposition holds (see [10] for the proof).

Proposition 2.1. *For every $u \in H^1_r(\mathbb{R}^3)$ denote by $(-\Delta)^{-1}[u^2]$ the unique solution in $D^1_r(\mathbb{R}^3)$ of*

$$-\Delta v = u^2.$$

Then the following representation formula holds :

$$(-\Delta)^{-1}[u^2](x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|}u^2(y)dy = \int_0^{+\infty} G(|x|,\rho)u^2(\rho)d\rho$$

where $G(s,\rho)$ has been defined in (1.4). Furthermore the functional $F : u \in H^1_r(\mathbb{R}^3) \mapsto \int_{\mathbb{R}^3} u^2(-\Delta)^{-1}[u^2]dx$ is compact and C^1 and

$$F'(u)[w] = 4\int_{\mathbb{R}^3} uw(-\Delta)^{-1}[u^2]dx \quad \forall u, w \in H^1_r(\mathbb{R}^3).$$

By suitably scaling, (1.1) can be reduced to the following system of equations:

$$\begin{cases} -\Delta u + u + \Gamma u\psi - u^p = 0 \text{ in } \mathbb{R}^3, \\ -\Delta\psi = \varepsilon u^2 \text{ in } \mathbb{R}^3, \\ u, \psi > 0, \ \ u, \psi \to 0 \text{ as } |x| \to +\infty, \end{cases} \tag{2.1}$$

where

$$\varepsilon = \hbar = \gamma^{\frac{2}{p-1}}, \ \ \Gamma = 4\pi\omega^2, \ \ u(x) = \sqrt{\varepsilon}v(\varepsilon x), \ \ \psi(x) = \frac{1}{4\pi\omega}\phi(\varepsilon x). \tag{2.2}$$

Associated with (2.1) is the following energy functional $E_\varepsilon \in C^1(H^1_r(\mathbb{R}^3), \mathbb{R})$:

$$E_\varepsilon[u] := \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + u^2\right)dx - \frac{1}{p+1}\int_{\mathbb{R}^3} u_+^{p+1}dx + \frac{\Gamma}{4}\varepsilon\int_{\mathbb{R}^3} u^2(-\Delta)^{-1}[u^2]dx.$$

By using proposition 2.1 the energy functional can be rewritten as

$$\begin{aligned} E_\varepsilon[u] =& 2\pi\int_0^{+\infty} r^2\left(|u'|^2 + |u|^2\right)dr - \frac{4\pi}{p+1}\int_0^{+\infty} r^2|u_+|^{p+1}dr \\ &+ \Gamma\pi\varepsilon\int_0^{+\infty}\int_0^{+\infty} r^2 G(r,\rho)u^2(r)u^2(\rho)drd\rho. \end{aligned}$$

Before going on we define in the following lemma two suitable functions which will be useful in order to locate the asymptotic peak of the solutions.

Lemma 2.1. *Assume $p < \frac{11}{7}$ and consider the two functions*

$$\rho(t) = (1+t)^{\frac{p-5}{2(p-1)}} t^2[(3p-7)t + 4(p-1)], \quad t > \hat{t} := \frac{2(p-1)}{7-3p},$$

and $t = t(s)$ the inverse function of

$$s(t) = \left(\Gamma \int_{\mathbb{R}} w^2 dy\right)^{-1} (1+t)^{\frac{p-5}{2(p-1)}} t, \quad t > \hat{t}.$$

Then the function $\rho(t(s))$ has a unique (and nondegenerate) minimum point s_0 for $s \in (0, s(\hat{t}))$ given by (1.6).

Proof. Since $p < \frac{11}{7}$, it is easy to see that s is a strictly decreasing function for $t > \hat{t}$. Thus the inverse function $t = t(s)$ exists for $s \in (0, s(\hat{t}))$. Now compute

$$\rho^{'}(t) = t(t+1)^{-\frac{p+3}{2(p-1)}} \left(\frac{7p-11}{2(p-1)} t + 4\right) [(3p-7)t + 2(p-1)].$$

For $t > \hat{t}$, $\rho(t)$ has a unique critical point:

$$t_0 = \frac{8(p-1)}{11-7p}. \tag{2.3}$$

Moreover $\rho^{''}(t_0) > 0$ which means that t_0 is also nondegenerate. Set $s_0 = s(t_0)$. Then the function $\rho(t(s))$ has a unique (and nondegenerate) minimum point s_0 for $s \in (0, s(\hat{t}))$. □

Now we are in position to introduce our approximate solutions. As we have already announced in the introduction, we construct the approximations by suitably rescaling the function w and then by using suitable truncations. Fix $s_1, s_2 \in (0, s(\hat{t}))$ such that $s_1 < s_2$ and, according to lemma 2.1,

$$s_0 \in (s_1, s_2), \quad \frac{2t(s)}{1+t(s)} \neq \frac{4(p-1)}{5-p} \quad \forall s \in [s_1, s_2]. \tag{2.4}$$

Let $\eta(r)$ be a cut-off function satisfying

$$\eta(r) = 1 \ \text{ for } r \in (-\delta, \delta), \ \ \eta(r) = 0 \ \text{ for } r \notin (-2\delta, +2\delta), \tag{2.5}$$

where $\delta > 0$ is a fixed number such that $s_1 - 2\delta > 0$. For every $s \in [s_1, s_2]$ define

$$w_s(y) := (1+t(s))^{\frac{1}{p-1}} w(\sqrt{1+t(s)}y),$$

and

$$U_{\varepsilon,s}(y) := w_s(y)\eta(\varepsilon y). \tag{2.6}$$

It is immediate to see that w_s satisfies

$$w_s^{''} - (1+t(s))w_s + w_s^p = 0, \ \ w_s > 0, \ \ w_s(y) = w_s(-y). \tag{2.7}$$

The definitions of $U_{\varepsilon,s}$ in (2.6) and w in (1.3) imply that for every $i \in \mathbb{N}$, denoting by $U^i_{\varepsilon,s}$ and $w^i_{\varepsilon,s}$ the derivatives of order i of the functions $U_{\varepsilon,s}$ and w_s respectively, we have

$$U^i_{\varepsilon,s}(y) = O(1)e^{-|y|}, \quad U^i_{\varepsilon,s}(y) = w^i_s(y) + o(\varepsilon)e^{-|y|} \tag{2.8}$$

uniformly for $s \in [s_1, s_2]$. We need to recall the following well-known facts on the functions w and w_s which will be frequently used later on.

Lemma 2.2. *(1) For every $s \in [s_1, s_2]$ the following identities hold*

$$\int_{\mathbb{R}} (w_s')^2 dy = \frac{p-1}{2(p+1)} \int_{\mathbb{R}} w_s^{p+1} dy, \ \ \int_{\mathbb{R}} w_s^2 dy = \frac{p+3}{2(1+t(s))(p+1)} \int_{\mathbb{R}} w_s^{p+1} dy. \tag{2.9}$$

(2) Let $L_0 : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the linear operator defined by

$$L_0[\phi] = \phi'' - \phi + pw^{p-1}\phi.$$

Then

$$L_0\Big[\frac{1}{p-1}w + \frac{1}{2}yw'\Big] = w \tag{2.10}$$

and $\phi \in H^2(\mathbb{R})$ satisfies

$$L_0[\phi] = 0$$

if and only if $\phi = cw'$ for some constant c. As a consequence, considered the kernel $\mathcal{K}_0$ and the cokernel $\mathcal{C}_0$ of the operator L_0:

$$\mathcal{K}_0 = \{cw' \mid c \in \mathbb{R}\} \subset H^2(\mathbb{R}), \ \ \mathcal{C}_0 = \{cw' \mid c \in \mathbb{R}\} \subset L^2(\mathbb{R}),$$

then L_0 is an invertible operator from $\mathcal{K}_0^\perp$ to $\mathcal{C}_0^\perp$.

Proof. Since each w_s satisfies (2.7), we have $\big((w_s')^2\big)' = (1+t(s))(w_s^2)' - \frac{2}{p+1}(w_s^{p+1})'$, by which $(w_s')^2 = (1+t(s))w_s^2 - \frac{2}{p+1}w_s^{p+1}$. Combining this with the following identity

$$\int_{\mathbb{R}} (w_s')^2 dy + (1+t(s)) \int_{\mathbb{R}} w_s^2 dy = \int_{\mathbb{R}} w_s^{p+1} dy$$

we deduce (2.9). To prove (2.10), it is sufficient to observe that $L_0[w] = (p-1)w^p$, $L_0[yw^{'}] = 2(w - w^p)$.

The last part of the lemma follows from the uniqueness of w. □

Fix $\lambda > 0$. We conclude this section by analyzing the following nonlocal operator

$$L : H^2(\mathbb{R}) \to L^2(\mathbb{R}), \ \ L[\phi] := L_0[\phi] - \lambda(p-1)\frac{\int_{\mathbb{R}} w\phi dy}{\int_{\mathbb{R}} w^2 dy} w.$$

The following lemma characterizes the kernel of L.

Lemma 2.3. *If* $\lambda \neq \frac{4}{5-p}$ *then* $\phi \in H^2(\mathbb{R})$ *satisfies*

$$L[\phi] = 0,$$

if and only if then $\phi = cw^{'}$ *for some constant* c.

Proof. Assume $L[\phi] = 0$. From (2.10) we deduce that

$$L_0\left[\phi - \lambda \frac{\int_{\mathbb{R}} w\phi dy}{\int_{\mathbb{R}} w^2 dy}\left(w + \frac{p-1}{2} yw^{'}\right)\right] = L[\phi] = 0.$$

By (2) of Lemma 2.2, we have

$$\phi - \lambda \frac{\int_{\mathbb{R}} w\phi dy}{\int_{R} w^2 dy}\left(w + \frac{p-1}{2} yw^{'}\right) = cw^{'} \tag{2.11}$$

for some c. Multiplying (2.11) by w and integrating over $\mathbb{R}$, we obtain that

$$(1-\lambda)\int_{\mathbb{R}} w\phi dy - \lambda \frac{p-1}{2} \frac{\int_{\mathbb{R}} w\phi dy}{\int_{\mathbb{R}} w^2 dy} \int_{\mathbb{R}} yww' dy = 0.$$

After an integration by parts we deduce $\int_{\mathbb{R}} yww'dy = -\frac{1}{2}\int_{\mathbb{R}} w^2 dy$, by which

$$\left(1 - \lambda\left(1 - \frac{p-1}{4}\right)\right)\int_{\mathbb{R}} w\phi dy = 0,$$

and hence $\int_{\mathbb{R}} w\phi dy = 0$ which implies $L_0[\phi] = L[\phi] = 0$; then lemma 2.2, leads to $\phi = cw^{'}$.

□

3. The Linearized Problem

For every $s \in [s_1, s_2]$, where s_1, s_2 have be chosen in (2.4), we set

$$I_{\varepsilon,s} := \left(-\frac{s}{\varepsilon}, +\infty\right).$$

Denote by $L^p_\varepsilon(I_{\varepsilon,s})$ the space of the functions $\phi : I_{\varepsilon,s} \to \mathbb{R}$ such that, setting $u_\phi(x) = \phi\big(|x| - \frac{s}{\varepsilon}\big)$, it results $u_\phi \in L^p(\mathbb{R}^3)$. $L^2_\varepsilon(I_{\varepsilon,s})$ can be equipped with the scalar product:

$$< \phi, \psi >_\varepsilon = \int_{I_{\varepsilon,s}} \phi\psi(s+\varepsilon y)^2 dy \quad \forall \phi, \psi \in L^2_\varepsilon(I_{\varepsilon,s})$$

(which is equivalent to the norm $\|u_\phi\|_{L^2(\mathbb{R}^3)}$).

We define the following operator:

$$T_{\varepsilon,s}[\phi](y) = \int_{I_{\varepsilon,s}} G(s+\varepsilon y, s+\varepsilon z)\phi(z)dz, \;\; y \in I_{\varepsilon,s}, \;\; \phi \in L^1_\varepsilon(I_{\varepsilon,s}).$$

Notice that

$$s - \varepsilon(2|z| + |y|) \leq G(s+\varepsilon y, s+\varepsilon z) \leq s + \varepsilon z \quad \forall y, z \in I_{\varepsilon,s}, \tag{3.12}$$

hence, by using (2.8) and lemma 2.1, we get

$$T_{\varepsilon,s}[U_{\varepsilon,s}^2](y) = s\int_{\mathbb{R}} U_{\varepsilon,s}^2 dz + O(\varepsilon)(|y|+1) = s\int_{\mathbb{R}} w_s^2 dz + O(\varepsilon)(|y|+1)$$
$$= s(1+t(s))^{\frac{5-p}{2(p-1)}}\int_{\mathbb{R}} w^2 dz + O(\varepsilon)(|y|+1) = \frac{1}{\Gamma}t(s) + O(\varepsilon)(|y|+1) \tag{3.13}$$

uniformly for $s\in[s_1,s_2]$. In the same way we define $H_\varepsilon^1(I_{\varepsilon,s})$ the space of the functions $\phi : I_{\varepsilon,s}\to\mathbb{R}$ such that $u_\phi\in H_r^1(\mathbb{R}^3)$. We can equip $H_\varepsilon^1(I_{\varepsilon,s})$ with the following scalar product:

$$(\phi,\psi)_\varepsilon = \int_{I_{\varepsilon,s}} [\phi'\psi' + (1+\Gamma T_{\varepsilon,s}[U_{\varepsilon,s}^2])\phi\psi](s+\varepsilon y)^2 dy \quad \forall\phi,\psi\in H_\varepsilon^1(I_{\varepsilon,s})$$

(which is equivalent to the norm $\|u_\phi\|_{H^1(\mathbb{R}^3)}$ since $0\le T_{\varepsilon,s}[U_{\varepsilon,s}^2](y)\le\int_{I_{\varepsilon,s}}(s+\varepsilon z)U_{\varepsilon,s}^2 dz$).

We introduce the following functions:

$$Z_{\varepsilon,s}(y) = U_{\varepsilon,s}'''(y) + \frac{2\varepsilon}{s+\varepsilon y}U_{\varepsilon,s}''(y) - (1+\Gamma T_{\varepsilon,s}[U_{\varepsilon,s}^2])U_{\varepsilon,s}'(y),\ \ y\in I_{\varepsilon,s}. \tag{3.14}$$

By integration by parts it is immediate to prove that

$$(\phi, U_{\varepsilon,s}')_\varepsilon = -<\phi, Z_{\varepsilon,s}>_\varepsilon \quad \forall\phi\in H_\varepsilon^1(I_{\varepsilon,s}), \tag{3.15}$$

then orthogonality to the functions $U_{\varepsilon,s}'$ in $H_\varepsilon^1(I_{\varepsilon,s})$ is equivalent to orthogonality to $Z_{\varepsilon,s}$ in $L_\varepsilon^2(I_{\varepsilon,s})$.

First we study a linear problem: given $h\in C(\bar I_{\varepsilon,s})\cap L_\varepsilon^2(I_{\varepsilon,s})$, find a function $\phi\in H_\varepsilon^1(I_{\varepsilon,s})$ and $c\in\mathbb{R}$ satisfying

$$\begin{cases} L_{\varepsilon,s}[\phi] = h + cZ_{\varepsilon,s}, \\ \phi'\Big(-\dfrac{s}{\varepsilon}\Big) = 0, \ \ <\phi, Z_{\varepsilon,s}>_\varepsilon = 0, \end{cases} \tag{3.16}$$

where

$$L_{\varepsilon,s}[\phi] := \phi'' + \frac{2\varepsilon}{s+\varepsilon y}\phi' - (1+\Gamma T_{\varepsilon,s}[U_{\varepsilon,s}^2])\phi + pU_{\varepsilon,s}^{p-1}\phi - 2\Gamma T_{\varepsilon,s}[U_{\varepsilon,s}\phi]U_{\varepsilon,s}.$$

In order to solve the system (3.16) we need the following result based on ODE estimates.

Lemma 3.1. *Let $\phi\in C^2(\bar I_{\varepsilon,s})$ satisfy*

$$\left|\phi''(y) + \frac{2\varepsilon}{s+\varepsilon y}\phi'(y) - (1+\Gamma T_{\varepsilon,s}[U_{\varepsilon,s}^2])\phi(y)\right| \le c_0 e^{-\mu|y|},\ \ \phi'\Big(-\frac{s}{\varepsilon}\Big) = 0,$$

for some $c_0>0$ and

$$\phi(y)\to 0 \quad \text{as } y\to+\infty.$$

Then, provided that $\mu>0$ is sufficiently small,

$$|\phi(y)| \le 2e(|\phi(0)|+c_0)e^{-\mu|y|} \quad \forall y\in I_{\varepsilon,s}.$$

Proof. We use a comparison principle. Take $\chi(t)$ a smooth cut-off function such that

$$\chi(t)=1 \text{ for } |t|\le 1,\ \ \chi(t)=0 \text{ for } \ |t|\ge 2,\ \ 0\le\chi\le 1.$$

Now consider the following auxiliary function:

$$\Phi(y)=A\Big[e^{\mu y}+(e^{\mu y_0}-e^{\mu y})\chi\Big(\mu\Big(y+\frac{s}{\varepsilon}\Big)\Big)\Big]$$

where

$$y_0=-\frac{s}{\varepsilon}+\frac{1}{\mu},\ \ A=2e(|\phi(0)|+c_0).$$

If $y\in(-\frac{s}{\varepsilon},y_0)$, $\Phi(y)=Ae^{\mu y_0}$ and hence

$$\Phi''+\frac{2\varepsilon}{s+\varepsilon y}\Phi'-(1+\Gamma T_{\varepsilon,s}[U^2_{\varepsilon,s}])\Phi=-A(1+T_{\varepsilon,s}[U^2_{\varepsilon,s}])e^{\mu y_0}\le -c_0e^{\mu y}.$$

If $y\in(-\frac{s}{\varepsilon}+\frac{2}{\mu},0)$, $\Phi(y)=Ae^{\mu y}$ and $\frac{\varepsilon}{s+\varepsilon y}\le\frac{\mu}{2}$, hence

$$\Phi''+\frac{2\varepsilon}{s+\varepsilon y}\Phi'-(1+\Gamma T_{\varepsilon,s}[U^2_{\varepsilon,s}])\Phi=A[2\mu^2-(1+T_{\varepsilon,s}[U^2_{\varepsilon,s}])]e^{\mu y}\le -c_0e^{\mu y}$$

provided that μ is sufficiently small. Finally it is easy to see that for $y\in(y_0,y_0+\frac{1}{\mu})$

$$e^{\mu y_0}\le e^{\mu y}\le ee^{\mu y_0},\ \ \Phi(y)\ge Ae^{\mu y_0},\ \ \frac{\mu}{2}\le\frac{\varepsilon}{s+\varepsilon y}\le\mu;$$

hence

$$\begin{aligned}\Phi''+\frac{2\varepsilon}{s+\varepsilon y}\Phi'-(1+\Gamma T_{\varepsilon,s}[U^2_{\varepsilon,s}])\Phi&\le AO(\mu^2)e^{\mu y}-A(1+\Gamma T_{\varepsilon,s}[U^2_{\varepsilon,s}])e^{\mu y_0}\\&\le AO(\mu^2)e^{\mu y}-\frac{A}{e}(1+\Gamma T_{\varepsilon,s}[U^2_{\varepsilon,s}])e^{\mu y}\le -c_0e^{\mu y}\end{aligned}$$

provided that μ is sufficiently small.

In any case, we have that for $y\in(-\frac{s}{\varepsilon},0)$, $\Phi(y)$ satisfies

$$\Phi^{''}+\frac{2\varepsilon}{s+\varepsilon y}\Phi^{'}-(1+\Gamma T_{\varepsilon,s}[U^2_{\varepsilon,s}])\Phi\le -c_0e^{\mu y},\ \ \Phi^{'}\Big(-\frac{s}{\varepsilon}\Big)=0,\ \ \Phi(0)\ge|\phi(0)|. \tag{3.17}$$

Combining (3.17) with the hypothesis we obtain

$$(\Phi-\phi)''(y)+\frac{2\varepsilon}{s+\varepsilon y}(\Phi-\phi)'(y)-(1+\Gamma T_{\varepsilon,s}[U^2_{\varepsilon,s}])(\Phi-\phi)(y)\le 0\ \ \forall y\in\Big[-\frac{s}{\varepsilon},0\Big] \tag{3.18}$$

and

$$(\Phi-\phi)(0)>0,\ \ (\Phi-\phi)'\Big(-\frac{s}{\varepsilon}\Big)=0;$$

we claim that $(\Phi-\phi)(y)\ge 0$ for $y\in[-\frac{s}{\varepsilon},0)$. Otherwise, if we call $\bar y$ the minimum point of $\Phi-\phi$ in $[-\frac{s}{\varepsilon},0)$, then it would be $(\Phi-\phi)(\bar y)<0$ and $(\Phi-\phi)'(\bar y)=0$, $(\Phi-\phi)''(\bar y)>0$, in contradiction with (3.18). Hence we have proved that $\phi\le\Phi$ in $[-\frac{s}{\varepsilon},0]$. On the other hand by (3.17) and the hypothesis we also get

$$(\Phi+\phi)^{''}(y)+\frac{2\varepsilon}{s+\varepsilon y}(\Phi+\phi)^{'}(y)-(1+\Gamma T_{\varepsilon,s}[U^2_{\varepsilon,s}])(\Phi+\phi)(y)\le 0\ \ \forall y\in\Big[-\frac{s}{\varepsilon},0\Big]$$

and $(\Phi+\phi)(0)>0$, $(\Phi+\phi)'(-\frac{s}{\varepsilon})=0$. Proceeding as before we conclude $\phi\geq-\Phi$ in $[-\frac{s}{\varepsilon},0]$.

For $y\in[0,+\infty)$, we use $\hat{\Phi}(y)=Ae^{-\mu y}$ and hence

$$\hat{\Phi}''+\frac{2\varepsilon}{s+\varepsilon y}\hat{\Phi}'-(1+\Gamma T_{\varepsilon,s}[U_{\varepsilon,s}^2])\hat{\Phi}\leq A(\mu^2-(1+\Gamma T_{\varepsilon,s}[U_{\varepsilon,s}^2])e^{-\mu y}\leq -c_0e^{-\mu y}$$

provided that μ is sufficiently small. By repeating the previous argument we obtain $|\phi|\leq\hat{\Phi}$ in $[0,+\infty)$ and the conclusion follows. □

Let $\mu\in(0,1)$ be a number sufficiently small such that lemma 3.1 holds and for every function $\phi: I_{\varepsilon,s}\to\mathbb{R}$ define

$$\|\phi\|_*=\|e^{\mu<y>}\phi(y)\|_{L^\infty(I_{\varepsilon,s})},$$

where $<y>=(1+y^2)^{\frac{1}{2}}$. Since $\frac{2\varepsilon}{s+\varepsilon y}U''_{\varepsilon,s}=O(\varepsilon)e^{-|y|}$, by using (2.7), (2.8) and (3.13) we obtain

$$Z_{\varepsilon,s}(y)=w_s'''(y)-(1+t(s))w_s'(y)+O(\varepsilon)e^{-\mu<y>}=-pw_s^{p-1}(y)w_s'(y)+O(\varepsilon)e^{-\mu<y>} \tag{3.19}$$

uniformly for $s\in[s_1,s_2]$.

Next we prove the following a priori estimate for (3.16).

Lemma 3.2. *There exists a constant $C>0$ such that, provided that ε is sufficiently small, if $s\in[s_1,s_2]$ and (ϕ,c,h) satisfy (3.16), the following holds:*

$$\|\phi\|_*\leq C\|h\|_*.$$

Proof. We argue by contradiction. Assume the existence of a sequence $\varepsilon_k\to0$, $(s_k)\subset[s_1,s_2]$ and $(\tilde{\phi}_k,\tilde{c}_k)\in H^1_\varepsilon(I_{\varepsilon_k,s_k})\times\mathbb{R}$, $h_k\in L^2_\varepsilon(I_{\varepsilon_k,s_k})\cap C(\bar{I}_{\varepsilon_k,s_k})$ satisfying (3.16) such that

$$\|\tilde{\phi}_k\|_*>k\|\tilde{h}_k\|_*.$$

In particular $\|\tilde{h}_k\|_*<+\infty$ for every k. For sake of simplicity, in the remaining part of the proof we replace the subscript (ε_k,s_k) by k. By standard regularity results $\tilde{\phi}_k\in C^2(\bar{I}_k)$; furthermore for every k $\|T_k(U_k\tilde{\phi}_k)\|_\infty\leq\int_{I_k}(s_k+\varepsilon_kz)U_k\tilde{\phi}_kdz<+\infty$, and, consequently $\|-pU_k^{p-1}\tilde{\phi}_k+2\Gamma T_k[U_k\tilde{\phi}_k]U_k+\tilde{h}_k+\tilde{c}_kZ_k\|_*<+\infty$. Then lemma 3.1 implies $\|\tilde{\phi}_k\|_*<+\infty$ for every k. Hence it makes sense to set $\phi_k=\frac{\tilde{\phi}_k}{\|\tilde{\phi}_k\|_*}$, $c_k=\frac{\tilde{c}_k}{\|\tilde{\phi}_k\|_*}$, $h_k=\frac{\tilde{h}_k}{\|\tilde{\phi}_k\|_*}$. We obtain that (ϕ_k,c_k,h_k) satisfies (3.16) and

$$\|\phi_k\|_*=1,\ \ \|h_k\|_*=o(1).$$

Assume without loss of generality $s_k \to s$ as $k \to +\infty$. By multiplying the equation $L_k[\phi_k] = h_k + c_k Z_k$ by $U_k^{'}$ and integrating over I_k, we get

$$c_k \int_{I_k} Z_k U_k^{'} dy = -\int_{I_k} h_k U_k^{'} dy + \int_{I_k} L_k[\phi_k] U_k^{'} dy. \tag{3.20}$$

First examine the left hand side of (3.20): by using (2.8) and (3.19) we deduce

$$\int_{I_k} Z_k U_k^{'} dy = -p \int_{\mathbb{R}} w_{s_k}^{p-1} (w_{s_k}')^2 dy + o(1) = -p \int_{\mathbb{R}} w_s^{p-1} (w_s')^2 dy + o(1). \tag{3.21}$$

The first term on the right hand side of (3.20) can be estimated as

$$\int_{I_k} |h_k U_k'| dy \leq \|h_k\|_* \int_{\mathbb{R}} |U_k'| e^{\mu<y>} dy = O(1)\|h_k\|_* = o(1). \tag{3.22}$$

The last term in (3.20) equals

$$\begin{aligned}
&\int_{I_{\varepsilon_k}} (L_k[\phi_k]) U_k' dy \\
&= \int_{I_k} \phi_k \Big[U_k''' + \frac{2\varepsilon_k^2}{(s_k + \varepsilon_k y)^2} U_k' - \frac{2\varepsilon_k}{s_k + \varepsilon_k y} U_k'' - (1 + \Gamma T_k[U_k^2]) U_k' + p U_k^{p-1} U_k' \Big] dy \\
&- 2\Gamma \int_{I_k} T_k[U_k \phi_k] U_k U_k^{'} dy \\
&= \int_{I_k} \phi_k \Big[U_k'''(y) - (1 + \Gamma T_k[U_k^2]) U_k'(y) + p U_k^{p-1} U_k' \Big] dy + O(\varepsilon_k)\|\phi_k\|_* \\
&- 2\Gamma \int_{I_k} T_k[U_k \phi_k] U_k U_k' dy
\end{aligned}$$

since, according to the definition of U_k in (2.6) and to (2.8), $\frac{2\varepsilon_k^2}{(s_k+\varepsilon_k y)^2} U_k' = O(\varepsilon_k) e^{-|y|}$ and $\frac{2\varepsilon_k}{s_k+\varepsilon_k y} U_k' = O(\varepsilon_k) e^{-|y|}$. By (2.7), (2.8) and (3.13) we obtain

$$\int_{I_k} \phi_k \big[U_k'''(y) - (1 + \Gamma T_k[U_k^2]) U_k' + p U_k^{p-1} U_k' \big] dy = O(\varepsilon_k)\|\phi_k\|_*. \tag{3.23}$$

On the other hand, by (2.8) and (3.12) we deduce,

$$T_k[U_k \phi_k](y) = s \int_{I_k} U_k \phi_k dy + O(\varepsilon_k)(|y| + 1)\|\phi_k\|_*, \tag{3.24}$$

by which

$$\int_{I_k} T_k[U_k \phi_k] U_k U_k^{'} dy = O(\varepsilon_k)\|\phi_k\|_* \int_{I_k} (|y| + 1)|U_k U_k^{'}| dy = O(\varepsilon_k)\|\phi_k\|_*.$$

Hence we have proved that

$$\int_{I_k} L_k[\phi_k] U_k' dz = O(\varepsilon_k)\|\phi_k\|_* = O(\varepsilon_k). \tag{3.25}$$

Combining (3.20), (3.21), (3.22) and (3.25) we achieve $|c_k| = o(1)$, by which $\|h_k + c_k Z_k\|_* = o(1)$.

Next we claim that $\phi_k \to 0$ uniformly in any compact interval of $\mathbb{R}$.

By multiplying equation (3.16) by $(s_k+\varepsilon_k y)^2\phi_k$ and integrating by parts we immediately get that the sequence $(\phi_k,\phi_k)_\varepsilon=\int_{I_k}(s_k+\varepsilon_k y)^2(|\phi_k'|^2+|\phi_k|^2)dy$ is bounded. Now we consider $\bar\phi_k(y)=\phi_k(y)\eta(\varepsilon_k y)\in H^1(\mathbb{R})$, where η has been defined in (2.5). Then it is easy to see that $\bar\phi_k$ is bounded in $H^1(\mathbb{R})$, and hence $\bar\phi_k\to\bar\phi_0$ weakly in $H^1(\mathbb{R})$. Taking into account of (3.13) and (3.24) $\bar\phi_0$ satisfies

$$\Delta\bar\phi_0-(1+t(s))\bar\phi_0+pw_s^{p-1}\bar\phi_0-2s\Gamma\Big(\int_{\mathbb{R}}w_s\bar\phi_0dy\Big)w_s=0 \text{ in } \mathbb{R},\ \ |\bar\phi_0|\le e^{-\mu<y>}.$$

By rescaling, setting $\phi_0(y)=\bar\phi_0\Big(\frac{y}{\sqrt{1+t(s)}}\Big)$ and recalling the definition of w_s in (2.6), we obtain

$$\phi_0''-\phi_0+pw^{p-1}\phi_0-2\frac{t(s)}{1+t(s)}\frac{\int_{\mathbb{R}}w\phi_0dy}{\int_{\mathbb{R}}w^2dy}w=0.$$

By (2.4), because of the choice of s,

$$2\frac{t(s)}{1+t(s)}\neq\frac{4(p-1)}{5-p};$$

Lemma 2.3 implies $\phi_0=cw'$. On the other hand, $\int_{I_k}\phi_k Z_k(s_k+\varepsilon_k y)^2dy=0$ and hence, since by (3.19) $Z_k(s_k+\varepsilon_k y)^2\to -s^2pw_s^{p-1}w_s'$ for every $y\in\mathbb{R}$, by Lebesgue's Dominated Convergence Theorem, $\int_{\mathbb{R}}\bar\phi_0w_s^{p-1}w_s'=0$, that is $\int_{\mathbb{R}}\phi_0w^{p-1}w'=0$, by which $c=0$; we have proved that $\phi_k\to0$ in any compact interval of $\mathbb{R}$. Taking into account of (2.8), this shows that

$$\|U_k^{p-1}\phi_k\|_*\le\sup_{y\in I_k}|e^{\mu<y>}w_{s_k}^{p-1}(y)\phi_k(y)|=o(1).$$

By using again Lebesgue's Dominated Convergence Theorem, we have that

$$\int_{I_k}U_k\phi_kdy\to0$$

which implies, using (3.24),

$$\Big\|\int_{I_k}T_k[U_k\phi_k]U_kdy\Big\|_*=o(1).$$

Thus we have arrived at the following situation: ϕ_k satisfies

$$\phi_k^{''}+\frac{2\varepsilon_k}{s_k+\varepsilon_k y}\phi_k^{'}-(1+\Gamma T_k[U_k^2])\phi_k=o(1)e^{-\mu<y>},$$

$$\phi_k^{'}\Big(-\frac{s_k}{\varepsilon_k}\Big)=0,\ \ \phi_k\to0 \text{ as } y\to\infty.$$

Furthermore $\phi_k(0)=o(1)$. Hence we can apply lemma 3.1 and obtain that $\phi_k(y)=o(1)e^{-\mu<y>}$, which is a contradiction since $\|\phi_k\|_*=1$. This proves the Lemma. ☐

Now we are in position to provide the existence of a solution for the system (3.16).

Lemma 3.3. *For $\varepsilon > 0$ sufficiently small, for every $s \in [s_1, s_2]$ and $h \in L^2_\varepsilon(I_{\varepsilon,s}) \cap C(\bar{I}_{\varepsilon,s})$, there exists a unique pair (ϕ, c) solving (3.16). Moreover, by lemma 3.2,*

$$\|\phi\|_* \leq C\|h\|_*. \tag{3.26}$$

Proof. The existence follows from Fredholm alternative. To this aim, let us consider $\mathcal{H}_\varepsilon$ the closed subset of $H^1_\varepsilon(I_{\varepsilon,s})$ defined by

$$\mathcal{H}_\varepsilon = \{u \in H^1_\varepsilon(I_{\varepsilon,s}) \,|\, (u, U^{'}_{\varepsilon,s})_\varepsilon = 0\}.$$

Notice that, by (3.15), ϕ solves the system (3.16) if and only if $\phi \in \mathcal{H}_\varepsilon$ and

$$(\phi, \psi)_\varepsilon - p < U^{p-1}_{\varepsilon,s}\phi, \psi >_\varepsilon + 2\Gamma < T_{\varepsilon,s}[U_{\varepsilon,s}\phi]U_{\varepsilon,s}, \psi >_\varepsilon = - < h, \psi >_\varepsilon, \;\; \forall \psi \in \mathcal{H}_\varepsilon. \tag{3.27}$$

Once we know ϕ we can determine a unique c from the equation

$$-p < U^{p-1}_{\varepsilon,s}\phi, U'_{\varepsilon,s} >_\varepsilon + 2\Gamma < T_{\varepsilon,s}[U_{\varepsilon,s}\phi]U_{\varepsilon,s}, U'_{\varepsilon,s} >_\varepsilon = - < h, U'_{\varepsilon,s} >_\varepsilon + c(U'_{\varepsilon,s}, U'_{\varepsilon,s})_\varepsilon.$$

Thus it remains to solve (3.27). According to Riesz's representation theorem, take $\mathcal{K}_\varepsilon(\phi)$, $\bar{h} \in \mathcal{H}_\varepsilon$ such that

$$(\mathcal{K}_\varepsilon(\phi), \psi)_\varepsilon = -p < U^{p-1}_{\varepsilon,s}\phi, \psi >_\varepsilon + 2\Gamma < T_{\varepsilon,s}[U_{\varepsilon,s}\phi]U_{\varepsilon,s}, \psi >_\varepsilon, \;\; \forall \psi \in \mathcal{H}_\varepsilon,$$

$$(\bar{h}, \psi)_\varepsilon = - < h, \psi >_\varepsilon \;\; \forall \psi \in \mathcal{H}_\varepsilon.$$

Then problem (3.27) consists in finding $\phi \in \mathcal{H}_\varepsilon$ such that

$$\phi + \mathcal{K}_\varepsilon(\phi) = \bar{h}. \tag{3.28}$$

It is easy to prove that $\mathcal{K}_\varepsilon$ is a linear compact operator form $\mathcal{H}_\varepsilon$ to $\mathcal{H}_\varepsilon$.

Using Fredholm's alternatives, (3.28) has a unique solution for each $\bar{h}$, if and only if (3.28) has a unique solution for $\bar{h} = 0$. Let $\phi \in \mathcal{H}_\varepsilon$ be a solution of $\phi + \mathcal{K}_\varepsilon(\phi) = 0$; then ϕ solves the system (3.16) with $h = 0$ for some $c \in \mathbb{R}$. Lemma 3.2 implies $\phi \equiv 0$. ☐

4. Finite Dimensional Reduction

This section is devoted to solve the following nonlinear system with the unknowns $(\phi, \beta) \in H^1_\varepsilon(I_{\varepsilon,s}) \times \mathbb{R}$:

$$\begin{cases} S_{\varepsilon,s}[U_{\varepsilon,s} + \phi] = \beta Z_{\varepsilon,s}, \;\; y \in I_{\varepsilon,s}, \\ \phi^{'}\Big(-\dfrac{s}{\varepsilon}\Big) = 0, \;\; < \phi, Z_{\varepsilon,s} >_\varepsilon = 0, \end{cases} \tag{4.29}$$

where

$$S_{\varepsilon,s}[\psi] = \psi'' + \frac{2\varepsilon}{s + \varepsilon y}\psi' - \psi + \psi^p_+ - \Gamma\psi T_{\varepsilon,s}[\psi^2].$$

Lemma 4.1. *Fix $\sigma \in (\frac{1}{2}, 1)$. Provided that $\varepsilon > 0$ is sufficiently small, for every $s \in [s_1, s_2]$ there is a unique pair $(\phi_{\varepsilon,s}, \beta_{\varepsilon,s}) \in H^1_\varepsilon(I_{\varepsilon,s}) \times \mathbb{R}$ satisfying (4.29) and*

$$\|\phi_{\varepsilon,s}\|_* \leq \varepsilon^\sigma, \ \ (\phi_{\varepsilon,s}, \phi_{\varepsilon,s})_\varepsilon \leq \varepsilon^\sigma. \tag{4.30}$$

Furthermore, setting $\Phi_{\varepsilon,s}(x) = \phi_{\varepsilon,s}\big(|x| - \frac{s}{\varepsilon}\big)$, the maps $\Phi_\varepsilon : s \in [s_1, s_2] \mapsto \Phi_{\varepsilon,s} \in H^1_r(\mathbb{R}^3)$ and $\beta_\varepsilon : s \in [s_1, s_2] \mapsto \beta_{\varepsilon,s} \in \mathbb{R}$ are continuous.

Proof. We write the equation in (4.29) in the following form:

$$L_{\varepsilon,s}[\phi] = E_{\varepsilon,s} + M_{\varepsilon,s}[\phi] + \beta Z_{\varepsilon,s} \tag{4.31}$$

and use contraction mapping theorem. Here

$$-E_{\varepsilon,s} = U^{''}_{\varepsilon,s} + \frac{2\varepsilon}{s + \varepsilon y} U^{'}_{\varepsilon,s} - (1 + \Gamma T_{\varepsilon,s}[U^2_{\varepsilon,s}]) U_{\varepsilon,s} + U^p_{\varepsilon,s} \tag{4.32}$$

and

$$M_{\varepsilon,s}[\phi] = -(U_{\varepsilon,s} + \phi)^p_+ + U^p_{\varepsilon,s} + pU^{p-1}_{\varepsilon,s}\phi + \Gamma\Big[(U_{\varepsilon,s} + \phi)T_{\varepsilon,s}[\phi^2] + 2\phi T_{\varepsilon,s}[U_{\varepsilon,s}\phi]\Big].$$

By (2.7), (2.8) and (3.13) we immediately obtain

$$\|E_{\varepsilon,s}\|_* = \|U''_{\varepsilon,s} - (1 + t(s))U_{\varepsilon,s} + U^p_{\varepsilon,s}\|_* + C\varepsilon \leq C\varepsilon \ \ \forall s \in [s_1, s_2] \tag{4.33}$$

where we have also used the estimate $\frac{2\varepsilon}{s+\varepsilon y} U'_{\varepsilon,s} = O(\varepsilon) e^{-|y|}$ uniformly for $s \in [s_1, s_2]$. Now we have to estimate the term $M_{\varepsilon,s}[\phi]$. Set $\mathcal{B}_{\varepsilon,s} = \{\phi \in C(\bar{I}_{\varepsilon,s}) \,|\, \|\phi\|_* \leq \varepsilon^\sigma\}$. Taken ϕ_1, $\phi_2 \in \mathcal{B}_{\varepsilon,s}$, we compute

$$\begin{aligned} &\| - (U_{\varepsilon,s} + \phi_1)^p_+ + pU^{p-1}_{\varepsilon,s}\phi_1 + (U_{\varepsilon,s} + \phi_2)^p_+ - pU^{p-1}_{\varepsilon,s}\phi_2\|_* \\ &\leq \|\phi_1 - \phi_2\|_* \sup_{\xi \in \mathcal{B}_{\varepsilon,s}} \|p(U_{\varepsilon,s} + \xi)^{p-1}_+ - pU^{p-1}_{\varepsilon,s}\|_\infty \\ &\leq C\varepsilon^{\sigma(p-1)} \|\phi_1 - \phi_2\|_* \ \ \forall s \in [s_1, s_2]. \end{aligned}$$

For every $\phi \in C(\bar{I}_{\varepsilon,s})$: $|T_{\varepsilon,s}[\phi](y)| \leq \|\phi\|_* \int_{I_{\varepsilon,s}} (s + \varepsilon z) e^{-\mu<z>} dz \leq C\|\phi\|_*$, then

$$\begin{aligned} &\|(U_{\varepsilon,s} + \phi_1)T_{\varepsilon,s}[\phi_1^2] - (U_{\varepsilon,s} + \phi_2)T_{\varepsilon,s}[\phi_2^2]\|_* \\ &\leq \|(U_{\varepsilon,s} + \phi_1)(T_{\varepsilon,s}[\phi_1^2 - \phi_2^2])\|_* + \|(\phi_1 - \phi_2)T_{\varepsilon,s}[\phi_2^2]\|_* \\ &\leq C\|\phi_1^2 - \phi_2^2\|_* + C\|\phi_1 - \phi_2\|_* \varepsilon^\sigma \leq C\|\phi_1 - \phi_2\|_* \varepsilon^\sigma \ \ \forall s \in [s_1, s_2]. \end{aligned}$$

In a similar way

$$\|\phi_1 T_{\varepsilon,s}[U_{\varepsilon,s}\phi_1] - \phi_2 T_{\varepsilon,s}[U_{\varepsilon,s}\phi_2]\|_* \leq C\|\phi_1 - \phi_2\|_* \varepsilon^\sigma \ \ \forall s \in [s_1, s_2],$$

by which

$$\|M_{\varepsilon,s}(\phi_1) - M_{\varepsilon,s}(\phi_2)\|_* \leq C(\varepsilon^\sigma + \varepsilon^{\sigma(p-1)})\|\phi_1 - \phi_2\|_* \ \ \forall \phi_1,\, \phi_2 \in \mathcal{B}_{\varepsilon,s}, \ \ \forall s \in [s_1, s_2]. \tag{4.34}$$

According to lemma 3.3, for every $\phi \in \mathcal{B}_{\varepsilon,s}$ we define $\mathcal{A}_{\varepsilon,s}[\phi] \in H^1_\varepsilon(I_{\varepsilon,s})$ to be the unique solution to the system (3.16) with $h = E_{\varepsilon,s} + M_{\varepsilon,s}[\phi]$. Then by lemma 3.3 we have

$$\|\mathcal{A}_{\varepsilon,s}[\phi]\|_* \le C\|E_{\varepsilon,s} + M_{\varepsilon,s}[\phi]\|_* \le C(\varepsilon + \varepsilon^{2\sigma} + \varepsilon^{\sigma p}) \le \varepsilon^\sigma \quad \forall s \in [s_1, s_2]$$

and hence $\mathcal{A}_{\varepsilon,s}[\phi] \in \mathcal{B}_{\varepsilon,s}$. Moreover, since $\mathcal{A}_{\varepsilon,s}[\phi_1] - \mathcal{A}_{\varepsilon,s}[\phi_2]$ solves the system (3.16) with $h = M_{\varepsilon,s}[\phi_1] - M_{\varepsilon,s}[\phi_2]$, by (4.34) we also have that

$$\|\mathcal{A}_\varepsilon[\phi_1] - \mathcal{A}_\varepsilon[\phi_2]\|_* \le C\|M_{\varepsilon,s}[\phi_1] - M_{\varepsilon,s}[\phi_2]\|_* < \|\phi_1 - \phi_2\|_* \quad \forall s \in [s_1, s_2], \tag{4.35}$$

i.e. the map $\mathcal{A}_{\varepsilon,s}$ is a contraction map from $\mathcal{B}_{\varepsilon,s}$ to $\mathcal{B}_{\varepsilon,s}$. By the contraction mapping theorem, (4.29) has a unique solution $(\phi_{\varepsilon,s}, \beta_{\varepsilon,s}) \in \mathcal{B}_{\varepsilon,s} \times \mathbb{R}$. By multiplying both members of (4.31) by $(s + \varepsilon z)^2 \phi_{\varepsilon,s}$ and integrating by parts we obtain

$$(\phi_{\varepsilon,s}, \phi_{\varepsilon,s})_\varepsilon =< pU^{p-1}_{\varepsilon,s}\phi_{\varepsilon,s} - 2\Gamma T_{\varepsilon,s}[U_{\varepsilon,s}\phi_{\varepsilon,s}]U_{\varepsilon,s} - E_{\varepsilon,s} - M_{\varepsilon,s}[\phi_{\varepsilon,s}], \phi_{\varepsilon,s} >_\varepsilon \le \varepsilon^\sigma \tag{4.36}$$

where we have used (4.33) and (4.34); in particular, since $\|\Phi_{\varepsilon,s}\|^2_{H^1(\mathbb{R}^3)} \le \varepsilon^{-2} 4\pi (\phi_{\varepsilon,s}, \phi_{\varepsilon,s})_\varepsilon$, fixed $\varepsilon > 0$, the family $\Phi_{\varepsilon,s}$ is bounded in $H^1_r(\mathbb{R}^3)$. On the other hand, by using (3.15), we have

$$\begin{aligned}(\phi_{\varepsilon,s}, U'_{\varepsilon,s})_\varepsilon = &< pU^{p-1}_{\varepsilon,s}\phi_{\varepsilon,s} - 2\Gamma T_{\varepsilon,s}[U_{\varepsilon,s}\phi_{\varepsilon,s}]U_{\varepsilon,s} - E_{\varepsilon,s} - M_{\varepsilon,s}[\phi_{\varepsilon,s}], U'_{\varepsilon,s} >_\varepsilon \\ &+ \beta_{\varepsilon,s}(U'_{\varepsilon,s}, U'_{\varepsilon,s})_\varepsilon\end{aligned}$$

by which we deduce that the family $\beta_{\varepsilon,s}$ is bounded too for fixed $\varepsilon > 0$.

Now consider $\{s_n\} \subset [s_1, s_2]$ such that $s_n \to \bar{s}$. Up to a subsequence, $\Phi_{\varepsilon,s_n} \rightharpoonup \bar{\Phi}$ weakly in $H^1_r(\mathbb{R}^3)$ and $\beta_{\varepsilon,s_n} \to \bar{\beta}$. Then, setting $\bar{\phi}(y) = \bar{\Phi}(y + \frac{\bar{s}}{\varepsilon})$ for $y \in I_{\varepsilon,\bar{s}}$, $(\bar{\phi}, \bar{\beta})$ solves the equation

$$L_{\varepsilon,\bar{s}}(\bar{\phi}) = E_{\varepsilon,\bar{s}} + M_{\varepsilon,\bar{s}}[\bar{\phi}] + \bar{\beta} Z_{\varepsilon,\bar{s}}, \quad \bar{\phi}'\left(\frac{\bar{s}}{\varepsilon}\right) = 0, \ < \bar{\phi}, Z_{\varepsilon,\bar{s}} >= 0, \ \|\bar{\phi}\|_* \le \varepsilon^\sigma.$$

From uniqueness it follows $\bar{\phi} = \phi_{\varepsilon,\bar{s}}$ and $\bar{\beta} = \beta_{\varepsilon,\bar{s}}$. By (4.36) we get

$$(\phi_{\varepsilon,s_n}, \phi_{\varepsilon,s_n})_\varepsilon \to < pU^{p-1}_{\varepsilon,\bar{s}}\bar{\phi} - 2\Gamma T_{\varepsilon,\bar{s}}[U_{\varepsilon,\bar{s}}\bar{\phi}]U_{\varepsilon,\bar{s}} - E_{\varepsilon,\bar{s}} - M_{\varepsilon,\bar{s}}[\bar{\phi}], \bar{\phi} >_\varepsilon = (\bar{\phi}, \bar{\phi})_\varepsilon,$$

hence (since the norm $(\phi, \phi)_\varepsilon$ is equivalent to $\|\Phi\|_{H^1(\mathbb{R}^3)}$), we deduce $\Phi_{\varepsilon,s_n} \to \Phi_{\varepsilon,\bar{s}}$ in $H^1(\mathbb{R}^3)$. □

Lemma 4.2. *For $\varepsilon > 0$ sufficiently small, the map $\Phi_\varepsilon : s \in [s_1, s_2] \mapsto \Phi_{\varepsilon,s} \in H^1_r(\mathbb{R}^3)$ constructed in lemma 4.1 is C^1.*

Proof. Consider the following map $H_\varepsilon : [s_1, s_2] \times H^1_r(\mathbb{R}^3) \times \mathbb{R} \to H^1_r(\mathbb{R}^3) \times \mathbb{R}$ of class C^1:

$$H_\varepsilon(s, \Phi, \beta) = \begin{pmatrix} \Big(\Delta - (1 + \Gamma T_{\varepsilon,s}[U^2_{\varepsilon,s}])\Big)^{-1}\Big(S_{\varepsilon,s}[U_{\varepsilon,s} + \phi] - \beta Z_{\varepsilon,s}\Big)(|x| - \frac{s}{\varepsilon}) \\ (\phi, U'_{\varepsilon,s})_\varepsilon \end{pmatrix},$$

where $\Phi(x) = \phi(|x| - \frac{s}{\varepsilon})$ and $v = \big(\Delta - (1 + \Gamma T_{\varepsilon,s}[U^2_{\varepsilon,s}])\big)^{-1}(h)$ is defined as the unique solution in $H^1_\varepsilon(I_{\varepsilon,s})$ of

$$v'' + \frac{2\varepsilon}{s + \varepsilon y} v' - \big(1 + \Gamma T_{\varepsilon,s}[U^2_{\varepsilon,s}]\big)v = h, \ \ v'\Big(-\frac{s}{\varepsilon}\Big) = 0, \ \ y \in I_{\varepsilon,s}. \tag{4.37}$$

It is immediate that (ϕ, β) solves the system (4.29) if and only if $H_\varepsilon(s, \Phi, \beta) = 0$. We are going to prove that, provided that ε is sufficiently small, for every $s \in [s_1, s_2]$ the linear operator

$$\frac{\partial H_\varepsilon(s, \Phi, \beta)}{\partial(\Phi, \beta)}\Big|_{(s, \Phi_{\varepsilon,s}, \beta_{\varepsilon,s})} : H^1_r(\mathbb{R}^3) \times \mathbb{R} \to H^1_r(\mathbb{R}^3) \times \mathbb{R}$$

is invertible. But first notice how, assuming this, the thesis easily follows: indeed the continuity of the map Φ_ε proved in lemma 4.1 implies that Φ_ε actually coincides with the implicit function associated to H_ε, hence the C^1-regularity of Φ_ε will follows from the Implicit Function Theorem.

Now we compute

$$\frac{\partial H_\varepsilon(s, \Phi, \beta)}{\partial(\Phi, \beta)}\Big|_{(s, \Phi_{\varepsilon,s}, \beta_{\varepsilon,s})}[\Phi, c]$$
$$= \begin{pmatrix} \Big(\Delta - (1 + \Gamma T_{\varepsilon,s}[U^2_{\varepsilon,s}])\Big)^{-1}\Big(S'_{\varepsilon,s}[U_{\varepsilon,s} + \phi_{\varepsilon,s}](\phi) - cZ_{\varepsilon,s}\Big)(|x| - \frac{s}{\varepsilon}) \\ (\phi, U'_{\varepsilon,s})_\varepsilon \end{pmatrix}.$$

Proceeding as in the proof of lemma 3.3, (Φ, c) solves the system

$$\frac{\partial H_\varepsilon(s, \Phi, \beta)}{\partial(\Phi, \beta)}\Big|_{(s, \Phi_{\varepsilon,s}, \beta_{\varepsilon,s})}[\Phi, c] = (\Theta, d)$$

if and only if $\phi = \bar\phi + \frac{d}{(U'_{\varepsilon,s}, U'_{\varepsilon,s})_\varepsilon} U'_{\varepsilon,s}$, where $\bar\phi \in \mathcal{H}_\varepsilon$ ($\mathcal{H}_\varepsilon$ is the closed subset of $H^1_\varepsilon(I_{\varepsilon,s})$ defined in lemma 3.3) and

$$\Big(\big(\Delta - \big(1 + \Gamma T_{\varepsilon,s}[U^2_{\varepsilon,s}]\big)\big)^{-1}\big(S'_{\varepsilon,s}[U_{\varepsilon,s} + \phi_{\varepsilon,s}](\phi) - cZ_{\varepsilon,s}\big), \psi\Big)_\varepsilon = (\theta, \psi)_\varepsilon \ \ \forall \psi \in \mathcal{H}_\varepsilon \tag{4.38}$$

where $\Theta(x) = \theta(|x| - \frac{s}{\varepsilon})$. By (4.37) we have $\Big(\big(\Delta - \big(1 + \Gamma T_{\varepsilon,s}[U^2_{\varepsilon,s}]\big)\big)^{-1}(h), \psi\Big)_\varepsilon = -< h, \psi >_\varepsilon$; hence, setting $d' = \frac{d}{(U'_{\varepsilon,s}, U'_{\varepsilon,s})_\varepsilon}$, (4.38) may be rewritten as

$$(\bar\phi, \psi)_\varepsilon - p < (U_{\varepsilon,s} + \phi_{\varepsilon,s})^{p-1}_+ (\bar\phi + d'U'_{\varepsilon,s}), \psi >_\varepsilon + \Gamma < T_{\varepsilon,s}[\phi^2_{\varepsilon,s} + 2U_{\varepsilon,s}\phi_{\varepsilon,s}](\bar\phi + d'U'_{\varepsilon,s}), \psi >_\varepsilon$$
$$+ 2\Gamma < (U_{\varepsilon,s} + \phi_{\varepsilon,s}) T_{\varepsilon,s}[(U_{\varepsilon,s} + \phi_{\varepsilon,s})(\bar\phi + d'U'_{\varepsilon,s})], \psi >_\varepsilon = (\theta, \psi)_\varepsilon, \ \ \forall \psi \in \mathcal{H}_\varepsilon.$$

Once we know $\bar{\phi}$, the related c is given by the identity

$$d - p < (U_{\varepsilon,s} + \phi_{\varepsilon,s})_+^{p-1}\phi, U'_{\varepsilon,s} >_\varepsilon + \Gamma d' < T_{\varepsilon,s}[\phi_{\varepsilon,s}^2 + 2U_{\varepsilon,s}\phi_{\varepsilon,s}]\phi, U'_{\varepsilon,s} >_\varepsilon$$
$$+ 2\Gamma d' < (U_{\varepsilon,s} + \phi_{\varepsilon,s})T_{\varepsilon,s}[(U_{\varepsilon,s} + \phi_{\varepsilon,s})\phi], U'_{\varepsilon,s} >_\varepsilon = c(U'_{\varepsilon,s}, U'_{\varepsilon,s})_\varepsilon + (\theta, U'_{\varepsilon,s})_\varepsilon.$$

According to Riesz's representation theorem, take $\mathcal{W}_\varepsilon(\bar{\phi}) \in \mathcal{H}_\varepsilon$ such that

$$(\mathcal{W}_\varepsilon(\bar{\phi}), \psi)_\varepsilon = < -p(U_{\varepsilon,s} + \phi_{\varepsilon,s})_+^{p-1}(\bar{\phi} + d'U'_{\varepsilon,s}) + \Gamma T_{\varepsilon,s}[\phi_{\varepsilon,s}^2 + 2U_{\varepsilon,s}\phi_{\varepsilon,s}](\bar{\phi} + d'U'_{\varepsilon,s}), \psi >_\varepsilon$$
$$+ 2\Gamma < (U_{\varepsilon,s} + \phi_{\varepsilon,s})T_{\varepsilon,s}[(U_{\varepsilon,s} + \phi_{\varepsilon,s})(\bar{\phi} + d'U'_{\varepsilon,s})], \psi >_\varepsilon, \quad \forall \psi \in \mathcal{H}_\varepsilon,$$

Then problem (4.38) consists in finding $\bar{\phi} \in \mathcal{H}_\varepsilon$ such that

$$\bar{\phi} + \mathcal{W}_\varepsilon(\bar{\phi}) = \theta. \tag{4.39}$$

On the other hand, by (2.8) and (3.12) we deduce,

$$T_{\varepsilon,s}[U_{\varepsilon,s}U'_{\varepsilon,s}](y) = s\int_{I_{\varepsilon,s}} U_{\varepsilon,s}U'_{\varepsilon,s}dy + O(\varepsilon)(|y| + 1) = O(\varepsilon)(|y| + 1),$$

by which, using (2.8),

$$< U_{\varepsilon,s}, T_{\varepsilon,s}[U_{\varepsilon,s}U'_{\varepsilon,s}], \psi >_\varepsilon = O(\varepsilon) < e^{-\mu<y>}, |\psi| >_\varepsilon \quad \forall \psi \in \mathcal{H}_\varepsilon. \tag{4.40}$$

Furthermore, by (2.8) and (3.19),

$$pU_{\varepsilon,s}^{p-1}U'_{\varepsilon,s} = -Z_{\varepsilon,s} + O(\varepsilon)e^{-\mu<y>},$$

hence

$$p < (U_{\varepsilon,s} + \phi_{\varepsilon,s})_+^{p-1}U'_{\varepsilon,s}, \psi >_\varepsilon = O(\varepsilon) < e^{-\mu<y>}, \psi >_\varepsilon + O(1) < |\phi_{\varepsilon,s}|^{p-1}, |\psi| >_\varepsilon \quad \forall \psi \in \mathcal{H}_\varepsilon. \tag{4.41}$$

Since $\|\phi_{\varepsilon,s}\|_* \le \varepsilon^\sigma$, combining (4.40) and (4.41) with the definition of $\mathcal{W}_\varepsilon$ and $\mathcal{K}_\varepsilon$ (see lemma 3.3), when $\varepsilon \to 0$ we have

$$\mathcal{W}_\varepsilon - \mathcal{K}_\varepsilon \to 0.$$

Since we have proved that $I + \mathcal{K}_\varepsilon$ is invertible, then the theory of the linear operator assures the invertibility of $I + \mathcal{W}_\varepsilon$ for small ε. This concludes the proof of the lemma. □

5. The Reduced Energy Functional: Proof of Theorem 1.1

For every $s \in [s_1, s_2]$ set

$$v_{\varepsilon,s}(x) = U_{\varepsilon,s}\Big(|x| - \frac{s}{\varepsilon}\Big) + \phi_{\varepsilon,s}\Big(|x| - \frac{s}{\varepsilon}\Big)$$

where $\phi_{\varepsilon,s}$ has been constructed in lemma 4.1 and consider the function

$$K_\varepsilon : [s_1, s_2] \to \mathbb{R}, \quad K_\varepsilon(s) := \frac{\varepsilon^2}{4\pi}E_\varepsilon[v_{\varepsilon,s}].$$

First we provide the following estimate.

Lemma 5.1. *For $s \in [s_1, s_2]$, we have*

$$K_\varepsilon(s) = \frac{1}{8(p+1)} \frac{\int_{\mathbb{R}} w^{p+1} dy}{(\int_{\mathbb{R}} w^2 dy)^2} \rho(t(s)) + o(1) \tag{5.1}$$

where $\rho(t)$ and $t(s)$ have been defined in lemma 2.1.

Proof. We compute

$$\begin{aligned} K_\varepsilon(s) &= \int_{-\frac{s}{\varepsilon}}^{+\infty} \Big[\frac{1}{2}|(U_{\varepsilon,s} + \phi_{\varepsilon,s})'|^2 + \frac{1}{2}|U_{\varepsilon,s} + \phi_{\varepsilon,s}|^2 - \frac{1}{p+1}(U_{\varepsilon,s} + \phi_{\varepsilon,s})_+^{p+1}\Big](s+\varepsilon y)^2 dy \\ &\quad + \frac{\Gamma}{4}\int_{-\frac{s}{\varepsilon}}^{\infty} (U_{\varepsilon,s} + \phi_{\varepsilon,s})^2 T_{\varepsilon,s}[(U_{\varepsilon,s} + \phi_{\varepsilon,s})^2](s+\varepsilon y)^2 dy \\ &= \frac{s^2}{2}\int_{\mathbb{R}} |w_s'|^2 dy + \frac{s^2}{2}\int_{\mathbb{R}} w_s^2 dy - \frac{s^2}{p+1}\int_{\mathbb{R}} w_s^{p+1} dy + \frac{s^2}{4} t(s) \int_{\mathbb{R}} w_s^2 dy + o(1), \end{aligned}$$

where we have used (2.8), (3.13) and the estimates $\|\phi_{\varepsilon,s}\|_*$, $(\phi_{\varepsilon,s}, \phi_{\varepsilon,s})_\varepsilon \le \varepsilon^\sigma$ given by lemma 4.1. Now we use (2.9) to obtain

$$\begin{aligned} K_\varepsilon(s) &= s^2 \frac{(3p-7)t(s) + 4(p-1)}{8(p+1)(1+t(s))} \int_{\mathbb{R}} w_s^{p+1} dy + o(1) \\ &= s^2 \frac{(3p-7)t(s) + 4(p-1)}{8(p+1)} (1+t(s))^{\frac{5-p}{2(p-1)}} \int_{\mathbb{R}} w^{p+1} dy + o(1) \\ &= s^2 \frac{\rho(t(s))}{8t(s)^2(p+1)} (1+t(s))^{\frac{5-p}{p-1}} \int_{\mathbb{R}} w^{p+1} dy + o(1) = \frac{1}{8(p+1)} \frac{\int_{\mathbb{R}} w^{p+1} dy}{\left(\Gamma \int_{\mathbb{R}} w^2 dy\right)^2} + o(1). \end{aligned}$$

□

Corollary 5.1. *For ε sufficiently small, the function $s \in [s_1, s_2] \mapsto K_\varepsilon(s)$ has a minimum $s_\varepsilon = s_0 + o(1)$.*

Proof. This follows immediately from the previous lemma. Indeed, since the function $\rho(t(s))$ has a unique local minimum point $s_0 \in [s_1, s_2]$ and moreover s_0 is nondegenerate, thus by (5.1) and by the continuity of $K_\varepsilon(s)$, for ε small enough the minimum point of K_ε is attained at some s_ε which tends to s_0 as $\varepsilon \to 0$. □

Proof of Theorem 1.1. Fix $\varepsilon > 0$ sufficiently small such that lemma 4.1 and lemma 4.2 hold for $\varepsilon \in (0, \varepsilon_0)$. According to Lemma 4.1 for every $\varepsilon \in (0, \varepsilon_0)$ and $s \in [s_1, s_2]$ the function $v_{\varepsilon,s}$ solves the equation

$$\Delta v_{\varepsilon,s}(x) - v_{\varepsilon,s}(x) + (v_{\varepsilon,s})_+^p(x) - \varepsilon\Gamma(-\Delta)^{-1}[v_{\varepsilon,s}^2](x) = \beta_{\varepsilon,s} Z_{\varepsilon,s}\Big(|x| - \frac{s}{\varepsilon}\Big) \text{ in } \mathbb{R}^3. \tag{5.2}$$

Set $u_\varepsilon = v_{\varepsilon,s_\varepsilon}$, where $s_\varepsilon \in (s_1, s_2)$ is the minimum point of K_ε, according to corollary 5.1. Then we have

$$\frac{d}{ds} E_\varepsilon(v_{\varepsilon,s})_{|_{s=s^\varepsilon}} = 0; \tag{5.3}$$

by using the C^1 regularity of the map $s \in [s_1, s_2] \mapsto \Phi_{\varepsilon,s} \in H^1_r(\mathbb{R}^3)$, (5.3) can be rewritten as

$$\int_{\mathbb{R}^3} \Big[\nabla u_\varepsilon \nabla \frac{d}{ds} v_{\varepsilon,s} + u_\varepsilon \frac{d}{ds} v_{\varepsilon,s} - (u_\varepsilon)_+^p \frac{d}{ds} v_{\varepsilon,s}\Big] dx + \varepsilon\Gamma \int_{\mathbb{R}^3} u_\varepsilon (-\Delta)^{-1}[u_\varepsilon^2] \frac{d}{ds} v_{\varepsilon,s}\Big|_{s=s^\varepsilon} dx = 0.$$

which is equivalent, by (5.2), to

$$\beta_{\varepsilon,s_\varepsilon} \int_{\mathbb{R}^3} Z_{\varepsilon,s_\varepsilon}\Big(|x| - \frac{s_\varepsilon}{\varepsilon}\Big) \frac{d}{ds} v_{\varepsilon,s} dx\Big|_{s=s_\varepsilon} = 0,$$

that is

$$\beta_{\varepsilon,s_\varepsilon} \int_{I_{\varepsilon,s}} Z_{\varepsilon,s_\varepsilon}(y) \frac{d}{ds}(v_{\varepsilon,s})\Big|_{s=s_\varepsilon} \Big(y + \frac{s_\varepsilon}{\varepsilon}\Big)(s_\varepsilon + \varepsilon y)^2 dy = 0. \tag{5.4}$$

Notice that, by (2.8), we easily compute

$$\begin{aligned} \frac{d}{ds}\Big[U_{\varepsilon,s}\Big(\cdot - \frac{s}{\varepsilon}\Big)\Big]\Big(y + \frac{s}{\varepsilon}\Big) =& \frac{t'(s)}{(p-1)(1+t(s))} w_s(y) + w_s'(y)\Big(\frac{t'(s)y}{2\sqrt{1+t(s)}} - \frac{\sqrt{1+t(s)}}{\varepsilon}\Big) \\ &+ o(\varepsilon)e^{-\mu<y>} = -\frac{\sqrt{1+t(s)}}{\varepsilon} w_s'(y) + O(1)e^{-\mu<y>}. \end{aligned} \tag{5.5}$$

Differentiating the equation $\int_{\mathbb{R}^3} \Phi_{\varepsilon,s} Z_{\varepsilon,s}\Big(|x| - \frac{s}{\varepsilon}\Big) dx = \varepsilon^{-2} < \phi_{\varepsilon,s}, Z_{\varepsilon,s} >_\varepsilon = 0$ with respect to s, we get

$$\int_{I_{\varepsilon,s}} Z_{\varepsilon,s} \frac{d}{ds}(\Phi_{\varepsilon,s})\Big(y + \frac{s}{\varepsilon}\Big)(s+\varepsilon y)^2 dy = -\int_{I_{\varepsilon,s}} \phi_{\varepsilon,s} \frac{d}{ds}\Big[Z_{\varepsilon,s}\Big(\cdot - \frac{s}{\varepsilon}\Big)\Big]\Big(y + \frac{s}{\varepsilon}\Big)(s+\varepsilon y)^2 dy.$$

By computing similarly as in (5.5) we get

$$\frac{d}{ds}\Big[Z_{\varepsilon,s}\Big(\cdot - \frac{s}{\varepsilon}\Big)\Big]\Big(y + \frac{s}{\varepsilon}\Big) = O(\varepsilon^{-1})e^{-\mu<y>},$$

by which

$$\int_{I_{\varepsilon,s}} Z_{\varepsilon,s} \frac{d}{ds}(\Phi_{\varepsilon,s})\Big(y + \frac{s}{\varepsilon}\Big)(s+\varepsilon y)^2 dy = O(\varepsilon^{\sigma-1}). \tag{5.6}$$

Combining (3.19), (5.5) and (5.6) we obtain

$$\int_{I_{\varepsilon,s}} Z_{\varepsilon,s} \frac{d}{ds}(v_{\varepsilon,s})\Big(y + \frac{s}{\varepsilon}\Big)(s+\varepsilon y)^2 dy = -\frac{s^2 p\sqrt{1+t(s)}}{\varepsilon} \int_{\mathbb{R}} w_s^{p-1}(w_s')^2 dy + O(\varepsilon^{\sigma-1}).$$

In particular $\int_{I_{\varepsilon,s_\varepsilon}} Z_{\varepsilon,s_\varepsilon} \frac{d}{ds}(v_{\varepsilon,s})\big|_{s=s_\varepsilon}\big(y + \frac{s_\varepsilon}{\varepsilon}\big)(s_\varepsilon + \varepsilon y)^2 dy \neq 0$ for ε sufficiently small, then (5.4) implies $\beta_{\varepsilon,s_\varepsilon} = 0$. Hence u_ε solves the equation

$$\Delta u_\varepsilon - u_\varepsilon + (u_\varepsilon)_+^p - \varepsilon\Gamma u_\varepsilon(-\Delta)^{-1}[u_\varepsilon^2] = 0 \text{ in } \mathbb{R}^3. \tag{5.7}$$

By multiplying both members of (5.7) by $(u_\varepsilon)_-$ and integrating by parts we get

$$\int_{\mathbb{R}^3} |\nabla(u_\varepsilon)_-|^2 dx + \int_{\mathbb{R}^3} |(u_\varepsilon)_-|^2 + \varepsilon\Gamma \int_{\mathbb{R}^3} (u_\varepsilon)_-^2 (-\Delta)^{-1}[u_\varepsilon^2] dx = 0,$$

by which $(u_\varepsilon)_- = 0$. From the strong Maximum Principle it follows that $u_\varepsilon > 0$. Hence u_ε and $\psi_\varepsilon = \varepsilon(-\Delta)^{-1}[u_\varepsilon^2]$ actually solve the system (2.1). In order to obtain (2) of theorem 1.1, using (2.8) and the estimate $\|\phi_{\varepsilon,s}\|_* \le \varepsilon^\sigma$, we obtain

$$\begin{aligned}\frac{1}{\sqrt{\varepsilon}}u_\varepsilon\Big(\frac{r}{\varepsilon}\Big) &= \frac{1}{\sqrt{\varepsilon}}U_{\varepsilon,s_\varepsilon}\Big(\frac{r-s_\varepsilon}{\varepsilon}\Big) + \frac{1}{\sqrt{\varepsilon}}\phi_{\varepsilon,s_\varepsilon}\Big(\frac{r-s_\varepsilon}{\varepsilon}\Big)\\ &= \frac{1}{\sqrt{\varepsilon}}(1+t(s_\varepsilon))^{\frac{1}{p-1}}w\Big(\sqrt{1+t(s_\varepsilon)}\frac{r-s_\varepsilon}{\varepsilon}\Big) + o(1)\exp\Big(-\mu\frac{|r-s_\varepsilon|}{\varepsilon}\Big).\end{aligned}$$

Finally notice that for every $r,\rho\in(0,+\infty)$ it results $|G(r,\rho)-G(r,s_\varepsilon)| \le |\rho-s_\varepsilon|\frac{1}{s_\varepsilon}\Big(2+\frac{\rho}{s_\varepsilon}\Big)G(r,s_\varepsilon)$, by which

$$\begin{aligned}\psi_\varepsilon\Big(\frac{r}{\varepsilon}\Big) &= \frac{1}{\varepsilon}\int_0^{+\infty} G(r,\rho)u_\varepsilon^2\Big(\frac{\rho}{\varepsilon}\Big)d\rho\\ &= \frac{1}{\varepsilon}G(r,s_\varepsilon)\int_0^{+\infty}u_\varepsilon^2\Big(\frac{\rho}{\varepsilon}\Big)d\rho + O(1)G(r,s_\varepsilon)\frac{1}{\varepsilon s_\varepsilon}\int_0^{+\infty}|\rho-s_\varepsilon|\Big(2+\frac{\rho}{s_\varepsilon}\Big)u_\varepsilon^2\Big(\frac{\rho}{\varepsilon}\Big)d\rho\\ &= G(r,s_\varepsilon)\Big((1+t(s_\varepsilon))^{\frac{5-p}{2(p-1)}}\int_{\mathbb{R}}w^2d\rho + o(1)\Big).\end{aligned}$$

Taking into account of (2.2), theorem 1.1 is proved. □

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