Bifurcation diagram of solutions to elliptic equation with exponential nonlinearity in higher dimensions

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Abstract

We consider the following semilinear elliptic equation:

$$\begin{cases} -\Delta u = \lambda e^{u^p} & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{0.1}$$

where B_1 is the unit ball in $\mathbb{R}^d$, $d \geq 3, \lambda > 0$ and $p > 0$. First, following Merle and Peletier [13], we show that there exists a unique eigenvalue $\lambda_{p,\infty}$ such that (0.1) has a solution $(\lambda_{p,\infty}, W_p)$ satisfying $\lim_{|x|\to 0} W_p(x) = \infty$. Secondly, we study a bifurcation diagram of regular solutions to (0.1). It follows from the result of Dancer [4] that (0.1) has an unbounded bifurcation branch of regular solutions which emanates from $(\lambda, u) = (0, 0)$. Here, using the singular solution, we show that the bifurcation branch has infinitely many turning points around $\lambda_{p,\infty}$ in case of $3 \leq d \leq 9$. We also investigate the Morse index of the singular solution in case of $d \geq 11$.

1 Introduction

In this paper, we study the following semilinear elliptic equation:

$$\begin{cases} -\Delta u = \lambda e^{u^p} & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{1.1}$$

where B_1 is the unit ball in $\mathbb{R}^d$, $d \geq 3, \lambda > 0$ and $p > 0$.

The purpose of this paper is to study the existence of a singular solution and a bifurcation diagram of regular solutions to (1.1) for general power $p > 0$. By a singular solution, we mean a positive regular solution to (1.1) in $B_1 \setminus \{0\}$ and tends to infinity at the origin $x = 0$. For example, putting $\lambda_{1,\infty} = 2(d-2)$ and $W_1(x) = -2\log|x|$, we see that $(\lambda_{1,\infty}, W_1)$ is a singular solution to (1.1) in case of $p = 1$.

Several studies have been made on (1.1) in case of $p = 1$. See [1, 3, 5, 6, 9, 10, 15, 17, 16] and references therein. We recall some of them. Gel'fand [6] showed that when $d = 3$, (1.1) has infinitely many solutions at $\lambda = \lambda_{1,\infty}$. Then, Joseph and Lundgren [10] gave a complete classification of solutions to (1.1). More precisely, they showed that (1.1) has infinitely many solutions at $\lambda = \lambda_{1,\infty}$ when $3 \leq d \leq 9$ and has a unique solution for

$0 < \lambda < \lambda_{1,\infty}$ and no solution for $\lambda > \lambda_{1,\infty}$ when $d \geq 10$. See Jacobsen and Schmitt [9] for the survey of this problem.

In this paper, we will treat general power $p > 0$ and show that (1.1) has a singular solution in the case where $p > 0$ and $d \geq 3$. In addition, we shall show that (1.1) has infinitely many regular solutions in the case where $p > 0$ and $3 \leq d \leq 9$.

First, we focus our attention on the existence of a singular solution. As we mentioned above, in case of $p = 1$, (1.1) has the explicit singular solution $(\lambda_{1,\infty}, W_1)$. The singular solution plays an important role in the bifurcation analysis of regular solutions to (1.1). However, we encounter difficulties when we seek a singular solution if the power p does not equal to 1. Therefore, it is worthwhile to investigate the existence of a singular solution for general power $p > 0$. Concerning this, we obtain the following.

Theorem 1.1. *Assume that $d \geq 3$ and $p > 0$. Then, there exists a unique eigenvalue $\lambda_{p,\infty} > 0$ such that the equation (1.1) has a singular solution $(\lambda_{p,\infty}, W_p)$ satisfying*

$$W_p(x) = \left[-2\log|x| - (1 - \frac{1}{p})\log(-\log|x|)\right]^{\frac{1}{p}} + O\left((\log|x|)^{-1+\frac{1}{p}}\right) \tag{1.2}$$

as $|x| \to 0$.

Once we obtain the singular solution, we investigate the relation between the singular solution and regular ones. Dancer [4] showed that for any $p > 0$, there exists an unbounded bifurcation branch $\mathcal{C} \subset \mathbb{R} \times L^\infty(B_1)$ which emanates from $(\lambda, u) = (0, 0)$. Let λ_1 be the first eigenvalue of the operator $-\Delta$ in B_1 with the Dirichlet boundary condition and ϕ_1 be the corresponding eigenfunction. By multiplying the equation in (1.1) by ϕ_1 and integrating the resulting equation, we see that if $(\lambda, u) \in \mathcal{C}$, we have $0 < \lambda < \lambda_1$. This yields that $\sup\{\|u\|_\infty \mid (\lambda, u) \in \mathcal{C}\} = \infty$. Moreover, from the result of Korman [12, Theorem 2.1] (see also Miyamoto [15, Proposition 6]), we see that the branch $\mathcal{C}$ can be parameterized by $\|u\|_\infty$. Namely, the branch $\mathcal{C}$ can be expressed by the following:

$$\mathcal{C} = \{(\lambda(\gamma), u(x,\gamma)) \mid \gamma = \|u\|_{L^\infty},\ 0 < \gamma < \infty\}. \tag{1.3}$$

Then, we obtain the following.

Theorem 1.2. *Assume that $d \geq 3$ and $p > 0$. Let $(\lambda_{p,\infty}, W_p)$ be the singular solution to equation (1.1) given by Theorem 1.1 and $(\lambda(\gamma), u(x,\gamma)) \in \mathcal{C}$. Then, we have $\lambda(\gamma) \to \lambda_{p,\infty}$ and*

$$u(x,\gamma) \to W_p(x) \qquad \text{in } C^1_{loc}(B_1 \setminus \{0\}) \text{ as } \gamma \to \infty.$$

From Theorem 1.2, we can obtain the following result.

Theorem 1.3. *Assume that $3 \leq d \leq 9$ and $p > 0$. Let $\lambda_{p,\infty} > 0$ be the eigenvalue given by Theorem 1.1. Then, for any integer k, there exist at least k regular positive solutions to (1.1) if λ is sufficiently close to $\lambda_{p,\infty}$. In particular, there exist infinitely many regular solutions to (1.1) at $\lambda = \lambda_{p,\infty}$.*

Finally, we estimate the Morse index of the singular solution W_p in case of $d \geq 11$. Here, we mean the Morse index by the number of the negative eigenvalues of the linearized operator $-\Delta - pW_p^{p-1}e^{W_p^p}$ with the domain $H^2(B_1) \cap H_0^1(B_1)$. It is well-known that the Morse index plays an important role in the bifurcation analysis for nonlinear elliptic equations (see e.g. [2], [8], [11] and references therein). In case of $9 \geq d \geq 3$, we see that the Morse index of the singular solution W_p is infinite by combining the argument of Guo and Wei [8, Proposition 2.1] with Proposition 4.1 below. However, concerning the case of $d \geq 11$, we find that the situation becomes different from the above. More precisely, we obtain the following result.

Theorem 1.4. *Assume that $d \geq 11$ and $p > 0$. Let W_p be the singular solution to* (1.1) *obtained in Theorem 1.1. Then, the Morse index of the singular solution W_p is finite.*

We prove Theorems 1.1 in the spirit of Merle and Peletier [13]. We first transform the equation (1.1) to a suitable one. From the result of Gidas, Ni and Nirenberg [7], we find that a positive solution to (1.1) is radially symmetric. Therefore, the equation (1.1) can be transformed into the following ordinary differential equation:

$$\begin{cases} u_{rr} + \frac{d-1}{r}u_r + \lambda e^{u^p} = 0 & 0 < r < 1, \\ u(r) = 0 & r = 1. \end{cases} \tag{1.4}$$

We put $s = \sqrt{\lambda}r$ and $\widehat{u}(s) = u(r)$. Then, we see that $\widehat{u}$ satisfies

$$\begin{cases} \widehat{u}_{ss} + \frac{d-1}{s}\widehat{u}_s + e^{\widehat{u}^p} = 0 & 0 < s < \sqrt{\lambda}, \\ \widehat{u}(s) = 0 & s = \sqrt{\lambda}. \end{cases} \tag{1.5}$$

We construct a local solution to the equation in (1.5) which has a singularity at the origin $s = 0$. To this end, we employ the Emden-Fowler transformation. Namely, we put $t = -\log s$ and $\overline{u}(t) = \widehat{u}(s)$. This yields that $\overline{u}$ satisfies the following:

$$\begin{cases} \overline{u}_{tt} - (d-2)\overline{u}_t + \exp[-2t + \overline{u}^p] = 0 & -\frac{\log\lambda}{2} < t < \infty, \\ \overline{u}(t) = 0 & t = -\frac{\log\lambda}{2}. \end{cases} \tag{1.6}$$

We give an approximate form of a singular solution near $t = \infty$. Then, we make an error estimate for the approximation. The proof of Theorem 1.2 is also based on that of Merle and Peletier [13]. We note that Dancer [4] already proved that there exists infinitely many regular positive solutions to (1.1) by by calculating the Morse index. Here, following Guo and Wei [8] and Miyamoto [14, 15], we shall show Theorem 1.3 by counting a intersection number of the singular solution and regular solutions. As a result, we can obtain a precise bifurcation diagram of solutions to (1.1). Let us explain this in detail. Let I be an interval in $\mathbb{R}$. For a function $v(s)$ on I, we define a number of zeros of v by

$$\mathcal{Z}_I[v(\cdot)] = \#\{s \in I \mid v(s) = 0\}.$$

We put $\widehat{W}_p(s) = W_p(r)$, where $s = \sqrt{\lambda} r$ and W_p is the singular solution given by Theorem 1.1. Let $(\lambda(\gamma), \widehat{u}(s,\gamma))$ be a regular solution to (1.5) with $\widehat{u}(0) = \gamma$. Then, we have

$$Z_{I_\lambda}[\widehat{u}(\cdot,\gamma) - \widehat{W}_p(\cdot)] \to \infty \qquad \text{as } \gamma \to \infty.$$

See Lemma 4.2 below in detail. From this, we can show that the bifurcation branch $\mathcal{C}$ given by (1.3) has infinitely many turning points, which yields Theorem 1.3.

This paper is organized as follows: In Section 2, we construct the singular solution to (1.1) in case of $d \geq 3$. In Section 3, we investigate the asymptotic behavior of the regular solutions $(\lambda(\gamma), u(r,\gamma))$ as γ goes to infinity. In Section 4, we count the intersection number and give a proof of Theorem 1.3. In Section 5, we show that the Morse index of the singular solution is finite in case of $d \geq 11$.

2 Existence of a singular solution

To prove Theorem 1.1, we first consider (1.6) and restrict ourselves to the case where $t > 0$ is sufficiently large. We seek a solution to (1.6) of the form

$$\overline{u}(t) = (\varphi(t) + \kappa)^{\frac{1}{p}} + \eta(t), \tag{2.1}$$

where

$$\varphi(t) = 2t - A_p \log t, \qquad A_p = 1 - \frac{1}{p}, \qquad \kappa = \log \frac{(d-2)2^{\frac{1}{p}}}{p}. \tag{2.2}$$

Then, the function η solves the following:

$$\eta_{tt} - (d-2)\eta_t + \exp[-2t + \overline{u}^p] - \frac{2(d-2)}{p}(\varphi + \kappa)^{-A_p} = f_1(t) \tag{2.3}$$

for sufficiently large $t > 0$, where

$$f_1(t) = \frac{(d-2)A_p(\varphi+\kappa)^{-A_p}}{pt} + \frac{1}{p}\left(1 - \frac{1}{p}\right)(\varphi+\kappa)^{\frac{1}{p}-2}(\varphi_t)^2 - \frac{1}{p}(\varphi+\kappa)^{-A_p}\varphi_{tt}. \tag{2.4}$$

Then, we show the following:

Theorem 2.1. *Let $d \geq 3$ and $p > 0$. There exist $T_\infty > 0$ and a solution $\eta_\infty \in C([T_\infty, \infty), \mathbb{R})$ to the equation (2.3) satisfying $\lim_{t\to\infty} \varphi^{A_p}\eta_\infty(t) = 0$.*

We show Theorem 2.1 by using the contraction mapping principle. To this end, we transform (2.3). First, we have

$$\begin{aligned} &\exp\left[-2t + \overline{u}^p\right] \\ &= \exp\left[-2t + \left\{(\varphi+\kappa)^{\frac{1}{p}} + \eta\right\}^p\right] \\ &= \exp\left[-2t + (\varphi+\kappa) + (\varphi+\kappa)\left\{\left(1 + (\varphi+\kappa)^{-\frac{1}{p}}\eta\right)^p - 1\right\}\right] \\ &= \frac{(d-2)2^{\frac{1}{p}}}{p} t^{-A_p} \exp\left[(\varphi+\kappa)\left\{\left(1 + (\varphi+\kappa)^{-\frac{1}{p}}\eta\right)^p - 1\right\}\right], \end{aligned} \tag{2.5}$$

Furthermore, we obtain

$$(\varphi+\kappa)\left\{\left(1+(\varphi+\kappa)^{-\frac{1}{p}}\eta\right)^{p}-1\right\}=p(\varphi+\kappa)^{A_p}\eta+(\varphi+\kappa)g_1(t,\eta) \tag{2.6}$$

where

$$g_1(t,\eta)=\left\{1+(\varphi+\kappa)^{-\frac{1}{p}}\eta\right\}^{p}-1-p(\varphi+\kappa)^{-\frac{1}{p}}\eta. \tag{2.7}$$

This yields that

$$\begin{aligned}&\exp[(\varphi+\kappa)\left\{\left(1+(\varphi+\kappa)^{-\frac{1}{p}}\eta\right)^{p}-1\right\}]\\&=\exp[p(\varphi+\kappa)^{A_p}\eta+(\varphi+\kappa)g_1(t,\eta)]\\&=\exp[p(\varphi+\kappa)^{A_p}\eta]+\exp[p(\varphi+\kappa)^{A_p}\eta]\left\{\exp[(\varphi+\kappa)g_1(t,\eta)]-1\right\}.\end{aligned} \tag{2.8}$$

By (2.5), (2.6), and (2.8), we have

$$\begin{aligned}\exp[-2t+\overline{u}^{p}]=&\frac{(d-2)2^{\frac{1}{p}}}{p}t^{-A_p}\exp[p(\varphi+\kappa)^{A_p}\eta]\\&+\frac{(d-2)2^{\frac{1}{p}}}{p}t^{-A_p}\exp[p(\varphi+\kappa)^{A_p}\eta]\left\{\exp[(\varphi+\kappa)g_1(t,\eta)]-1\right\}.\end{aligned}$$

Therefore, (2.3) can be written by the following:

$$\begin{aligned}&\eta_{tt}-(d-2)\eta_t+2(d-2)\eta\\&=f_1(t)-\frac{(d-2)2^{\frac{1}{p}}}{p}t^{-A_p}+\frac{2(d-2)}{p}(\varphi+\kappa)^{-A_p}\\&\quad+2(d-2)\eta-\frac{(d-2)2^{\frac{1}{p}}}{p}t^{-A_p}\times p(\varphi+\kappa)^{A_p}\eta\\&\quad-\frac{(d-2)2^{\frac{1}{p}}}{p}t^{-A_p}\exp[p(\varphi+\kappa)^{A_p}\eta]\left\{\exp[(\varphi+\kappa)g_1(t,\eta)]-1\right\}\\&\quad-\frac{(d-2)2^{\frac{1}{p}}}{p}t^{-A_p}\left\{\exp[p(\varphi+\kappa)^{A_p}\eta]-1-p(\varphi+\kappa)^{A_p}\eta\right\}\\&=f_1(t)+f_2(t)+f_3(t,\eta)+f_4(t,\eta)+f_5(t,\eta),\end{aligned}$$

where

$$\begin{aligned}f_2(t)&=-\frac{(d-2)2^{\frac{1}{p}}}{p}t^{-A_p}+\frac{2(d-2)}{p}(\varphi+\kappa)^{-A_p}\\&=\frac{(d-2)2^{\frac{1}{p}}}{p}t^{-A_p}\left(1-(2t)^{A_p}(\varphi+\kappa)^{-A_p}\right),\end{aligned} \tag{2.9}$$

$$\begin{aligned}f_3(t,\eta)&=2(d-2)\eta-\frac{(d-2)2^{\frac{1}{p}}}{p}t^{-A_p}\times p(\varphi+\kappa)^{A_p}\eta\\&=2(d-2)\left\{1-(2t)^{-A_p}(\varphi+\kappa)^{A_p}\right\}\eta,\end{aligned} \tag{2.10}$$

$$f_4(t,\eta)=-\frac{(d-2)2^{\frac{1}{p}}}{p}t^{-A_p}\exp[p(\varphi+\kappa)^{A_p}\eta]\left\{\exp[(\varphi+\kappa)g_1(t,\eta)]-1\right\}, \tag{2.11}$$

$$f_5(t,\eta) = -\frac{(d-2)2^{\frac{1}{p}}}{p} t^{-A_p} \left\{ \exp[p(\varphi+\kappa)^{A_p}\eta] - 1 - p(\varphi+\kappa)^{A_p}\eta \right\}. \tag{2.12}$$

Thus, we seek a solution to the following equation:

$$\eta_{tt} - (d-2)\eta_t + 2(d-2)\eta = f_1(t) + f_2(t) + f_3(t,\eta) + f_4(t,\eta) + f_5(t,\eta)$$

We estimate the inhomogeneous terms $f_i(t)(1 \le i \le 5)$. We obtain the following.

Lemma 2.1. (i) $f_1(t) = O(t^{-A_p-1})$, $f_2(t) = O(t^{-A_p-1}\log t)$ *as* $t \to \infty$,

(ii) *If* η *satisfies* $\eta(t) \le \varepsilon t^{-A_p}$ *for sufficiently small* $\varepsilon > 0$, *we have*

$$f_3(t,\eta) = O(t^{-A_p-1}\log t), \qquad f_4(t) = O(t^{-A_p-1}), \qquad |f_5(t)| \le \varepsilon^2 t^{-A_p}.$$

for sufficiently large $t > 0$

Proof. By (2.4) and (2.9), we obtain (i). It follows from (2.2) that

$$\left|1 - (2t)^{-A_p}(\varphi+\kappa)^{A_p}\right| \lesssim \frac{\log t}{t} \tag{2.13}$$

for sufficiently large $t > 0$. Thus, by (2.10), we have

$$|f_3(t,\eta)| = \left|2(d-2)\left\{1 - (2t)^{-A_p}(\varphi+\kappa)^{A_p}\right\}\eta\right| \lesssim t^{-A_p-1}\log t.$$

From (2.7), we have

$$|g_1(t,\eta)| \lesssim |\varphi+\eta|^{-\frac{2}{p}}\eta^2. \tag{2.14}$$

This yields that

$$|(\varphi+\eta)g_1(t,\eta)| \lesssim t^{-1}.$$

It follows that

$$|\exp[(\varphi+\kappa)g_1(t,\eta)] - 1| \lesssim |(\varphi+\kappa)g_1(t,\eta)| \lesssim t^{-1}. \tag{2.15}$$

From (2.11), we have $f_4(t) = O(t^{-A_p-1})$. Similarly, we see that

$$|\exp[p(\varphi+\kappa)^{A_p}\eta] - 1 - p(\varphi+\kappa)^{A_p}\eta| \lesssim (\varphi+\kappa)^{2A_p}\eta^2 \lesssim \varepsilon^2.$$

Thus, we obtain $|f_5(t)| \le \varepsilon^2 t^{-A_p}$ from (2.12). □

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. We set

$$F(t,\eta) = f_1(t) + f_2(t) + f_3(t,\eta) + f_4(t,\eta) + f_5(t,\eta).$$

In order to prove Theorem 2.1, it is enough to solve the following final value problem:

$$\begin{cases} \eta_{tt} - (d-2)\eta_t + 2(d-2)\eta = F(t,\eta) & T < t < +\infty, \\ \varphi^{A_p}(t)\eta(t) \to 0 & \text{as } t \to +\infty. \end{cases} \tag{2.16}$$

for some $T > 0$. We note that

$$(d-2)^2 - 8(d-2) = (d-2)(d-10) \begin{cases} < 0 & \text{if } 3 \leq d \leq 9, \\ = 0 & \text{if } d = 10, \\ > 0 & \text{if } d \geq 11. \end{cases}$$

We consider the case where $3 \leq d \leq 9$ only because we can prove similarly in the other cases. Let $\mu = \sqrt{-(d-2)(d-10)}$. Then, the final value problem (2.16) is transformed into the following integral equation:

$$\eta(t) = \mathcal{T}[\eta](t)$$

in which

$$\mathcal{T}[\eta](t) = \frac{e^{\frac{d-2}{2}t}}{\mu} \int_t^\infty e^{-\frac{(d-2)}{2}\sigma} \sin(\mu(\sigma - t)) F(\sigma, \eta) d\sigma.$$

Fix $T > 0$ large enough and let X be a space of continuous function on (T, ∞) equipped with the following norm:

$$\|\xi\| = \sup \left\{ |t|^{A_p} |\xi(t)| \;\middle|\; t > T \right\}.$$

We fix arbitrary $\varepsilon > 0$ and set

$$\Sigma = \left\{ \xi \in X \;\middle|\; \|\xi\| < \varepsilon \right\}. \tag{2.17}$$

First, we shall show that $\mathcal{T}$ maps from Σ to itself. It follows from Lemma 2.1 that $|F(t, \eta)| \leq \varepsilon^2 t^{-A_p}$ for sufficiently large $t > 0$. This yields that

$$|\mathcal{T}[\eta](t)| \lesssim e^{\frac{d-2}{2}t} \int_t^\infty e^{-\frac{d-2}{2}\sigma} \varepsilon^2 \sigma^{-A_p} d\sigma \leq \varepsilon^2 t^{-A_p} e^{\frac{d-2}{2}t} \int_t^\infty e^{-\frac{d-2}{2}} d\sigma \lesssim \varepsilon^2 t^{-A_p}$$

for $\eta \in \Sigma$. It follows that $T[\eta] \in \Sigma$. Thus, we have proved the claim.

Next, we shall show that $\mathcal{T}$ is a contraction mapping. For $\eta_1, \eta_2 \in \Sigma$, we have

$$\left| \mathcal{T}[\eta_1](t) - \mathcal{T}[\eta_2](t) \right| \leq C e^{\frac{(d-2)}{2}t} \sum_{i=3}^{5} \int_t^\infty e^{-\frac{(d-2)}{2}\sigma} |f_i(\sigma, \eta_1) - f_i(\sigma, \eta_2)| d\sigma.$$

From the definition, we obtain

$$|f_3(t, \eta_1) - f_3(t, \eta_2)| \lesssim t^{-1} \log t |\eta_1 - \eta_2| \lesssim t^{-A_p - 1} \log t \|\eta_1 - \eta_2\|. \tag{2.18}$$

Thus, we see that

$$|f_3(t, \eta_1) - f_3(t, \eta_2)| \leq \varepsilon t^{-A_p} \|\eta_1 - \eta_2\|. \tag{2.19}$$

Next, we estimate the term $|f_5(t,\eta_1) - f_5(t,\eta_2)|$. It follows that

$$
\begin{aligned}
&|f_5(t,\eta_1) - f_5(t,\eta_2)| \\
&\lesssim t^{-A_p}\left|\exp[p(\varphi+\kappa)^{A_p}\eta_1] - \exp[p(\varphi+\kappa)^{A_p}\eta_2] - p(\varphi+\kappa)^{A_p}(\eta_1-\eta_2)\right| \\
&= t^{-A_p}\left|\exp[p(\varphi+\kappa)^{A_p}\eta_2]\left\{\exp[p(\varphi+\kappa)^{A_p}(\eta_2-\eta_1)] - 1\right\} - p(\varphi+\kappa)^{A_p}(\eta_1-\eta_2)\right| \\
&\lesssim t^{-A_p}\left|\exp[p(\varphi+\kappa)^{A_p}\eta_2]\left\{\exp[p(\varphi+\kappa)^{A_p}(\eta_2-\eta_1)] - 1 - p(\varphi+\kappa)^{A_p}(\eta_1-\eta_2)\right\}\right| \\
&\quad + t^{-A_p}\left|\exp[p(\varphi+\kappa)^{A_p}\eta_2] - 1\right| p(\varphi+\kappa)^{A_p}|\eta_1-\eta_2| \\
&\lesssim t^{-A_p}|p(\varphi+\kappa)^{A_p}(\eta_1-\eta_2)|^2 + t^{-A_p}|p(\varphi+\kappa)^{A_p}\eta_2|\|\eta_1-\eta_2\| \\
&\lesssim \varepsilon t^{-A_p}\|\eta_1-\eta_2\|.
\end{aligned}
$$

Therefore, for sufficiently large $t > 0$, we have

$$
|f_5(t,\eta_1) - f_5(t,\eta_2)| \leq \varepsilon t^{-A_p}\|\eta_1-\eta_2\|. \tag{2.20}
$$

Finally, we estimate the term $|f_4(t,\eta_1) - f_4(t,\eta_2)|$. We can compute that

$$
\begin{aligned}
|f_4(t,\eta_1) - f_4(t,\eta_2)| &\lesssim t^{-A_p}|\exp[p\varphi^{A_p}\eta_1] - \exp[p\varphi^{A_p}\eta_2]||\exp[g_1(t,\eta_2)] - 1| \\
&\quad + t^{-A_p}\exp[p\varphi^{A_p}\eta_2]|\exp[g_1(t,\eta_1)] - \exp[g_1(t,\eta_2)]| \\
&=: I + II.
\end{aligned} \tag{2.21}
$$

By the Taylor expansion together with (2.15), we have

$$
\begin{aligned}
I &\lesssim t^{-A_p-2}\exp[p\varphi^{A_p}\eta_2]\left\{\exp[p\varphi^{A_p}(\eta_2-\eta_1)] - 1\right\} \\
&\lesssim t^{-A_p-2}\exp[p\varepsilon]|\varphi^{A_p}(\eta_2-\eta_1)| \\
&\lesssim t^{-A_p-2}\varphi^{A_p}|\eta_1-\eta_2| \\
&\lesssim t^{-A_p-2}\|\eta_1-\eta_2\|.
\end{aligned} \tag{2.22}
$$

Similarly, by (2.14), we obtain

$$
\begin{aligned}
II &\lesssim t^{-A_p}\exp[p\varphi^{A_p}\eta_2]|\exp[g_1(t,\eta_1)] - \exp[g_1(t,\eta_2)]| \\
&\lesssim t^{-A_p}\exp[p\varphi^{A_p}\eta_2]\exp[g_1(t,\eta_2)]|\exp[g_1(t,\eta_1) - g_1(t,\eta_2)] - 1| \\
&\lesssim t^{-A_p}|g_1(t,\eta_1) - g_1(t,\eta_2)|.
\end{aligned} \tag{2.23}
$$

From (2.7), we obtain

$$
\begin{aligned}
&|g_1(t,\eta_1) - g_1(t,\eta_2)| \\
&\lesssim \left|\left\{1 + p(\varphi+\kappa)^{-\frac{1}{p}}\eta_1\right\}^p - \left\{1 + p(\varphi+\kappa)^{-\frac{1}{p}}\eta_2\right\}^p\right| + (\varphi+\kappa)^{-\frac{1}{p}}|\eta_1-\eta_2| \\
&\lesssim |\varphi+\kappa|^{-\frac{1}{p}}|\eta_1-\eta_2| \\
&\lesssim t^{-1}\|\eta_1-\eta_2\|.
\end{aligned} \tag{2.24}
$$

It follows from (2.21)–(2.24) that

$$|f_4(t,\eta_1) - f_4(t,\eta_2)| \leq \varepsilon t^{-A_p}\|\eta_1 - \eta_2\|. \tag{2.25}$$

By (2.18), (2.20) and (2.25), we see that

$$\Big|\mathcal{T}[\eta_1](t) - \mathcal{T}[\eta_2](t)\Big| \leq C\varepsilon t^{-A_p}\|\eta_1 - \eta_2\| \leq \frac{1}{2}t^{-A_p}\|\eta_1 - \eta_2\|. \tag{2.26}$$

Thus, we find that $\mathcal{T}$ is a contraction mapping. This completes the proof. □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. It follows from Theorem 2.1 that there exist a constant $T_\infty > 0$ and a solution $\eta_\infty(t)$ to the equation (2.3) for $t \in (T_\infty, +\infty)$ satisfying $|t|^{A_p}|\eta_\infty(t)| \leq \varepsilon$. For such a solution η_∞, we put

$$\overline{u}_\infty(t) = (\varphi(t) + \kappa)^{\frac{1}{p}} + \eta_\infty(t).$$

Then we see that $\overline{u}_\infty(t)$ satisfies

$$\overline{u}_{tt} - (d-2)\overline{u}_t + \exp[-2t + \overline{u}^p] = 0 \tag{2.27}$$

for $t \in (T_\infty, +\infty)$. We shall show that $\overline{u}_\infty(t)$ has a zero for some $T_0 \in (-\infty, \infty)$. Suppose the contradiction that $\overline{u}_\infty(t)$ is positive for all $t \in (-\infty, \infty)$. Then, we see that $\overline{u}_\infty$ is monotone increasing. Indeed, if not, there exists a local minimum point $t_* \in (-\infty, \infty)$. It follows that $(d^2\overline{u}_\infty/dt^2)(t_*) \geq 0$ and $(d\overline{u}_\infty/dt)(t_*) = 0$. Then, from the equation (2.27), we obtain

$$0 \leq \frac{d^2\overline{u}_\infty}{dt^2}(t_*) - (d-2)\frac{d\overline{u}_\infty}{dt}(t_*) = -\exp[-2t_* + \overline{u}_\infty^p(t_*)] < 0,$$

which is a contradiction. Since $\overline{u}_\infty$ is positive and monotone increasing, there exists a constant $C \geq 0$ such that $\overline{u}_\infty(t) \to C$ as $t \to -\infty$. This together with (2.27) yields that

$$0 = \lim_{t\to-\infty}\left\{\frac{d^2\overline{u}_\infty}{dt^2}(t) - (d-2)\frac{d\overline{u}_\infty}{dt}(t)\right\} = \lim_{t\to-\infty} -\exp[-2t + \overline{u}_\infty^p(t)] = -\infty,$$

which is absurd. Therefore, we see that $\overline{u}_\infty$ has a zero for some $T_0 \in (-\infty, \infty)$. Then, $\overline{u}_\infty$ satisfies

$$\begin{cases} \overline{u}_{tt} + (d-2)\overline{u}_t = -e^{-2t+\overline{u}^p}, & t \in (T_0, \infty), \\ \overline{u}(t) = 0, & t = T_0, \\ \overline{u}(t) > 0, & t \in (T_0, \infty). \end{cases}$$

If we choose $\lambda_{p,\infty} > 0$ so that $-\log\lambda_{p,\infty} = 2T_0$, that is, $\lambda_{p,\infty} = e^{-2T_0}$, we find that $\overline{u}_\infty(s)$ is a solution to (1.6) with $\lambda = \lambda_{p,\infty}$. This completes the proof. □

3 Asymptotic behavior of a regular solution

In this section, we give a proof of Theorem 1.2. We denote by $\widehat{u}(s,\gamma)$ a positive solution to (1.5) with $\widehat{u}(0)=\|\widehat{u}\|_{L^\infty}=\gamma$. If there is no confusion, we just denote by $\widehat{u}(s)$. We set

$$\widehat{u}(s,\gamma)=\gamma+\frac{\gamma^{1-p}}{p}\widetilde{u}(\rho,\gamma), \qquad \rho=\sqrt{\gamma^{p-1}\exp(\gamma^p)}\; s. \tag{3.1}$$

Then, we see that $\widetilde{u}(\rho,\gamma)$ satisfies

$$\begin{cases} \widetilde{u}_{\rho\rho}+\frac{d-1}{\rho}\widetilde{u}_\rho+p\exp\left[-\gamma^p+\gamma^p\left(1+\frac{\gamma^{-p}}{p}\widetilde{u}\right)^p\right]=0, & 0<\rho<\sqrt{\lambda\gamma^{p-1}\exp(\gamma^p)}, \\ \widetilde{u}(0)=0, & \\ \widetilde{u}(\rho)<0, & 0<\rho<\sqrt{\lambda\gamma^{p-1}\exp(\gamma^p)}. \end{cases} \tag{3.2}$$

Concerning the solutions to (3.2), the following lemma holds:

Lemma 3.1. *Let $\widetilde{u}(\rho,\gamma)$ be a solution to (3.2). Then, we have $\widetilde{u}(\cdot,\gamma)\to U(\cdot)$ in $C^1_{loc}([0,\infty))$ as $\gamma\to\infty$, where $U(\rho)$ is a solution to the following equation:*

$$\begin{cases} U_{\rho\rho}+\frac{d-1}{\rho}U_\rho+p\exp[U]=0, & 0<\rho<\infty, \\ U(\rho)=0, & \rho=0, \\ U(\rho)<0, & 0<\rho<\infty. \end{cases} \tag{3.3}$$

Remark 3.1. *We note that Dancer [4] already gave the proof of Lemma 3.1 in more general situations. Here, using an ODE approach, we shall give an alternative proof.*

Proof of Lemma 3.1. First, for each $\rho_0>0$, we shall show that $\widetilde{u}(\rho,\gamma)$ is uniformly bounded for $\rho\in[0,\rho_0)$. Since $\gamma=\|\widehat{u}\|_{L^\infty}$ and $\widehat{u}(\rho,\gamma)$ is positive, (3.1) yields that

$$-p\gamma^p<\widetilde{u}(\rho,\gamma)\le 0. \tag{3.4}$$

By (3.4), we have

$$0<1+\frac{\gamma^{-p}}{p}\widetilde{u}\le 1.$$

This yields that

$$\exp\left[-\gamma^p+\gamma^p\left(1+\frac{\gamma^{-p}}{p}\widetilde{u}\right)^p\right]\le\exp[-\gamma^p+\gamma^p]=1.$$

It follows from the first equation in (3.2) that

$$\widetilde{u}_{\rho\rho}+\frac{d-1}{\rho}\widetilde{u}_\rho\ge -p.$$

This yields that

$$(\rho^{d-1}\widetilde{u}_\rho)_\rho\ge -p\rho^{d-1}.$$

Integrating the above inequality, we have $\rho^{d-1}\widetilde{u}_\rho(\rho) \geq -p\rho^d/d$. Thus, we obtain $\widetilde{u}_\rho(\rho) \geq -p\rho/d$ for $\rho \in [0,\rho)$. Integrating the inequality yields that

$$\widetilde{u}(\rho) \geq \widetilde{u}(0) - \frac{p}{d}\int_0^\rho \tau d\tau = -\frac{p}{2d}\rho^2.$$

Therefore, for $\rho \in [0,\rho_0)$, we have

$$-\frac{p}{2d}\rho_0^2 \leq \widetilde{u}(\rho) \leq 0. \tag{3.5}$$

This together with the equation in (3.2) gives the uniform boundedness of $\widetilde{u}_\rho$ and $\widetilde{u}_{\rho\rho}$ for $\rho \in [0,\rho_0)$. Then, by the Ascoli-Arzela theorem, there exists a function U such that $\widetilde{u}(\rho,\gamma)$ converges to U in $C^1_{\mathrm{loc}}([0,\rho_0))$ as γ goes to infinity. Moreover, by the Taylor expansion, there exists $\theta \in (0,1)$ such that

$$\begin{aligned}
&\left|\exp\left[-\gamma^p + \gamma^p\left(1+\frac{\gamma^{-p}}{p}\widetilde{u}(\rho,\gamma)\right)^p\right] - \exp[U]\right| \\
&= \left|\exp\left[\widetilde{u} + \frac{p-1}{2p}\left(1+\theta\frac{\gamma^{-p}}{p}\widetilde{u}\right)^{p-2}\gamma^{-p}\widetilde{u}^2\right] - \exp[U]\right| \\
&\leq \exp[\widetilde{u}]\left|\exp\left[\frac{p-1}{2p}\left(1+\theta\frac{\gamma^{-p}}{p}\widetilde{u}\right)^{p-2}\gamma^{-p}\widetilde{u}^2\right] - 1\right| + |\exp[\widetilde{u}] - \exp[U]|.
\end{aligned}$$

Therefore, by (3.5), we have

$$\left|\exp\left[-\gamma^p + \gamma^p\left(1+\frac{\gamma^{-p}}{p}\widetilde{u}(\rho,\gamma)\right)^p\right] - \exp[U]\right| \to 0 \qquad \text{as } \gamma \to \infty.$$

This yields that U satisfies (3.3). This completes the proof. □

Next, we put $t = -\log s$. We define $y(t,\gamma)$ by

$$\widehat{u}(s,\gamma) = \varphi^{1/p}(t) + \frac{\varphi^{-A_p}(t)}{p}(\kappa + y(t,\gamma)). \tag{3.6}$$

We see that $y(t,\gamma)$ satisfies the following:

$$\begin{aligned}
&y_{tt} - \left\{(d-2) + 2A_p\varphi^{-1}\varphi_t\right\}y_t - 2(d-2) + p\varphi^{A_p}\exp[-2t + \varphi(1+\frac{\varphi^{-1}}{p}(\kappa+y))^p] \\
&= f_6(t,y)
\end{aligned} \tag{3.7}$$

for sufficiently large $t > 0$, where

$$\begin{aligned}
f_6(t,y) = &A_p\varphi^{-1}(\varphi_t)^2 - \varphi_{tt} - A_p(A_p+1)\varphi^{-2}(\varphi_t)^2(\kappa+y) + A_p\varphi^{-1}\varphi_{tt}(\kappa+y) \\
&+ (d-2)A_p\varphi^{-1}\varphi_t(\kappa+y) + \frac{(d-2)A_p}{t}
\end{aligned} \tag{3.8}$$

For the function $y(t,\gamma)$, we make the following spatial translation:

$$\tau = -\log\rho = t - \frac{\gamma^p}{2} - \frac{(p-1)\log\gamma}{2}, \qquad \widehat{y}(\tau,\gamma) = y(t,\gamma), \qquad \widehat{\varphi}(\tau) = \varphi(t). \tag{3.9}$$

Let U be the solution to (3.3). We put $U_*(\tau) = U(\rho)$ and

$$Y(\tau) = U_*(\tau) - 2\tau - \log \frac{2(d-2)}{p}. \tag{3.10}$$

Then, Y satisfies

$$\begin{cases} Y_{\tau\tau} - (d-2)Y_\tau + 2(d-2)\left\{\exp[Y] - 1\right\} = 0, & -\infty < \tau < \infty, \\ \lim_{\tau\to\infty}\left\{Y(\tau) + 2\tau + \log\frac{2(d-2)}{p}\right\} = 0, & \\ Y(\tau) + 2\tau + \log\frac{2(d-2)}{p} < 0, & -\infty < \tau < \infty. \end{cases} \tag{3.11}$$

Then, the following lemma holds:

Lemma 3.2. *Let $\widehat{y}$ and Y be the functions defined by* (3.10) *and* (3.9), *respectively. Then, we have $\widehat{y}(\tau,\gamma) \to Y(\tau)$ in $C^1_{loc}((-\infty,\infty))$ as $\gamma \to \infty$.*

Proof. It follows from (3.1) and (3.6) that

$$\begin{aligned} \widetilde{u}(\rho,\gamma) &= -p\gamma^p + p\gamma^{p-1}\widehat{u}(s,\gamma) \\ &= -p\gamma^p + p\gamma^{p-1}\left\{\varphi^{1/p}(t) + \frac{\varphi^{-A_p}(t)}{p}(\kappa + y(t,\gamma))\right\} \\ &= p(-\gamma^p + \gamma^{p-1}\widehat{\varphi}^{1/p}(\tau)) + \gamma^{p-1}\widehat{\varphi}^{\,-A_p}(\tau)(\kappa + \widehat{y}(\tau,\gamma)). \end{aligned} \tag{3.12}$$

By (2.2), (3.9) and the Taylor expansion, we have

$$\begin{aligned} &-\gamma^p + \gamma^{p-1}\widehat{\varphi}^{1/p}(\tau) \\ &= -\gamma^p + \gamma^{p-1}\left\{2\tau + \gamma^p + (p-1)\log\gamma - A_p\log\left(\tau + \frac{\gamma^p}{2} + \frac{p-1}{2}\log\gamma\right)\right\}^{\frac{1}{p}} \\ &= -\gamma^p + \gamma^p\left\{\frac{2\tau}{\gamma^p} + 1 - \frac{A_p}{\gamma^p}\log\gamma^{-p} - \frac{A_p}{\gamma^p}\log\left(\tau + \frac{\gamma^p}{2} + \frac{p-1}{2}\log\gamma\right)\right\}^{\frac{1}{p}} \\ &= -\gamma^p + \gamma^p\left\{1 + \frac{2\tau}{\gamma^p} - \frac{A_p}{\gamma^p}\log\left(\frac{\tau}{\gamma^p} + \frac{1}{2} + \frac{(p-1)\log\gamma}{2\gamma^p}\right)\right\}^{\frac{1}{p}} \\ &= \frac{1}{p}\left(2\tau - A_p\log\left(\frac{1}{2} + \frac{\tau}{\gamma^p} + \frac{(p-1)\log\gamma}{2\gamma^p}\right)\right) \\ &\quad + \frac{p-1}{2p^2\gamma^p}\left(1 + \theta\left(\frac{2\tau}{\gamma^p} - \frac{A_p}{\gamma^p}\log\left(\frac{1}{2} + \frac{\tau}{\gamma^p} + \frac{(p-1)\log\gamma}{2\gamma^p}\right)\right)\right)^{\frac{1}{p}-2} \times \\ &\quad \times \left(2\tau + A_p\log\left(\frac{1}{2} + \frac{\tau}{\gamma^p} + \frac{(p-1)\log\gamma}{2\gamma^p}\right)\right)^2 \end{aligned} \tag{3.13}$$

for some $\theta \in (0,1)$. This yields that

$$-\gamma^p + \gamma^{p-1}\widehat{\varphi}^{1/p}(\tau) \to \frac{2\tau}{p} + \frac{A_p}{p}\log 2 \qquad \text{as } \gamma \to \infty \tag{3.14}$$

for each $\tau \in (-\infty,\infty)$. Similarly, we obtain

$$\gamma^{p-1}\widehat{\varphi}^{\,-A_p}(\tau) = \left\{1 + \frac{2\tau}{\gamma^p} - \frac{A_p}{\gamma^p}\log\left(\frac{1}{2} + \frac{\tau}{\gamma^p} + \frac{(p-1)\log\gamma}{2\gamma^p}\right)\right\}^{-A_p} \to 1 \qquad \text{as } \gamma \to \infty. \tag{3.15}$$

(3.12)–(3.15) imply that

$$\lim_{\gamma\to\infty} \widetilde{u}(\rho,\gamma) = 2\tau + A_p \log 2 + \kappa + \lim_{\gamma\to\infty} \widehat{y}(\tau,\gamma). \tag{3.16}$$

It follows from Lemma 3.1 that $\lim_{\gamma\to\infty} \widetilde{u}(\rho,\gamma) = U(\rho) = U_*(\tau)$. Thus, by (2.2), (3.10) and (3.16), we see that

$$\begin{aligned}\lim_{\gamma\to\infty} \widehat{y}(\tau,\gamma) &= -2\tau - A_p \log 2 - \kappa + U_*(\tau)\\ &= -2\tau - A_p \log 2 - \kappa + Y(\tau) + 2\tau + \log \frac{2(d-2)}{p}\\ &= Y(\tau) - \kappa + \log \frac{(d-2)2^{\frac{1}{p}}}{p} = Y(\tau).\end{aligned}$$

This completes the proof. □

Lemma 3.3. *Let Y be a solution to* (3.11). *Then, Y satisfies $(Y, Y_\tau) \to (0,0)$ as $\tau \to -\infty$.*

Proof. We set $Z_1(\tau) = Y(\tau)$ and $Z_2(\tau) = Y_\tau(\tau)$. Then, the pair of functions (Z_1, Z_2) satisfies

$$\begin{cases} \frac{dZ_1}{d\tau} = Z_2, \\ \frac{dZ_2}{d\tau} = (d-2)Z_2 - 2(d-2)\left[\exp[Z_1] - 1\right]. \end{cases} \tag{3.17}$$

We define an energy E by

$$E(\tau) = \frac{(Z_2)^2}{2} + 2(d-2)\left[\exp[Z_1] - 1 - Z_1\right].$$

From the equation (3.17), we have $\frac{dE}{d\tau}(\tau) = (d-2)(Z_2)^2 > 0$. Moreover, $(0,0)$ is an equiblium point of (3.17) and a minimum of the energy E. This yields that $(Z_1(\tau), Z_2(\tau)) \to (0,0)$ as $\tau \to -\infty$. □

We set

$$z_1(t,\gamma) = y(t,\gamma), \qquad z_2(t,\gamma) = y_t(t,\gamma), \tag{3.18}$$

where $y(t,\gamma)$ is the function defined by (3.6). Then, $(z_1(t,\gamma), z_2(t,\gamma))$ satisfies

$$\begin{cases} \frac{dz_1}{dt} &= z_2 & \text{for } t \in (-\frac{\log\lambda(\gamma)}{2}, \infty), \\ \frac{dz_2}{dt} &= (d-2-2A_p\varphi^{-1}\varphi_t)z_2 + 2(d-2) + f_6(t,z_1) & \text{for } t \in (-\frac{\log\lambda(\gamma)}{2}, \infty). \\ & \quad -p\varphi^{A_p}\exp[-2t + \varphi(1+\frac{\varphi^{-1}}{p}(\kappa + z_1(t)))^p] \end{cases} \tag{3.19}$$

From Lemma 3.3, we see that for any $\varepsilon > 0$, there exists $\tau_\varepsilon \in (-\infty, 0)$ such that $|(Z_1(\tau_\varepsilon), Z_2(\tau_\varepsilon))| < \varepsilon/2$, where (Z_1, Z_2) is a solution to (3.17). We fix $\tau_\varepsilon \in (-\infty, 0)$ and put

$$t_\varepsilon = \tau_\varepsilon + \frac{\gamma^p}{2} + \frac{(p-1)\log\gamma}{2}.$$

Then, by Lemma 3.2, we have

$$|(z_1(t_\varepsilon,\gamma), z_2(t_\varepsilon,\gamma))| < \varepsilon \tag{3.20}$$

for sufficiently large $\gamma > 0$. We shall show the following.

Lemma 3.4. *Let $(z_1(t,\gamma), z_2(t,\gamma))$ be the function defined by (3.18). For arbitrary $\varepsilon > 0$, we set*

$$\Gamma_\varepsilon = \left\{(\xi_1,\xi_2) \in \mathbb{R}^2 \mid 2(d-2)\left\{\exp[\xi_1] - 1 - \xi_1\right\} + \frac{\xi_2^2}{2} < \varepsilon\right\}.$$

There exists T_ε which does no depend on γ and t_ε but on ε such that $(z_1(t,\gamma), z_2(t,\gamma)) \in \Gamma_{2\varepsilon}$ for $t \in (T_\varepsilon, t_\varepsilon)$.

Proof. We define an energy by

$$E_1(t) = \frac{z_2^2}{2} + 2(d-2)\left\{\exp[z_1] - 1 - z_1\right\}.$$

By (3.19), we have

$$\begin{aligned}\frac{dE_1}{dt}(t) &= z_2 z_{2t} + 2(d-2)\left\{\exp[z_1] - 1\right\} z_2 \\ &= (d-2-2A_p\varphi^{-1}\varphi_t)z_2^2 \\ &\quad - p\varphi^{A_p}\exp[-2t + \varphi(1+\frac{\varphi^{-1}}{p}(\kappa+z_1)^p]z_2 + f_6(t,z_1)z_2 \\ &\quad + 2(d-2)\exp[z_1]z_2.\end{aligned}$$

Similarly as in (2.5), by the Taylor expansion, we obtain

$$\begin{aligned}&p\varphi^{A_p}\exp[-2t + \varphi(1+\frac{\varphi^{-1}}{p}(\kappa+z_1))^p] \\ &= (d-2)2^{\frac{1}{p}}\varphi^{A_p}t^{-A_p}\exp[z_1]\exp\left[\widetilde{g}_1(t,z_1)\right] \\ &= 2(d-2)\exp[z_1] \\ &\quad - \left(2(d-2)\exp[z_1] - (d-2)2^{\frac{1}{p}}\varphi^{A_p}t^{-A_p}\exp[z_1]\exp[\widetilde{g}_1(t,z_1)]\right),\end{aligned}$$

where

$$\widetilde{g}_1(t,z_1) = \varphi(1+\frac{\varphi^{-1}}{p}(\kappa+z_1))^p - \varphi(t) - \kappa - z_1.$$

Therefore, we have

$$\begin{aligned}\frac{dE_1}{dt}(t) &= (d-2-2A_p\varphi^{-1}\varphi_t)z_2^2 + f_6(t,z_1)z_2 \\ &\quad + \left(2(d-2)\exp[z_1] - (d-2)2^{\frac{1}{p}}\varphi^{A_p}t^{-A_p}\exp[z_1]\exp[\widetilde{g}_1(t,z_1)]\right) z_2.\end{aligned} \tag{3.21}$$

Since Γ_ε is a neighborhood of $(0,0)$, we can take $\varepsilon > 0$ so small such that $\Gamma_{2\varepsilon} \subset \{(x_1,x_2) \mid |x_1| + |x_2| < 1\}$. We choose $T_\varepsilon > 0$ so that

$$0 < \frac{C_*}{\sqrt{T_\varepsilon}} < \frac{\varepsilon}{2}, \tag{3.22}$$

where the constant $C_* > 0$ which does not depend on ε and is defined by (3.26) below. We shall show that $(z_1(t), z_2(t)) \in \Gamma_{2\varepsilon}$ for $t \in (T_\varepsilon, t_\varepsilon)$ by contradiction. Suppose the contrary that $(z_1(t), z_2(t)) \in \Gamma_{2\varepsilon}$ for $t \in (T_\varepsilon, t_\varepsilon]$ and $(z_1(T_\varepsilon), z_2(T_\varepsilon)) \notin \Gamma_{2\varepsilon}$. Then, by (3.21), we have

$$\begin{aligned} &E_1(t_\varepsilon) - E_1(T_\varepsilon) \\ =& \int_{T_\varepsilon}^{t_\varepsilon} (d-2-2A_p\varphi^{-1}\varphi_t) z_2^2 ds + \int_{T_\varepsilon}^{t_\varepsilon} f_6(s, z_1) z_2 ds \\ &+ \int_{T_\varepsilon}^{t_\varepsilon} \Big(2(d-2)\exp[z_1] - (d-2)2^{\frac{1}{p}}\varphi^{A_p}(s)s^{-A_p}\exp[z_1]\exp[\widetilde{g}_1(s,z_1)]\Big) z_2 ds. \end{aligned} \tag{3.23}$$

Since $|z_1(t)| + |z_2(t)| < 1$, we see from (3.8) that there exists a constant $C_1 > 0$ satisfying $|f_6(s, z_1)| \le C_1/|s|$. Furthermore, from (2.2), we have

$$\begin{aligned} &\Big|2(d-2)\exp[z_1] - (d-2)2^{\frac{1}{p}}\varphi^{A_p}(s)s^{-A_p}\exp[z_1]\exp[\widetilde{g}_1(s,z_1)]\Big| \\ &= 2(d-2)\exp[z_1]\Big|1 - \Big(\frac{\varphi(s)}{2}\Big)^{A_p} s^{-A_p}\exp[\widetilde{g}_1(s,z_1)]\Big| \\ &= 2(d-2)\exp[z_1]\Big|1 - \Big(1 - \frac{A_p}{2}\frac{\log s}{s}\Big)^{A_p}\exp[\widetilde{g}_1(s,z_1)]\Big| \\ &\le C\Big|1 - \exp[\widetilde{g}_1(s,z_1)]\Big| + C\Big|1 - \Big(1 - \frac{A_p}{2}\frac{\log s}{s}\Big)^{A_p}\Big|\exp[\widetilde{g}_1(s,z_1)]. \end{aligned} \tag{3.24}$$

Similarly as in the proof of Lemma 2.1, there exists a constant $C > 0$ such that

$$\Big|1 - \Big(1 - \frac{A_p}{2}\frac{\log s}{s}\Big)^{A_p}\Big| \le C\frac{\log s}{s}, \qquad |\widetilde{g}_1(s, z_1)| \le \frac{C}{s}$$

for sufficiently large $s > 0$. This yields together with (3.24) that

$$\Big|2(d-2)\exp[z_1] - (d-2)2^{\frac{1}{p}}\varphi^{A_p}(s)s^{-A_p}\exp[z_1]\exp[g_1(s,z_1)]\Big| \le \frac{C}{s^{\frac{3}{4}}}$$

for some constant $C > 0$. Therefore, by the Young inequality, we have

$$\begin{aligned} &\Big|\int_{T_\varepsilon}^{t_\varepsilon} \Big(2(d-2)\exp[z_1] - (d-2)2^{\frac{1}{p}}\varphi^{A_p}s^{-A_p}\exp[z_1]\exp[g_1(s,z_1)]\Big) z_2 ds\Big| \\ &+ \Big|\int_{T_\varepsilon}^{t_\varepsilon} f_6(s,z_1) z_2 ds\Big| \\ \le& \int_{T_\varepsilon}^{t_\varepsilon} \frac{C}{s^{\frac{3}{4}}} z_2 ds \\ \le& \frac{2C^2}{d-2}\int_{T_\varepsilon}^{t_\varepsilon} \frac{1}{s^{\frac{3}{2}}} ds + \frac{(d-2)}{2}\int_{T_\varepsilon}^{t_\varepsilon} |z_2|^2 ds \\ \le& \frac{4C^2}{(d-2)\sqrt{T_\varepsilon}} + \frac{d-2}{2}\int_{T_\varepsilon}^{t_\varepsilon} |z_2|^2 ds. \end{aligned} \tag{3.25}$$

We set

$$C_* = \frac{4C^2}{d-2}. \tag{3.26}$$

Then, it follows from (3.22) and (3.25) that

$$\begin{aligned}
&\left|\int_{T_\varepsilon}^{t_\varepsilon} \left(2(d-2)\exp[z_1] - (d-2)2^{\frac{1}{p}}\varphi^{A_p}s^{-A_p}\exp[z_1]\exp[g_1(s,z_1)]\right) z_2 ds\right| \\
&+ \left|\int_{T_\varepsilon}^{t_\varepsilon} f_1(s,z_1)z_2 ds\right| \\
&\le \frac{C_*}{\sqrt{T_\varepsilon}} + \frac{d-2}{2}\int_{T_\varepsilon}^{t_\varepsilon} |z_2|^2 ds \le \frac{\varepsilon}{2} + \frac{d-2}{2}\int_{T_\varepsilon}^{t_\varepsilon} |z_2|^2 ds.
\end{aligned} \tag{3.27}$$

Moreover, we take $T_\varepsilon > 0$ so that $|2A_p\varphi^{-1}(t)\varphi_t(t)| < (d-2)/2$ for $t > T_\varepsilon$. Then, we have

$$\int_{T_\varepsilon}^{t_\varepsilon} (d-2-2A_p\varphi^{-1}\varphi_t)z_2^2 ds \ge \frac{d-2}{2}\int_{T_\varepsilon}^{t_\varepsilon} |z_2|^2 ds. \tag{3.28}$$

It follows from (3.23), (3.27) and (3.28) that

$$E_1(t_\varepsilon) - E_1(T_\varepsilon) \ge \frac{d-2}{2}\int_{T_\varepsilon}^{t_\varepsilon} |z|_2^2 ds - \frac{\varepsilon}{2} - \frac{d-2}{2}\int_{t_\varepsilon}^{T_\varepsilon} |z_2|^2 ds > -\frac{\varepsilon}{2}.$$

This together with (3.20) and $(z_1(T_\varepsilon), z_2(T_\varepsilon)) \notin \Gamma_{2\varepsilon}$ implies that

$$2\varepsilon \le E(T_\varepsilon) < E(t_\varepsilon) + \frac{\varepsilon}{2} = \frac{3}{2}\varepsilon,$$

which is a contradiction. Therefore, our assertion holds. □

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Let $\{\gamma_n\}_{n=1}^\infty \subset \mathbb{R}_+$ be a sequence satisfying $\lim_{n\to\infty}\gamma_n = \infty$. Let $(z_1(t,\gamma_n), z_2(t,\gamma_n))$ be the function defined by (3.18). By Lemma 3.4, we find that $(z_1(t,\gamma_n), z_2(t,\gamma_n))$ is uniformly bounded in the interval $(T_\varepsilon, t_\varepsilon)$. This together with (3.7) implies that $y_{tt}(t,\gamma_n)$ is also uniformly bounded in the interval $(T_\varepsilon, t_\varepsilon)$. Differentiating the equation (3.7) implies that $y_{ttt}(t,\gamma)$ is also uniformly bounded in $(T_\varepsilon, t_\varepsilon)$. This yields that $(z_1(t,\gamma_n), z_2(t,\gamma_n))$ and $(z_{1t}(t,\gamma_n), z_{2t}(t,\gamma_n))$ are equicontinuous. Thus, it follows from the Ascoli-Arzela theorem that there exists a subsequence $\{(z_1(t,\gamma_n), z_2(t,\gamma_n))\}$ (we still denote by the same letter) and a pair of functions $(z_{*,1}(t), z_{*,2}(t))$ in $(C^1(T_\varepsilon, t_\varepsilon))^2$ as n tends to infinity, Since $t_\varepsilon (> T_\varepsilon)$ is arbitrary, we find that $(z_1(t,\gamma_n), z_2(t,\gamma_n))$ converges to $(z_{*,1}(t), z_{*,2}(t))$ in $(C^1(T_\varepsilon, \infty))^2$ as n goes to infinity. We note $0 < \lambda(\gamma_n) < \lambda_1$, where λ_1 is the first eigenvalue of the operator $-\Delta$ in B_1 with the Dirichlet boundary condition. Thus, there exists $\lambda_* \ge 0$ such that $\lambda(\gamma_n) \to \lambda_*$ as n tends to infinity. By the result of Dancer [4], we see that $\lambda_* > 0$. From these, we see that $(z_{*,1}, z_{*,2}, \lambda_*)$ satisfies

$$\begin{cases}
\frac{dz_1}{dt} = z_2 \qquad \text{for } t \in (-\frac{\log\lambda_*}{2}, \infty), \\
\frac{dz_2}{dt} = (d-2-2A_p\varphi^{-1}\varphi_t)z_2 + 2(d-2) + f_6(t,z_1) \\
\qquad -p\varphi^{A_p}\exp[-2t + \varphi(1 + \frac{\varphi^{-1}}{p}(\kappa + z_1(t)))^p \qquad \text{for } t \in (-\frac{\log\lambda_*}{2}, \infty)
\end{cases}$$

We shall show that

$$z_{*,1}(t) \to 0 \qquad \text{as } t \to \infty. \tag{3.29}$$

Let us admit (3.29) for a moment and continue to prove. We set

$$\eta_*(t) = \varphi^{\frac{1}{p}}(t) + \frac{\varphi^{-A_p}(t)}{p}(\kappa + z_*(t)) - (\varphi(t) + \kappa)^{\frac{1}{p}}.$$

Then, we see that η_* satisfies (2.3). Moreover, it follows that

$$\begin{aligned}\eta_*(t) &= \varphi^{\frac{1}{p}}(t) + \frac{\varphi^{-A_p}(t)}{p}(\kappa + z_*(t)) - \varphi^{\frac{1}{p}}(t) - \kappa\frac{\varphi^{-A_p}(t)}{p} \\ &\quad - \frac{1}{2p}\left(\frac{1}{p} - 1\right)(1 + \theta_*\kappa\varphi^{-1}(t))^{\frac{1}{p}-2}(\kappa\varphi^{-1}(t))^2 \\ &= \frac{\varphi^{-A_p}(t)}{p}z_*(t) - \frac{1}{2p}\left(\frac{1}{p} - 1\right)(1 + \theta_*\kappa\varphi^{-1}(t))^{\frac{1}{p}-2}(\kappa\varphi^{-1}(t))^2\end{aligned}$$

for some $\theta_* \in (0,1)$. This together with (3.29) implies that $\eta_* \in \Sigma$, where the function space Σ is defined by (2.17). From Theorem 2.1, there exists a unique solution η_∞ to (2.3) in Σ. Therefore, we have $\eta_*(t) = \eta_\infty(t)$. This yields that $\lambda_* = \lambda_{p,\infty}$.

Thus, all we have to do is to prove (3.29). Suppose the contrary that there exists $\delta > 0$ and $\{t_k\} \subset \mathbb{R}_+$ such that $|z_{*,1}(t_k)| \geq \delta$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} t_k = \infty$. Then, there exists $k_0 \in \mathbb{N}$ such that $t_{k_0} > T_\varepsilon$. Then, we see that $|z_1(t_{k_0}, \gamma)| \geq \delta/2$ for sufficiently large $\gamma > 0$. We choose $\varepsilon = \delta/4$. It follows from (3.20) that $(z_1(\tau_\varepsilon + \frac{\gamma^p}{2} + \frac{p-1}{2}\log\gamma, \gamma), z_2(\tau_\varepsilon + \frac{\gamma^p}{2} + \frac{p-1}{2}\log\gamma, \gamma)) \in \Gamma_\varepsilon$. By Lemma 3.4, we see that $(z_1(t,\gamma), z_2(t,\gamma)) \in \Gamma_{2\varepsilon} = \Gamma_{\delta/2}$ for $t \in (T_\varepsilon, \tau_\varepsilon + \frac{\gamma^p}{2} + \frac{p-1}{2}\log\gamma)$. We can take $\gamma > 0$ sufficiently large so that $t_{k_0} \in (T_\varepsilon, \tau_\varepsilon + \frac{\gamma^p}{2} + \frac{p-1}{2}\log\gamma)$, which is a contradiction. This completes the proof. □

4 Infinitely many regular solutions in case of $3 \leq d \leq 9$

In this section, following Guo and Wei [8] and Miyamoto [14, 15], we shall give a proof of Theorem 1.3. More precisely, we count a intersection number of the singular solution and regular ones. Let I be an interval in $\mathbb{R}$. For a function $v(s)$ on I, we define a number of zeros of v by

$$\mathcal{Z}_I[v(\cdot)] = \#\{s \in I \mid v(s) = 0\}.$$

Then the following result is known.

Proposition 4.1. *Let $U(\rho)$ be a solution to* (3.3). *We define a function V by*

$$V(\rho) = -2\log\rho + \log\frac{2(d-2)}{p}. \tag{4.1}$$

Then, in case of $3 \leq d \leq 9$, we have

$$Z_{[0,\infty)}[U(\rho) - V(\rho)] = \infty.$$

See Nagasaki and Suzuki [17] or Miyamoto [15] for a proof of Proposition 4.1.

Remark 4.1. *We can easily check that V defined by* (4.1) *is a singular solution to the equation in* (3.3).

We set

$$\widehat{W}_p(s) = \varphi^{\frac{1}{p}}(t) + \frac{\varphi^{-A_p}}{p}(\kappa + y_\infty(t)), \tag{4.2}$$

where $t = -\log s$ and

$$y_\infty(t) = p\varphi^{A_p}\left((\varphi+\kappa)^{\frac{1}{p}} - \varphi^{\frac{1}{p}}\right) + p\varphi^{A_p}\eta_\infty - \kappa.$$

Here, η_∞ is the solution to (2.3) given by Theorem 2.1. Then, it follows from Theorem 2.1 that $\lim_{t\to\infty} y_\infty(t) = 0$. Thus, we see that $\widehat{W}_p$ is a singular solution to (1.5) with $\lambda = \lambda_{p,\infty}$. Using Proposition 4.1, we shall show the following:

Lemma 4.2. *Let $\widehat{u}(s,\gamma)$ be a regular solution to* (1.5) *with $\widehat{u}(0) = \gamma$. Then, we have*

$$Z_{I_\gamma}\left[\widehat{u}(\cdot,\gamma) - \widehat{W}_p(\cdot)\right] \to \infty \qquad \text{as } \gamma \to \infty, \tag{4.3}$$

where $I_\gamma = [0, \min\{\sqrt{\lambda_{p,\infty}}, \sqrt{\lambda(\gamma)}\})$.

Proof. We put

$$\widetilde{u}_*(\rho,\gamma) = -p\gamma^p + p\gamma^{p-1}\widehat{W}_p(s), \qquad \rho = \sqrt{\gamma^{p-1}\exp(\gamma^p)}s, \tag{4.4}$$

where $\widehat{W}_p$ is defined by (4.2). We claim that

$$\widetilde{u}_*(\rho,\gamma) \to V(\rho) \quad \text{in } C^1_{\text{loc}}([0,\infty)) \qquad \text{as } \gamma \to \infty. \tag{4.5}$$

It follows from (4.2) and (4.4) that

$$\widetilde{u}_*(\rho,\gamma) = -p\gamma^p + p\gamma^{p-1}\widehat{W}_p(s) = -p\gamma^p + p\gamma^{p-1}\varphi^{\frac{1}{p}}(t) + \gamma^{p-1}\varphi^{-A_p}(t)(\kappa + y_\infty(t)).$$

We fix $\rho > 0$. Then, it follows that

$$t = -\log s = -\log\rho + \frac{\gamma^p}{2} + \frac{(p-1)\log\gamma}{2} \to \infty \qquad \text{as } \gamma \to \infty.$$

This implies that

$$y_\infty(t) \to 0 \qquad \text{as } \gamma \to \infty. \tag{4.6}$$

Similarly as in (3.14), (3.15) together with (4.6), we obtain

$$\begin{aligned}\widetilde{u}_*(\rho,\gamma) &= -p\gamma^p + p\gamma^{p-1}\varphi^{\frac{1}{p}}(t) + \gamma^{p-1}\varphi^{-A_p}(t)(\kappa + y_\infty(t)) \\ &\to -2\log\rho + \log\frac{2(d-2)}{p} = V(\rho) \qquad \text{as } \gamma \to \infty.\end{aligned}$$

Therefore, (4.5) holds.

It follows from (3.1) and (4.4) that

$$Z_{I_\gamma}\left[\widehat{u}(s,\gamma) - \widehat{W}_p(s)\right] = Z_{J_\gamma}\left[\widetilde{u}(\rho,\gamma) - \widetilde{u}_*(\rho,\gamma)\right] \tag{4.7}$$

where $J_\gamma = [0, \sqrt{\gamma^{p-1}\exp(\gamma^p)\min\{\sqrt{\lambda_{p,\infty}}, \sqrt{\lambda(\gamma)}\}})$. Combining Lemma 3.1, Proposition 4.1 and (4.5), we find that

$$\lim_{\gamma\to\infty} Z_{J_\gamma}\left[\widetilde{u}(\rho,\gamma) - \widetilde{u}_*(\rho,\gamma)\right] = Z_{[0,\infty)}\left[U(\rho) - V(\rho)\right] = \infty, \tag{4.8}$$

From (4.7) and (4.8), we obtain the desired result. □

Once we obtain Lemma 4.2, we can prove Theorem 1.3 by employing the same argument as Miyamoto [15, Lemma 5]. However, for the sake of reader's convenience, we shall give a proof.

Proof of Theorem 1.3. Let $\widehat{u}(s,\gamma)$ be a solution to (1.5) with $\widehat{u}(0)=\gamma$ and $\widehat{W_p}(s)$ be the singular solution defined by (4.2). We put $\widehat{v}(s,\gamma)=\widehat{u}(s,\gamma)-\widehat{W_p}(s)$. Then, $\widehat{v}(s,\gamma)$ satisfies the following ordinary differential equation:

$$\widehat{v}_{ss}+\frac{d-1}{s}\widehat{v}_s+e^{(\widehat{v}+W_p)^p}-e^{W_p^p}=0, \qquad 0<s<\widehat{\lambda}(\gamma),$$

where $\widehat{\lambda}(\gamma)=\min\{\sqrt{\lambda_{p,\infty}},\sqrt{\lambda(\gamma)}\}$. Then, if $\widehat{v}(s,\gamma)$ has a zero at s_0, we have

$$\widehat{v}(s_0,\gamma)=0, \qquad \widehat{v}_s(s_0,\gamma)\neq 0 \tag{4.9}$$

from the uniqueness of a solution. Moreover, for each $\gamma>0$, $\widehat{v}(s,\gamma)$ has at most finitely many zeros in $(0,\widehat{\lambda}(\gamma))$. Indeed, if it is not, there exist a sequence of $\{s_n\}\subset[0,\widehat{\lambda}(\gamma)]$ and $s_*>0$ such that $\lim_{n\to\infty}s_n=s_*$. Then, we see that $\widehat{v}(s_*,\gamma)=\widehat{v}_s(s_*,\gamma)=0$, which is a contradiction. In addition, it follows from (4.9) and the implicit function theorem that each zeros depends continuously on γ. Therefore, we find that the number of zeros of $\widehat{v}(s,\gamma)$ does not change unless another zero enters from the boundary of the interval $[0,\widehat{\lambda}(\gamma)]$. We note that $\widehat{v}(0,\gamma)=\widehat{u}(0,\gamma)-\widehat{W_p}(0)=-\infty$. From this, we find that zero of $\widehat{v}(s,\gamma)$ enter the interval $[0,\widehat{\lambda}(\gamma)]$ from $s=\widehat{\lambda}(\gamma)$ only.

In order to prove Theorem 1.3, it is enough to show that the function $\lambda(\gamma)$ oscillates infinitely many times around $\lambda_{p,\infty}$ as $\gamma\to\infty$. Suppose that there exists $\gamma_0>0$ such that $\lambda(\gamma)>\lambda_{p,\infty}$ for all $\gamma>\gamma_0$. Then, we have $\widehat{\lambda}(\gamma)=\sqrt{\lambda_{p,\infty}}$ for all $\gamma>\gamma_0$. Then we see that $\widehat{v}(\sqrt{\lambda_{p,\infty}})=\widehat{u}(\sqrt{\lambda_{p,\infty}},\gamma)-W_p(\sqrt{\lambda_{p,\infty}})=\widehat{u}(\sqrt{\lambda_{p,\infty}},\gamma)>0$. This implies that the number of zeros cannot increase. This contradicts with (4.3). Next, suppose that there exists $\gamma_1>0$ such that $\lambda(\gamma)<\lambda_{p,\infty}$ for all $\gamma>\gamma_1$. By the same argument as above, we can derive a contradiction. These imply that the function $\lambda(\gamma)$ oscillates infinitely many times around $\lambda_{p,\infty}$. □

5 Finiteness of the Morse index in case of $d\geq 11$

In this section, we investigate the Morse index of the singular solution in case of $d\geq 11$. It is enough to restrict ourselves to radially symmetric functions. Let $\widehat{W_p}$ be the singular solution to (1.5). The following lemma is a key for the proof of Theorem 1.4.

Lemma 5.1. *Assume that $d\geq 11$ and $p>0$. Then, there exists $\rho_1>0$ such that*

$$p\widehat{W_p}^{p-1}(s)e^{\widehat{W_p}^p(s)}<\frac{(d-2)^2}{4s^2} \qquad \text{for } 0<s<\rho_1. \tag{5.1}$$

Proof. We set $\overline{W}_p(t) = \widehat{W}_p(s)$ and $t = -\log s$. From the proof of Theorems 1.1 and 1.2, the singular solution $\overline{W}_p(t)$ can be written as follows:

$$\overline{W}_p(t) = \varphi^{\frac{1}{p}}(t) + \frac{\varphi^{-A_p}(t)}{p}(\kappa + y_*(t)),$$

where $\lim_{t\to\infty} y_*(t) = 0$. Then, for any $\varepsilon > 0$, there exists $t_1 = t_1(\varepsilon) > 0$ such that

$$\overline{W}_p^{\,p}(t) \le 2t - A_p \log t + \kappa + \varepsilon, \qquad \overline{W}_p^{\,p-1}(t) \le (2t)^{A_p}(1+\varepsilon) \qquad \text{for } t \ge t_1.$$

This yields that

$$\begin{aligned} p\overline{W}_p^{p-1}(t)e^{\overline{W}_p^p(t)} \le p(2t)^{A_p}(1+\varepsilon)e^{2t-A_p\log t+\kappa+\varepsilon} &= p2^{A_p}(1+\varepsilon)e^{2t}\frac{(d-2)2^{\frac{1}{p}}}{p}e^{\varepsilon} \\ &= 2(d-2)(1+\varepsilon)e^{\varepsilon}e^{2t}. \end{aligned}$$

We note that $2(d-2) < (d-2)^2/4$ if $d \ge 11$. Therefore, we can take $\varepsilon > 0$ sufficiently small so that

$$p\overline{W}_p^{\,p-1}(t)e^{\overline{W}_p^{\,p}(t)} < \frac{(d-2)^2}{4}e^{2t}.$$

Thus, we see that (5.1) holds for $0 < s < \rho_1$ with $\rho_1 = e^{-t_1}$. □

We are now in a position to prove Theorem 1.4.

Proof of Theorem 1.4. It is enough to show that the number of negative eigenvalues of the operator L_∞ on $H^1_{0,\mathrm{rad}}(B_{\sqrt{\lambda_*}})$ is finite, where $L_\infty = -\Delta - p\widehat{W}_p^{p-1}(s)e^{\widehat{W}_p^p}$. We define smooth functions χ_1 and χ_2 on $[0, \sqrt{\lambda_*})$ by

$$\chi_1(s) = \begin{cases} 1 & (0 \le s < \rho_1/2), \\ 0 & (\rho_1 < s < \sqrt{\lambda_*}), \end{cases} \qquad 0 \le \chi_1(s) \le 1 \quad (0 \le s \le \sqrt{\lambda_*})$$

and $\chi_2(s) = 1 - \chi_1(s)$. For each $\widehat{\phi} \in H^1_{0,\mathrm{rad}}(B_{\sqrt{\lambda_*}})$, we have

$$\begin{aligned} \langle L_\infty\widehat{\phi}, \widehat{\phi}\rangle &= \omega_{d-1}\int_0^{\sqrt{\lambda_*}} \left\{ \left|\frac{d\widehat{\phi}}{ds}\right|^2 - p\widehat{W}_p^{p-1}e^{\widehat{W}_p^p(s)}|\widehat{\phi}|^2 \right\} s^{d-1}ds \\ &= \omega_{d-1}\int_0^{\sqrt{\lambda_*}} \left\{ \left|\frac{d\widehat{\phi}}{ds}\right|^2 - p(\chi_1(s)+\chi_2(s))\widehat{W}_p^{p-1}e^{\widehat{W}_p^p(s)}|\widehat{\phi}|^2 \right\} s^{d-1}ds \\ &\ge \omega_{d-1}\int_0^{\rho_1} \left\{ \left|\frac{d\widehat{\phi}}{ds}\right|^2 - p\widehat{W}_p^{p-1}e^{\widehat{W}_p^p(s)}|\widehat{\phi}|^2 \right\} s^{d-1}ds \\ &\quad + \omega_{d-1}\int_0^{\sqrt{\lambda_*}} \left\{ \left|\frac{d\widehat{\phi}}{ds}\right|^2 - p\chi_2(s)\widehat{W}_p^{p-1}e^{\widehat{W}_p^p(s)}|\widehat{\phi}|^2 \right\} s^{d-1}ds, \end{aligned} \tag{5.2}$$

where ω_{d-1} is the volume of the unit ball in $\mathbb{R}^{d-1}$. By (5.1) and the Hardy inequality, we obtain

$$\int_0^{\rho_1}\left\{\left|\frac{d\widehat{\phi}}{ds}\right|^2-p\widehat{W}_p^{p-1}e^{\widehat{W}_p^p(s)}|\widehat{\phi}|^2\right\}s^{d-1}ds\geq\int_0^{\rho_1}\left\{\left|\frac{d\widehat{\phi}}{ds}\right|^2-\frac{(d-2)^2}{4s^2}|\widehat{\phi}|^2\right\}s^{d-1}ds\geq 0.$$

This together with (5.2) yields that

$$\langle\widehat{L}\widehat{\phi},\widehat{\phi}\rangle\geq\omega_{d-1}\int_0^{\sqrt{\lambda_*}}\left\{\left|\frac{d\widehat{\phi}}{ds}\right|^2-p\chi_2(s)\widehat{W}_p^{p-1}e^{\widehat{W}_p^p(s)}|\widehat{\phi}|^2\right\}s^{d-1}ds. \tag{5.3}$$

We note that the potential $p\chi_2(s)\widehat{W}_p^{p-1}e^{\widehat{W}_p^p(s)}$ is bounded. Therefore, we find that

$$\inf_{\phi\in H^1_{0,\mathrm{rad}}(B_{\sqrt{\lambda_*}}),\ \|\phi\|_{L^2}=1}\left\{\omega_{d-1}\int_0^{\sqrt{\lambda_*}}\left[\left|\frac{d\widehat{\phi}}{ds}\right|^2-p\chi_2(s)\widehat{W}_p^{p-1}e^{\widehat{W}_p^p(s)}|\widehat{\phi}|^2\right]s^{d-1}ds\right\}>-\infty.$$

This together with (5.3) implies that the number of the negative eigenvalues of the operator L_∞ is finite. This completes the proof. □

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