An introduction to the finite and infinite dimensional reduction method

Manuel del Pino
Departamento de Ingeniería Matemática and CMM,
Universidad de Chile, Casilla 170 Correo 3,
Santiago, Chile
delpino@dim.uchile.cl

Juncheng Wei
Department of Mathematics,
University of British Columbia,
Vancouver, BC V6T 1Z2, Canada
jcwei@math.ubc.ca

We give an introductory description of the two gluing methods: finite dimensional and infinite dimensional. In each case we use a model problem to illustrate the ideas.

1. *Part I: Finite-dimensional reduction method*

1.1. *Introduction: What is finite dimensional Liapunov-Schmidt reduction method?*

We briefly introduce the abstract set-up of the finite dimensional Lyapunov-Schmidt reduction (although it is always used in a framework that occurs often in bifurcation theory).

Let X, Y be Banach spaces and $S(u)$ be a C^1 nonlinear map from X to Y. To find a solution to the nonlinear equation

$$S(u) = 0, \tag{1.1}$$

a natural way is to find approximations first and then to look for genuine solutions as (small) perturbations of approximations. Assume that U_λ are the approximations, where $\lambda \in \Lambda$ is the parameter (we think of Λ as the configuration space). Writing $u = U_\lambda + \phi$, then solving $S(u) = 0$ amounts to solving

$$L[\phi] + E + N(\phi) = 0, \tag{1.2}$$

where

$$L[\phi] = S'(U_\lambda)[\phi], \; E = S(U_\lambda), \text{ and } N(\phi) = S(U_\lambda + \phi) - S(U_\lambda) - S'(U_\lambda)[\phi].$$

Here $S'(U_\lambda)$ stands for the Fréchet derivative of S at U_λ, E denotes the error of approximation, and $N(\phi)$ denotes the nonlinear term. In order to solve (1.2), we try to invert the linear operator L so that we can rephrase the problem as a fixed point problem. That is, when L has a uniformly bounded inverse in a suitable space, one can rewrite the equation (1.2) as

$$\phi = -L^{-1}[E + N(\phi)] = \mathcal{A}(\phi).$$

What is left is to use fixed point theorems such as contraction mapping theorem.

The finite dimensional Lyapunov-Schmidt reduction deals with the situation when the linear operator L is Fredholm and its eigenfunction space associated to small eigenvalues has finite dimensional. Assuming that $\{\mathcal{Z}_1, \ldots, \mathcal{Z}_n\}$ is a basis of the eigenfunction space associated to small eigenvalues of L, we can divide the procedure of solving (1.2) into two steps:
[(i)] solving the projected problem for any $\lambda \in \Lambda$,

$$\begin{cases} L[\phi] + E + N(\phi) = \sum\limits_{j=1}^{n} c_j \mathcal{Z}_j, \\ \langle \phi, \mathcal{Z}_j \rangle = 0, \; \forall\, j = 1, \ldots, n, \end{cases}$$

where c_j may be constant or function depending on the form of $\langle \phi, \mathcal{Z}_j \rangle$.

[(ii)] solving the reduced problem

$$c_j(\lambda) = 0, \; \forall\, j = 1, \ldots, n,$$

by adjusting λ in the configuration space.

The original finite dimensional Liapunov-Schmidt reduction method was first introduced in a seminal paper by Floer and Weinstein [27] in their construction of single bump solutions to one dimensional nonlinear Schrodinger equations (Oh [54] generalized to high dimensional case)

$$\epsilon^2 \Delta u - V(x)u + u^p = 0, u > 0, \; u \in H^1(\mathbb{R}^N). \tag{1.3}$$

On the other hand, Bahri [3] and Bahri-Coron [4] developed the reduction method for critical exponent problems. In the last fifteen years, there are renewed efforts in refining the finite dimensional reduction method by many authors. When combined with variational methods, this reduction becomes "localized energy method". For subcritical exponent problems, we refer to Ambrosetti-Malchiodi [1], Gui-Wei [28], Malchiodi [48], Li-Nirenberg [41], Lin-Ni-Wei [42], Ao-Wei-Zeng [2], Wei-Yan [63] and the references therein. The localized energy method in degenerate setting is done by Byeon-Tanaka [6, ?]. For critical exponents, we refer to Bahri-Li-Rey [5], Del Pino-Felmer-Musso [17], Del Pino-Kowalczyk-Musso [18], Li-Wei-Xu [40], Rey-Wei [56, ?] and Wei-Yan [64] and the references therein. Many new features of the finite dimensional reduction are found in the references mentioned.

In the following we shall use the model problem (1.3) to give an introductory description of this method.

1.2. *Model Problem: Schrodinger equation in dimension N*

We start with the following model problem to illustrate the idea of finite dimensional reduction method:

$$\begin{cases} \varepsilon^2\Delta u - V(x)u + u^p = 0 & \text{in } \mathbb{R}^N \\ 0 < u \ \text{ in } \mathbb{R}^N, \quad u(x)\to 0, & \text{as } |x|\to\infty. \end{cases} \tag{1.4}$$

The basic assumption on the exponent is that $1 < p < \infty$ if $N \le 2$, and $1 < p < \frac{N+2}{N-2}$ if $N \ge 3$. (More general nonlinearity can be dealt with similarly.) Without loss of generality we assume that the function $V(x)$ is a positive function satisfying

$$0 < \alpha \le V(x) \le \beta < +\infty. \tag{1.5}$$

The basic building block that we consider is

$$\begin{cases} \Delta w - w + w^p = 0 & \text{in } \mathbb{R}^N \\ 0 < w \text{ in } \mathbb{R}^N, \quad w(x)\to 0, & \text{as } |x|\to\infty \end{cases} \tag{1.6}$$

We look for a solution $w = w(|x|)$, a radially symmetric solution. $w(r)$ satisfies the ordinary differential equation

$$\begin{cases} w'' + \frac{N-1}{r}w' - w + w^p = 0 & r\in(0,\infty) \\ w'(0) = 0,\ 0 < w \text{ in } (0,\infty) \ w(|x|)\to 0, & \text{as } |x|\to\infty \end{cases} \tag{1.7}$$

We collect the following basic properties of w, whose proof can be found in the appendix of the book [62].

Proposition 1.1: *(a) There exist a solution $w(r)$ to (1.7);*

(b) $w(r)$ satisfies the decay estimate $w(r) = A_0 r^{-\frac{N-1}{2}} e^r (1 + O(\frac{1}{r}))$;

(c) $w(r)$ is nondegenerate, i.e., the only bounded solution to

$$L(\phi) = \Delta\phi + pw(x)^{p-1}\phi - \phi = 0, \quad \phi \in L^\infty(\mathbb{R}^N) \tag{1.8}$$

is a linear combination of the functions $\frac{\partial w}{\partial x_j}(x)$, $j = 1, \dots, N$.

We want to solve the problem

$$\begin{cases} \varepsilon^2 \Delta\tilde{u} - V(x)\tilde{u} + \tilde{u}^p = 0 & \text{in} \quad \mathbb{R}^N \\ \quad 0 < \tilde{u} \quad \text{in } \mathbb{R}^N & \tilde{u}(x) \to 0, \quad \text{as} \quad |x| \to \infty \end{cases} \tag{1.9}$$

We fix a point $\xi \in \mathbb{R}^N$. Observe that $U_{\varepsilon,\xi}(y) := V(\xi)^{\frac{1}{p-1}} w\left(\sqrt{V(\xi)}\frac{y-\xi}{\varepsilon}\right)$, is a solution of the rescaled equation

$$\varepsilon^2 \Delta u - V(\xi)u + u^p = 0.$$

We will look for a solution of (1.9) such $u_\varepsilon(x) \approx U_{\varepsilon,\xi}(y)$ for some $\xi \in \mathbb{R}^N$. We define $w_\lambda = \lambda^{\frac{1}{p-1}} w(\sqrt{\lambda}x)$.

Let us observe that if $\tilde{u}$ satisfies (1.9), then $u(x) = \tilde{u}(\varepsilon z)$ satisfies the problem

$$\begin{cases} \Delta u - V(\varepsilon z)u + u^p = 0 & \text{in} \quad \mathbb{R}^N \\ \quad 0 < u \quad \text{in } \mathbb{R}^N & u(x) \to 0, \ \text{as} \ |x| \to \infty \end{cases} \tag{1.10}$$

Let $\xi' = \frac{\xi}{\varepsilon}$. We want a solution of (1.10) with the form $u(z) = w_\lambda(z - \xi') + \tilde{\phi}(z)$, with $\lambda = V(\xi)$ and $\tilde{\phi}$ being small compared with $w_\lambda(z - \xi')$.

1.3. *Equation in terms of* ϕ.

Let $\phi(x) = \tilde{\phi}(x - \xi')$. Then ϕ satisfies the equation

$$\Delta_x[w_\lambda(x) + \phi(x)] - V(\xi + \varepsilon x)[w_\lambda(x) + \phi(x)] + [w_\lambda(x) + \phi(x)]^p = 0.$$

We can write this equation as

$$\Delta\phi - V(\xi)\phi + pw_\lambda^{p-1}(x)\phi - E + B(\phi) + N(\phi) = 0 \tag{1.11}$$

where $E = (V(\xi + \varepsilon x) - V(\xi))w_\lambda(x)$, $B(\phi) = (V(\xi) - V(\xi + \varepsilon x))\phi$ and $N(\phi) = (w_\lambda + \phi)^p - w_\lambda^p - pw_\lambda^{p-1}\phi$.

We first consider the linear problem for $\lambda = V(\xi)$,

$$\begin{cases} L(\phi) = \Delta\phi - V(\xi + \varepsilon x)\phi + pw_\lambda(x)\phi = g - \sum_{i=1}^N c_i \frac{\partial w}{\partial x_i} \\ \qquad\qquad\qquad\qquad \int_{\mathbb{R}^N} \phi \frac{\partial w_\lambda}{\partial x_i} = 0, \quad i = 1, \dots, N \end{cases} \tag{1.12}$$

The $c_i' s$ are defined such that

$$\int_{\mathbb{R}^N} (L(\phi) - g)\frac{\partial w_\lambda}{\partial x_i} dx = 0, i = 1, \dots, N \tag{1.13}$$

which is equivalent to

$$\int_{\mathbb{R}^N} (L(\frac{\partial w_\lambda}{\partial x_i})\phi - g\frac{\partial w_\lambda}{\partial x_i})dx = 0, i = 1, \dots, N \tag{1.14}$$

Denoting

$$L_0(\phi) = \Delta\phi - V(\xi)\phi + pw_\lambda(x)\phi$$

and using the fact that

$$L_0(\frac{\partial w_\lambda}{\partial x_i}) = 0$$

we see that (1.14) can be further simplified as follows

$$\int_{\mathbb{R}^N} ((V(\xi) - V(\xi + \epsilon x))\frac{\partial w_\lambda}{\partial x_i}\phi - g\frac{\partial w_\lambda}{\partial x_i})dx = 0, i = 1, \dots, N \tag{1.15}$$

Since

$$\int_{\mathbb{R}^N} \frac{\partial w_\lambda}{\partial x_i}\frac{\partial w_\lambda}{\partial x_j} = \int_{\mathbb{R}^N} (\frac{\partial w}{\partial x_1})^2 \delta_{ij}$$

we find that

$$c_i = \frac{\int_{\mathbb{R}^N} ((V(\xi) - V(\xi + \epsilon x))\frac{\partial w_\lambda}{\partial x_i}\phi - g\frac{\partial w_\lambda}{\partial x_i})dx}{\int_{\mathbb{R}^N} (\frac{\partial w_\lambda}{\partial x_1})^2}, i = 1, \dots, N \tag{1.16}$$

In the following we shall solve the following:

Problem: Given $g \in L^\infty(\mathbb{R}^N)$ we want to find $\phi \in L^\infty(\mathbb{R}^N)$ solution to the problem (1.12)-(1.16).

1.4. *A priori estimates of a linear problem*

Let us assume that $V \in C^1(\mathbb{R}^N)$, $\|V\|_{C^1} < \infty$. We assume in addition that $|\xi| \le M_0$ and $0 < \alpha \le V$. Then we have

Proposition 1.2: *There exists ε_0, $C_0 > 0$ such that $\forall 0 < \varepsilon \le \varepsilon_0$, $\forall |\xi| \le M_0$, $\forall g \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, there exist a unique solution $\phi \in L^\infty(\mathbb{R}^N)$ to (1.12), $\phi = T[g]$ satisfies*

$$\|\phi\|_{C^1} \le C_0\|g\|_\infty$$

Proof:

We divide the proof into two steps.

Step 1-a priori estimates: We first obtain *a priori estimates* of the problem (1.12) on bounded domains $B_R(0)$: There exist R_0, ε_0, C_0 such that $\forall\varepsilon < \varepsilon_0$, $R > R_0$, $|\xi| \le M_0$ such that $\forall\phi, g \in L^\infty$ solving $L(\phi) = g - \sum_i c_i \frac{\partial w_\lambda}{\partial x_i}$ in B_R, $\int_{B_R} \phi \frac{\partial w_\lambda}{\partial x_i} = 0$ and $\phi = 0$ on ∂B_R, we have

$$\|\phi\|_{C^1(B_R)} \le C_0 \|g\|_\infty$$

We prove first $\|\phi\|_\infty \le C_0\|g\|_\infty$. Assuming the opposite, then there exist sequences ϕ_n, g_n, $\varepsilon \to 0$, $R_n \to \infty$, $|\xi_n| \le M_0$ such that

$$L(\phi_n) = g_n - \sum_i c_i^n \frac{\partial w_\lambda}{\partial x_i}.$$

The first fact is that $c_i^n \to 0$ as $n \to \infty$. This fact follows just after multiplying the equation against $\frac{\partial w_\lambda}{\partial x_i}$ and integrating by parts, as we did in (1.16).

We observe that if $\Delta\phi = g$ in B_2 then there exist C such that

$$\|\nabla\phi\|_{L^\infty(B_1)} \le C[\|g\|_{L^\infty(B_2)} + \|\phi\|_{L^\infty(B_2)}]$$

where B_1 and B_2 are concentric balls. This implies that $\|\nabla\phi_n\|_{L^\infty(B)} \le C$ a given bounded set B, $\forall n \ge n_0$. Hence passing to a subsequence we obtain $\phi_n \to \phi$ uniformly on compact sets, and $\phi \in L^\infty(\mathbb{R}^N)$. Observe that $\|\phi_n\|_\infty = 1$, and this implies that $\|\phi\|_\infty \le 1$. We can also assume that up to a subsequence $\xi_n \to \xi_0$.

Since ϕ satisfies the equation $\Delta\phi - V(\xi_0)\phi + pw_{\lambda_0}^{p-1}(x)\phi = 0$, where $\lambda_0 = V(\xi_0)$, we have that $\phi \in \text{Span}\left\{\frac{\partial w_{\lambda_0}}{\partial x_1}, \dots, \frac{\partial w_{\lambda_0}}{\partial x_N}\right\}$. Taking limits in the orthogonality condition (1.12) we obtain that $\int_{\mathbb{R}^N} \phi(w_{\lambda_0})_{\partial x_i} = 0$, $i = 1, \dots, N$. This implies that $\phi = 0$ and hence $\|\phi_n\|_{L^\infty(B_M(0))} \to 0, \forall M < \infty$. Maximum principle yields that $\|\phi_n\|_{L^\infty(B_{R_n} \setminus B_{M_0}} \to 0$, since $|\phi_n| = o(1)$ on $\partial B_{R_n} \setminus B_{M_0}$ and $\|g_n\|_\infty \to 0$. Therefore we arrive at $\|\phi_n\|_\infty \to 0$, which is a contradiction. This implies that $\|\phi\|_{L^\infty(B_R)} \le C_0\|g\|_{L^\infty(B_R)}$ uniformly on large R. The C^1 estimate follows from elliptic local boundary estimates for elliptic operators.

Step 2-Existence: Recall that $g \in L^\infty$. We want to solve (1.12). We claim that solving (1.12) is equivalent to finding

$$\phi \in X = \{\psi \in H_0^1(B_R) : \int \psi \frac{\partial w_\lambda}{\partial x_i} = 0,\, i = 1, \dots, N\}$$

such that

$$\int \nabla\phi\nabla\psi + \int V(\xi + \varepsilon x)\phi\psi - pw^{p-1}\phi\psi + \int g\psi = 0, \quad \forall\psi \in X.$$

Take general $\Psi \in H_0^1$. We can decompose into $\Psi = \psi - \sum_i \alpha_i \frac{\partial w_\lambda}{\partial x_i}$, with $\alpha_i = \frac{\int \Psi \frac{\partial w_\lambda}{\partial x_i}}{\int (\frac{\partial w_\lambda}{\partial x_i})^2}$. We have

$$-\int \Delta(\sum_i \alpha_i \frac{\partial w_\lambda}{\partial x_i})\nabla\phi + \int V(\xi)(\sum_i \alpha_i (\frac{\partial w_\lambda}{\partial x_i})\phi - pw^{p-1}(\sum_i \alpha_i \frac{\partial w_\lambda}{\partial x_i})\phi = 0$$

which implies that

$$\int \nabla\phi\nabla\Psi + \int V(\xi)\phi\Psi - pw^{p-1}\phi\Psi$$

$$-\int (V(\xi) - V(\xi + \varepsilon x))(\Psi - \sum_i \alpha_i \frac{\partial w_\lambda}{\partial x_i}) + \int g(\Psi - \sum_i \alpha_i \frac{\partial w_\lambda}{\partial x_i})$$

$$= \int [(V(\xi + \varepsilon x) - V(\xi))\phi + g](\Psi - \sum_i \alpha_i \frac{\partial w_\lambda}{\partial x_i})$$

Let $\Pi_X(\Psi) = \sum_i \alpha_i \frac{\partial w_\lambda}{\partial x_i}$. Then the above integral equals

$$\int \Pi_X([(V(\xi + \varepsilon x) - V(\xi))\phi + g]\phi)\Psi$$

This implies that

$$-\Delta\phi + V(\xi)\phi - pw^{p-1}\phi + \Pi_X([(V(\xi + \varepsilon x) - V(\xi))\phi + g]\phi) = 0.$$

The problem is formulated weakly as

$$\int \nabla\phi\nabla\psi + \int (V(\xi + \varepsilon x) - pw^{p-1})\phi\psi + \int g\psi = 0, \phi \in X, \forall\psi \in X$$

which can be written as $\phi = A[\phi] + \tilde{g}$, where A is a compact operator. The a priori estimate implies that the only solution when $g = 0$ of this equation is $\phi = 0$. We conclude existence by Fredholm alternative. Finally we let $R \to +\infty$ and obtain the existence in the whole space, thanks to the a priori estimate in Step 1. □

Next we consider the assembly of multiple spikes. We look for a solution of (1.10) which near $x_j = \xi_j' = \xi_j/\varepsilon$, $j = 1, \ldots, k$ looks like $v(x) \approx w_{\lambda_j}(x - \xi_j')$, $\lambda_j = V(\xi_j)$, where $w_\lambda = \lambda^{1/(p-1)} w(\sqrt{\lambda} y)$.

Let $\xi_1, \xi_2, \ldots \xi_k \in \mathbb{R}^N$ be such that $|\xi_j' - \xi_l'| \gg 1$, if $j \neq l$. We look for a solution $v(x) \approx \sum_{j=1}^k w_{\lambda_j}(x - \xi_j')$, $\lambda_j = V(\xi_j)$. We assume $V \in C^2(R^N)$ and $\|V\|_{C^2} < \infty$, $0 < \alpha \leq V$. We use the notation $W_j = w_{\lambda_j}(x - \xi_j')$, $\lambda_j = V(\xi_j)$ and $W = \sum_{j=1}^k W_j$.

Setting $v = W + \phi$, then ϕ solves the problem

$$\Delta\phi - V(\varepsilon x)\phi + pW^{p-1}\phi + E + N(\phi) = 0 \tag{1.17}$$

where

$$E = \Delta W - VW + W^p, \quad N(\phi) = (W+\phi)^p - W^p - pW^{p-1}\phi.$$

Observe that $\Delta W = \sum_j \Delta W_j = \sum_j \lambda_j W_j - W_j^p$. So we can write

$$E = \sum_j (\lambda_j - V(\varepsilon x))W_j + (\sum_j W_j)^p - \sum_j W_j^p.$$

Our next objective is to solve the approximate linearized projected problem.

1.5. *Linearized (projected) problem*

We use the following notation $Z_j^i = \frac{\partial W_j}{\partial x_i}$. The linearized projected problem is the following

$$\Delta\phi - V(\varepsilon x)\phi + pW^{p-1}\phi + g = \sum_{i,j} c_j^i Z_j^i, \tag{1.18}$$

with the orthogonality condition $\int \phi Z_j^i = 0$, $\forall i, j$. The Z_j^i's are "nearly orthogonal" if the centers ξ_j' are far away one to each other. The c_j^i's are, by definition, the solution of the linear system

$$\int_{\mathbb{R}^N} (\Delta\phi - V(\varepsilon x)\phi + pW^{p-1}\phi + g) Z_{j_0}^{i_0} = \sum_{i,j} c_j^i \int_{\mathbb{R}^N} Z_j^i Z_{j_0}^{i_0},$$

for $i_0 = 1, \dots, N$, $j_0 = 1, \dots, k$. The c_j^i's are indeed uniquely determined provided that $|\xi_l' - \xi_j'| > R_0 \gg 1$, because the matrix with coefficients $\alpha_{i,j,i_0,j_0} = \int Z_j^i Z_{j_0}^{i_0}$ is "nearly diagonal", which means

$$\alpha_{i,j,i_0,j_0} = \begin{cases} \frac{1}{N}\int |\nabla W_j|^2 & \text{if } (i,j) = (i_0, j_0), \\ o(1) & \text{if not} \end{cases}$$

Moreover by a similar argument leading to (1.15) we have

$$|c_{j_0}^{i_0}| \le C \sum_{i,j} \int |\phi|[|\lambda_j - V| + p|W^{p-1} - W_j^{p-1}|]|Z_j^i| + \int |g||Z_j^i| \le C(\|\phi\|_\infty + \|g\|_\infty)$$

with C is uniform for large R_0. Furthermore if we rescale $x = \xi' + y$, we get

$$|(\lambda_j - V(\varepsilon x))Z_j^i| \le |(V(\xi_j) - V(\xi_j + \varepsilon y))|\left|\frac{\partial w_{\lambda_j}}{\partial y_i}\right| \le C\varepsilon e^{-\frac{\sqrt{\alpha}}{2}|y|},$$

because $|\frac{\partial w_{\lambda_j}}{\partial y_i}| \le Ce^{-|y|\sqrt{\lambda_j}}|y|^{-(N-1)/2}$. Observe also that

$$|(W^{p-1} - W_j^{p-1})Z_j^i| = |((1 - \sum_{l\neq j} \frac{W_l}{W_j})^{p-1} - 1)|W_j^{p-1}Z_j^i.$$

We estimate the interactions at each spike in two regions.

Observe that if $|x - \xi'_j| < \delta_0 \min_{j_1\neq j_2} |\xi'_{j_1} - \xi'_{j_2}|$, then

$$\frac{W_l(x)}{W_j(x)} \approx \frac{e^{-\sqrt{\lambda_l}|x-\xi'_l|}}{e^{-\sqrt{\lambda_j}|x-\xi'_j|}} < \frac{e^{-\sqrt{\lambda_l}|x-\xi'_l|}}{e^{-\sqrt{\lambda_j}\delta_0 \min_{j_1\neq j_2}|\xi'_{j_1}-\xi'_{j_2}|}}$$

If $\delta_0 \ll 1$ but fixed, we conclude that $e^{-\sqrt{\lambda_l}|\xi'_j-\xi'_l|+\delta_0(\sqrt{\lambda_l}-\sqrt{\lambda_j})\min_{j_1\neq j_2}|\xi'_{j_1}-\xi'_{j_2}|} < e^{-\rho \min_{j_1\neq j_2}|\xi'_{j_1}-\xi'_{j_2}|} \ll 1$. Thus we conclude that if $|x - \xi'_j| < \delta_0 \min_{j_1\neq j_2} |\xi'_{j_1} - xi'_{j_2}|$ then

$$|(W^{p-1} - W_j^{p-1})Z_j^i| \le e^{-\rho \min_{j_1\neq j_2}|\xi'_{j_1}-\xi'_{j_2}|} e^{-\frac{\alpha}{2}|x-\xi'_j|}.$$

On the other hand if $|x - \xi'_j| > \delta_0 \min_{j_1\neq j_2} |\xi'_{j_1} - \xi'_{j_2}|$, then

$$|(W^{p-1} - W_j^{p-1})Z_j^i| \le C|Z_j^i| \le Ce^{-\rho \min_{j_1\neq j_2}|\xi'_{j_1}-\xi'_{j_2}|} e^{-\frac{\alpha}{2}|x-\xi'_j|}$$

As a conclusion we obtain the following estimate

$$|c_{j_0}^{i_0}| \le C(\varepsilon + e^{-\rho \min_{j_1\neq j_2}|\xi'_{j_1}-\xi'_{j_2}|})\|\phi\|_\infty + \|g\|_\infty \tag{1.19}$$

Lemma 1.1: *Given $k \ge 1$, there exist R_0, C_0, ε_0 such that for all points ξ'_j with $|\xi'_{j_1} - \xi'_{j_2}| > R_0$, $j = 1, \ldots, k$ and all $\varepsilon < \varepsilon_0$ then exist a unique solution ϕ to the linearized projected problem with*

$$\|\phi\|_\infty \le C_0\|g\|_\infty.$$

Proof: As before we first prove the a priori estimate $\|\phi\|_\infty \le C_0\|g\|_\infty$. If not there exist $\varepsilon_n \to 0$, $\|\phi_n\|_\infty = 1$, $\|g_n\| \to 0$, ξ'^n_j with $\min_{j_1\neq j_2} |\xi'^n_{j_1} - \xi'^n_{j_2}| \to \infty$. We denote $W_n = \sum_j W_{j_n}$, and we have

$$\Delta\phi_n - V(\varepsilon_n x)\phi_n + pW_n^{p-1}\phi_n + g_n = \sum_{i,j}(c_j^i)_n(z_j^i)_n$$

Our first observation is that $(c_j^i)_n \to 0$ (which follows from the same estimate for $c_{j_0}^{i_0}$). Next we claim that $\forall R > 0$ $\|\phi_n\|_{L^\infty(B(\xi'^n_j,R))} \to 0$, $j = 1, \ldots, k$. If not, there exist j_0 $\|\phi_n\|_{L^\infty(B(\xi'^n_{j_0},R))} \ge \gamma > 0$. We denote $\tilde{\phi}_n(y) := \phi_n(\xi'^n_{j_0}+y)$. We have $\|\tilde{\phi}_n\|_{L^\infty(B(0,R))} \ge \gamma > 0$. Since $|\Delta\tilde{\phi}_n| \le C$, $\|\tilde{\phi}_n\|_\infty \le 1$. This implies that $\|\nabla\tilde{\phi}_n\| \le C$. Passing to a subsequence we may assume

$\tilde{\phi}_n \to \tilde{\phi}$ uniformly on compacts sets. Observe that also $V(\varepsilon_n(\xi'^n_{j_0}+y)) = V(\varepsilon_n\xi'^n_{j_0}) + O(\varepsilon_n|y|) \to \lambda_{j_0}$ over compact sets and $W_n(\xi'^n_{j_0}+y) \to W_{\lambda_{j_0}}(y)$ uniformly on compact sets. This implies that $\tilde{\phi}$ is a solution of the problem

$$\Delta\tilde{\phi} - \lambda_{j_0}\tilde{\phi} + pw^{p-1}_{\lambda_0}p\,\tilde{-}\,1 = 0, \quad \int \tilde{\phi}\frac{\partial W_{\lambda_{j_0}}}{\partial y_i}dy = 0, i = 1, \ldots, N$$

Nondegeneracy of $w_{\lambda_{j_0}}$ implies that $\tilde{\phi} = \sum_i \alpha_i \frac{\partial w_{\lambda_{j_0}}}{\partial y_i}$. The orthogonality condition implies that $\alpha_i = 0$, $\forall i = 1, \ldots, N$. This implies that $\tilde{\phi} = 0$ but $\|\tilde{\phi}\|_{L^\infty(B(0,R))} \geq \gamma > 0$, a contradiction.

Now we prove: $\|\phi_n\|_{L^\infty}(\mathbb{R}^N \setminus \cup_n B(\xi'^n_j, R)) \to 0$, provided that $R \gg 1$ and fixed so that $\phi_n \to 0$ in the sense of $\|\phi_n\|_\infty$ (again a contradiction). We will denote $\Omega_n = \mathbb{R}^N \setminus \cup_n B(\xi'^n_j, R)$. For $R \gg 1$ the equation for ϕ_n has the form

$$\Delta\phi_n - Q_n\phi_n + g_n = 0$$

where $Q_n = V(\varepsilon x) - pW_n^{p-1} \geq \frac{\alpha}{2} > 0$ for some R sufficiently large (but fixed).

Let us take for $\sigma^2 < \alpha/2$

$$\bar{\phi} = \delta \sum_j e^{\sigma|x-\xi'^n_j|} + \mu_n.$$

We denote $\varphi(y) = e^{\sigma|y|}$, $r = |y|$. Observe that $\Delta\varphi - \alpha/2\varphi = e^{\sigma|y|}(\sigma^2 + \frac{N-1}{|y|} - \alpha/2) < 0$ if $|y| > R \gg 1$. Then

$$-\Delta\bar{\phi} + Q_n\bar{\phi} - g_n > -\Delta\bar{\phi} + \frac{\alpha}{2}\bar{\phi} - \|g_n\|_\infty > \frac{\alpha}{2}\mu_n - \|g_n\|_\infty > 0 \qquad (1.20)$$

if we choose $\mu_n \geq \|g_n\|_\infty \frac{2}{\alpha}$. In addition we take $\mu_n = \sum_j \|\phi_n\|_{L^\infty(B(\xi^n_j,R))} + \|g_n\|_\infty \frac{2}{\alpha}$. Maximum principle implies that $\phi_n(x) \leq \bar{\phi}$ for all $x \in \Omega_n$. Taking $\delta \to 0$ this implies that $\phi_n(x) \leq \mu_n$, for all $x \in \Omega_n$. It is also true that $|\phi_n(x)| \leq \mu_n$ for all $x \in \Omega_n^c$, and this implies that $\|\phi_n\|_{L^\infty(\mathbb{R}^N)} \to 0$. □

Remark: If in addition we have the following decay for the error

$$\theta_n = \|g_n \left(\sum_j e^{-\rho|x-\xi'^n_j|}\right)^{-1} \|_\infty \to 0$$

with $\rho < \alpha/2$, then we can use as a barrier function

$$\bar{\phi} = \delta \sum_j e^{\sigma|x-\xi'^n_j|} + \mu_n \sum_j e^{-\rho|x-\xi'^n_j|}$$

with $\mu_n = e^{\rho R}\sum_j \|\phi_n\|_{L^\infty(B(\xi_j'^n,R))} + \theta_n$. It is easy to see that $\bar{\phi}$ is a super solution of the equation in $(\cup_j B(\xi_j,R))^c$ and we have $|\phi_n| \leq \bar{\phi}$. Letting $\delta \to 0$ we get $|\phi_n(x)| \leq \mu_n \sum_j e^{-\rho|x-\xi_j'^n|}$. As a conclusion we also get the a priori estimate

$$\|\phi\left(\sum_{j=1}^k e^{-\rho|x-\xi_j'|}\right)^{-1}\|_\infty \leq C\|g\left(\sum_{j=1}^k e^{-\rho|x-\xi_j'|}\right)^{-1}\|_\infty$$

provided that $0 \leq \rho < \alpha/2$, $|\xi_{j_1}' - \xi_{j_2}'| > R_0 \gg 1$, $\varepsilon < \varepsilon_0$.

We now give the proof of existence.

Proof: Let g be compactly supported smooth functions. The weak formulation for

$$\Delta\phi - V(\varepsilon x)\phi + pW^{p-1}\phi + g = \sum_{i,j} c_j^i Z_j^i, \quad \int \phi Z_j^i = 0, \forall i,j \tag{1.21}$$

is to find $\phi \in X = \{\phi \in H^1(\mathbb{R}^N) : \int \phi Z_j^i = 0, \forall i,j\}$ such that

$$\int_{\mathbb{R}^N} \nabla\phi\nabla\psi + V\phi\psi - pw^{p-1}\phi\psi - g\psi = 0, \quad \forall\psi \in X. \tag{1.22}$$

Assume ϕ solves (1.21). For $g \in L^2$, we decompose $g = \tilde{g} + \Pi[g]$ where $\int \tilde{g} Z_j^i = 0$ for all i,j, and Π is the orthogonal projection of g onto the space spanned by the Z_j^i's.

Let $\psi \in H^1(\mathbb{R}^N)$. We now use $\psi - \Pi[\psi]$ as a test function in (1.22). Then if $\varphi \in C_c^\infty(\mathbb{R}^N)$, then we have

$$\int_{\mathbb{R}^N} \nabla\varphi\nabla(\Pi[\psi]) = -\int_{\mathbb{R}^N} \Delta\varphi\Pi[\psi] = -\int_{\mathbb{R}^N} \Pi[\Delta\varphi]\psi. \tag{1.23}$$

On the other hand, we have $\Pi[\Delta\varphi] = \sum_{i,j} \alpha_{i,j} Z_j^i$, where

$$\sum \alpha_{i,j} \int Z_{i,j} Z_{i_0,j_0} = \int \Delta\varphi Z_{i_0}^{j_0} = \int \varphi\Delta Z_{i_0}^{j_0} \tag{1.24}$$

Then $\|\Pi[\Delta\varphi]\|_{L^2} \leq C\|\varphi\|_{H^1}$. By density argument it is also true for $\varphi \in H^1$ where $\Delta\varphi \in H^{-1}$. Therefore

$$\int \nabla\phi\nabla\psi + \int (V\phi - pW^{p-1}\phi - g)\psi = \int \Pi(V\phi - pW^{p-1}\phi + g)\psi \tag{1.25}$$

It follows that ϕ solves in weak sense

$$-\Delta\phi + V\phi - pW^{p-1}\phi - g = \Pi[-\Delta\phi + V\phi - pW^{p-1}\phi - g] \tag{1.26}$$

and $\Pi[-\Delta\phi + V\phi - pW^{p-1}\phi - g] = \sum_{i,j} c_i^j Z_i j$. Therefore by definition ϕ solves (1.22) implies that ϕ solves (1.26). Classical regularity gives that this weak solution is solution of (1.26) in strong sense, in particular $\phi \in L^\infty$ so that

$$\|\phi\|_\infty \leq C\|g\|_\infty. \tag{1.27}$$

Now we give the proof of existence for (1.21). We take g compactly supported. The equation (1.26) can be written in the following way (using Riesz theorem):

$$\langle \phi, \psi \rangle_{H^1} + \langle B[\phi], \psi \rangle_{H^1} = \langle \tilde{g}, \psi \rangle_{H^1} \tag{1.28}$$

or $\phi + B[\phi] = \tilde{g}$, $\phi \in X$. We claim that B is a compact operator. Indeed if $\phi_n \rightharpoonup 0$ in X, then $\phi_n \to 0$ in L^2 over compacts and

$$|\langle B[\phi_n], \psi \rangle| \leq |\int pW^{p-1}\phi_n\psi| \leq (\int pw^{p-1}\phi_n^2)^{1/2}(\int pW^{p-1}\psi^2)^{1/2} \tag{1.29}$$

which yields

$$|\langle B[\phi_n], \psi \rangle| \leq c(\int pW^{p-1}\phi_n^2)^{1/2}\|\psi\|_{H^1} \tag{1.30}$$

Take $\psi = B[\phi_n]$, which implies

$$\|B[\phi_n]\|_{H^1} \leq c(\int pW^{p-1}\phi_n^2)^{1/2} \to 0. \tag{1.31}$$

This gives that B is a compact operator.

Now we prove existence with the aid of Fredholm alternative. Problem (1.21) is solvable if for $\tilde{g} = 0$ the only solution to (1.22) is $\phi = 0$. But $\phi + B[\phi] = 0$ implies solve (1.21)(strongly) with $g = 0$. This implies $\phi \in L^\infty$, and the a priori estimate implies $\phi = 0$. Considering $g\Xi_{B_R(0)}$ we conclude that

$$\|\phi_R\|_\infty \leq \|g\|_\infty \tag{1.32}$$

Taking $R \to \infty$ then along a subsequence $\phi_R \to \phi$ uniform over compacts we obtain a solution to (1.21). □

Next we want to study the dependence and regularity of the solution with respect to the parameters. Let $g \in L^\infty$. We denote $\phi = T_{\xi'}[g]$, where $\xi' = (\xi_1', \ldots, \xi_k')$. We want to analyze derivatives $\partial_{\xi_{ji}'} T_{\xi'}[g]$. We know that $\|T_{\xi'}[g]\| \leq C_0\|g\|_\infty$. First we make a formal differentiation. We denote $\Phi = \frac{\partial\phi}{\partial\xi_{i_0j_0}'}$.

We have $\Delta\phi - V\phi + pW^{p-1}\phi + g = \sum_{i,j} c^i_j Z^i_j$ and $\int \phi Z^i_j = 0$, for all i, j. Formal differentiation yields

$$\Delta\Phi - V\Phi + pW^{p-1}\Phi + +\partial_{\xi_{i_0j_0}}(W^{p-1})\phi - \sum_{i,j} c^i_j \partial_{\xi_{i_0j_0}} Z^j_i = \sum_{i,j} \tilde{c}^i_j Z^i_j \quad (1.33)$$

where formally $\tilde{c}^j_i = \partial_{\xi_{i_0j_0}} c^j_i$. The orthogonality conditions is reduced to

$$\int_{\mathbb{R}^N} \Phi Z^i_j = \begin{cases} 0 & \text{if } j \neq j_0 \\ -\int \phi \partial_{\xi_{i_0j_0}} Z^i_{j_0} & \text{if } j = j_0 \end{cases} \quad (1.34)$$

Let us define $\tilde{\Phi} = \Phi - \sum_{i,j} \alpha_{i,j} Z^i_j$. We want $\int \tilde{\Phi} Z^i_j = 0$, for all i, j. We need

$$\sum_{i,j} \alpha_{i,j} \int Z^i_j Z^{\bar{i}}_{\bar{j}} = \begin{cases} 0 & \text{if } \bar{j} \neq j_0 \\ -\int \phi \partial_{\xi_{i_0j_0}} Z^i_{j_0} & \text{if } \bar{j} = j_0 \end{cases} \quad (1.35)$$

The system has a unique solution and $|\alpha_{i,j}| \leq C\|\phi\|_\infty$ (since the system is almost diagonal). So we have the condition $\int \tilde{\Phi} Z^i_j = 0$, for all i, j. We add to the equation the term $\sum_{i,j} \alpha_{i,j}(\Delta - V + pW^{p-1})Z^i_j$, so $\tilde{\Phi}$ satisfies the equation$\Delta\phi - V\phi + pW^{p-1}\phi + g = \sum_{i,j} c^i_j Z^i_j$

$$\Delta\tilde{\Phi} - V\tilde{\Phi} + pW^{p-1}\tilde{\Phi} + \partial_{\xi_{i_0j_0}}(W^{p-1})\phi - \sum_{i,j} c^i_j \partial_{\xi_{i_0j_0}} Z^j_i = \sum_{i,j} \tilde{c}^i_j Z^i_j - \sum_{i,j} \alpha_{i,j}(\Delta - V + pW^{p-1}) Z^i_j \quad (1.36)$$

This implies $\|\tilde{\Phi}\| \leq C(\|h\| + \|g\|) \leq C\|g\|_\infty$ and hence $\|\Phi\| \leq C\|g\|_\infty$.

The above formal procedure can be made rigorous by performing the analysis discretely, namely we consider solutions corresponding to ξ and $\xi + h$ respectively. Then we consider the quotient and pass the limit in h. We omit the details. In conclusion the map $\xi \to \partial_\xi \phi$ is well defined and continuous (into L^∞). Besides we also have $\|\partial_\xi \phi\|_\infty \leq C\|g\|_\infty$, and this implies

$$\|\partial_\xi T_\xi[\phi]\| \leq C\|g\| \quad (1.37)$$

1.6. *Nonlinear projected problem*

Consider now the nonlinear projected problem

$$\Delta\phi - V\phi + pw^{p-1}\phi + E + N(\phi) = \sum_{i,j} c^j_i Z^i_j, \quad \int \phi Z^j_i = 0, \forall i, j \quad (1.38)$$

We solve this by fixed point. We have $\phi = T(E + N(\phi)) =: M(\phi)$. We define $\Lambda = \{\phi \in C(\mathbb{R}^N) \cap L^\infty(R^N) : \|\phi\|_\infty \leq M\|E\|_\infty\}$. Remember that

$E = \sum_i (\lambda_j - V(\varepsilon x))W_j + (\sum_j W_j)^p - \sum_j W_j^p$. Observe that

$$|E| \le \varepsilon \sum_i e^{-\sigma|x-\xi'_j|} + ce^{-\delta_0 \min_{j_1 \neq j_2} |\xi'_{j_1} - \xi'_{j_2}|} \sum_j e^{-\sigma|x-\xi'_j|} \tag{1.39}$$

so, for existence we have $\|E\| \le C[\varepsilon + e^{-\delta_0 \min_{j_1 \neq j_2} |\xi'_{j_1} - \xi'_{j_2}|}] =: \rho$ (see that ρ is small). Contraction mapping implies there exists a unique solution $\phi = \Phi(\xi)$ and $\|\Phi(\xi)\| \le M\rho$. The proof is standard and hence omitted.

1.7. *Differentiability in ξ' of $\Phi(\xi')$*

As before the solutions obtained for the nonlinear projected problem has more regularity. In fact we can write the equation for Φ as

$$\Phi - T'_\xi(E'_\xi + N'_\xi(\phi)) = A(\Phi, \xi') = 0 \tag{1.40}$$

If $(D_\Phi A)(\Phi(\xi'), \xi')$ is invertible in L^∞, then $\Phi(\xi')$ turns out to be of class C^1. This is a consequence of the fixed point characterization, i.e., $D_\Phi A(\Phi(\xi'), \xi') = I + o(1)$ (the order $o(1)$ is a direct consequence of fixed point characterization). Then it is invertible. Contraction mapping theorem yields the existence of C^1 derivative of $A(\Phi, \xi')$ in (ϕ, ξ'). This implies $\Phi(\xi')$ is C^1. With a little bit of more work we can show that $\|D'_\xi \Phi(\xi')\| \le C\rho$ (just using the derivative given by the implicit function theorem).

1.8. *Solving the reduced problem: direct method*

By (1.38), to solve (1.17), we need to find ξ' such that the reduced problem

$$c_j^i = 0, \forall i, j \tag{1.41}$$

to get a solution to the original problem (1.10). There are two ways to solve the reduced problem (1.41): the first one is the direct method, and the second one is the variational reduction method. We describe the first method first by proving the following

Theorem 1: *(Oh [54]) Assume that $\xi_j^0, j = 1, ..., k$ are k distinct non-degenerate critical points of V. Then there exist a solution u_ε to the original problem with*

$$u_\varepsilon(x) \approx \sum_{j=1}^k w_{V(\xi_j^\varepsilon)}(x - \xi_j^\varepsilon/\varepsilon), \quad \xi_j^\varepsilon \to \xi_j^0$$

Proof: To solve the problem (1.41) we first obtain the asymptotic formula for c_j^i. To this end we multiply the equation (1.38) by $Z_{j_0}^{i_0}$ and integrate by parts. We obtain

$$\int_{\mathbb{R}^N} Z_j^i Z_{j_0}^{i_0} c_j^i = \int_{\mathbb{R}^N} (V(\xi_j + \epsilon x) - V(\xi_j)) w_{\xi_j} Z_{j_0}^{i_0} + O(\epsilon^2)$$

and thus

$$c_{j_0}^{i_0} \sim \partial_{i_0} V(\xi_j^0) + O(\epsilon)$$

The nondegeneracy of the critical point $\nabla V(\xi_j^0)$ and implicit function theorem yields the existence of $\xi_j = \xi_j^0 + O(\epsilon)$ such that (1.41) holds. □

The direct method can be used to construct multiple spike solutions for problems *without variational structure*, such as Gierer-Meinhardt system. For this application we refer to [62].

1.9. *Solving the reduced problem: variational reduction*

If the problem concerned has a variational structure, it is more appropriate to use a variational reduction method to solve (1.41). This method gives much stronger results under very weak assumptions.

We now describe the procedure that we call Variational Reduction in which the problem of finding ξ' with $c_j^i = 0$, for all i, j, is equivalent to finding a critical point of a reduced functional of ξ'.

Define an energy functional

$$J(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + V(\varepsilon x)v^2 - \frac{1}{p+1}\int_{\mathbb{R}^{N+1}} v_+^{p+1} \tag{1.42}$$

where $v \in H^1(\mathbb{R}^N)$ and $1 < p < \frac{N+2}{N-2}$. Since p is subcritical, by standard elliptic regularity arguments and Maximum Principle v is a solution of the problem

$$\Delta v - Vv + v^p = 0, v \to 0 \tag{1.43}$$

if and only if $v \in H^1(\mathbb{R}^N)$ and $J'(v) = 0$. Observe that $\langle J'(v), \varphi\rangle = \int \nabla v \nabla\varphi + Vv\varphi - v_+^p\varphi$.

We will prove the following Variational Reduction Principle

Theorem 2: *$v = W_{\xi'_*} + \phi(\xi')$ is a solution of the original problem (for $\rho \ll 1$) if and only if*

$$\partial_{\xi'} J(W_{\xi'} + \phi(\xi'))|_{\xi'=\xi'_*} = 0. \tag{1.44}$$

Proof: Indeed, observe that $v(\xi') := W_{\xi'} + \phi(\xi')$ solves the problem $\Delta v(\xi') - V(\varepsilon x)v(\xi') + v(\xi')^p = \sum_{i,j} c_j^i Z_j^i$ and also that

$$\partial_{\xi'_{j_0 i_0}} J(v(\xi')) = \langle J'(v(\xi')), \partial_{\xi'_{j_0 i_0}} v(\xi')\rangle = -\sum_{j,i} c_j^i \int Z_j^i \partial_{\xi'_{j_0 i_0}} v = -\sum_{i,j} c_j^i \int Z_i^j (\partial_{\xi'_{j_0 i_0}} W_{\xi'} + \partial_{\xi'_{j_0 i_0}} \phi(\xi')). \tag{1.45}$$

Recall that $W_{\xi'} = \sum_{j=1}^k w_{\lambda_j}(x - \xi'_j)$,

$$\partial_{\xi'_{j_0 i_0}} W'_\xi = \partial_{\xi'_{j_0 i_0}} w_{\lambda_{j_0(\xi')}}(x - \xi'_j) = (\partial_\lambda w_\lambda (x - \xi'_{j_0}))|_{\lambda = \lambda_{j_0}} - \partial_{x_{i_0}} w_{\lambda_{j_0}}(x - \xi'_{j_0}) = O(e^{-\delta|x - \xi'_0|})o(\varepsilon) - Z_{j_0 i_0 (x)} \tag{1.46}$$

This is because $\partial_\lambda w_\lambda = O(e^{-\delta|x-\xi'_0|})$. On the other hand since $\int Z_i^j \phi(\xi') = 0$ we have

$$\int Z_i^j \partial_{\xi'_{j_0 i_0}} \phi(\xi') = -\int \phi(\xi') \partial_{\xi'_{j_0 i_0}} Z_i^j$$

which is small thanks to the fact that $|\phi| \le C\rho e^{-\delta|x-\xi'_{j_0}|}$. Finally, observe that

$$-\int Z_j^i (\partial_{\xi'_{j_0 i_0}} W'_\xi + \partial_{\xi'_{j_0 i_0}} \phi) = \int Z_j^i Z_{j_0}^{i_0} + O(\rho) \tag{1.47}$$

The matrix of these numbers is invertible provided $\rho \ll 1$. □

We now discuss several applications of the reduction principle.

Theorem 3: *(del Pino and Felmer [15]) Assume that there exists an open, bounded set $\Lambda \subset \mathbb{R}^N$ such that*

$$\inf_{\partial\Lambda} V > \inf_{\Lambda} V, \tag{1.48}$$

then there exist a solution to the original problem, v_ε with $v_\varepsilon(x) = w_{V(\xi_\varepsilon)}((x - \xi_\varepsilon)/\varepsilon) + o(1)$ and $V(\xi_\varepsilon) \to \min_\Lambda V$, $\xi = \xi_\varepsilon$.

Theorem 4: *(del Pino-Felmer [16]) Assume that $\Lambda_1, \ldots, \Lambda_k$ are disjoint bounded sets with*

$$\inf_{\Lambda_j} V < \inf_{\partial\Lambda_j} V, j = 1, \cdots, k.$$

Then there exist a solution u_ε to the original problem with

$$u_\varepsilon(x) \approx \sum_{j=1}^k w_{V(\xi_j^\varepsilon)}(x - \xi_j^\varepsilon/\varepsilon), \quad \xi_j^\varepsilon \in \Lambda_j$$

and $V(\xi_j^\varepsilon) \to \inf_{\Lambda_j} V$. The same result holds if the minimum is replaced by maximum.

Theorem 5: *(Kang-Wei* [39]*) Let* Γ *be a bounded open set such that*

$$\max_{\Gamma} V(x) > \max_{\partial\Gamma} V(x)$$

Then for any positive integer K *there exists a solution* u_ϵ *such that*

$$u_\varepsilon(x) \approx \sum_{j=1}^{k} w_{V(\xi_j^\varepsilon)}(x - \xi_j^\varepsilon/\varepsilon), \quad \xi_j^\varepsilon \in \Lambda, V(\xi_j^\varepsilon) \to \max_{\Lambda} V(x)$$

Proof:

Assume that $j = 1$ first so that $v(\xi') = W_{\xi'} + \phi(\xi')$. Then we can compute the reduced energy as follows:

$$J(v(\xi')) = J(W_{\xi'}+\phi(\xi'))+\langle J'(W'_\xi+\phi), -\phi\rangle+\frac{1}{2}J''(W'_\xi+(1-t)\phi)[\phi]^2 \quad (1.49)$$

(This follows from Taylor expansion of the function $\alpha(t) = J(W_{\xi'} + (1 - t)\phi)$.) Observe that $\langle J'(W'_\xi + \phi), -\phi\rangle = \sum_{i,j} c_j^i \int Z_i^j \phi = 0$. Also observe that

$$J''(W'_\xi + (1-t)\phi)[\phi]^2 = \int |\nabla\phi|^2 + V(\varepsilon x)\phi^2 - p(W'_\xi + (1-t)\phi)\phi^2 = O(\varepsilon^2) \quad (1.50)$$

uniformly on ξ' because $\nabla\phi, \phi = O(\varepsilon e^{-\delta|x-\xi'|})$. We call $\Phi(\xi) := J(v(\xi')) = J(W_{\xi'}) + O(\varepsilon^2)$, and

$$J(W_{\xi'}) = \frac{1}{2}\int |\nabla W_{\xi'}|^2 + V(\xi)W_{\xi'}^2 - \frac{1}{p+1}\int W_{\xi'}^{p+1} + \int (V(\varepsilon x) - V(\xi'))W_{\xi'}^2 \quad (1.51)$$

Taking $\lambda = V(\xi)$, we have that

$$\int |\nabla w_\lambda(x)|^2 = \lambda^{-N/2}\int |\nabla w(\lambda^1/2x)|^2 \lambda^{1+2/(p-1)}\lambda^{N/2}dx = \lambda^{-N/2+p+1/p-1}|\nabla w(y)|^2 dy \quad (1.52)$$

and

$$\lambda \int w_\lambda^2(x) = \lambda^{-N/2p+1/p-1}\int w(y)^{p+1}dy \quad (1.53)$$

This implies that

$$\frac{1}{2}\int |\nabla W_{\xi'}|^2 + V(\xi')W_{\xi'}^2 - \frac{1}{p+1}\int W_{\xi'}^{p+1} = V(\xi')^{p+1/p-1-N/2}c_{p,N} \quad (1.54)$$

and we also have

$$\int (V(\varepsilon x) - V(\xi'))w_\lambda(x - \xi')^2 = O(\varepsilon) \quad (1.55)$$

uniformly in ξ'.

In summary we have the following asymptotic expansion of the reduced energy

$$\Phi(\xi) = J(v(\xi')) = V(\xi)^{p+1/p-1-N/2} c_{p,N} + O(\varepsilon) \tag{1.56}$$

To prove Theorem 3 we observe that $\frac{p+1}{p-1} - \frac{N}{2} > 0$. Then $\forall \varepsilon \ll 1$ we have

$$\inf_{\xi \in \Lambda} \Phi(\xi) < \inf_{\xi \in \partial\Lambda} \Phi(\xi) \tag{1.57}$$

and therefore Φ has a local minimum $\xi_\varepsilon \in \Lambda$ and $V(\xi_\varepsilon) \to \min_\Lambda V$. The same procedure also works for local maximums.

For several separated local minimums, the proof is similar. In fact when $|\xi_{j_1} - \xi_{j_2}| > \delta$, for all $j_1 \neq j_2$, we have $\rho = e^{-\delta_0 \min_{j_1 \neq j_2} |\xi'_{j_1} - \xi'_{j_2}|} + \varepsilon \leq e^{-\delta_0 \delta/\varepsilon} + \varepsilon < 2\varepsilon$. So we obtain

$$|\nabla_x \phi(\xi')| + |\phi(\xi')| \leq C\varepsilon \sum_j e^{-\delta_0 |x - \xi'_j|} \tag{1.58}$$

Now we get

$$J(v(\xi')) = \sum_j V(\varepsilon \xi_j^{'})^{p+1/p-1-N/2} c_{p,N} + O(\varepsilon) \tag{1.59}$$

$\varepsilon\xi' = (\xi_1, \ldots, \xi_k)$ implies for several minimal points on the Λ_j we have the result desired.

Finally we prove the existence of multiply interacting spikes. The computations are little bit involved since we have to measure precisely the interactions. The reduced energy functional takes the following form:

$$J(v(\xi')) = \sum_j V(\varepsilon \xi_j)^{p+1/p-1-N/2} (c_{p,N} + o(1)) - (1 + o(1)) \sum_{i \neq j} e^{-\min_{i \neq j} (\sqrt{V(\xi_i), V(\xi_j)}) |\xi' - \xi_j^{'}|}. \tag{1.60}$$

We shall take the following configuration space

$$\Sigma = \{(\xi_1, ..., \xi_k) \mid \xi_i \in \Lambda, \min_{i \neq j} |\xi_i - \xi_j| > \rho \epsilon \log \frac{1}{\epsilon}\}$$

and prove that the following maximization problem attains a solution in the interior part of the set Σ:

$$\min_{(\xi_1, \ldots, \xi_k) \in \Sigma} J(v(\xi')) \qquad \square$$

2. *Part II: Infinite-dimensional reduction method*

2.1. *An introduction*

In Chapter one, we have dealt with the problem of constructing solutions with finitely many bumps. The idea is to first sum up these finite many bumps and solve it in the space orthogonal to the translations. Then we adjust the points to obtain a true solution. The concentrating solutions concentrate at finite number of points which accounts for zero Lebesgue measure. In this Chapter we generalize this idea to the problem of constructing solutions concentrating on higher dimensional sets, such as curves, surfaces, or minimal surfaces of codimension k. As in the finite dimensional case, we proceed in two steps. In the first step, we solve the problem along each tangent fibre. This amounts to imposing *infinitely many* orthogonal conditions. In the second step, we move the higher dimensional object in the normal direction to find a true solution. We will encounter at least three problems: the first is the uniform estimate of the error in the first step. Sometimes there may be resonances due the combined effect of tangential and instability of the profile. The second problem is the adjustment of the higher dimensional subjects, which typically involves a second order nonlocal nonlinear reduced equation. The third problem is the non-compactness of the higher dimensional object.

In the following we take the model problem of Allen-Cahn equation in $\mathbb{R}^3$ and the higher dimensional concentration object is minimal surfaces. For higher dimensional concentration problems with resonances we refer to papers [19], [21] and [22].

2.2. *Model problem: the Allen-Cahn equation and minimal surfaces*

We consider the following so-called Allen-Cahn equation in $\mathbb{R}^N$

$$\Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^N, \tag{2.1}$$

where $f(s) = -W'(s)$ and W is a "double-well potential", bi-stable and balanced, namely

$$W(s) > 0 \text{ if } s \neq 1, -1, \quad W(1) = 0 = W(-1), \quad W''(\pm 1) = f'(\pm 1) =: \sigma_\pm^2 > 0. \tag{2.2}$$

A typical example of such a nonlinearity is

$$f(u) = (1-u^2)u \quad \text{for } W(u) = \frac{1}{4}(1-u^2)^2, \tag{2.3}$$

while we will not make use of the special symmetries enjoyed by this example.

Equation (2.1) is a prototype for the continuous modeling of phase transition phenomena. Let us consider the energy in a subregion region Ω of $\mathbb{R}^N$

$$J_\alpha(v) = \int_\Omega \frac{\alpha}{2} |\nabla v|^2 + \frac{1}{4\alpha} W(v),$$

whose Euler-Lagrange equation is a scaled version of (2.1),

$$\alpha^2 \Delta v \, + \, f(v) \, = \, 0 \quad \text{in } \Omega \, . \tag{2.4}$$

We observe that the constant functions $u = \pm 1$ minimize J_α. They are idealized as two *stable phases* of a material in Ω. It is of interest to analyze stationary configurations in which the two phases coexist. Given any subset Λ of Ω, any discontinuous function of the form

$$u_* = \chi_\Lambda - \chi_{\Omega \setminus \Lambda} \tag{2.5}$$

minimizes the second term in J_ε. The introduction of the gradient term in J_α makes an α-regularization of u_* a test function for which the energy gets bounded and proportional to the surface area of the *interface* $M = \partial\Lambda$, so that in addition to minimizing approximately the second term, stationary configurations should also select asymptotically interfaces M that are stationary for surface area, namely (generalized) minimal surfaces. This intuition on the Allen-Cahn equation gave important impulse to the calculus of variations, motivating the development of the theory of Γ-*convergence* in the 1970's. Modica [46] proved that a family of local minimizers u_α of J_α with uniformly bounded energy must converge in suitable sense to a function of the form (2.5) where $\partial\Lambda$ minimizes perimeter. Thus, intuitively, for each given $\lambda \in (-1, 1)$, the level sets $[v_\alpha = \lambda]$, collapse as $\alpha \to 0$ onto the interface $\partial\Lambda$. Similar result holds for critical points not necessarily minimizers, see [60]. For minimizers this convergence is known in very strong sense, see [10, 11].

If, on the other hand, we take such a critical point u_α and scale it around an interior point $0 \in \Omega$, setting $u_\alpha(x) = v_\alpha(\alpha x)$, then u_α satisfies equation (2.1) in an expanding domain,

$$\Delta u_\alpha + f(u_\alpha) = 0 \quad \text{in } \alpha^{-1}\Omega$$

so that letting formally $\alpha \to 0$ we end up with equation (2.1) in entire space. The "interface" for u_α should thus be around the (asymptotically flat) minimal surface $M_\alpha = \alpha^{-1}M$. Modica's result is based on the intuition that if M happens to be a smooth surface, then the transition from the equilibria -1 to 1 of u_α along the normal direction should take place in the approximate form $u_\alpha(x) \approx w(z)$ where z designates the normal coordinate to M_α. Then w should solve the ODE problem

$$w'' + f(w) = 0 \quad \text{in } \mathbb{R}, \quad w(-\infty) = -1, \; w(+\infty) = 1 \; . \tag{2.6}$$

This solution indeed exists thanks to assumption (2.2). It is strictly increasing and unique up to constant translations. We fix in what follows the unique w for which

$$\int_{\mathbb{R}} t\, w'(t)^2 \, dt \;=\; 0 \; . \tag{2.7}$$

For example (2.3), we have $w(t) = \tanh\big(t/\sqrt{2}\big)$. In general w approaches its limits at exponential rates,

$$w(t) - \pm 1 \;=\; O(\, e^{-\sigma_\pm |t|}\,) \quad \text{as } t \to \pm\infty \; .$$

Observe then that

$$J_\alpha(u_\alpha) \approx Area\,(M) \int_{\mathbb{R}} [\frac{1}{2}{w'}^2 + W(w)]$$

which is what makes it plausible that M is critical for area, namely a minimal surface.

The above considerations led E. De Giorgi [24] to formulate in 1978 a celebrated conjecture on the Allen-Cahn equation (2.1), parallel to Bernstein's theorem for minimal surfaces: The level sets $[u = \lambda]$ of a bounded entire solution u to (2.1), which is also monotone in one direction, must be hyperplanes, at least for dimension $N \leq 8$. Equivalently, up to a translation and a rotation, $u = w(x_1)$. This conjecture has been proven in dimensions $N = 2$ by Ghoussoub and Gui [29], $N = 3$ by Ambrosio and Cabré [9], and under a mild additional assumption by Savin [58]. A counterexample was built for $N \geq 9$ by M. del Pino, M.Kowalczyk and Wei in [25], see also [14, 43]. See [26] for a recent survey on the state of the art of this question.

The counter-example in [25] was built on the counter-example to the Bernstein conjecture for minimal graphs: Bernstein conjectured that all minimal graphs, i.e. graphs $\{x_N = F(x^{'})\}$ for which F satisfies additionally the minimal graph equation

$$\nabla(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}}) = 0, \quad x^{'} \in \mathbb{R}^{N-1} \tag{2.8}$$

In 1969, Bombierie, De Giorgi and Giusti [8] built a nontrivial solution to (2.8) in dimension $N = 9$. In [25, ?] we took the opposite view of Γ-convergence: for a given nondegenerate minimal surface it is possible to build a solution to the Allen-Cahn equation which concentrates on this minimal surface. The class of minimal surfaces will include the Bombierie-De Giorgi-Giusti minimal graph, and the complete embedded minimal surfaces in $\mathbb{R}^3$.

In this following we construct a new class of entire solutions to the Allen-Cahn equation in $\mathbb{R}^3$ whose level sets resemble a large dilation of a given complete, embedded minimal surface M, asymptotically flat in the sense that it has *finite total curvature*, namely

$$\int_M |K|\, dV < +\infty$$

where K denotes Gauss curvature of the manifold, which is also *non-degenerate* in a sense that we will make precise below.

2.3. *Embedded minimal surfaces of finite total curvature*

The theory of embedded, minimal surfaces of finite total curvature in $\mathbb{R}^3$, has reached a notable development in the last 25 years. For more than a century, only two examples of such surfaces were known: the plane and the catenoid. The first nontrivial example was found in 1981 by C. Costa, [12, ?]. The *Costa surface* is a genus one minimal surface, complete and properly embedded, which outside a large ball has exactly three components (its *ends*), two of which are asymptotically catenoids with the same axis and opposite directions, the third one asymptotic to a plane perpendicular to that axis. The complete proof of embeddedness is due to Hoffman and Meeks [34]. In [35, 37] these authors generalized notably Costa's example by exhibiting a class of three-end, embedded minimal surface, with the same look as Costa's far away, but with an array of tunnels that provides arbitrary genus $k \geq 1$. This is known as the Costa-Hoffman-Meeks surface with genus k.

Many other examples of multiple-end embedded minimal surfaces have been found since, see for instance [44, ?] and references therein. In general all these surfaces look like parallel planes, slightly perturbed at their ends by asymptotically logarithmic corrections with a certain number of catenoidal links connecting their adjacent sheets. In reality this intuitive picture is not a coincidence.

Using the Eneper-Weierstrass representation, Osserman [51] established that any embedded, complete minimal surface with finite total curvature can be described by a conformal diffeomorphism of a compact surface (actually of a Riemann surface), with a finite number of its points removed. These points correspond to the ends. Moreover, after a convenient rotation, the ends are asymptotically all either catenoids or plane, all of them with parallel axes, see Schoen [59]. The topology of the surface is thus characterized by the genus of the compact surface and the number of ends, having therefore "finite topology".

2.4. *Main results*

In what follows M designates a complete, embedded minimal surface in $\mathbb{R}^3$ with finite total curvature (to which below we will make a further nondegeneracy assumption). As pointed out in [38], M is orientable and the set $\mathbb{R}^3 \setminus M$ has exactly two components S_+, S_-.

In what follows we fix a continuous choice of unit normal field $\nu(y)$, which conventionally we take it to point towards S_+.

For $x = (x_1, x_2, x_3) = (x', x_3) \in \mathbb{R}^3$, we denote

$$r = r(x) = |(x_1, x_2)| = \sqrt{x_1^2 + x_2^2}.$$

After a suitable rotation of the coordinate axes, outside the infinite cylinder $r < R_0$ with sufficiently large radius R_0, then M decomposes into a finite number m of unbounded components $M_1, \dots, M_m$, its *ends*. From a result in [59], we know that asymptotically each end of M_k either resembles a plane or a catenoid. More precisely, M_k can be represented as the graph of a function F_k of the first two variables,

$$M_k = \{\, y \in \mathbb{R}^3 \ / \ r(y) > R_0,\ y_3 = F_k(y')\,\}$$

where F_k is a smooth function which can be expanded as

$$F_k(y') = a_k \log r + b_k + b_{ik}\frac{y_i}{r^2} + O(r^{-3}) \quad \text{as } r \to +\infty, \tag{2.9}$$

for certain constants a_k, b_k, b_{ik}, and this relation can also be differentiated. Here

$$a_1 \le a_2 \le \ldots \le a_m\ , \qquad \sum_{k=1}^{m} a_k \ = \ 0\,. \tag{2.10}$$

The direction of the normal vector $\nu(y)$ for large $r(y)$ approaches on the ends that of the x_3 axis, with alternate signs. We use the convention that for $r(y)$ large we have

$$\nu(y) \;=\; \frac{(-1)^k}{\sqrt{1+|\nabla F_k(y')|^2}}\,(\nabla F_k(y')\,,\,-1\,) \quad \text{if } y\in M_k. \tag{2.11}$$

Let us consider the Jacobi operator of M

$$\mathcal{J}(h) \;:=\; \Delta_M h + |A|^2 h \tag{2.12}$$

where $|A|^2 = -2K$ is the Euclidean norm of the second fundamental form of M. $\mathcal{J}$ is the linearization of the mean curvature operator with respect to perturbations of M measured along its normal direction. A smooth function $z(y)$ defined on M is called a *Jacobi field* if $\mathcal{J}(z) = 0$. Rigid motions of the surface induce naturally some bounded Jacobi fields: Associated to respectively translations along coordinates axes and rotation around the x_3-axis, are the functions

$$z_1(y) = \nu(y)\cdot e_i, \quad y\in M, \quad i=1,2,3,$$

$$z_4(y) = (-y_2, y_1, 0)\cdot\nu(y), \quad y\in M. \tag{2.13}$$

We assume that M is *non-degenerate* in the sense that these functions are actually *all* the bounded Jacobi fields, namely

$$\{\; z\in L^\infty(M)\;/\;\mathcal{J}(z)=0\,\} \;=\; \text{span}\,\{\,z_1,z_2,z_3,z_4\,\}\,. \tag{2.14}$$

We denote in what follows by J the dimension (≤ 4) of the above vector space.

This assumption, expected to be generic for this class of surfaces, is known in some important cases, most notably the catenoid and the Costa-Hoffmann-Meeks surface which is an example of a three ended M whose genus may be of any order. See Nayatani [49, 50] and Morabito [47]. Note that for a catenoid, $z_{04} = 0$ so that $J = 3$. Non-degeneracy has been used as a tool to build new minimal surfaces for instance in Hauswirth and Pacard [33], and in Pérez and Ros [53]. It is also the basic element, in a compact-manifold version, to build solutions to the small-parameter Allen-Cahn equation in Pacard and Ritoré [52].

Let us consider a large dilation of M,

$$M_\alpha \;:=\; \alpha^{-1}M.$$

This dilated minimal surface has ends parameterized as

$$M_{k,\alpha} \;=\; \{\, y \in \mathbb{R}^3 \;/\; r(\alpha y) > R_0,\; y_3 = \alpha^{-1} F_k(\alpha y')\,\}\;.$$

Let β be a vector of given m real numbers with

$$\beta \;=\; (\beta_1, \dots, \beta_m), \qquad \sum_{i=1}^{m} \beta_i = 0\;. \tag{2.15}$$

Our main result asserts the existence of a solution $u = u_\alpha$ defined for all sufficiently small $\alpha > 0$ such that given $\lambda \in (-1,1)$, its level set $[u_\alpha = \lambda]$ defines an embedded surface lying at a uniformly bounded distance in α from the surface M_α, for points with $r(\alpha y) = O(1)$, while its k-th end, $k = 1, \dots, m$, lies at a uniformly bounded distance from the graph

$$r(\alpha y) > R_0,\; y_3 \;=\; \alpha^{-1}\, F_k(\alpha y') + \beta_k \log |\alpha y'|\,. \tag{2.16}$$

The parameters β must satisfy an additional constraint. It is clear that if two ends are parallel, say $a_{k+1} = a_k$, we need at least that $\beta_{k+1} - \beta_k \geq 0$, for otherwise the ends would eventually intersect. Our further condition on these numbers is that these ends in fact diverge at a sufficiently fast rate. We require

$$\beta_{k+1} - \beta_k \;>\; 4\, \max\{\sigma_-^{-1}, \sigma_+^{-1}\} \quad \text{if} \quad a_{k+1} = a_k\;. \tag{2.17}$$

Let us consider the smooth map

$$X(y,z) = y + z\nu(\alpha y), \quad (y,t) \in M_\alpha \times \mathbb{R}. \tag{2.18}$$

$x = X(y,z)$ defines coordinates inside the image of any region where the map is one-to-one. In particular, let us consider a function $p(y)$ with

$$p(y) = (-1)^k \beta_k \log|\alpha y'| + O(1), \quad k = 1, \dots, m,$$

and β satisfying $\beta_{k+1} - \beta_k > \gamma > 0$ for all k with $a_k = a_{k+1}$. Then the map X is one-to-one for all small α in the region of points (y,z) with

$$|z - q(y)| < \frac{\delta}{\alpha} + \gamma \log(1 + |\alpha y'|)$$

provided that $\delta > 0$ is chosen sufficiently small.

Theorem 6: *(del Pino-Kowalczyk-Wei* [20]*) Let $N = 3$ and M be a minimal surface embedded, complete with finite total curvature which is nondegenerate. Then, given β satisfying relations* (2.15) *and* (2.17)*, there exists* *a bounded solution u_α of equation* (2.1)*, defined for all sufficiently small α, such that*

$$u_\alpha(x) = w(z - q(y)) + O(\alpha) \quad \text{for all} \quad x = y + z\nu(\alpha y), \quad |z - q(y)| < \frac{\delta}{\alpha}, \tag{2.19}$$

where the function q satisfies

$$q(y) \;=\; (-1)^k \beta_k \,\log |\alpha y'| + O(1) \quad y \in M_{k,\alpha}, \quad k = 1, \dots, m.$$

In particular, for each given $\lambda \in (-1, 1)$, the level set $[u_\alpha = \lambda]$ is an embedded surface that decomposes for all sufficiently small α into m disjoint components (ends) outside a bounded set. The k-th end lies at $O(1)$ distance from the graph

$$y_3 \;=\; \alpha^{-1} F_k(\alpha y) + \beta_k \log |\alpha y'|.$$

We will devote the rest of this part to the proofs of Theorems 6. For the full proofs we refer to [20] in which more detailed behavior of the solutions constructed, such as finite Morse index, can be found.

3. *Geometric Background*

In this section we present the geometric backgrounds on the expansion of the Laplacian operator near a manifold.

3.1. *Parametrization of M and its Laplace-Betrami Operator*

Let D be the set

$$D \;=\; \{\mathtt{y} \in \mathbb{R}^2 \;/\; |\mathtt{y}| > R_0\}.$$

We can parameterize the end M_k of M as

$$\mathtt{y} \in D \longmapsto y := Y_k(\mathtt{y}) \;=\; \mathtt{y}_i e_i + F_k(\mathtt{y}) e_3\,. \tag{3.1}$$

and F_k is the function in (2.9). In other words, for $y = (y', y_3) \in M_k$ the coodinate $\mathtt{y}$ is just defined as $\mathtt{y} = y'$. We want to represent Δ_M–the Laplace-Beltrami operator of M–with respect to these coordinates. For the coefficients of the metric g_{ij} on M_k we have

$$\partial_{\mathtt{y}_i} Y_k = e_i + O\left(r^{-1}\right) e_3$$

so that

$$g_{ij}(\mathtt{y}) = \langle \partial_i Y_k, \partial_j Y_k \rangle = \delta_{ij} + O\left(r^{-2}\right), \tag{3.2}$$

where $r = |\mathtt{y}|$. The above relations "can be differentiated" in the sense that differentiation makes the terms $O(r^{-j})$ gain corresponding negative powers of r. Then we find the representation

$$\Delta_M = \frac{1}{\sqrt{\det g_{ij}}}\partial_i(\sqrt{\det g_{ij}}\, g^{ij}\partial_j) = \Delta_{\mathtt{y}} + O(r^{-2})\partial_{ij} + O(r^{-3})\ \partial_i \quad \text{on } M_k\,. \tag{3.3}$$

The normal vector to M at $y \in M_k$ $k = 1, \dots, m$, corresponds to

$$\nu(y) \ = \ (-1)^k \frac{1}{\sqrt{1+|\nabla F_k(\mathtt{y})|^2}}\ (\ \partial_i F_k(\mathtt{y}) e_i - e_3\ )\ , \quad y = Y_k(\mathtt{y}) \in M_k$$

so that

$$\nu(y) \ = \ (-1)^k e_3 + \alpha_k r^{-2}\, \mathtt{y}_i e_i + O(r^{-2})\ , \quad y = Y_k(\mathtt{y}) \in M_k\ . \tag{3.4}$$

Let us observe for later reference that since $\partial_i \nu = O(r^{-2})$, then the principal curvatures of M, k_1, k_2 satisfy $k_l = O(r^{-2})$. In particular, we have that

$$|A(y)|^2 = k_1^2 + k_2^2 = O(r^{-4}). \tag{3.5}$$

To describe the entire manifold M we consider a finite number $N \geq m+1$ of local parametrizations

$$\mathtt{y} \in \mathcal{U}_k \subset \mathbb{R}^2 \longmapsto y = Y_k(\mathtt{y}), \quad Y_k \in C^\infty(\bar{\mathcal{U}}_k), \quad k = 1, \dots, N. \tag{3.6}$$

For $k = 1, \dots, m$ we choose them to be those in (3.1), with $\mathcal{U}_k = D$, so that $Y_k(\mathcal{U}_k) = M_k$, and $\bar{\mathcal{U}}_k$ is bounded for $k = m+1, \dots, N$. We require then that

$$M = \bigcup_{k=1}^{N} Y_k(\mathcal{U}_k).$$

We remark that the Weierstrass representation of M implies that we can actually take $N = m+1$, namely only one extra parametrization is needed to describe the bounded complement of the ends in M. We will not use this fact. In general, we represent for $y \in Y_k(\mathcal{U}_k)$,

$$\Delta_M = a^0_{ij}(y)\partial_{ij} + b^0_i(y)\partial_i, \quad y = Y_k(\mathtt{y}), \quad \mathtt{y} \in \mathcal{U}_k, \tag{3.7}$$

where a^0_{ij} is a uniformly elliptic matrix and the index k is not made explicit in the coefficients. For $k = 1, \dots, m$ we have

$$a^0_{ij}(y) = \delta_{ij} + O(r^{-2}), \quad b^0_i = O(r^{-3}), \quad \text{as } r(y) = |\mathtt{y}| \to \infty. \tag{3.8}$$

The parametrizations set up above induce naturally a description of the expanded manifold $M_\alpha = \alpha^{-1}M$ as follows. Let us consider the functions

$$Y_{k\alpha} : \mathcal{U}_{k\alpha} := \alpha^{-1}\mathcal{U}_k \to M_\alpha, \quad \mathtt{y} \mapsto Y_{k\alpha}(\mathtt{y}) := \alpha^{-1}Y_k(\alpha \mathtt{y}), \quad k = 1, \dots, N. \tag{3.9}$$

Obviously we have

$$M_\alpha = \bigcup_{k=1}^{N} Y_{k\alpha}(\mathcal{U}_{k\alpha}).$$

The computations above lead to the following representation for the operator Δ_{M_α}:

$$\Delta_{M_\alpha} = a^0_{ij}(\alpha y)\partial_{ij} + b^0_i(\alpha y)\partial_i, \quad y = Y_{k\alpha}(\mathtt{y}), \quad \mathtt{y} \in \mathcal{U}_{k\alpha}, \tag{3.10}$$

where a^0_{ij}, b^0_i are the functions in (3.7), so that for $k = 1, \dots, m$ we have

$$a^0_{ij} = \delta_{ij} + O(r_\alpha^{-2}), \quad b^0_i = O(r_\alpha^{-3}), \quad \text{as } r_\alpha(y) := |\alpha \mathtt{y}| \to \infty. \tag{3.11}$$

3.2. *Coordinates near M and the Euclidean Laplacian: Fermi coordinates*

Next we shall consider the parametrization of a neighborhood of M. Let us consider the smooth map

$$(y, z) \in M \times \mathbb{R} \ \longmapsto \ x = \tilde{X}(y, z) = y + z\nu(y) \in \mathbb{R}^3. \tag{3.12}$$

Let us consider an open subset $\tilde{\mathcal{O}}$ of $M \times \mathbb{R}$ and assume that the map $X|_{\tilde{\mathcal{O}}}$ is one to one, and that it defines a diffeomorphism onto its image $\mathcal{N} = X(\tilde{\mathcal{O}})$. Certainly we can choose $\tilde{\mathcal{O}}$ such that

$$\{(y, z) \in M \times \mathbb{R} \ / \ |z| < \delta \ \log(1 + r(y))\} \subset \tilde{\mathcal{O}}.$$

Since along ends $\partial_i \nu = O(r^{-2})$ so that $z\partial_i\nu$ is uniformly small in $\tilde{O}$, it follows that $\tilde{X}$ is actually a diffeomorphism onto is image.

The Euclidean Laplacian Δ_x can be computed in such a region by the well-known formula in terms of the coordinates $(y, z) \in \tilde{\mathcal{O}}$ as

$$\Delta_x = \partial_{zz} + \Delta_{M_z} - H_{M_z}\partial_z, \quad x = \tilde{X}(y, z), \quad (y, z) \in \mathcal{O} \tag{3.13}$$

where M_z is the manifold

$$M_z = \{y + z\nu(y) \ / \ y \in M\}.$$

To see the formula (3.13) we observe that

$$X_i = Y_i + z\nu_i, i = 1, 2, X_z = \nu$$

and hence for $i, j = 1, 2$

$$g_{ij}(x, z) = g_{0ij} + 2z\nu_i Y_j + z^2\nu_i\nu_j$$

and $g_{iz} = 0, g_{zz} = 1$. Hence the Euclidean laplacian in $\tilde{\mathcal{O}}$ becomes

$$\Delta_{M_z} h(y)|_{X=X(y,z)} = \frac{1}{\sqrt{\det g_z}}\partial_i(\sqrt{\det(g_z)}g_z^{ij}\partial_i h)(y, z)$$

$$= \partial_{zz}h + \Delta_{M_z}h + \partial_z \log(\sqrt{\det g_z})\partial_z h$$

We note that by direct computations $\det(g_z) = \Pi_{j=1}^2(1 - zk_j)^2 \det g_0$. This gives the formula (3.13).

Local coordinates $y = Y_k(\mathtt{y})$, $\mathtt{y} \in \mathbb{R}^2$ as in (3.1) induce natural local coordinates in M_z. The metric $g_{ij}(z)$ in M_z can then be computed as

$$g_{ij}(z) = \langle\partial_i Y, \partial_j Y\rangle + z(\langle\partial_i Y, \partial_j \nu\rangle + \langle\partial_j Y, \partial_i \nu\rangle) + z^2\langle\partial_i \nu, \partial_j \nu\rangle \qquad (3.14)$$

or

$$g_{ij}(z) = g_{ij} + z\,O(r^{-2}) + z^2 O(r^{-4})\,.$$

where these relations can be differentiated. Thus we find from the expression of Δ_{M_z} in local coordinates that

$$\Delta_{M_z} = \Delta_M + za_{ij}^1(y, z)\partial_{ij} + zb_i^1(y, z)\partial_i, \quad y = Y(\mathtt{y}) \qquad (3.15)$$

where a_{ij}^1, b_i^1 are smooth functions of their arguments. Let us examine this expansion closer around the ends of M_k where $y = Y_k(\mathtt{y})$ is chosen as in (3.1). In this case, from (3.14) and (3.2) we find

$$g^{ij}(z) = g^{ij} + z\,O(r^{-2}) + z^2 O(r^4) + \dots$$

Then we find that for large r,

$$\Delta_{M_z} = \Delta_M + z\,O(r^{-2})\partial_{ij} + zO(r^{-3})\partial_i. \qquad (3.16)$$

Let us consider the remaining term in the expression for the Laplacian, the mean curvature H_{M_z}. We have the validity of the formula

$$H_{M_z} = \sum_{i=1}^2 \frac{k_i}{1 - k_i z} = \sum_{i=1}^2 k_i + k_i^2 z + k_i^3 z^2 + \cdots$$

where k_i, $i = 1, 2$ are the principal curvatures. Since M is a minimal surface, we have that $k_1 + k_2 = 0$. Thus

$$|A|^2 = k_1^2 + k_2^2 = -2k_1k_2 = -2K$$

where $|A|$ is the Euclidean norm of the second fundamental form, and K the Gauss curvature. As $r \to +\infty$ we have seen that $k_i = O(r^{-2})$ and hence $|A|^2 = O(r^{-4})$. More precisely, we find for large r,

$$H_{M_z} = |A|^2 z + z^2 O(r^{-6}).$$

Thus we have found the following expansion for the Euclidean Laplacian,

$$\Delta_x = \partial_{zz} + \Delta_M - z|A|^2 \partial_z + B \tag{3.17}$$

where expressed in local coordinates in M the operator B has the form

$$B = z\, a^1_{ij}(y,z)\partial_{ij} + z\, b^1_i(y,z)\partial_i + z^2 b^1_3(y,z)\partial_z \tag{3.18}$$

with $a^1_{ij},\ b^1_i,\ b^1_3$ smooth functions. Besides, we find that

$$a^1_{ij}(y,z) = O(r^{-2}), \quad b^1_i(y,z) = O(r^{-3}), \quad b^1_i(y,z) = O(r^{-6}), \tag{3.19}$$

uniformly in z for $(y,z) \in \tilde{\mathcal{O}}$. Moreover, the way these coefficients are produced from the metric yields for instance that

$$a^1_{ij}(y,z) = a^1_{i,j}(y,0) + z a^{(2)}_{i,j}(y,z), \quad a^2_{i,j}(y,z) = O(r^{-3}),$$

$$b^1_i(y,z) = b^1_i(y,0) + z b^{(2)}_i(y,z), \quad b^{(2)}_i(y,z) = O(r^{-4})\,.$$

We summarize the discussion above. Let us consider the parametrization in (3.12) of the region $\tilde{\mathcal{N}}$.

Lemma 3.1: *The Euclidean Laplacian can be expanded in* $\tilde{\mathcal{N}}$ *as*

$$\Delta_x = \partial_{zz} + \Delta_{M_z} - H_{M_z}\partial_z \ =$$

$$\partial_{zz} + \Delta_M - z\,|A|^2 \partial_z \ + \ z\,[a^1_{ij}(\mathtt{y},z)\partial_{ij} + b^1_i(\mathtt{y},z)\partial_i] \ + z^2 b^1_3(\mathtt{y},z)\partial_z,$$

$$\Delta_M = a^0_{ij}\partial_{ij} + b^0_i\partial_i, \quad x = \tilde{X}(y,z), \quad (y,z) \in \tilde{\mathcal{O}},$$

where a^l_{ij}, b^l_j *are smooth, bounded functions, with the index* k *omitted. In addition, for* $k = 1, \ldots, m$,

$$a^l_{ij} = \delta_{ij}\delta_{0l} + O(r^{-2}), \quad b^l_i = O(r^{-3}), \quad b^1_3 = O(r^{-6})\ ,$$

as $r = |\mathtt{y}| \to \infty$, *uniformly in* z *variable.*

3.3. *Laplacian in expanded variables*

Now we consider the expanded minimal surface $M_\alpha = \alpha^{-1}M$ for a small number α. We have that $\mathcal{N} = \alpha^{-1}\tilde{\mathcal{N}}$. We describe $\mathcal{N}$ via the coordinates

$$x = X(y,z) := y + z\nu_\alpha(y), \quad (y,z) \in \alpha^{-1}\tilde{\mathcal{O}}. \tag{3.20}$$

Let us observe that

$$X(y,z) = \alpha^{-1}\tilde{X}(\alpha y, \alpha z)$$

where $\tilde{x} = \tilde{X}(\tilde{y},\tilde{z}) = \tilde{y}+\tilde{z}\nu(\tilde{y})$, where the coordinates in $\mathcal{N}_\delta$ previously dealt with. We want to compute the Euclidean Laplacian in these coordinates associated to M_α. Observe that

$$\Delta_x[u(x)]\,|_{x=X(y,z)} = \alpha^2\Delta_{\tilde{x}}[u(\alpha^{-1}\tilde{x})\,]\,|_{\tilde{x}=\tilde{X}(\alpha y,\alpha z)}$$

and that the term in the right hand side is the one we have already computed. In fact setting $v(y,z) := u(y + z\nu_\alpha(y))$, we get

$$\Delta_x u\,|_{x=X(y,z)} \;=\; \alpha^2(\Delta_{\tilde{y},M_{\tilde{z}}} + \partial_{\tilde{z}\tilde{z}} - H_{M_{\tilde{z}}}\partial_{\tilde{z}})\,[v(\alpha^{-1}\tilde{y},\alpha^{-1}\tilde{z})]\;|_{(\tilde{y},\tilde{z})=(\alpha y,\alpha z)}\;. \tag{3.21}$$

We can then use the discussion summarized in Lemma 3.1 to obtain a representation of Δ_x in $\mathcal{N}$ via the coordinates $X(y,t)$ in (3.20). Let us consider the local coordinates $Y_{k\alpha}$ of M_α in (3.9). .

Lemma 3.2: *In $\mathcal{N}$ we have*

$$\Delta_x = \partial_{zz} + \Delta_{M_{\alpha,z}} - H_{M_{\alpha,z}}\partial_z \;=$$

$$\partial_{zz}+\Delta_{M_\alpha}-\alpha^2 z\,|A(\alpha y)|^2\partial_z + \alpha z\,[a^1_{ij}(\alpha \mathrm{y},\alpha z)\partial_{ij}+\alpha b^1_i(\alpha \mathrm{y},\alpha z)\partial_i] + \alpha^3 z^2 b^1_3(\alpha \mathrm{y},\alpha z)\partial_z,$$

$$\Delta_{M_\alpha} = a^0_{ij}(\alpha y)\partial_{ij} + b^1_i(\alpha y)\partial_i, \quad (y,z)\in\alpha^{-1}\tilde{\mathcal{O}}, \quad y = Y_{k\alpha}(\mathrm{y})$$

where a^l_{ij}, b^l_j are smooth, bounded functions. In addition, for $k = 1,\ldots,m$,

$$a^l_{ij} = \delta_{ij}\delta_{0l} + O(r_\alpha^{-2}), \quad b^l_i = O(r_\alpha^{-3}), \quad b^1_3 = O(r_\alpha^{-6})\;,$$

as $r_\alpha(y) = |\alpha \mathrm{y}| \to \infty$, uniformly in z variable.

3.4. *The Euclidean Laplacian near M_α under a perturbation*

We now describe in coordinates relative to M_α the Euclidean Laplacian Δ_x, $x \in \mathbb{R}^3$, in a setting needed for the proof of our main results. The main idea is to introduce a smooth perturbation of the minimal surfaces, a priori unknown. We will need to compute the Euclidean Laplacian under this perturbation.

Let us consider a smooth function $h : M \to \mathbb{R}$, and the smooth map X_h defined as

$$X_h \, : \, M_\alpha \times \mathbb{R} \to \mathbb{R}^3, \quad (y,t) \longmapsto X_h(y,t) := y + (t + h(\alpha y)\,)\,\nu(\alpha y) \tag{3.22}$$

where ν is the unit normal vector to M. Let us consider an open subset $\mathcal{O}$ of $M_\alpha \times \mathbb{R}$ and assume that the map $X_h|_{\mathcal{O}}$ is one to one, and that it defines a diffeomorphism onto its image $\mathcal{N} = X_h(\mathcal{O})$. Then

$$x = X_h(y,t), \quad (y,t) \in \mathcal{O},$$

defines smooth coordinates to describe the open set $\mathcal{N}$ in $\mathbb{R}^3$. Moreover, the maps

$$x \;=\; X_h(\,Y_{k\alpha}(\mathtt{y})\,,t), \quad (\mathtt{y},t) \in (\mathcal{U}_{k\alpha} \times \mathbb{R}) \cap \mathcal{O}, \quad k = 1, \dots, N,$$

define local coordinates $(\mathtt{y}, t)$ to describe the region $\mathcal{N}$. We shall assume in addition that for certain small number $\delta > 0$, we have

$$\mathcal{O} \subset \{(y,t) \;/\; |t + h(\alpha y)| < \frac{\delta}{\alpha}\, \log(2 + r_\alpha(y)\,)\,\}. \tag{3.23}$$

We have the validity of the following expression for the Euclidean Laplacian operator in $\mathcal{N}$.

Lemma 3.3: *For $x = X_h(y,t)$, $(y,t) \in \mathcal{O}$ with $y = Y_{k\alpha}(\mathtt{y})$, $\mathtt{y} \in \mathcal{U}_{k\alpha}$, we have the validity of the identity*

$$\Delta_x = \partial_{tt} + \Delta_{M_\alpha} - \alpha^2[(t+h)|A|^2 + \Delta_M h]\partial_t \; - 2\alpha\, a^0_{ij}\, \partial_j h \partial_{it} \; +$$

$$\alpha(t+h)\,[a^1_{ij}\partial_{ij} - 2\alpha\, a^1_{ij}\, \partial_i h \partial_{jt} \; + \; \alpha\, b^1_i\, (\partial_i - \alpha \partial_i h \partial_t)\;] \; +$$

$$\alpha^3(t+h)^2 b^1_3 \partial_t \; + \; \alpha^2[\,a^0_{ij} + \alpha(t+h)a^1_{ij}\,]\partial_i h \partial_j h\, \partial_{tt}\; . \tag{3.24}$$

Here, in agreement with (3.10), $\Delta_{M_\alpha} = a^0_{ij}(\alpha y)\partial_{ij} + b^0_i(\alpha y)\partial_i$.
The functions a^1_{ij}, b^1_i, b^1_3 in the above expressions appear evaluated at the pair

$(\alpha y, \alpha(t + h(\alpha y))$, while the functions h, $\partial_i h$, $\Delta_M h$, $|A|^2$, a^0_{ij}, b^0_i are evaluated at αy In addition, for $k = 1, \dots, m$, $l = 0, 1$,

$$a^l_{ij} = \delta_{ij}\delta_{0l} + O(r_\alpha^{-2}), \quad b^l_i = O(r_\alpha^{-3}), \quad b^1_3 = O(r_\alpha^{-6}) \ ,$$

as $r_\alpha(y) = |\alpha \mathrm{y}| \to \infty$, uniformly in their second variables. The notation $\partial_j h$ refers to $\partial_j [h \circ Y_k]$.

Proof:

Let us consider a function u defined in $\mathcal{N}$, expressed in coordinates $x = X(y, z)$, and consider the expression of u in the coordinates $x = X_h(y, t)$, namely the function $v(\mathrm{y}, t)$ defined by the relation in local coordinates $y = Y_k(\mathrm{y})$,

$$v(\mathrm{y}, z - h(\alpha \mathrm{y})) = u(\mathrm{y}, z),$$

(by slight abuse of notation we are denoting just by h the function $h \circ Y_k$). Then we compute

$$\partial_i u = \partial_i v - \alpha \partial_t v \partial_i h, \quad \partial_z u = \partial_t v,$$

$$\partial_{ij} u = \partial_{ij} v - \alpha \partial_{it} v \partial_j h - \alpha \partial_{jt} v \partial_i h + \alpha^2 \partial_{tt} v \partial_i h \partial_j h - \alpha^2 \partial_t v \partial_{ij} h \, .$$

Observe that, in the notation for coefficients in Lemma 3.2,

$$a^0_{ij} \partial_{ij} h + b^0_i \partial_i h = \Delta_M h, \quad a^0_{ij} \partial_{ij} v + \alpha b^0_i \partial_i v = \Delta_{M_\alpha} v \ .$$

We find then

$$\Delta_x = \partial_{tt} + \Delta_{M_\alpha} - \alpha^2 [(t+h)|A|^2 + \Delta_M h] \partial_t \ - 2\alpha \, a^0_{ij} \, \partial_j h \partial_{it} \ +$$

$$\alpha(t+h) \, [a^1_{ij} \partial_{ij} - 2\alpha \, a^1_{ij} \, \partial_i h \partial_{jt} \ + \ \alpha (b^1_i \partial_i - \alpha b^1_i \partial_i h \partial_t) \ ] \ +$$

$$\alpha^3 (t+h)^2 b^1_3 \partial_t \ + \ \alpha^2 [\, a^0_{ij} + \alpha(t+h) a^1_{ij} \,] \partial_i h \partial_j h \, \partial_{tt} \qquad (3.25)$$

where all the coefficients are understood to be evaluated at $\alpha \mathrm{y}$ or $(\alpha \mathrm{y}, \alpha(t + h(\alpha \mathrm{y}))$. The desired properties of the coefficients have already been established. The proof of Lemma 3.3 is concluded. □

The proof actually yields that the coefficients a^1_{ij} and b^1_i can be further expanded as follows:

$$a^1_{ij} = a^1_{ij}(\alpha y, 0) + \alpha(t+h) \, a^{(2)}_{ij}(\alpha y, \alpha(t+h)) =: a^{1,0}_{ij} + \alpha(t+h) a^2_{ij},$$

with $a_{ij}^{(2)} = O(r_\alpha^{-3})$, and similarly

$$b_j^1 = b_j^1(\alpha y, 0) + \alpha(t+h)\, b_j^{(2)}(\alpha y, \alpha(t+h)) =: b_j^{1,0} + \alpha(t+h) b_j^2,$$

with $b_j^{(2)} = O(r_\alpha^{-4})$. As an example of the previous formula, let us compute the Laplacian of a function that separates variables t and y, that will be useful in §4.

Lemma 3.4: *Let $v(x) = k(y)\,\psi(t)$. Then the following holds.*

$$\Delta_x v \;=\; k\psi'' + \psi\,\Delta_{M_\alpha} k \,-\, \alpha^2[(t+h)|A|^2 + \Delta_M h]\, k\,\psi' \,-\, 2\alpha\, a_{ij}^0\, \partial_j h \partial_i k\, \psi' \,+$$

$$\alpha(t+h)\,[a_{ij}^{1,0}\partial_{ij} k\psi - 2\alpha\, a_{ij}^{1,0}\,\partial_j h \partial_i k\,\psi' \,+ \alpha(b_i^{1,0}\partial_i k\,\psi - \alpha b_i^{1,0}\partial_i h\, k\,\psi')\,] \,+$$

$$\alpha^2(t+h)^2\,[a_{ij}^2\partial_{ij} k\psi - 2\alpha\, a_{ij}^2\,\partial_j h \partial_i k\,\psi' \,+ \alpha(b_i^2\partial_i k\,\psi - \alpha b_i^2\partial_i h\, k\,\psi')\,] \,+$$

$$\alpha^3(t+h)^2 b_3^1\, k\,\psi' \,+\, \alpha^2[\,a_{ij}^0 + \alpha(t+h)a_{ij}^1\,]\partial_i h \partial_j h\, k\,\psi''\,. \qquad (3.26)$$

4. *Approximation of the solution and preliminary discussion*

4.1. *Approximation of order zero and its projection*

Let us consider a function h and sets $\mathcal{O}$ and $\mathcal{N}$ as in §3.4. Let $x = X_h(y,t)$ be the coordinates introduced in (3.22). At this point we shall make a more precise assumption about the function h. We need the following preliminary result whose proof we postpone for §7.2.

We consider a fixed m-tuple of real numbers $\beta = (\beta_1, \dots, \beta_m)$ such that

$$\sum_{i=1}^{m} \beta_j = 0. \qquad (4.1)$$

Lemma 4.1: *Given any real numbers $\beta_1, \dots, \beta_m$ satisfying (4.1), there exists a smooth function $h_0(y)$ defined on M such that*

$$\mathcal{J}(h_0) = \Delta_M h_0 + |A|^2 h_0 = 0 \quad \textit{in } M,$$

$$h_0(y) = (-1)^j \beta_j \log r + \theta \ \textit{as}\ r \to \infty \quad \textit{in}\ M_j \quad \textit{for all} \quad y \in M_j \,,$$

where θ satisfies

$$\|\theta\|_\infty + \|r^2 D\theta\|_\infty < +\infty \ . \tag{4.2}$$

We fix a function h_0 as in the above lemma and consider a function h in the form

$$h = h_0 + h_1.$$

We allow h_1 to be a parameter which we will adjust. For now we will assume that for a certain constant $\mathcal{K}$ we have

$$\|h_1\|_{L^\infty(M)} + \|(1+r^2)Dh_1\|_{L^\infty(M)} \ \le \ \mathcal{K}\alpha\,. \tag{4.3}$$

We want to find a solution to

$$S(u) := \Delta_x u + f(u) = 0.$$

We consider in the region $\mathcal{N}$ the approximation

$$u_0(x) := w(t) = w(z - h_0(\alpha y) - h_1(\alpha y))$$

where z designates the normal coordinate to M_α. Thus, whenever $\beta_j \neq 0$, the level sets $[u_0 = \lambda]$ for a fixed $\lambda \in (-1,1)$ departs logarithmically from the end $\alpha^{-1}M_j$ being still asymptotically catenoidal, more precisely it is described as the graph

$$y_3 = (\alpha^{-1}a_j + \beta_j)\ \log r \ + \ O(1) \text{ as } r \to \infty.$$

Note that, just as in the minimal surface case, the coefficients of the ends are balanced in the sense that they add up to zero.

It is clear that if two ends are parallel, say $a_{j+1} = a_j$, we need at least that $\beta_{j+1} - \beta_j \ \geq \ 0$, for otherwise the ends of this zero level set would eventually intersect. We recall that our further condition on these numbers is that these ends in fact diverge at a sufficiently fast rate:

$$\beta_{j+1} - \beta_j \ > \ 4 \ \max\{\sigma_-^{-1}, \sigma_+^{-1}\} \quad \text{if} \quad a_{j+1} = a_j \ . \tag{4.4}$$

We will explain later the role of this condition. Let us evaluate the error of approximation $S(u_0)$. Using Lemma 3.4 and the fact that $w'' + f(w) = 0$, we find

$$S(u_0) := \Delta_x u_0 + f(u_0) \ =$$

$$-\alpha^2[|A|^2 h_1 + \Delta_M h_1]\, w' +$$

$$-\alpha^2 |A|^2\, tw' \; + 2\; \alpha^2 a^0_{ij}\, \partial_i h_0 \partial_j h_0\, w'' +$$

$$\alpha^2\; a^0_{ij}\, (2\partial_i h_0 \partial_j h_1 + \partial_i h_1 \partial_j h_1\,)\, w'' +$$

$$2\alpha^3 (t + h_0 + h_1) a^1_{ij}\, \partial_i (h_0 + h_1) \partial_j (h_0 + h_1)\, w'' +$$

$$\alpha^3 (t + h_0 + h_1) b^1_i \partial_i (h_0 + h_1)\, w' + \alpha^3 (t + h_0 + h_1)^3 b^1_3 w' \qquad (4.5)$$

where the formula above has been broken into "sizes", keeping in mind that h_0 is fixed while $h_1 = O(\alpha)$. Since we want that u_0 be as close as possible to be a solution of (2.1), then we would like to choose h_1 in such a way that the quantity (4.5) be as small as possible. Examining the above expression, it does not look like we can do that in absolute terms. However part of the error could be made smaller by adjusting h_1. Let us consider the "L^2-projection" onto $w'(t)$ of the error for each fixed y, given by

$$\Pi(y) := \int_{-\infty}^{\infty} S(u_0)(y,t)\, w'(t)\, dt$$

where for now, and for simplicity we assume the coordinates are defined for all t, the difference with the integration is taken in all the actual domain for t produces only exponentially small terms in α^{-1}. Then we find

$$\Pi(y) = \alpha^2 (\Delta_M h_1 + h_1 |A|^2) \int_{-\infty}^{\infty} w'^2 dt + \alpha^3 \partial_i (h_0 + h_1) \int_{-\infty}^{\infty} b^1_i (t + h_0 + h_1) w'^2 dt +$$

$$\alpha^3 \partial_i (h_0 + h_1) \partial_j (h_0 + h_1) \int_{-\infty}^{\infty} (t + h_0 + h) a^1_{ij} w'' w' dt + \alpha^3 \int_{-\infty}^{\infty} (t + h_0 + h_1)^3 b^1_3 w'^2 dt \qquad (4.6)$$

where we have used $\int_{-\infty}^{\infty} tw'^2\, dt = \int_{-\infty}^{\infty} w'' w'\, dt = 0$ to get rid in particular of the terms of order α^2.

Making all these "projections" equal to zero amounts to a nonlinear differential equation for h of the form

$$\mathcal{J}(h_1) = \Delta_M h_1 + h_1 |A(y)|^2 = G_0(h_1) \quad y \in M \qquad (4.7)$$

where G_0 is easily checked to be a contraction mapping of small constant in h_1, in the ball radius $O(\alpha)$ with the C^1 norm defined by the expression in the left hand side of inequality (4.3). This is where the nondegeneracy assumption on the Jacobi operator $\mathcal{J}$ enters, since we would like to invert it, in such a way to set up equation (4.7) as a fixed point problem for a contraction mapping of a ball of the form (4.3).

4.2. *Improvement of approximation*

The previous considerations are not sufficient since even after adjusting optimally h, the error in absolute value does not necessarily decrease. As we observed, the "large" term in the error,

$$-\alpha^2|A|^2tw' + \alpha^2 a^0_{ij}\partial_i h_0 \partial_j h_0\, w''$$

did not contribute to the projection. In order to eliminate, or reduce the size of this remaining part $O(\alpha^2)$ of the error, we improve the approximation through the following argument. Let us consider the differential equation

$$\psi_0''(t) + f'(w(t))\psi_0(t) = tw'(t),$$

which has a unique bounded solution with $\psi_0(0) = 0$, given explicitly by the formula

$$\psi_0(t) = w'(t)\int_0^t w'(t)^{-2}\int_{-\infty}^s sw'(s)^2 ds\,.$$

Observe that this function is well defined and it is bounded since $\int_{-\infty}^{\infty} sw'(s)^2 ds = 0$ and $w'(t) \sim e^{-\sigma_\pm|t|}$ as $t \to \pm\infty$, with $\sigma_\pm > 0$. Note also that $\psi_1(t) = \frac{1}{2}tw'(t)$ solves

$$\psi_1''(t) + f'(w(t))\psi_1(t) = w''(t)\,.$$

We consider as a second approximation

$$u_1 = u_0 + \phi_1, \quad \phi_1(y,t) := \alpha^2|A(\alpha y)|^2\psi_0(t) - \alpha^2 a^0_{ij}\partial_i h_0\partial_j h_0(\alpha y)\,\psi_1(t)\ . \tag{4.8}$$

Let us observe that

$$S(u_0+\phi) = S(u_0)+\Delta_x\phi+f'(u_0)\phi+N_0(\phi), \quad N_0(\phi) = f(u_0+\phi)-f(u_0)-f'(u_0)\phi\,.$$

We have that

$$\partial_{tt}\phi_1 + f'(u_0)\phi_1 = \alpha^2|A(\alpha y)|^2tw' - \alpha^2 a^0_{ij}\partial_i h_0\partial_j h_0(\alpha y)\, w''\,.$$

Hence we get that the largest remaining term in the error is canceled. Indeed, we have

$$S(u_1) = S(u_0) - (2\alpha^2 a^0_{ij}\partial_i h_0 \partial_j h_0\, w'' - \alpha^2 |A(\alpha y)|^2 t w') + [\Delta_x - \partial_{tt}]\phi_1 + N_0(\phi_1).$$

Since ϕ_1 has size of order α^2, a smooth dependence in αy and it is of size $O(r_\alpha^{-2} e^{-\sigma|t|})$ using Lemma 3.4, we readily check that the "error created"

$$[\Delta_x - \partial_{tt}]\phi_1 + N_0(\phi_1) := -\alpha^4\,(\,|A|^2 t\psi_0' - a^0_{ij}\partial_i h_0 \partial_j h_0\, t\psi_1'\,)\,\Delta h_1 \; + R_0$$

satisfies

$$|R_0(y,t)| \le C\alpha^3 (1 + r_\alpha(y))^{-4} e^{-\sigma|t|}.$$

Hence we have eliminated the h_1-independent term $O(\alpha^2)$ that did not contribute to the projection $\Pi(y)$, and replaced it by one smaller and with faster decay. Let us be slightly more explicit for later reference. We have

$$S(u_1) := \Delta u_1 + f(u_1) \; =$$

$$-\alpha^2 [|A|^2 h_1 + \Delta_M h_1]\, w' + \alpha^2\, a^0_{ij}\,(\partial_i h_0 \partial_j h_1 + \partial_i h_1 \partial_j h_0 + \partial_i h_1 \partial_j h_1\,)\, w''$$

$$-\alpha^4\,(\,|A|^2 t\psi_0' - a^0_{ij}\partial_i h_0 \partial_j h_0\, t\psi_1'\,)\,\Delta_M h_1 \; + 2\alpha^3 (t+h) a^1_{ij}\,\partial_i h \partial_j h\, w'' + R_1 \tag{4.9}$$

where

$$R_1 = R_1(y, t, h_1(\alpha y), \nabla_M h_1(\alpha y))$$

with

$$|D_\imath R_1(y,t,\imath,\jmath)| + |D_\jmath R_1(y,t,\imath,\jmath)| + |R_1(y,t,\imath,\jmath)| \le C\alpha^3 (1 + r_\alpha(y))^{-4} e^{-\sigma|t|}$$

and the constant C above possibly depends on the number $\mathcal{K}$ of condition (4.3).

The above arguments are in reality the way we will actually solve the problem: two separate, but coupled steps are involved: (1) Eliminate the parts of the error that do not contribute to the projection Π and (2) Adjust h_1 so that the projection Π becomes identically zero.

4.3. *The condition of diverging ends*

Let us explain the reason to introduce condition (4.4) in the parameters β_j. To fix ideas, let us assume that we have two consecutive planar ends of M, M_j and M_{j+1}, namely with $a_j = a_{j+1}$ and with $d = b_{j+1} - b_j > 0$. Assuming that the normal in M_j points upwards, the coordinate t reads approximately as

$$t = x_3 - \alpha^{-1} b_j - h \quad \text{near } M_{j\alpha}, \quad t = \alpha^{-1} b_{j+1} - x_3 - h \quad \text{near } M_{j+1\alpha}.$$

If we let $h_0 \equiv 0$ both on $M_{j\alpha}$ and $M_{j+1\alpha}$ which are separated at distance d/α, then a good approximation in the entire region between $M_{j\alpha}$ and $M_{j+1\alpha}$ that matches the parts of $w(t)$ coming both from M_j and M_{j+1} should read near M_j approximately as

$$w(t) + w(\alpha^{-1}d - t) - 1.$$

When computing the error of approximation, we observe that the following additional term arises near $M_{j\alpha}$:

$$E := f(\,w(t) + w(\alpha^{-1}d - t) - 1\,) \; - \; f(w(t)) - f(\,w(\alpha^{-1}d - t)\,) \;\sim$$

$$\sim \; [f'(w(t)) - f'(1)\,]\,(\,w(\alpha^{-1}d - t) - 1\,)\,.$$

Now in the computation of the projection of the error this would give rise to

$$\int_{-\infty}^{\infty} [\,f'(w(t)) - f'(1)\,]\,(\,w(\alpha^{-1}d - t) \; - \; 1\,)\,w'(t)\,dt \;\sim c_* e^{-\sigma + \frac{d}{\alpha}}.$$

where $c_* \neq 0$ is a constant. Thus equation (4.7) for h_1 gets modified with a term which even though very tiny, it has no decay as $|y| \to +\infty$ on M_j, unlike the others involved in the operator G_0 in (4.7). That terms eventually dominates and the equation for h_1 for very large r would read in M_j as

$$\Delta_M h_1 \sim e^{-\frac{\sigma}{\alpha}} \neq 0,$$

which is inconsistent with the assumption that h is bounded. Worse yet, its solution would be quadratic thus eventually intersecting another end. This nuisance is fixed with the introduction of h_0 satisfying condition (4.4). In that case the term E created above will now read near $M_{j\alpha}$ as

$$E \sim C e^{-\sigma + \frac{d}{\alpha}}\, e^{-(\beta_{j+1} - \beta_j)\log r_\alpha}\, e^{-\sigma|t|} \; = \; O(e^{-\frac{\sigma}{\alpha}} r_\alpha^{-4} e^{-\sigma|t|})$$

which is qualitatively of the same type of the other terms involved in the computation of the error.

4.4. *The global first approximation*

The approximation $u_1(x)$ in (4.2) will be sufficient for our purposes, however it is so far defined only in a region of the type $\mathcal{N}$ which we have not made precise yet. Since we are assuming that M_α is connected, the fact that M_α is properly embedded implies that $\mathbb{R}^3 \setminus M_\alpha$ consists of precisely two components S_- and S_+. Let us use the convention that ν points in the direction of S_+. Let us consider the function $\mathbb{H}$ defined in $\mathbb{R}^3 \setminus M_\alpha$ as

$$\mathbb{H}(x) := \begin{cases} 1 & \text{if } x \in S_+ \\ -1 & \text{if } x \in S_- \end{cases}. \tag{4.10}$$

Then our approximation $u_1(x)$ approaches $\mathbb{H}(x)$ at an exponential rate $O(e^{-\sigma_\pm |t|})$ as $|t|$ increases. The global approximation we will use consists simply of interpolating u_1 with $\mathbb{H}$ sufficiently well-inside $\mathbb{R}^3 \setminus M_\alpha$ through a cut-off in $|t|$. In order to avoid the problem described in §4.3 and having the coordinates (y,t) well-defined, we consider this cut-off to be supported in a region y-dependent that expands logarithmically in r_α. Thus we will actually consider a region $\mathcal{N}_\delta$ expanding at the ends, thus becoming wider as $r_\alpha \to \infty$ than the set $\mathcal{N}_\delta^\alpha$ previously considered, where the coordinates are still well-defined.

We consider the open set $\mathcal{O}$ in $M_\alpha \times \mathbb{R}$ defined as

$$\mathcal{O} = \{ (y,t) \in M_\alpha \times \mathbb{R}, \quad |t + h_1(\alpha y)| < \frac{\delta}{\alpha} + 4 \max\{\sigma_-^{-1}, \sigma_+^{-1}\} \log(1 + r_\alpha(y)) =: \rho_\alpha(y) \} \tag{4.11}$$

where δ is small positive number. We consider the the region $\mathcal{N} =: \mathcal{N}_\delta$ of points x of the form

$$x = X_h(y,t) = y + (t + h_0(\alpha y) + h_1(\alpha y))\, \nu(\alpha y), \quad (y,t) \in \mathcal{O},$$

namely $\mathcal{N}_\delta = X_h(\mathcal{O})$. The coordinates (y,t) are well-defined in $\mathcal{N}_\delta$ for any sufficiently small δ: indeed the map X_h is one to one in $\mathcal{O}$ thanks to assumption (4.4) and the fact that $h_1 = O(\alpha)$. Moreover, Lemma 3.3 applies in $\mathcal{N}_\delta$.

Let $\eta(s)$ be a smooth cut-off function with $\eta(s) = 1$ for $s < 1$ and $= 0$ for $s > 2$. and define

$$\eta_\delta(x) := \begin{cases} \eta(\,|t + h_1(\alpha y)| - \rho_\alpha(y) - 3) & \text{if } x \in \mathcal{N}_\delta\,, \\ 0 & \text{if } x \notin \mathcal{N}_\delta \end{cases} \tag{4.12}$$

where ρ_α is defined in (4.11). Then we let our global approximation $\mathtt{w}(x)$ be simply defined as

$$\mathtt{w} := \eta_\delta u_1 + (1 - \eta_\delta)\mathbb{H} \tag{4.13}$$

where $\mathbb{H}$ is given by (4.10) and $u_1(x)$ is just understood to be $\mathbb{H}(x)$ outside $\mathcal{N}_\delta$.

Since $\mathbb{H}$ is an exact solution in $\mathbb{R}^3 \setminus M_\delta$, the global error of approximation is simply computed as

$$S(\mathtt{w}) \;=\; \Delta \mathtt{w} + f(\mathtt{w}) \;=\; \eta_\delta S(u_1) \quad + E \tag{4.14}$$

where

$$E = 2\nabla\eta_\delta \nabla u_1 + \Delta\eta_\delta(u_1 - \mathbb{H}) \,+\, f(\eta_\delta u_1 + (1-\eta_\delta)\mathbb{H})\,) \,-\, \eta_\delta f(u_1)\,.$$

The new error terms created are of exponentially small size $O(e^{-\frac{\sigma}{\alpha}})$ but have in addition decay with r_α. In fact we have

$$|E| \le Ce^{-\frac{\delta}{\alpha}}\, r_\alpha^{-4}.$$

Let us observe that $|t + h_1(\alpha y)| = |z - h_0(\alpha y)|$ where z is the normal coordinate to M_α, hence η_δ does not depend on h_1, in particular the term $\Delta\eta_\delta$ does involves second derivatives of h_1 on which we have not made assumptions yet.

5. *The proof of Theorem 6*

The proof of Theorem 6 involves various ingredients whose detailed proofs are fairly technical. In order to keep the presentation as clear as possible, in this section we carry out the proof, skimming it from several (important) steps, which we state as lemmas or propositions, with complete proofs postponed for the subsequent sections.

We look for a solution u of the Allen Cahn equation (2.1) in the form

$$u = \mathtt{w} + \varphi \tag{5.1}$$

where $\mathtt{w}$ is the global approximation defined in (4.13) and φ is in some suitable sense small. Thus we need to solve the following problem

$$\Delta\varphi + f'(\mathtt{w})\varphi = -S(\mathtt{w}) - N(\varphi) \tag{5.2}$$

where

$$N(\varphi) = f(\mathtt{w} + \varphi) - f(\mathtt{w}) - f'(\mathtt{w})\varphi.$$

Next we introduce various norms that we will use to set up a suitable functional analytic scheme for solving problem (5.2). For a function $g(x)$ defined in $\mathbb{R}^3$, $1 < p \le +\infty$, $\mu > 0$, and $\alpha > 0$ we write

$$\|g\|_{p,\mu,*} := \sup_{x\in\mathbb{R}^3} (1 + r(\alpha x))^{\mu} \|g\|_{L^p(B(x,1))}, \quad r(x', x_3) = |x'| \ .$$

On the other hand, given numbers $\mu \ge 0$, $0 < \sigma < \min\{\sigma_+, \sigma_-\}$, $p > 3$, and functions $g(y,t)$ and $\phi(y,t)$ defined in $M_\alpha \times \mathbb{R}$ we consider the norms

$$\|g\|_{p,\mu,\sigma} \ := \sup_{(y,t)\in M_\alpha\times\mathbb{R}} r_\alpha(y)^{\mu}\, e^{\sigma|t|} \left(\int_{B((y,t),1)} |f|^p \, dV_\alpha \right)^{\frac{1}{p}} . \tag{5.3}$$

Consistently we set

$$\|g\|_{\infty,\mu,\sigma} \ := \sup_{(y,t)\in M_\alpha\times\mathbb{R}} r_\alpha(y)^{\mu}\, e^{\sigma|t|}\, \|f\|_{L^\infty(B((y,t),1))} \tag{5.4}$$

and let

$$\|\phi\|_{2,p,\mu,\sigma} := \|D^2\phi\|_{p,\mu,\sigma} + \|D\phi\|_{\infty,\mu,\sigma} + \|\phi\|_{\infty,\mu,\sigma} \, . \tag{5.5}$$

We consider also for a function $g(y)$ defined in M the L^p-weighted norm

$$\|f\|_{p,\beta} \ := \left(\int_M |f(y)|^p \, (1 + |y|^{\beta}\,)^p \, dV(y) \right)^{1/p} \ = \ \|\, (1 + |y|^{\beta})\, f\, \|_{L^p(M)} \tag{5.6}$$

where $p > 1$ and $\beta > 0$.

We assume in what follows, that for a certain constant $\mathcal{K} > 0$ and $p > 3$ we have that the parameter function $h_1(y)$ satisfies

$$\|h_1\|_* := \|h_1\|_{L^\infty(M)} + \|(1+r^2)Dh_1\|_{L^\infty(M)} + \|D^2 h_1\|_{p,4-\frac{4}{p}} \ \le \ \mathcal{K}\alpha \, . \tag{5.7}$$

Next we reduce problem (5.2) to solving one qualitatively similar (equation (5.20) below) for a function $\phi(y,t)$ defined in the whole space $M_\alpha \times \mathbb{R}$.

5.1. *Step 1: the gluing reduction*

We will follow the following procedure. Let us consider again $\eta(s)$, a smooth cut-off function with $\eta(s) = 1$ for $s < 1$ and $= 0$ for $s > 2$, and define

$$\zeta_n(x) \ := \begin{cases} \eta(\,|t + h_1(\alpha y)| - \frac{\delta}{\alpha} + n) & \text{if } x \in \mathcal{N}_\delta \\ 0 & \text{if } x \notin \mathcal{N}_\delta \end{cases} . \tag{5.8}$$

We look for a solution $\varphi(x)$ of problem (5.2) of the following form

$$\varphi(x) = \zeta_2(x)\phi(y,t) + \psi(x) \tag{5.9}$$

where ϕ is defined in entire $M_\alpha \times \mathbb{R}$, $\psi(x)$ is defined in $\mathbb{R}^3$ and $\zeta_2(x)\phi(y,t)$ is understood as zero outside $\mathcal{N}_\delta$.

We compute, using that $\zeta_2 \cdot \zeta_1 = \zeta_1$,

$$S(\mathtt{w}+\varphi) \;=\; \Delta\varphi + f'(\mathtt{w})\varphi + N(\varphi) + S(\mathtt{w}) =$$

$$\zeta_2\;[\,\Delta\phi + f'(u_1)\phi \;+\; \zeta_1(f'(u_1)+H(t))\psi \;+\zeta_1 N(\psi+\phi) + S(u_1)\,] \;+$$

$$\Delta\psi - [\,(1-\zeta_1)f'(u_1) + \zeta_1 H(t)\,]\psi \,+$$

$$(1-\zeta_2)S(\mathtt{w}) + (1-\zeta_1)N(\psi+\zeta_2\phi) + 2\nabla\zeta_1\nabla\phi + \phi\Delta\zeta_1 \tag{5.10}$$

where $H(t)$ is any smooth, strictly negative function satisfying

$$H(t) \;=\; \begin{cases} f'(+1) & \text{if } t>1\,, \\ f'(-1) & \text{if } t<-1 \end{cases}\;.$$

Thus, we will have constructed a solution $\varphi = \zeta_2\phi + \psi$ to problem (5.2) if we require that the pair (ϕ,ψ) satisfies the following coupled system

$$\Delta\phi + f'(u_1)\phi + \zeta_1(f'(u_1) - H(t))\psi + \zeta_1 N(\psi+\phi) + S(u_1) = 0 \text{ for } |t| < \frac{\delta}{\alpha} + 3 \tag{5.11}$$

$$\Delta\psi + [\,(1-\zeta_1)f'(u_1) + \zeta_1 H(t)\,]\psi \,+$$

$$(1-\zeta_2)S(\mathtt{w}) + (1-\zeta_1)N(\psi+\zeta_2\phi) + 2\nabla\zeta_1\nabla\phi + \phi\Delta\zeta_1 \;=\; 0 \quad \text{in } \mathbb{R}^3\,. \tag{5.12}$$

In order to find a solution to this system we will first extend equation (5.11) to entire $M_\alpha \times \mathbb{R}$ in the following manner. Let us set

$$\mathtt{B}(\phi) = \zeta_4[\Delta_x - \partial_{tt} - \Delta_{y,M_\alpha}\,]\,\phi \tag{5.13}$$

where Δ_x is expressed in (y,t) coordinates using expression (3.24) and $\mathtt{B}(\phi)$ is understood to be zero for $|t+h_1| > \frac{\delta}{\alpha} + 5$. The other terms in equation (5.11) are simply extended as zero beyond the support of ζ_1. Thus we consider the extension of equation (5.11) given by

$$\partial_{tt}\phi + \Delta_{y,M_\alpha}\phi + \mathtt{B}(\phi) + f'(w(t))\phi = -\tilde S(u_1)$$

$$-\{[f'(u_1) - f'(w)]\phi + \zeta_1(f'(u_1) - H(t))\psi + \zeta_1 N(\psi + \phi)\} \ \text{in} \ \in M_\alpha \times \mathbb{R}, \tag{5.14}$$

where we set, with reference to expression (4.9),

$$\tilde S(u_1) = -\alpha^2[|A|^2 h_1 + \Delta_M h_1]\, w' + \alpha^2\, a^0_{ij}\,(2\partial_i h_0 \partial_j h_1 + \partial_i h_1 \partial_j h_1\,)\, w''$$

$$-\alpha^4\,(\,|A|^2 t\psi_0' - a^0_{ij}\partial_i h_0 \partial_j h_0\, t\psi_1'\,)\,\Delta h_1 + \zeta_4\,[\,\alpha^3(t+h)a^1_{ij}\,\partial_i h \partial_j h\, w'' + R_1(y,t)\,] \tag{5.15}$$

and, we recall

$$R_1 = R_1(y, t, h_1(\alpha y), \nabla_M h_1(\alpha y))$$

with

$$|D_\imath R_1(y,t,\imath,\jmath)| + |D_\jmath R_1(y,t,\imath,\jmath)| + |R_1(y,t,\imath,\jmath)| \le C\alpha^3(1+r_\alpha(y))^{-4} e^{-\sigma|t|}. \tag{5.16}$$

In summary $\tilde S(u_1)$ coincides with $S(u_1)$ if $\zeta_4 = 1$ while outside the support of ζ_4, their parts that are not defined for all t are cut-off.

To solve the resulting system (5.12)-(5.14), we find first solve equation (5.12) in ψ for a given ϕ a small function in absolute value. Noticing that the potential $[\,(1-\zeta_1)f'(u_1) + \zeta_1 H(t)\,]$ is uniformly negative, so that the linear operator is qualitatively like $\Delta - 1$ and using contraction mapping principle, a solution $\psi = \Psi(\phi)$ is found according to the following lemma, whose detailed proof we carry out in §8.1.2.

Lemma 5.1: *For all sufficiently small α the following holds. Given ϕ with $\|\phi\|_{2,p,\mu,\sigma} \le 1$, there exists a unique solution $\psi = \Psi(\phi)$ of problem* (5.12) *such that*

$$\|\psi\|_X := \|D^2\psi\|_{p,\mu,*} + \|\psi\|_{p,\mu,*} \le Ce^{-\frac{\sigma\delta}{\alpha}}. \tag{5.17}$$

Besides, Ψ satisfies the Lipschitz condition

$$\|\Psi(\phi_1) - \Psi(\phi_2)\|_X \le C\, e^{-\frac{\sigma\delta}{\alpha}}\, \|\phi_1 - \phi_2\|_{2,p,\mu,\sigma}\ . \tag{5.18}$$

Thus we replace $\psi = \Psi(\phi)$ in the first equation (5.11) so that by setting

$$\mathtt{N}(\phi) := \mathtt{B}(\phi) + [f'(u_1) - f'(w)]\phi + \zeta_1(f'(u_1) - H(t))\Psi(\phi) + \zeta_1 N(\Psi(\phi) + \phi), \tag{5.19}$$

our problem is reduced to finding a solution ϕ to the following nonlinear, nonlocal problem in $M_\alpha \times \mathbb{R}$.

$$\partial_{tt}\phi \, + \, \Delta_{y,M_\alpha}\phi \, + f'(w)\phi \, = \, -\tilde{S}(u_1) - \mathtt{N}(\phi) \quad \text{in } M_\alpha \times \mathbb{R}. \tag{5.20}$$

Thus, we concentrate in the remaining of the proof in solving equation (5.20). As we hinted in §4.2, we will find a solution of problem (5.20) by considering two steps: (1) "Improving the approximation", roughly solving for ϕ that eliminates the part of the error that does not contribute to the "projections" $\int[\tilde{S}(U_1) + \mathtt{N}(\phi)]w'(t)dt$, which amounts to a nonlinear problem in ϕ, and (2) Adjust h_1 in such a way that the resulting projection is actually zero. Let us set up the scheme for step (1) in a precise form.

5.2. *Step 2: Eliminating terms not contributing to projections*

Let us consider the problem of finding a function $\phi(y,t)$ such that for a certain function $c(y)$ defined in M_α, we have

$$\begin{aligned} &\partial_{tt}\phi \, + \, \Delta_{y,M_\alpha}\phi \, = \, -\tilde{S}(u_1) - \mathtt{N}(\phi) \, + \, c(y)w'(t) \quad \text{in } M_\alpha \times \mathbb{R}, \\ &\int_{\mathbb{R}} \phi(y,t)\, w'(t)\, dt = 0, \quad \text{for all} \quad y \in M_\alpha\,. \end{aligned} \tag{5.21}$$

Solving this problem for ϕ amounts to "eliminating the part of the error that does not contribute to the projection" in problem (5.20). To justify this phrase let us consider the associated linear problem in $M_\alpha \times \mathbb{R}$

$$\begin{aligned} \partial_{tt}\phi + \Delta_{y,M_\alpha}\phi + f'(w(t))\phi = g(y,t) + c(y)w'(t), \quad \text{for all} \quad (y,t) \in M_\alpha \times \mathbb{R}, \\ \int_{-\infty}^{\infty} \phi(y,t)\, w'(t)\, dt = 0, \quad \text{for all} \quad y \in M_\alpha\,. \end{aligned} \tag{5.22}$$

Assuming that the corresponding operations can be carried out, let us multiply the equation by $w'(t)$ and integrate in t for fixed y. We find that

$$\Delta_{y,M_\alpha} \int_{\mathbb{R}} \phi(y,t)\, w'\, dt + \int_{\mathbb{R}} \phi(y,t)\, [w''' + f'(w)w']\, dt = \int_{\mathbb{R}} g\, w' + c(y) \int_{\mathbb{R}} {w'}^2\,.$$

The left hand side of the above identity is zero and then we find that

$$c(y) = -\frac{\int_{\mathbb{R}} g(y,t) w' dt}{\int_{\mathbb{R}} {w'}^2 dt}\,, \tag{5.23}$$

hence a ϕ solving problem (5.22). ϕ *precisely* solves or *eliminates* the part of g which does not contribute to the projections in the equation $\Delta\phi + f'(w)\phi = g$, namely the same equation with g replaced by $\tilde{g}$ given by

$$\tilde{g}(y,t) = g(y,t) - \frac{\int_{\mathbb{R}} f(y,\cdot) w'}{\int_{\mathbb{R}} {w'}^2}\, w'(t)\,. \tag{5.24}$$

The term $c(y)$ in problem (5.21) has a similar role, except that we cannot find it so explicitly.

In order to solve problem (5.21) we need to devise a theory to solve problem (5.22) where we consider a class of right hand sides g with a qualitative behavior similar to that of the error $S(u_1)$. As we have seen in (5.15), typical elements in this error are of the type $O((1+r_\alpha(y))^{-\mu} e^{-\sigma|t|})$, so this is the type of functions $g(y,t)$ that we want to consider. This is actually the motivation to introduce the norms (5.3), (5.4) and (5.5). We will prove that problem (5.22) has a unique solution ϕ which respects the size of g in norm (5.3) up to its second derivatives, namely in the norm (5.5). The following fact holds.

Proposition 5.1: *Given $p > 3$, $\mu \geq 0$ and $0 < \sigma < \min\{\sigma_-, \sigma_+\}$, there exists a constant $C > 0$ such that for all sufficiently small $\alpha > 0$ the following holds. Given f with $\|g\|_{p,\mu,\sigma} < +\infty$, then Problem* (5.22) *with $c(y)$ given by* (5.23)*, has a unique solution ϕ with $\|\phi\|_{\infty,\mu,\sigma} < +\infty$. This solution satisfies in addition that*

$$\|\phi\|_{2,p,\mu,\sigma} \leq C \|g\|_{p,\mu,\sigma}\ . \tag{5.25}$$

We will prove this result in §6 . After Proposition 5.1, solving Problem (5.21) for a small ϕ is easy using the small Lipschitz character of the terms involved in the operator $\mathtt{N}(\phi)$ in (5.19) and contraction mapping principle. The error term $\tilde{S}(u_1)$ satisfies

$$\|\tilde{S}(u_1) + \alpha^2 \Delta h_1 w'\|_{p,4,\sigma} \leq C\alpha^3. \tag{5.26}$$

Using this, and the fact that $\mathtt{N}(\phi)$ defines a contraction mapping in a ball center zero and radius $O(\alpha^3)$ in $\|\ \|_{2,p,4,\sigma}$, we conclude the existence of a unique small solution ϕ to problem (5.21) whose size is $O(\alpha^3)$ for this norm. This solution ϕ turns out to define an operator in h_1 $\phi = \Phi(h_1)$

which is Lipschitz in the norms $\| \, \|_*$ appearing in condition (5.7). In precise terms, we have the validity of the following result, whose detailed proof we postpone for §8.2.

Proposition 5.2: *Assume* $p > 3$, $0 \le \mu \le 3$, $0 < \sigma < \min\{\sigma_+, \sigma_-\}$. *There exists a* $K > 0$ *such that problem* (8.8) *has a unique solution* $\phi = \Phi(h_1)$ *such that*

$$\|\phi\|_{2,p,\mu,\sigma} \le K\alpha^3 .$$

Besides, Φ *has a Lipschitz dependence on* h_1 *satisfying* (5.7) *in the sense that*

$$\|\Phi(h_1) - \Phi(h_2)\|_{2,p,\mu,\sigma} \le C\alpha^2 \|h_1 - h_2\|_* . \tag{5.27}$$

5.3. *Step 3: Adjusting h_1 to make the projection zero*

In order to conclude the proof of the theorem, we have to carry out the second step, namely adjusting h_1, within a region of the form (5.7) for suitable $\mathcal{K}$ in such a way that the "projections" are identically zero, namely making zero the function $c(y)$ found for the solution $\phi = \Phi(h_1)$ of problem (5.21). Using expression (5.23) for $c(y)$ we find that

$$c(y) \int_{\mathbb{R}} {w'}^2 = \int_{\mathbb{R}} \tilde{S}(u_1) w' \, dt + \int_{\mathbb{R}} \mathtt{N}(\Phi(h_1)\,)\, w' \, dt . \tag{5.28}$$

Now, setting $c_* := \int_{\mathbb{R}} {w'}^2 dt$ and using same computation employed to derive formula (4.6), we find from expression (5.15) that

$$\int_{\mathbb{R}} \tilde{S}(u_1)(y,t)\, w'(t)\, dt \ = \ -c_*\, \alpha^2 (\Delta_M h_1 + h_1 |A|^2) \ + \ c_* \alpha^2 G_1(h_1)$$

where

$$c_* G_1(h_1) \ = \ -\alpha^2\, \Delta h_1 \, (\, |A|^2 \int_{\mathbb{R}} t\psi_0' w' \, dt - a_{ij}^0 \partial_i h_0 \partial_j h_0 \int_{\mathbb{R}} t\psi_1' w' \, dt \,) \ +$$

$$\alpha\, \partial_i (h_0 {+} h_1) \partial_j (h_0 {+} h_1) \int_{\mathbb{R}} \zeta_4 (t {+} h) a_{ij}^1 w'' w' \, dt + \alpha^{-2} \int_{\mathbb{R}} \zeta_4 \, R_1(y,t,h_1, \nabla_M h_1\,)\, w' \, dt \tag{5.29}$$

and we recall that R_1 is of size $O(\alpha^3)$ in the sense (5.16). Thus, setting

$$c_* G_2(h_1) \ := \ \alpha^{-2} \int_{\mathbb{R}} \mathtt{N}(\Phi(h_1)\,)\, w' \, dt, \quad G(h_1) := G_1(h_1) + G_2(h_1), \tag{5.30}$$

we find that the equation $c(y) = 0$ is equivalent to the problem

$$\mathcal{J}(h_1) = \Delta_M h_1 + |A|^2 h_1 = G(h_1) \quad \text{in } M. \tag{5.31}$$

Therefore, we will have proven Theorem 6 if we find a function h_1 defined on M satisfying constraint (5.7) for a suitable $\mathcal{K}$ that solves equation (5.31). Again, this is not so direct since the operator $\mathcal{J}$ has a nontrivial bounded kernel. Rather than solving directly (5.31), we consider first a projected version of this problem, namely that of finding h_1 such that for certain scalars $c_1, \dots, c_J$ we have

$$\mathcal{J}(h_1) = G(h_1) + \sum_{i=1}^{J} \frac{c_i}{1+r^4}\,\hat{z}_i \quad \text{in } M,$$

$$\int_M \frac{\hat{z}_i h}{1+r^4}\, dV \;=\; 0, \quad i = 1, \dots J. \tag{5.32}$$

Here $\hat{z}_1, ..., \hat{z}_J$ is a basis of the vector space of bounded Jacobi fields.

In order to solve problem (5.32) we need a corresponding linear invertibility theory. This leads us to consider the linear problem

$$\mathcal{J}(h) = f + \sum_{i=1}^{J} \frac{c_i}{1+r^4}\,\hat{z}_i \quad \text{in } M,$$

$$\int_M \frac{\hat{z}_i h}{1+r^4}\, dV \;=\; 0, \quad i = 1, \dots J. \tag{5.33}$$

Here $\hat{z}_1, ..., \hat{z}_J$ are bounded, linearly independent Jacobi fields, and J is the dimension of the vector space of bounded Jacobi fields.

We will prove in §7.1 the following result.

Proposition 5.3: *Given $p > 2$ and f with $\|f\|_{p,4-\frac{4}{p}} < +\infty$, there exists a unique bounded solution h of problem* (5.33). *Moreover, there exists a positive number $C = C(p, M)$ such that*

$$\|h\|_* := \|h\|_\infty + \|\,(1+|y|^2)\,Dh\|_\infty + \|\,D^2 h\,\|_{p,4-\frac{4}{p}} \;\le\; C\|f\|_{p,4-\frac{4}{p}}\,. \tag{5.34}$$

Using the fact that G is a small operator of size $O(\alpha)$ uniformly on functions h_1 satisfying (5.7), Proposition 5.3 and contraction mapping principle yield the following result, whose detailed proof we carry out in §9.

Proposition 5.4: *Given $p > 3$, there exists a number $\mathcal{K} > 0$ such that for all sufficiently small $\alpha > 0$ there is a unique solution h_1 of problem* (5.32) *that satisfies constraint* (5.7).

5.4. *Step 3: Conclusion*

At the last step we prove that the constants c_i found in equation (5.32) are in reality all zero, without the need of adjusting any further parameters but rather as a consequence of the natural invariances of the of the full equation. The key point is to realize what equation has been solved so far.

First we observe the following. For each h_1 satysfying (5.7), the pair (ϕ, ψ) with $\phi = \Phi(h_1)$, $\psi = \Psi(\phi)$, solves the system

$$\Delta\phi + f'(u_1)\phi + \zeta_1(f'(u_1) - H(t))\psi + \zeta_1 N(\psi + \phi) + S(u_1) = c(y)w'(t) \text{ for } |t| < \frac{\delta}{\alpha} + 3$$

$$\Delta\psi + [\,(1-\zeta_1)f'(u_1) + \zeta_1 H(t)\,]\psi +$$

$$(1-\zeta_2)S(\mathtt{w}) + (1-\zeta_1)N(\psi + \zeta_2\phi) + 2\nabla\zeta_1\nabla\phi + \phi\Delta\zeta_1 \;=\; 0 \quad \text{ in } \mathbb{R}^3 .$$

Thus setting

$$\varphi(x) = \zeta_2(x)\phi(y,t) + \psi(x), \quad u = \mathtt{w} + \varphi\,,$$

we find from formula (5.10) that

$$\Delta u + f(u) = S(\mathtt{w} + \varphi) \;=\; \zeta_2 c(y)\, w'(t)\ .$$

On the other hand choosing h_1 as that given in Proposition 5.4 which solves problem (5.32), amounts precisely to making

$$c(y) = c_*\alpha^2 \sum_{i=1}^{J} c_i \frac{\hat z_i(\alpha y)}{1 + r_\alpha(y)^4}$$

for certain scalars c_i. In summary, we have found h_1 satisfying constraint (5.7) such that

$$u = \mathtt{w} + \zeta_2(x)\Phi(h_1) + \Psi(\Phi(h_1)\,) \tag{5.35}$$

solves the equation

$$\Delta u + f(u) = \sum_{j=1}^{J} \frac{\tilde c_i}{1 + r_\alpha^4}\, \hat z_i(\alpha y) w'(t) \tag{5.36}$$

where $\tilde c_i = c_*\alpha^2 c_i$. Testing equation (5.36) against the generators of the rigid motions $\partial_i u$ $i = 1,2,3$, $-x_2\partial_1 u + x_1\partial_2 u$, and using the balancing formula for the minimal surface and the zero average of the numbers β_j in the definition of h_0, we find a system of equations that leads us to $c_i = 0$ for all i, thus conclude the proof. We will carry out the details in §10.

In sections §6-10 we will complete the proofs of the intermediate steps of the program designed in this section.

6. *The linearized operator*

In this section we will prove Proposition 5.1. At the core of the proof of the stated a priori estimates is the fact that the one-variable solution w of (2.1) is *nondegenerate* in $L^\infty(\mathbb{R}^3)$ in the sense that the linearized operator

$$L(\phi) = \Delta_y \phi + \partial_{tt}\phi + f'(w(t))\phi, \quad (y,t) \in \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R},$$

is such that the following property holds.

Lemma 6.1: *Let ϕ be a bounded, smooth solution of the problem*

$$L(\phi) = 0 \quad \text{in } \mathbb{R}^2 \times \mathbb{R}. \tag{6.1}$$

Then $\phi(y,t) = Cw'(t)$ for some $C \in \mathbb{R}$.

Proof: We begin by reviewing some known facts about the one-dimensional operator $L_0(\psi) = \psi'' + f'(w)\psi$. Assuming that $\psi(t)$ and its derivative decay sufficiently fast as $|t| \to +\infty$ and defining $\psi(t) = w'(t)\rho(t)$, we get that

$$\int_{\mathbb{R}} [|\psi'|^2 - f'(w)\psi^2]\, dt \;=\; \int_{\mathbb{R}} L_0(\psi)\psi\, dt \;=\; \int_{\mathbb{R}} {w'}^2 |\rho'|^2\, dt,$$

therefore this quadratic form is positive unless ψ is a constant multiple of w'. Using this and a standard compactness argument we get that there is a constant $\gamma > 0$ such that whenever $\int_{\mathbb{R}} \psi w' = 0$ with $\psi \in H^1(\mathbb{R})$ we have that

$$\int_{\mathbb{R}} (\, |\psi'|^2 - f'(w)\psi^2 \,)\, dt \;\geq\; \gamma \int_{\mathbb{R}} (\, |\psi'|^2 + |\psi|^2 \,)\, dt. \tag{6.2}$$

Now, let ϕ be a bounded solution of equation (6.1). We claim that ϕ has exponential decay in t, uniform in y. Let us consider a small number $\sigma > 0$ so that for a certain $t_0 > 0$ and all $|t| > t_0$ we have that

$$f'(w) < -2\sigma^2.$$

Let us consider for $\varepsilon > 0$ the function

$$g_\varepsilon(t,y) = e^{-\sigma(|t|-t_0)} + \varepsilon \sum_{i=1}^{2} \cosh(\sigma y_i)$$

Then for $|t| > t_0$ we get that

$$L(g_\delta) < 0 \quad \text{if } |t| > t_0.$$

As a conclusion, using maximum principle, we get

$$|\phi| \le \|\phi\|_\infty \, g_\varepsilon \quad \text{if } |t| > t_0,$$

and letting $\varepsilon \to 0$ we then get

$$|\phi(y,t)| \ \le \ C\|\phi\|_\infty e^{-\sigma|t|} \quad \text{if } |t| > t_0 \ .$$

Let us observe the following fact: the function

$$\tilde\phi(y,t) = \phi(y,t) - \left(\int_{\mathbb{R}} w'(\zeta)\, \phi(y,\zeta)\, d\zeta \right) \frac{w'(t)}{\int_{\mathbb{R}} {w'}^2}$$

also satisfies $L(\tilde\phi) = 0$ and, in addition,

$$\int_{\mathbb{R}} w'(t)\, \tilde\phi(y,t)\, dt = 0 \quad \text{for all} \quad y \in \mathbb{R}^2. \tag{6.3}$$

In view of the above discussion, it turns out that the function

$$\varphi(y) := \int_{\mathbb{R}} \tilde\phi^2(y,t)\, dt$$

is well defined. In fact so are its first and second derivatives by elliptic regularity of ϕ, and differentiation under the integral sign is thus justified. Now, let us observe that

$$\Delta_y \varphi(y) = 2 \int_{\mathbb{R}} \Delta_y \tilde\phi \cdot \tilde\phi \, dt + 2 \int_{\mathbb{R}} |\nabla_y \tilde\phi|^2$$

and hence

$$\begin{aligned} 0 &= \int_{\mathbb{R}} (L(\tilde\phi) \cdot \tilde\phi) \\ &= \frac{1}{2} \Delta_y \varphi - \int_{\mathbb{R}} |\nabla_y \tilde\phi|^2 \, dz - \int_{\mathbb{R}} (\, |\tilde\phi_t|^2 - f'(w) \tilde\phi^2 \,)\, dt \,. \end{aligned} \tag{6.4}$$

Let us observe that because of relations (6.3) and (6.2), we have that

$$\int_{\mathbb{R}} (\, |\tilde\phi_t|^2 - f'(w)\tilde\phi^2 \,)\, dt \ \ge \gamma \varphi.$$

It follows then that

$$\frac{1}{2} \Delta_y \varphi - \gamma \varphi \ge 0.$$

Since φ is bounded, from maximum principle we find that φ must be identically equal to zero. But this means

$$\phi(y,t) = \left(\int_{\mathbb{R}} w'(\zeta)\,\phi(y,\zeta)\,d\zeta \right) \frac{w'(t)}{\int_{\mathbb{R}} w'^2}. \tag{6.5}$$

Then the bounded function

$$g(y) = \int_{\mathbb{R}} w_\zeta(\zeta)\,\phi(y,\zeta)\,d\zeta$$

satisfies the equation

$$\Delta_y g = 0, \quad \text{in } \mathbb{R}^2. \tag{6.6}$$

Liouville's theorem implies that $g \equiv$ constant and relation (6.5) yields $\phi(y,t) = Cw'(t)$ for some C. This concludes the proof. □

6.1. *A priori estimates*

We shall consider problem (5.22) in a slightly more general form, also in a domain finite in y-direction. For a large number $R > 0$ let us set

$$M_\alpha^R := \{ y \in M_\alpha \ / \ r(\alpha y) < R \}$$

and consider the variation of Problem (5.22) given by

$$\begin{aligned} \partial_{tt}\phi + \Delta_{y,M_\alpha}\phi + f'(w(t))\phi &= g(y,t) + c(y)w'(t) \quad \text{in } M_\alpha^R \times \mathbb{R}, \\ \phi &= 0, \quad \text{on } \partial M_\alpha^R \times \mathbb{R}, \\ \int_{-\infty}^{\infty} \phi(y,t)\, w'(t)\, dt = 0 & \quad \text{for all} \quad y \in M_\alpha^R, \end{aligned} \tag{6.7}$$

where we allow $R = +\infty$ and

$$c(y) \int_{\mathbb{R}} {w'}^2 dt = - \int_{\mathbb{R}} g(y,t)\, w'\, dt \ .$$

We begin by proving a priori estimates.

Lemma 6.2:

Let us assume that $0 < \sigma < \min\{\sigma_-, \sigma_+\}$ *and* $\mu \geq 0$. *Then there exists a constant* $C > 0$ *such that for all small* α *and all large* R, *and every solution* ϕ *to Problem* (6.13) *with* $\|\phi\|_{\infty,\mu,\sigma} < +\infty$ *and right hand side* g *satisfying* $\|g\|_{p,\mu,\sigma} < +\infty$ *we have*

$$\|D^2\phi\|_{p,\mu,\sigma} + \|D\phi\|_{\infty,\mu,\sigma} + \|\phi\|_{\infty,\mu,\sigma} \leq C\|g\|_{p,\mu,,\sigma}. \tag{6.8}$$

Proof: For the purpose of the a priori estimate, it clearly suffices to consider the case $c(y) \equiv 0$. By local elliptic estimates, it is enough to show that

$$\|\phi\|_{\infty,\mu,\sigma} \le C\|g\|_{p,\mu,\sigma}. \tag{6.9}$$

Let us assume by contradiction that (6.9) does not hold. Then we have sequences $\alpha = \alpha_n \to 0$, $R = R_n \to \infty$, g_n with $\|g_n\|_{p,\mu,\sigma} \to 0$, ϕ_n with $\|\phi_n\|_{\infty,\mu,\sigma} = 1$ such that

$$\begin{aligned} \partial_{tt}\phi_n + \Delta_{y,M_\alpha}\phi_n + f'(w(t))\phi_n &= g_n \quad \text{in } M_\alpha^R \times \mathbb{R},\\ \phi_n &= 0 \quad \text{on } \partial M_\alpha^R \times \mathbb{R},\\ \int_{-\infty}^{\infty} \phi_n(y,t)\, w'(t)\, dt &= 0 \quad \text{for all} \quad y \in M_\alpha^R\,. \end{aligned} \tag{6.10}$$

Then we can find points $(y_n, t_n) \in M_\alpha^R \times \mathbb{R}$ such that

$$e^{-\sigma|t_n|}(1 + r(\alpha_n y_n))^\mu\, |\phi_n(y_n, t_n)| \ge \frac{1}{2}.$$

We will consider different possibilities. We may assume that either $r_\alpha(y_n) = O(1)$ or $r_\alpha(y_n) \to +\infty$.

6.1.1. *Case $r(\alpha_n y_n)$ bounded.*

We have $\alpha_n y_n$ lies within a bounded subregion of M, so we may assume that

$$\alpha_n y_n \to \tilde{y}_0 \in M.$$

Assume that $\tilde{y}_0 \in Y_k(\mathcal{U}_k)$ for one of the local parametrization of M. We consider $\tilde{\mathrm{y}}_n, \tilde{\mathrm{y}}_0 \in \mathcal{U}_k$ with $Y_k(\tilde{\mathrm{y}}_n) = \alpha_n y_n$, $Y_k(\tilde{\mathrm{y}}_0) = \tilde{y}_0$.

On $\alpha_n^{-1} Y_k(\mathcal{U}_k)$, M_α is parameterized by $Y_{k,\alpha_n}(\mathrm{y}) = \alpha_n^{-1} Y_k(\alpha_n \mathrm{y})$, $\mathrm{y} \in \alpha_n^{-1}\mathcal{U}_k$. Let us consider the local change of variable,

$$\mathrm{y} = \alpha^{-1}\tilde{\mathrm{y}}_n + \mathbf{y}.$$

6.1.2. *Subcase t_n bounded*

Let us assume first that $|t_n| \le C$. Then, setting

$$\tilde{\phi}_n(\mathbf{y}, t) := \tilde{\phi}_n(\alpha^{-1}\tilde{\mathrm{y}}_n + \mathbf{y}, t),$$

the local equation becomes

$$a^0_{ij}(\tilde{\mathbf{y}}_n+\alpha_n\mathbf{y})\partial_{ij}\tilde\phi_n+\alpha_n b^0_j(\tilde{\mathbf{y}}_n+\alpha_n\mathbf{y})\partial_j\tilde\phi_n+\partial_{tt}\tilde\phi_n+f'(w(t))\tilde\phi_n=\tilde g_n(\mathbf{y},t)$$

where $\tilde g_n(\mathbf{y},t):=g_n(\tilde{\mathbf{y}}_n+\alpha\mathbf{y},t)$. We observe that this expression is valid for $\mathbf{y}$ well-inside the domain $\alpha^{-1}\mathcal{U}_k$ which is expanding to entire $\mathbb{R}^2$. Since $\tilde\phi_n$ is bounded, and $\tilde g_n\to 0$ in $L^p_{loc}(\mathbb{R}^2)$, we obtain local uniform $W^{2,p}$-bound. Hence we may assume, passing to a subsequence, that $\tilde\phi_n$ converges uniformly in compact subsets of $\mathbb{R}^3$ to a function $\tilde\phi(\mathbf{y},t)$ that satisfies

$$a^0_{ij}(\tilde{\mathbf{y}})\partial_{ij}\tilde\phi+\partial_{tt}\tilde\phi+f'(w(t))\tilde\phi=0\,.$$

Thus $\tilde\phi$ is non-zero and bounded. After a rotation and stretching of coordinates, the constant coefficient operator $a^0_{ij}(\tilde{\mathbf{y}})\partial_{ij}$ becomes $\Delta_{\mathbf{y}}$. Hence Lemma 6.1 implies that, necessarily, $\tilde\phi(\mathbf{y},t)=Cw'(t)$. On the other hand, we have

$$0=\int_{\mathbb{R}}\tilde\phi_n(\mathbf{y},t)\,w'(t)\,dt\longrightarrow\int_{\mathbb{R}}\tilde\phi(\mathbf{y},t)\,w'(t)\,dt\qquad\text{as } n\to\infty.$$

Hence, necessarily $\tilde\phi\equiv 0$. But we have $(1+r(\alpha_n y_n))^{\mu}\,|\tilde\phi_n(0,t_n)|\ge\frac12$, and since t_n and $r(\alpha_n y_n)$ were bounded, the local uniform convergence implies $\tilde\phi\ne 0$. We have reached a contradiction.

6.1.3. *Subcase t_n unbounded*

If y_n is in the same range as above, but, say, $t_n\to+\infty$, the situation is similar. The variation is that we define now

$$\tilde\phi_n(\mathbf{y},t)=e^{\sigma(t_n+t)}\phi_n(\alpha_n^{-1}\mathbf{y}_n+\mathbf{y},t_n+t),\quad \tilde g_n(\mathbf{y},t)=e^{\sigma(t_n+t)}g_n(\alpha_n^{-1}\mathbf{y}_n+\mathbf{y},t_n+t).$$

Then $\tilde\phi_n$ is uniformly bounded, and $\tilde g_n\to 0$ in $L^p_{loc}(\mathbb{R}^3)$. Now $\tilde\phi_n$ satisfies

$$a^0_{ij}(\mathbf{y}_n+\alpha_n\mathbf{y})\,\partial_{ij}\tilde\phi_n\,+\,\partial_{tt}\tilde\phi_n\,+\,\alpha_n b_j(\mathbf{y}_n+\alpha_n\mathbf{y})\,\partial_j\tilde\phi_n$$

$$-2\sigma\,\partial_t\tilde\phi_n\,+\,(f'(w(t+t_n))+\sigma^2)\,\tilde\phi_n\,=\,\tilde g_n.$$

We fall into the limiting situation

$$a^*_{ij}\,\partial_{ij}\tilde\phi\,+\,\partial_{tt}\tilde\phi\,-\,2\sigma\,\partial_t\tilde\phi\,-\,(\sigma_+^2-\sigma^2)\,\tilde\phi\,=\,0\quad\text{in }\mathbb{R}^3 \tag{6.11}$$

where a^*_{ij} is a positive definite, constant matrix and $\tilde\phi\ne 0$. But since, by hypothesis $\sigma_+^2-\sigma^2>0$, maximum principle implies that $\tilde\phi\equiv 0$. We obtain a contradiction.

6.1.4. *Case* $r(\alpha_n y_n) \to +\infty$.

In this case we may assume that the sequence $\alpha_n y_n$ diverges along one of the ends, say M_k. Considering now the parametrization associated to the end, $y = \psi_k(\mathrm{y})$, given by (3.1), which inherits that for $M_{\alpha_n,k}$, $y = \alpha_n^{-1}\psi_k(\alpha_n \mathrm{y})$. Thus in this case $a^0_{ij}(\tilde{\mathbf{y}}_n + \alpha_n \mathbf{y}) \to \delta_{ij}$, uniformly in compact subsets of $\mathbb{R}^2$.

6.1.5. *Subcase* t_n *bounded*

Let us assume first that the sequence t_n is bounded and set

$$\tilde{\phi}_n(\mathbf{y}, t) = (1 + r(\tilde{\mathbf{y}}_n + \alpha_n \mathbf{y}))^\mu \, \phi_n(\alpha_n^{-1}\tilde{\mathbf{y}}_n + \mathbf{y}, t_n + t).$$

Then

$$\partial_j(r_{\alpha_n}^{-\mu}\tilde{\phi}_n) = -\mu\alpha\, r^{-\mu-1}\partial_j r\tilde{\phi} + r^{-\mu}\partial_j\tilde{\phi}$$

$$\partial_{ij}(r_{\alpha_n}^{-\mu}\tilde{\phi}_n) = \mu(\mu+1)\alpha^2 r^{-\mu-2}\partial_i r\partial_j r\phi - \mu\alpha^2 r^{-\mu-1}\partial_{ij} r\tilde{\phi} - \mu\alpha r^{-\mu-1}\partial_j r\partial_i\tilde{\phi}$$
$$+ r^{-\mu}\partial_{ij}\tilde{\phi} - \mu\alpha r^{-\mu-1}\partial_i r\partial_j\tilde{\phi} \; .$$

Now $\partial_i r = O(1)$, $\partial_{ij} r = O(r^{-1})$, hence we have

$$\partial_j(r_{\alpha_n}^{-\mu}\tilde{\phi}_n) = r^{-\mu}\, \left[\, \partial_j\tilde{\phi} + O(\alpha r_\alpha^{-1})\tilde{\phi}\,\right] \, ,$$

$$\partial_{ij}(r_{\alpha_n}^{-\mu}\tilde{\phi}_n) = r_\alpha^{-\mu}\, \left[\, \partial_{ij}\tilde{\phi} + O(\alpha r_\alpha^{-1})\partial_i\tilde{\phi} + O(\alpha^2 r_\alpha^{-2})\tilde{\phi}\,\right] ,$$

and the equation satisfied by $\tilde{\phi}_n$ has therefore the form

$$\Delta_{\mathbf{y}}\tilde{\phi}_n \, + \, \partial_{tt}\tilde{\phi}_n \, + \, o(1)\partial_{ij}\tilde{\phi}_n + \, o(1)\,\partial_j\tilde{\phi}_n \, + \, o(1)\,\tilde{\phi}_n + f'(w(t))\tilde{\phi}_n = \tilde{g}_n.$$

where $\tilde{\phi}_n$ is bounded, $\tilde{g}_n \to 0$ in $L^p_{loc}(\mathbb{R}^3)$. From elliptic estimates, we also get uniform bounds for $\|\partial_j\tilde{\phi}_n\|_\infty$ and $\|\partial_{ij}\tilde{\phi}_n\|_{p,0,0}$. In the limit we obtain a $\tilde{\phi} \neq 0$ bounded, solution of

$$\Delta_{\mathbf{y}}\tilde{\phi} + \partial_{tt}\tilde{\phi} + f'(w(t))\tilde{\phi} = 0, \quad \int_{\mathbb{R}} \tilde{\phi}(\mathbf{y}, t)\, w'(t)\, dt \;=\; 0\, , \tag{6.12}$$

a situation which is discarded in the same way as before if $\tilde{\phi}$ is defined in $\mathbb{R}^3$. There is however, one more possibility which is that $r(\alpha_n y_n) - R_n = O(1)$. In such a case we would see in the limit equation (6.12) satisfied in a half-space, which after a rotation in the $\mathbf{y}$-plane can be assumed to be

$$H = \{(\mathbf{y}, t) \in \mathbb{R}^2 \times \mathbb{R} \,/\, \mathrm{y}_2 < 0\,\}, \quad \text{with } \phi(\mathrm{y}_1, 0, t) = 0 \quad \text{for all} \quad (\mathrm{y}_1, t) \in \mathbb{R}^2.$$

By Schwarz's reflection, the odd extension of $\tilde{\phi}$, which achieves for $y_2 > 0$, $\tilde{\phi}(y_1, y_2, t) = -\tilde{\phi}(y_1, -y_2, t)$, satisfies the same equation, and thus we fall into one of the previous cases, again finding a contradiction.

6.1.6. *Subcase t_n unbounded*

Let us assume now $|t_n| \to +\infty$. If $t_n \to +\infty$ we define

$$\tilde\phi_n(\mathbf{y},t) = (1 + r(\tilde{\mathbf{y}}_n + \alpha_n \mathbf{y}))^\mu \, e^{t_n + t} \, \phi_n(\alpha_n^{-1} \tilde{\mathbf{y}}_n + \mathbf{y}, t_n + t).$$

In this case we end up in the limit with a $\tilde\phi \neq 0$ bounded and satisfying the equation

$$\Delta_{\mathbf{y}} \tilde\phi \, + \, \partial_{tt} \tilde\phi \, - \, 2\sigma \, \partial_t \tilde\phi \, - \, (\sigma_+^2 - \sigma^2) \, \tilde\phi \, = \, 0$$

either in entire space or in a Half-space under zero boundary condition. This implies again $\tilde\phi = 0$, and a contradiction has been reached that finishes the proof of the a priori estimates.

6.2. *Existence: conclusion of proof of Proposition 5.1*

Let us prove now existence. We assume first that g has compact support in $M_\alpha \times \mathbb{R}$.

$$\begin{aligned} \partial_{tt}\phi + \Delta_{y,M_\alpha}\phi + f'(w(t))\phi = g(y,t) + c(y)w'(t) \quad \text{in } M_\alpha^R \times \mathbb{R}, \\ \phi = 0, \quad \text{on } \partial M_\alpha^R \times \mathbb{R}, \\ \int_{-\infty}^{\infty} \phi(y,t)\, w'(t)\, dt = 0 \quad \text{for all} \quad y \in M_\alpha^R, \end{aligned} \tag{6.13}$$

where we allow $R = +\infty$ and

$$c(y) \int_{\mathbb{R}} {w'}^2 dt = - \int_{\mathbb{R}} g(y,t)\, w' \, dt \ .$$

Problem (6.13) has a weak formulation which is the following. Let

$$H = \{ \phi \in H_0^1(M_\alpha^R \times \mathbb{R}) \ / \ \int_{\mathbb{R}} \phi(y,t)\, w'(t)\, dt = 0 \quad \text{for all} \quad y \in M_\alpha^R \ \} \, .$$

H is a closed subspace of $H_0^1(M_\alpha^R \times \mathbb{R})$, hence a Hilbert space when endowed with its natural norm,

$$\|\phi\|_H^2 = \int_{M_\alpha^R} \int_{\mathbb{R}} (\, |\partial_t \phi|^2 + |\nabla_{M_\alpha} \phi|^2 - f'(w(t)\, \phi^2 \,) \, dV_\alpha \, dt \ .$$

ϕ is then a weak solution of Problem (6.13) if $\phi \in H$ and satisfies

$$a(\phi,\psi) := \int_{M_\alpha^R \times \mathbb{R}} (\, \nabla_{M_\alpha} \phi \cdot \nabla_{M_\alpha} \psi \, - \, f'(w(t)) \, \phi \, \psi \,) \ dV_\alpha \, dt \ =$$

$$- \int_{M_\alpha^R \times \mathbb{R}} g \, \psi \, dV_\alpha \, dt \quad \text{for all} \quad \psi \in H.$$

It is standard to check that a weak solution of problem (6.13) is also classical provided that g is regular enough. Let us observe that because of the orthogonality condition defining H we have that

$$\gamma \int_{M_\alpha^R \times \mathbb{R}} \psi^2 \, dV_\alpha \, dt \ \leq \ a(\psi, \psi) \quad \text{for all} \quad \psi \in H.$$

Hence the bilinear form a is coercive in H, and existence of a unique weak solution follows from Riesz's theorem. If g is regular and compactly supported, ψ is also regular. Local elliptic regularity implies in particular that ϕ is bounded. Since for some $t_0 > 0$, the equation satisfied by ϕ is

$$\Delta\phi + f'(w(t))\,\phi = c(y)w'(t), \quad |t| > t_0, \quad y \in M_\alpha^R, \tag{6.14}$$

and $c(y)$ is bounded, then enlarging t_0 if necessary, we see that for $\sigma < \min\{\sigma_+, \sigma_-\}$, the function $v(y,t) := Ce^{-\sigma|t|} + \varepsilon e^{\sigma|t|}$ is a positive supersolution of equation (6.14), for a large enough choice of C and arbitrary $\varepsilon > 0$. Hence $|\phi| \leq Ce^{-\sigma|t|}$, from maximum principle. Since M_α^R is bounded, we conclude that $\|\phi\|_{p,\mu,\sigma} < +\infty$. From Lemma 6.2 we obtain that if R is large enough then

$$\|D^2\phi\|_{p,\mu,\sigma} + \|D\phi\|_{\infty,\mu,\sigma} + \|\phi\|_{\infty,\mu,\sigma} \leq C\|g\|_{p,\mu,\sigma} \tag{6.15}$$

Now let us consider Problem (6.13) for $R = +\infty$, allowed above, and for $\|g\|_{p,\mu,\sigma} < +\infty$. Then solving the equation for finite R and suitable compactly supported g_R, we generate a sequence of approximations ϕ_R which is uniformly controlled in R by the above estimate. If g_R is chosen so that $g_R \to g$ in $L^p_{loc}(M_\alpha \times \mathbb{R})$ and $\|g_R\|_{p,\mu,\sigma} \leq C\|g\|_{p,\mu,\sigma}$, We obtain that ϕ_R is locally uniformly bounded, and by extracting a subsequence, it converges uniformly locally over compacts to a solution ϕ to the full problem which respects the estimate (5.25). This concludes the proof of existence, and hence that of the proposition.

7. *Theory of the Jacobi operator*

We consider this section the problem of finding a function h such that for certain constants $c_1, \ldots, c_J$,

$$\mathcal{J}(h) = \Delta_M h + |A|^2 h = f + \sum_{j=1}^{J} \frac{c_i}{1+r^4}\,\hat{z}_i \quad \text{in } M, \tag{7.1}$$

$$\int_M \frac{\hat{z}_i h}{1+r^4} \ = \ 0, \quad i = 1, \ldots, J \tag{7.2}$$

and prove the result of Proposition 5.3. We will also deduce the existence of Jacobi fields of logarithmic growth as in Lemma 4.1. We recall the definition of the norms $\| \ \|_{p,\beta}$ in (5.6).

Outside of a ball of sufficiently large radius R_0, it is natural to parameterize each end of M, $y_3 = F_k(y_1, y_2)$ using the Euclidean coordinates $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$. The requirement in f on each end amounts to $\tilde{f} \in L^p(B(0, 1/R_0))$ where

$$\tilde{f}(\mathbf{y}) := |\mathbf{y}|^{-4} f(|\mathbf{y}|^{-2}\mathbf{y})\,. \tag{7.3}$$

Indeed, observe that

$$\|\tilde{f}\|^p_{L^p(B(0,1/R_0))} = \int_{B(0,1/R_0)} |\mathbf{y}|^{-4p} |\, f(|\mathbf{y}|^{-2}\mathbf{y})\,|^p \, d\mathbf{y} \ = \int_{\mathbb{R}^2 \setminus B(0,R_0)} |\mathrm{y}|^{4(p-1)} |f(\mathrm{y})|^p \, d\mathbf{y}\,.$$

In order to prove the proposition we need some a priori estimates.

Lemma 7.1: *Let $p > 2$. For each $R_0 > 0$ sufficiently large there exists a constant $C > 0$ such that if*

$$\|f\|_{p,4-\frac{4}{p}} + \|h\|_{L^\infty(M)} < +\infty$$

and h solves

$$\Delta_M h + |A|^2 h = f, \quad y \in M, \quad |y| > R_0\,,$$

then

$$\|h\|_{L^\infty(|y|>2R_0)} + \|\,|y|^2 Dh\|_{L^\infty(|y|>2R_0)} + \|\,|y|^{4-\frac{4}{p}} D^2 h\|_{L^p(|y|>2R_0)} \ \leq$$

$$C\,[\,\|f\|_{p,4-\frac{4}{p}} \ + \|h\|_{L^\infty(R_0<|y|<3R_0)} \ ]\,.$$

Proof: Along each end M_k of M, Δ_M can be expanded in the coordinate y as

$$\Delta_M = \Delta + O(|\mathrm{y}|^{-2})D^2 + O(|\mathrm{y}|^{-3})D.$$

A solution of h of equation (7.1) satisfies

$$\Delta_M h + |A|^2 h = f, \quad |\mathrm{y}| > R_0$$

for a sufficiently large R_0. Let us consider a Kelvin's transform

$$h(\mathrm{y}) = \tilde{h}(\mathrm{y}/|\mathrm{y}|^2).$$

Then we get

$$\Delta h(\mathrm{y}) = |\mathrm{y}|^{-4}(\Delta\tilde{h})(\mathrm{y}/|\mathrm{y}|^2)\,.$$

Besides

$$O(|y|^{-2})D^2h(y)+O(|y|^{-3})Dh(y) = O(|y|^{-6})D^2\tilde h(y/|y|^2)+O(|y|^{-5})D\tilde h(y/|y|^2)\,.$$

Hence

$$(\Delta_M h)(\mathbf{y}/|\mathbf{y}|^2) = |\mathbf{y}|^4\left[\,\Delta\tilde h(\mathbf{y}) + O(|\mathbf{y}|^2)D^2\tilde h(\mathbf{y}) + O(|\mathbf{y}|)D\tilde h(\mathbf{y})\right]\,.$$

Then $\tilde h$ satisfies the equation

$$\Delta\tilde h + O(|\mathbf{y}|^2)D^2\tilde h + O(|\mathbf{y}|)D\tilde h + O(1)h = \tilde f(\mathbf{y}),\quad 0<|\mathbf{y}|<\frac{1}{R_0}$$

where $\tilde f$ is given by (7.3). The operator above satisfies maximum principle in $B(0,\frac{1}{R_0})$ if R_0 is fixed large enough. This, the fact that $\tilde h$ is bounded, and L^p-elliptic regularity for $p>2$ in two dimensional space imply that

$$\|\tilde h\|_{L^\infty(B(0,1/2R_0))} + \|D\tilde h\|_{L^\infty(B(0,1/2R_0))} + \|D^2\tilde h\|_{L^p(B(0,1/2R_0))} \ \le$$

$$C[\|\tilde f\|_{L^p((B(0,1/R_0))} + \|\tilde h\|_{L^\infty(1/3R_0<|y|<1/R_0)}] \ \le$$

$$C\,[\,\|f\|_{p,4-\frac{4}{p}} + \|h\|_{L^\infty(B(R_0<|y|<3R_0))}\,]\,.$$

Let us observe that

$$\|\tilde h\|_{L^\infty(B(0,1/2R_0))} = \|h\|_{L^\infty(|y|>2R_0)},$$

$$\|D\tilde h\|_{L^\infty(B(0,1/2R_0))} = \|\,|y|^2\,Dh\|_{L^\infty(|y|>2R_0)}.$$

Since

$$|D^2h(y)| \le C(\,|y|^{-4}\,|D^2\tilde h(\,|y|^{-2}y)| + |y|^{-3}|D\tilde h(\,|y|^{-2}y)|\,)$$

then

$$|y|^{4-\frac{4}{p}}|D^2h(y)| \ \le\ C(\,|y|^{-4/p}|D^2\tilde h(\,|y|^{-2}y)| + |y|^{-\frac{4}{p}-1}|D\tilde h(|y|^{-2}y)|\,).$$

Hence

$$\int_{|y|>2R_0}|y|^{4p-4}|D^2h|^p dy \ \le$$

$$C(\int_{B(0,1/2R_0)}|D^2\tilde h(\mathbf{y})|^p\,d\mathbf{y} \ +\ \|D\tilde h\|^p_{L^\infty(B(0,1/2R_0))}\int_{|y|>2R_0}|y|^{-4-p}dy\,).$$

It follows that

$$\|h\|_{L^\infty(|y|>2R_0)} + \|\,|y|^2Dh\|_{L^\infty(|y|>2R_0)} + \|\,|y|^{4-\frac{4}{p}}D^2h\|_{L^p(|y|>2R_0)} \ \le$$

$$C\,[\,\|f\|_{p,4-\frac{4}{p}} + \|h\|_{L^\infty(B(R_0<|y|<3R_0))}\,]\,.$$

Since this estimate holds at each end, the result of the lemma follows, after possibly changing slightly the value R_0. □

Lemma 7.2: *Under the conditions of Lemma 7.1, assume that h is a bounded solution of Problem* (7.1)-(7.2)*. Then the a priori estimate (5.34) holds.*

Proof:

Let us observe that this a priori estimate in Lemma 7.1 implies in particular that the Jacobi fields $\hat z_i$ satisfy

$$\nabla \hat z_i(y) = O(|y|^{-2}) \quad \text{as } |y| \to +\infty.$$

Using $\hat z_i$ as a test function in a ball $B(0,\rho)$ in M we obtain

$$\int_{\partial B(0,\rho)} (h\partial_\nu \hat z_i - \hat z_i \partial_\nu \hat z_i) + \int_{|y|<\rho} (\Delta_M \hat z_i + |A|^2 \hat z_i)\, h =$$

$$\int_{|y|<\rho} f\hat z_i \; + \sum_{j=1}^{J} c_j \int_M \frac{\hat z_i \hat z_j}{1+r^4}.$$

Since the boundary integral in the above identity is of size $O(\rho^{-1})$ we get

$$\int_M f\hat z_i \; + \sum_{j=1}^{J} c_j \int_M \frac{\hat z_i \hat z_j}{1+r^4} = 0 \tag{7.4}$$

so that in particular

$$|c_j| \le C\|f\|_{p,4-\frac{4}{p}} \quad \text{for all} \quad j = 1, \dots, J. \tag{7.5}$$

In order to prove the desired estimate, we assume by contradiction that there are sequences h_n, f_n with $\|h_n\|_\infty = 1$ and $\|f_n\|_{p,4-\frac{4}{p}} \to 0$, such that

$$\Delta_M h_n + |A|^2 h_n = f_n + \sum_{j=1}^{J} \frac{c_i^n \hat z_i}{1+r^4}$$

$$\int_M \frac{h_n \hat z_i}{1+r^4} = 0 \quad \text{for all} \quad i = 1, \dots, J.$$

Thus according estimate (7.5), we have that $c_i^n \to 0$. From Lemma 7.1 we find

$$\|h_n\|_{L^\infty(|y|>2R_0)} \le C[o(1) + \|h_n\|_{L^\infty(B(0,3R_0))}]\,.$$

The latter inequality implies that

$$\|h_n\|_{L^\infty(B(0,3R_0))} \ge \gamma > 0.$$

Local elliptic estimates imply a C^1 bound for h_n on bounded sets. This implies the presence of a subsequence h_n which we denote the same way such that $h_n \to h$ uniformly on compact subsets of M, where h satisfies

$$\Delta_M h + |A|^2 h = 0\,.$$

h is bounded hence, by the nondegeneracy assumption, it is a linear combination of the functions $\hat z_i$. Besides $h \ne 0$ and satisfies

$$\int_M \frac{h\hat z_i}{1+r^4} = 0 \quad \text{for all} \quad i = 1, \dots, J\,.$$

The latter relations imply $h = 0$, hence a contradiction that proves the validity of the a priori estimate. □

7.1. *Proof of Proposition 5.3*

Thanks to Lemma 7.2 it only remains to prove existence of a bounded solution to problem (7.1)-(7.2). Let f be as in the statement of the proposition. Let us consider the Hilbert space H of functions $h \in H^1_{loc}(M)$ with

$$\|h\|_H^2 := \int_M |\nabla h|^2 + \frac{1}{1+r^4}|h|^2 < +\infty\,,$$

$$\int_M \frac{1}{1+r^4} h\hat z_i = 0 \quad \text{for all} \quad i = 1, \dots, J\,.$$

Problem (7.1)-(7.2) can be formulated in weak form as that of finding $h \in H$ with

$$\int_M \nabla h \nabla \psi - |A|^2 h\,\psi = -\int_M f\psi \quad \text{for all} \quad \psi \in H\,.$$

In fact, a weak solution $h \in H$ of this problem must be bounded thanks to elliptic regularity, with the use of Kelvin's transform in each end for the control at infinity. Using that $|A|^2 \le Cr^{-4}$, Riesz representation theorem and the fact that H is compactly embedded in $L^2((1+r^4)^{-1}dV)$ (which follows for instance by inversion at each end), we see that this weak problem can be written as an equation of the form

$$h - T(h) = \tilde{f}$$

where T is a compact operator in H and $\tilde{f} \in H$ depends linearly on f. When $f = 0$, the a priori estimates found yield that necessarily $h = 0$. Existence of a solution then follows from Fredholm's alternative. The proof is complete.

7.2. *Jacobi fields of logarithmic growth. The proof of Lemma 4.1*

We will use the theory developed above to construct Jacobi fields with logarithmic growth as $r \to +\infty$, whose existence we stated and use to set up the initial approximation in Lemma 4.1. One of these Jacobi fields is the generator of dilations of the surface, $z_0(y) = y \cdot \nu(y)$. We will prove next that there are another $m - 2$ linearly independent logarithmically growing Jacobi fields.

Let us consider an m-tuple of numbers $\beta_1, \ldots, \beta_m$ with $\sum_j \beta_j = 0$, and any smooth function $p(y)$ in M such that on each end M_j we have that for sufficiently large $r = r(y)$,

$$p(y) = (-1)^j \beta_j \log r(y), \quad y \in M_j$$

for certain numbers $\beta_1, \ldots, \beta_m$ that we will choose later. To prove the result of Lemma 4.1 we need to find a solution h_0 of the equation $\mathcal{J}(h_0) = 0$ of the form $h_0 = p + h$ where h is bounded. This amounts to solving

$$\mathcal{J}(h) = -\mathcal{J}(p)\,. \tag{7.6}$$

Let us consider the cylinder $C_R = \{x \in \mathbb{R}^3 \;/\; r(x) < R\}$ for a large R. Then

$$\int_{M\cap C_R} \mathcal{J}(p)\, z_3 dV = \int_{M\cap C_R} \mathcal{J}(z_3) z_3 dV + \int_{\partial C_R \cap M} (z_3 \partial_n p - p \partial_n z_3)\, d\sigma(y)\,.$$

Hence

$$\int_{M\cap C_R} \mathcal{J}(p)\, z_3 dV = \sum_{j=1}^{m} \int_{\partial C_R \cap M_j} (z_3 \partial_n p - p \partial_n z_3)\, d\sigma(y)\,.$$

Thus using the graph coordinates on each end, we find

$$\int_{M\cap C_R} \mathcal{J}(p)\, z_3 dV \;=$$

$$\sum_{j=1}^{m}(-1)^j \left[\frac{\beta_j}{R} \int_{|\mathtt{y}|=R} \nu_3 d\sigma(\mathtt{y}) \, - \, \beta_j \log R \int_{|\mathtt{y}|=R} \partial_r \nu_3 d\sigma(\mathtt{y}) \right] \, + \, O(R^{-1}).$$

We have that, on each end M_j,

$$\nu_3(\mathtt{y}) = \frac{(-1)^j}{\sqrt{1+|\nabla F_k(\mathtt{y})|^2}} = (-1)^j + O(r^{-2}), \quad \partial_r \nu_3(\mathtt{y}) = O(r^{-3}).$$

Hence we get

$$\int_{M\cap C_R} \mathcal{J}(p)\, z_3 dV \;=\; 2\pi \sum_{j=1}^{m} \beta_j + O(R^{-1})\,.$$

It is easy to see, using the graph coordinates that $\mathcal{J}(p) = O(r^{-4})$ and it is hence integrable. We pass to the limit $R \to +\infty$ and get

$$\int_{M} \mathcal{J}(p)\, z_3 dV \;=\; 2\pi \sum_{j=1}^{m} \beta_j = 0\ . \tag{7.7}$$

We make a similar integration for the remaining bounded Jacobi fields. For $z_i = \nu_i(y)$ $i = 1, 2$ we find

$$\int_{M\cap C_R} \mathcal{J}(p)\, z_2 dV = \sum_{j=1}^{m}(-1)^j \left[\frac{\beta_j}{R} \int_{|\mathtt{y}|=R} \nu_2 d\sigma(\mathtt{y}) \, - \, \beta_j \log R \int_{|\mathtt{y}|=R} \partial_r \nu_2 d\sigma(\mathtt{y}) \right] + O(R^{-1}).$$

Now, on M_j,

$$\nu_2(\mathtt{y}) = \frac{(-1)^j}{\sqrt{1+|\nabla F_k(\mathtt{y})|^2}} = (-1)^j a_j \frac{x_i}{r^2} + O(r^{-3}), \quad \partial_r \nu_2(\mathtt{y}) = O(r^{-2}).$$

Hence

$$\int_{M} \mathcal{J}(p)\, z_i dV \;=\; 0 \;\; i = 1, 2.$$

Finally, for $z_4(y) = (-y_2, y_1, 0) \cdot \nu(y)$ we find on M_j,

$$(-1)^j z_4(\mathtt{y}) = -\mathbf{y}_2 \partial_2 F_j + \mathbf{y}_1 \partial_1 F_j \;=\; b_{j1} \frac{\mathtt{y}_2}{r^2} - b_{j2} \frac{\mathtt{y}_1}{r^2} + O(r^{-2}), \quad \partial_r z_4 = O(r^{-2})$$

and hence again

$$\int_{M} \mathcal{J}(p)\, z_4 dV \;=\; 0\ .$$

From the solvability theory developed, we can then find a bounded solution to the problem

$$\mathcal{J}(h) = -\mathcal{J}(p) + \sum_{j=1}^{J} q c_j \hat{z}_j\ .$$

Since $\int_M \mathcal{J}(p) z_i dV = 0$ and hence $\int_M \mathcal{J}(p)\hat{z}_i dV = 0$, relations (7.4) imply that $c_i = 0$ for all i.

We have thus found a bounded solution to equation (7.6) and the proof is concluded.

Remark 7.3: *Observe that, in particular, the explicit Jacobi field* $z_0(y) = y \cdot \nu(y)$ *satisfies that*

$$z(y) = (-1)^j a_j \log r + O(1) \quad \textit{for all} \quad y \in M_j$$

and we have indeed $\sum_j a_j = 0$. *Besides this one, we thus have the presence of another* $m-2$ *linearly independent Jacobi fields with* $|z(y)| \sim \log r$ *as* $r \to +\infty$, *where* m *is the number of ends.*

These are in reality all *Jacobi fields with exact logarithmic growth. In fact if* $\mathcal{J}(z) = 0$ *and*

$$|z(y)| \le C \log r\,, \tag{7.8}$$

then the argument in the proof of Lemma 7.1 shows that the Kelvin's inversion $\tilde{z}(\mathrm{y})$ *as in the proof of Lemma 7.2 satisfies near the origin* $\Delta \tilde{z} = \tilde{f}$ *where* $\tilde{f}$ *belongs to any* L^p *near the origin, so it must equal a multiple of* $\log|\mathrm{y}|$ *plus a regular function. It follows that on* M_j *there is a number* β_j *with*

$$z(\mathrm{y}) = (-1)^j \beta_j \log|\mathrm{y}| + h$$

where h *is smooth and bounded. The computations above force* $\sum_j \beta_j = 0$. *It follows from Lemma 4.1 that then* z *must be equal to one of the elements there predicted plus a bounded Jacobi field. We conclude in particular that the dimension of the space of Jacobi fields satisfying (7.8) must be at most* $m - 1 + J$, *thus recovering a fact stated in Lemma 5.2 of* [53].

8. *Reducing the gluing system and solving the projected problem*

In this section we prove Lemma 5.1, which reduces the gluing system (5.12)-(5.14) to solving the nonlocal equation (5.20) and prove Proposition 5.2 on solving the nonlinear projected problem (5.21), in which the basic element is linear theory stated in Proposition 5.1. In what follows we refer to notation and objects introduced in §5.1, §5.2.

8.1. *Reducing the gluing system*

Let us consider equation (5.12) in the gluing system (5.12)-(5.14),

$$\Delta\psi - W_\alpha(x)\psi + (1-\zeta_2)S(\mathtt{w}) + (1-\zeta_1)N(\psi+\zeta_2\phi) + 2\nabla\zeta_1\nabla\phi + \phi\Delta\zeta_1 \ = \ 0 \quad \text{in } \mathbb{R}^3 \tag{8.1}$$

where

$$W_\alpha(x) := [\,(1-\zeta_1)f'(u_1) + \zeta_1 H(t)\,] \ .$$

8.1.1. *Solving the linear outer problem*

We consider first the linear problem

$$\Delta\psi - W_\alpha(x)\psi + g(x) \ = \ 0 \quad \text{in } \mathbb{R}^3 \tag{8.2}$$

We observe that globally we have $0 < a < W_\alpha(x) < b$ for certain constants a and b. In fact we can take $a = \min\{\sigma_-^2, \sigma_+^2\} - \tau$ for arbitrarily small $\tau > 0$.

We consider for the purpose the norms for $1 < p \le +\infty$,

$$\|g\|_{p,\mu} := \sup_{x\in\mathbb{R}^3} (1 + r(\alpha x))^\mu \|g\|_{L^p(B(x,1))}, \quad r(x', x_3) = |x'| \ .$$

Lemma 8.1: *Given $p > 3$, $\mu \ge 0$, there is a $C > 0$ such that for all sufficiently small α and any g with $\|g\|_{p,\mu} < +\infty$ there exists a unique ψ solution to Problem* (8.2) *with $\|\psi\|_{\infty,\mu} < +\infty$. This solution satisfies in addition,*

$$\|D^2\psi\|_{p,\mu} + \|\psi\|_{\infty,\mu} \ \le \ C\|g\|_{p,\mu}. \tag{8.3}$$

Proof:

We claim that the a priori estimate

$$\|\psi\|_{\infty,\mu} \ \le \ C\|g\|_{p,\mu} \tag{8.4}$$

holds for solutions ψ with $\|\psi\|_{\infty,\mu} < +\infty$ to problem (8.2) with $\|g\|_{p,\mu} < +\infty$ provided that α is small enough. This and local elliptic estimates in turn implies the validity of (8.3). To see this, let us assume the opposite, namely the existence $\alpha_n \to 0$, and solutions ψ_n to equation (8.2) with $\|\psi_n\|_{\infty,\mu} = 1$, $\|g_n\|_{p,\mu} \to 0$. Let us consider a point x_n with

$$(1 + r(\alpha_n x_n))^\mu \psi_n(x_n) \ge \frac{1}{2}$$

and define

$$\tilde\psi_n(x) = (1+r(\alpha_n(x_n+x))^\mu \psi_n(x_n+x), \quad \tilde g_n(x) = (1+r(\alpha_n(x_n+x))^\mu g_n(x_n+x),$$

$$\tilde{W}_n(x) = W_{\alpha_n}(x_n + x).$$

Then, similarly to what was done in the previous section, we check that the equation satisfied by $\tilde{\psi}_n$ has the form

$$\Delta\tilde{\psi}_n - \tilde{W}_n(x)\tilde{\psi}_n + o(1)\nabla\tilde{\psi}_n + o(1)\tilde{\psi}_n = \tilde{g}_n.$$

$\tilde{\psi}_n$ is uniformly bounded. Then elliptic estimates imply L^∞-bounds for the gradient and the existence of a subsequence uniformly convergent over compact subsets of $\mathbb{R}^3$ to a bounded solution $\tilde{\psi} \neq 0$ to an equation of the form

$$\Delta\tilde{\psi} - W_*(x)\tilde{\psi} = 0 \quad \text{in } \mathbb{R}^3$$

where $0 < a \leq W_*(x) \leq b$. But maximum principle makes this situation impossible, hence estimate (8.4) holds.

Now, for existence, let us consider g with $\|g\|_{p,\mu} < +\infty$ and a collection of approximations g_n to g with $\|g_n\|_{\infty,\mu} < +\infty$, $g_n \to g$ in $L^p_{loc}(\mathbb{R}^3)$ and $\|g_n\|_{p,\mu} \leq C\|g\|_{p,\mu}$. The problem

$$\Delta\psi_n - W_n(x)\psi_n = g_n \quad \text{in } \mathbb{R}^3$$

can be solved since this equation has a positive supersolution of the form $C_n(1 + r(\alpha x)\,)^{-\mu}$, provided that α is sufficiently small, but independently of n. Let us call ψ_n the solution thus found, which satisfies $\|\psi_n\|_{\infty,\mu} < +\infty$. The a priori estimate shows that

$$\|D^2\psi_n\|_{p,\mu} + \|\psi_n\|_{\infty,\mu} \;\leq\; C\|g\|_{p,\mu}.$$

and passing to the local uniform limit up to a subsequence, we get a solution ψ to problem (8.2), with $\|\psi\|_{\infty,\mu} < +\infty$. The proof is complete. □

8.1.2. *The proof of Lemma 5.1*

Let us call $\psi := \Upsilon(g)$ the solution of Problem (8.2) predicted by Lemma 8.1. Let us write Problem (8.1) as fixed point problem in the space X of $W^{2,p}_{loc}$-functions ψ with $\|\psi\|_X < +\infty$,

$$\psi = \Upsilon(g_1 + K(\psi)\,) \tag{8.5}$$

where

$$g_1 = (1 - \zeta_2)S(\mathtt{w}) + \,2\nabla\zeta_1\nabla\phi + \phi\Delta\zeta_1\,, \quad K(\psi) = (1 - \zeta_1)N(\psi + \zeta_2\phi)\ .$$

Let us consider a function ϕ defined in $M_\alpha \times \mathbb{R}$ such that $\|\phi\|_{2,p,\mu,\sigma} \le 1$. Then,

$$|\, 2\nabla\zeta_1 \nabla\phi + \phi\Delta\zeta_1 \,| \;\le\; Ce^{-\sigma\frac{\delta}{\alpha}}\,(1 + r(\alpha x))^{-\mu}\|\phi\|_{2,p,\mu,\sigma}.$$

We also have that $\|S(\mathtt{w})\|_{p,\mu,\sigma} \le C\alpha^3$, hence

$$|(1-\zeta_2)S(\mathtt{w})| \le Ce^{-\sigma\frac{\delta}{\alpha}}\,(1 + r(\alpha x))^{-\mu}$$

and

$$\|g_1\|_{p,\mu} \le Ce^{-\sigma\frac{\delta}{\alpha}}.$$

Let consider the set

$$\Lambda \;=\; \{\psi \in X \;/\; \|\psi\|_X \le Ae^{-\sigma\frac{\delta}{\alpha}}\},$$

for a large number $A > 0$. Since

$$|\, K(\psi_1) - K(\psi_2) \,| \;\le\; C(1-\zeta_1) \sup_{t\in(0,1)} |t\psi_1 + (1-t)\psi_2 + \zeta_2\phi|\, |\psi_1 - \psi_2|\,,$$

we find that

$$\|\, K(\psi_1) - K(\psi_2) \,\|_{\infty,\mu} \;\le\; C\, e^{-\sigma\frac{\delta}{\alpha}} \|\, \psi_1 - \psi_2 \,\|_{\infty,\mu}$$

while $\|K(0)\|_{\infty,\mu} \;\le\; C\, e^{-\sigma\frac{\delta}{\alpha}}$. It follows that the right hand side of equation (8.5) defines a contraction mapping of Λ, and hence a unique solution $\psi = \Psi(\phi) \in \Lambda$ exists, provided that the number A in the definition of Λ is taken sufficiently large and $\|\phi\|_{2,p,\mu,\sigma} \le 1$. In addition, it is direct to check the Lipschitz dependence of Ψ (5.18) on $\|\phi\|_{2,p,\mu,\sigma} \le 1$.

Thus, we replace replace $\psi = \Psi(\phi)$ into the equation (5.14) of the gluing system (5.12)-(5.14) and get the (nonlocal) problem,

$$\partial_{tt}\phi \,+\, \Delta_{y,M_\alpha}\phi \;=\; -\tilde{S}(u_1) - \mathtt{N}(\phi) \quad \text{ in } M_\alpha \times \mathbb{R} \tag{8.6}$$

where

$$\mathtt{N}(\phi) := \underbrace{\mathtt{B}(\phi) + [f'(u_1) - f'(w)]\phi}_{\mathtt{N}_1(\phi)} + \underbrace{\zeta_1(f'(u_1) - H(t))\Psi(\phi)}_{\mathtt{N}_2(\phi)} + \underbrace{\zeta_1 N(\Psi(\phi) + \phi)}_{\mathtt{N}_3(\phi)}, \tag{8.7}$$

which is what we concentrate in solving next.

8.2. *Proof of Proposition 5.2*

We recall from §5.2 that Proposition 5.2 refers to solving the projected problem

$$
\begin{aligned}
&\partial_{tt}\phi + \Delta_{y,M_\alpha}\phi = -\tilde{S}(u_1) - \mathtt{N}(\phi) + c(y)w'(t) \quad \text{in } M_\alpha \times \mathbb{R},\\
&\int_{\mathbb{R}} \phi(y,t)\, w'(t)\, dt = 0, \quad \text{for all} \quad y \in M_\alpha,
\end{aligned} \tag{8.8}
$$

and then adjust h_1 so that $c(y) \equiv 0$. Let $\phi = T(g)$ be the linear operator providing the solution in Proposition 5.1. Then Problem (8.8) can be reformulated as the fixed point problem

$$
\phi \;=\; T(-\tilde{S}(u_1) - \mathtt{N}(\phi)\,) =: \mathcal{T}(\phi), \quad \|\phi\|_{2,p,\mu,\sigma} \le 1 \tag{8.9}
$$

which is equivalent to

$$
\phi \;=\; T(-\tilde{S}(u_1) + \alpha^2 \Delta h_1\, w' - \mathtt{N}(\phi)\,), \quad \|\phi\|_{2,p,\mu,\sigma} \le 1, \tag{8.10}
$$

since the term added has the form $\rho(y)w'$ which thus adds up to $c(y)w'$. The reason to absorb this term is that because of assumption (5.7), $\|\alpha^2 \Delta h_1\, w'\|_{p,4,\sigma} = O(\alpha^{3-\frac{2}{p}})$ while the remainder has a priori size slightly smaller, $O(\alpha^3)$.

8.2.1. *Lipschitz character of* N

We will solve Problem (8.10) using contraction mapping principle, so that we need to give account of a suitable Lipschitz property for the operator $\mathcal{T}$. We claim the following.

Claim. *We have that for a certain constant $C > 0$ possibly depending on $\mathcal{K}$ in* (5.7) *but independent of $\alpha > 0$, such that for any ϕ_1, ϕ_2 with*

$$
\|\phi_l\|_{2,p,\mu,\sigma} \le K\alpha^3,
$$

$$
\|\mathtt{N}(\phi_1) - \mathtt{N}(\phi_2)\|_{p,\mu+1,\sigma} \;\le\; C\,\alpha\,\|\phi_1 - \phi_2\|_{2,p,\mu,\sigma} \tag{8.11}
$$

where the operator N *is defined in* (8.7).

We study the Lipschitz character of the operator N through analyzing each of its components. Let us start with N_1. This is a second order linear operator with coefficients of order α plus a decay of order at least $O(r_\alpha^{-1})$. We recall that $\mathtt{B} = \zeta_2 B$ where in coordinates

$$B \,=\, (f'(u_1) - f'(w)) - \alpha^2[(t+h_1)|A|^2 + \Delta_M h_1]\partial_t \,\, - 2\alpha\, a^0_{ij}\partial_j h \partial_{it} \,+$$

$$\alpha(t+h)\,[a^1_{ij}\partial_{ij} - \alpha\, a^1_{ij}(\,\partial_j h\partial_{it} + \partial_i h\partial_{jt}) + \alpha(b^1_i\partial_i - \alpha b^1_i\partial_i h\partial_t)\,)\,] \,+$$

$$\alpha^3(t+h)^2 b^1_3\partial_t \,+\, \alpha^2[\,a^0_{ij} + \alpha(t+h)a^1_{ij})\,]\partial_i h\partial_j h\,\partial_{tt} \tag{8.12}$$

where, we recall,

$$a^1_{ij} = O(r_\alpha^{-2}), \quad a^1_{ij} = O(r_\alpha^{-2}), \quad b^1_i = O(r_\alpha^{-3}), \quad b^3_i = O(r_\alpha^{-6}),$$

$$f'(u_1) - f'(w) = O(\alpha^2 r_\alpha^{-2} e^{-\sigma|t|}) \quad \partial_j h = O(r_\alpha^{-1}), \quad |A|^2 = O(r_\alpha^{-4})\,.$$

We claim that

$$\|\mathtt{N}_1(\phi)\|_{p,\mu+1,\sigma} \,\le\, C\,\alpha\,\|\phi\|_{2,p,\mu,\sigma}. \tag{8.13}$$

The only term of $N_1(\phi)$ that requires a bit more attention is $\alpha^2(\Delta h_1)(\alpha y)\partial_t\phi$. We have

$$\int_{B((y,t),1)} |\alpha^2(\Delta h_1)(\alpha z)\partial_t\phi|^p\, dV_\alpha(z)\, d\tau \,\,\le$$

$$C\,\alpha^{2p}\|\partial_t\phi\|_{L^\infty(B((y,t),1)}\,(1{+}r_\alpha(y)\,)^{-4p+4}\int_{B((y,t),1)}|(1{+}r_\alpha(z)\,)^{4-\frac{4}{p}}(\Delta h_1)(\alpha z)|^p\,|dV_\alpha(z) \,\,\le$$

$$C\,\alpha^{2p-2}\|\Delta h_1\|^p_{L^p(M)} e^{-p\sigma|t|}(1+r_\alpha(y))^{-p\mu-4p+4}\|\nabla\phi\|_{\infty,\mu,\sigma},$$

and hence in particular for $p \ge 3$,

$$\|\alpha^2(\Delta h_1)(\alpha y)\partial_t\phi\|_{p,\mu+2,\sigma} \le\, C\,\alpha^{2-\frac{2}{p}}\,\|h_1\|_*\,\|\phi\|_{2,p,\mu,\sigma} \le C\,\alpha^{3-\frac{2}{p}}\,\|\phi\|_{2,p,\mu,\sigma}.$$

Let us consider now functions ϕ_l with

$$\|\phi_l\|_{2,p,\mu,\sigma} \le 1, \quad l = 1, 2.$$

Now, according to Lemma 5.1, we get that

$$\|\mathtt{N}_2(\phi_1) - \mathtt{N}_2(\phi_2)\|_{p,\mu,\sigma} \,\le\, C\,e^{-\sigma\frac{\delta}{\alpha}}\,\|\phi_1 - \phi_2\|_{p,\mu,\sigma}\,. \tag{8.14}$$

Finally, we also have that

$$|\mathtt{N}_3(\phi_1) - \mathtt{N}_3(\phi_2)\,| \,\,\le$$

$$C\zeta_1 \sup_{t\in(0,1)} |t(\Psi(\phi_1)+\phi_1)+(1-t)(\Psi(\phi_2)+\phi_2)|\,[\,|\phi_1-\phi_2|+|\Psi(\phi_1)-\Psi(\phi_2)|\,],$$

hence

$$\|\mathtt{N}_3(\phi_1)-\mathtt{N}_3(\phi_2)\|_{p,2\mu,\sigma} \le C\,(\,\|\phi_1\|_{\infty,\mu,\sigma}+\|\phi_2\|_{\infty,\mu,\sigma}+e^{-\sigma\frac{\delta}{\alpha}})\,\|\phi_1-\phi_2\|_{\infty,\mu,\sigma}. \tag{8.15}$$

From (8.13), (8.14) and (8.15), inequality (8.11) follows. The proof of the claim is concluded.

8.2.2. *Conclusion of the proof of Proposition 5.2*

The first observation is that choosing $\mu \le 3$, we get

$$\|\tilde S(u_1)+\alpha^2\Delta h_1 w'\|_{p,\mu,\sigma} \le C\alpha^3. \tag{8.16}$$

Let us assume now that $\phi_1,\phi_2\in B_\alpha$ where

$$B_\alpha = \{\phi \;/\; \|\phi\|_{2,p,\mu,\sigma} \le K\alpha^3\}$$

where K is a constant to be chosen. Then we observe that for small α

$$\|\mathtt{N}(\phi)\|_{p,\mu+1,\sigma} \le C\alpha^4, \quad \text{for all} \quad \phi\in B_\alpha,$$

where C is independent of K. Then, from relations (8.16)-(8.15) we see that if K is fixed large enough independent of α, then the right hand side of equation (8.5) defines an operator that applies B_α into itself, which is also a contraction mapping of B_α endowed with the norm $\|\ \|_{p,\mu\sigma}$, provided that $\mu\le 3$. We conclude, from contraction mapping principle, the existence of ϕ as required.

The Lipschitz dependence (5.27) is a consequence of series of lengthy but straightforward considerations of the Lipschitz character in h_1 of the operator in the right hand side of equation (8.5) for the norm $\|\ \|_*$ defined in (5.34). Let us recall expression (8.12) for the operator B, and consider as an example, two terms that depend linearly on h_1:

$$A(h_1,\phi) := \alpha\, a^0_{ij}\,\partial_j h_1 \partial_{it}\phi\,.$$

Then

$$|A(h_1,\phi)| \le C\alpha|\partial_j h_1|\,|\partial_{it}\phi\,.$$

Hence

$$\|A(h_1,\phi)\|_{p,\mu+2,\sigma} \le C\alpha\|(1+r_\alpha^2)\,\partial_j h_1\|_\infty\,\|\partial_{it}\phi\,\|_{p,\mu,\sigma} \le C\alpha^4\|h_1\|_*\,\|\phi\|_{2,p,\mu,\sigma}.$$

Similarly, for $A(\phi, h_1) = \alpha^2 \Delta_M h_1 \, \partial_t \phi$ we have

$$| A(\phi, h_1) | \ \le \ C\alpha^2 |\Delta_M h_1(\alpha y)| \, (1 + r_\alpha)^{-\mu} e^{-\sigma|t|} \|\phi\|_{2,p,\mu,\sigma} \, .$$

Hence

$$\|\alpha^2 \Delta_M h_1 \, \partial_t \phi \, \|_{p,\mu+2,\sigma} \ \le \ C\alpha^{5-\frac{2}{p}} \|h_1\|_* \, \|\phi\|_{2,p,\mu,\sigma} .$$

We should take into account that some terms involve nonlinear, however mild dependence, in h_1. We recall for instance that $a^1_{ij} = a^1_{ij}(\alpha y, \alpha(t + h_0 + h_1))$. Examining the rest of the terms involved we find that the whole operator $\mathtt{N}$ produces a dependence on h_1 which is Lipschitz with small constant, and gaining decay in r_α,

$$\|\mathtt{N}(h_1, \phi) - \mathtt{N}(h_2, \phi)\|_{p,\mu+1,\sigma} \le C\alpha^2 \|h_1 - h_2\|_* \, \|\phi\|_{2,p,\mu,\sigma} . \qquad (8.17)$$

Now, in the error term

$$\mathcal{R} = -\tilde{S}(u_1) + \alpha^2 \Delta h_1 w' ,$$

we have that

$$\|\mathcal{R}(h_1) - \mathcal{R}(h_2)\|_{p,3,\sigma} \le C \, \alpha^2 \, \|h_1 - h_2\|_* \, . \qquad (8.18)$$

To see this, again we go term by term in expansion (5.15). For instance the linear term $\alpha^2 \ a^0_{ij} \partial_i h_0 \partial_j h_1 \, w''$. We have

$$|\alpha^2 \ a^0_{ij} \, \partial_i h_0 \partial_j h_1| \ \le \ C \, \alpha^2 \, (1 + r_\alpha)^{-3} \, e^{-\sigma|t|} \, \|h_1\|_*$$

so that

$$\|\alpha^2 \ a^0_{ij} \, \partial_i h_0 \, \partial_j h_1\|_{p,3,\sigma} \ \le \ C \, \alpha^2 \, \|h_1\|_* ,$$

the remaining terms are checked similarly.

Combining estimates (8.17), (8.18) and the fixed point characterization (8.5) we obtain the desired Lipschitz dependence (5.27) of Φ.

This concludes the proof.

9. *The reduced problem: proof of Proposition 5.4*

In this section we prove Proposition 5.4 based on the linear theory provided by Proposition 5.3. Thus, we want to solve the problem

$$\mathcal{J}(h_1) = \Delta_M h_1 + h_1 |A|^2 = G(h_1) + \sum_{i=1}^{J} \frac{c_i}{1 + r^4} \, \hat{z}_i \quad \text{in } M \, , \qquad (9.1)$$

$$\int_M \frac{h_1 \hat{z}_i}{1 + r^4} \, dV \ = \ 0 \quad \text{for all} \quad i = 1, \cdots , J \, ,$$

where the linearly independent Jacobi fields $\hat{z}_i$ will be chosen in (10.1) and (10.2) of §8, and $G = G_1 + G_2$ was defined in (5.29), (5.30). We will use contraction mapping principle to determine the existence of a unique solution h_1 for which constraint (5.7), namely

$$\|h_1\|_* := \|h_1\|_{L^\infty(M)} + \|(1+r^2)Dh_1\|_{L^\infty(M)} + \|D^2 h_1\|_{p,4-\frac{4}{p}} \ \leq \ \mathcal{K}\alpha \ , \quad (9.2)$$

is satisfied after fixing $\mathcal{K}$ sufficiently large.

We need to analyze the size of the operator G, for which the crucial step is the following estimate.

Lemma 9.1: *Let $\psi(y,t)$ be a function defined in $M_\alpha \times \mathbb{R}$ such that*

$$\|\psi\|_{p,\mu,\sigma} \ := \sup_{(y,t)\in M_\alpha\times\mathbb{R}} e^{\sigma|t|}(1 + r_\alpha^\mu)\, \|\psi\|_{L^p(B((y,t),1)} < +\infty$$

for $\sigma, \mu \geq 0$. The function defined in M as

$$q(y) := \int_{\mathbb{R}} \psi(y/\alpha, t)\, w'(t)\, dt$$

satisfies

$$\|q\|_{p,a} \ \leq \ C\, \|\psi\|_{p,\mu,\sigma} \quad (9.3)$$

provided that

$$\mu > \frac{2}{p} + a\,.$$

In particular, for any $\tau > 0$,

$$\|q\|_{p,2-\frac{2}{p}-\tau} \ \leq \ C\, \|\psi\|_{p,2,\sigma} \quad (9.4)$$

and

$$\|q\|_{p,4-\frac{4}{p}} \ \leq \ C\, \|\psi\|_{p,4,\sigma}\,. \quad (9.5)$$

Proof: We have that for $|y| > R_0$

$$\int_{|y|>R_0} |y|^{ap} \left| \int_{\mathbb{R}} \psi(y/\alpha, t)\, w'(t)\, dt \right|^p dV \ \leq \ C \int_{\mathbb{R}} w'(t)\, dt \int_{|y|>R_0} |y|^{ap}\, |\psi(y/\alpha, t)|^p\, dV\,.$$

Now

$$\int_{|y|>R_0} |y|^{ap}\, |\psi(y/\alpha, t)|^p\, dV \ = \ \alpha^{ap+2} \int_{|y|>R_0/\alpha} |y|^{ap}\, |\psi(y,t)|^p\, dV_\alpha$$

and

$$\int_{|y|>R_0/\alpha} |y|^{ap}\, |\psi(y,t)|^p\, dV_\alpha \ \leq \ C \sum_{i\geq[R_0/\alpha]} i^{ap} \int_{i<|y|<i+1} |\psi(y,t)|^p\, dV_\alpha\ .$$

Now, $i < |y| < i+1$ is contained in $O(i)$ balls with radius one centered at points of the annulus, hence

$$\int_{i<|y|<i+1} |\psi(y,t)|^p \, dV_\alpha \; \leq \; Ce^{-\sigma p|t|} i^{1-\mu p} \, \|\psi\|^p_{p,\mu}$$

$$\leq \; Ce^{-\sigma p|t|} \|\psi\|^p_{p,\mu} \int_{i<|y|<i+1} (1+r_\alpha)^{-\mu p} dV_\alpha$$

$$\leq \; Ce^{-\sigma p|t|} \|\psi\|^p_{p,\mu} \int_{i<|y|<i+1} |\alpha y|^{-\mu p} dV_\alpha$$

$$\leq \; Ce^{-\sigma p|t|} \|\psi\|^p_{p,\mu} \alpha^{-\mu p} i^{1-\mu p} \, .$$

Then we find

$$\| \, |y|^a \, q\|^p_{L^p(|y|>R_0)} \; \leq \; C \, \alpha^{ap-\mu p+2} \|\psi\|^p_{p,\mu} \sum_{i \geq [R_0/\alpha]} i^{ap-\mu p+1} \, .$$

The sum converges if $\mu > \frac{2}{p} + a$ and in this case

$$\| \, |y|^a \, q\|^p_{L^p(|y|>R_0)} \; \leq \; C \, \alpha^{ap-\mu p+2} \alpha^{-ap+\mu p-2} \|\psi\|^p_{p,\mu} \; = \; C \, \|\psi\|^p_{p,\mu}$$

so that

$$\| \, |y|^a \, q\|_{L^p(|y|>R_0)} \; \leq \; C \, \|\psi\|_{p,\mu}.$$

Now, for the inner part $|y| < R_0$ in M, the weights play no role. We have

$$\int_{|y|<R_0} |\psi(\mathtt{y}/\alpha, t)|^p \, dV \; = \; \alpha^2 \int_{|y|<R_0/\alpha} |\psi(y,t)|^p \, dV_\alpha \leq$$

$$C\alpha^2 \sum_{i \leq R_0/\alpha} \int_{i<|y|<i+1} |\psi(y,t)|^p \, dV_\alpha \; \leq C\alpha^2 \, \|\psi\|^p_{p,\mu} e^{-\sigma p|t|} \sum_{i \leq R_0/\alpha} i$$

$$\leq C \|\psi\|^p_{p,\mu} e^{-\sigma p|t|} \; .$$

Hence if $\mu > \frac{2}{p} + a$ we finally get

$$\|q\|_{p,a} \; \leq \; C \, \|\psi\|_{p,\mu}$$

and the proof of (9.3) is concluded. Letting $(\mu, a) = (2, 2 - \frac{2}{p} - \tau), (\mu, a) = (4, 4 - \frac{4}{p})$ respectively in (9.3), we obtain (9.4) and (9.5).

□

Let us apply this result to $\psi(y,t) = \mathtt{N}(\Phi(h_1))$ to estimate the size of the operator G_2 in (5.30). For $\phi = \Phi(h_1)$ we have that

$$G_2(h_1)(y) := c_*^{-1}\alpha^{-2}\int_{\mathbb{R}} \mathtt{N}(\phi)(y/\alpha,t)\, w'\, dt$$

satisfies

$$\|G_2(h_1)\|_{p,4-\frac{4}{p}} \le C\alpha^{-2}\|\mathtt{N}(\phi)\|_{p,4,\sigma} \ \le \ C\,\alpha^2.$$

On the other hand, we have that, similarly, for $\phi_l = \Phi(h_l)$, $l = 1,2$,

$$\|G_2(h_1) - G_2(h_2)\|_{p,4-\frac{4}{p}} \ \le \ C\alpha^{-2}\|\mathtt{N}(\phi_1,h_1) - \mathtt{N}(\phi_2,h_2)\|_{p,4,\sigma}.$$

Now,

$$\|\mathtt{N}(\phi_1,h_1) - \mathtt{N}(\phi_1,h_2)\|_{p,4,\sigma} \le C\alpha^2\|h_1-h_2\|_*\|\phi_1\|_{2,p,3,\sigma}, \le C\alpha^5\|h_1-h_2\|_*,$$

according to inequality (8.17), and

$$\|\mathtt{N}(\phi_1,h_1) - \mathtt{N}(\phi_2,h_1)\|_{p,4,\sigma} \le C\alpha^2\|\phi_1-\phi_2\|_{p,3,\sigma} \le C\alpha^4\|h_1-h_2\|_*\ .$$

We conclude then that

$$\|G_2(h_1) - G_2(h_2)\|_{p,4-\frac{4}{p}} \ \le \ C\,\alpha^2\|h_1-h_2\|_*\ .$$

In addition, we also have that

$$\|G_2(0)\|_{p,4-\frac{4}{p}} \le C\alpha^2.$$

for some $C>0$ possibly dependent of $\mathcal{K}$. On the other hand, it is similarly checked that the remaining small operator $G_1(h_1)$ in (5.29) satisfies

$$\|G_1(h_1) - G_1(h_2)\|_{p,4-\frac{4}{p}} \ \le \ C_1\,\alpha\|h_1-h_2\|_*\ .$$

A simple but crucial observation we make is that

$$c_*G_1(0) \ = \ \alpha\,\partial_i h_0\partial_j h_0\int_{\mathbb{R}}\zeta_4(t+h_0)a^1_{ij}w''w'\,dt + \alpha^{-2}\int_{\mathbb{R}}\zeta_4\,R_1(y,t,0,0\,)\,w'\,dt$$

so that for a constant C_2 independent of $\mathcal{K}$ in (9.2) we have

$$\|G_1(0)\|_{p,4-\frac{4}{p}} \le C_2\alpha\,.$$

In all we have that the operator $G(h_1)$ has an $O(\alpha)$ Lipschitz constant, and in addition satisfies

$$\|G(0)\|_{p,4-\frac{4}{p}} \le 2C_2\alpha.$$

Let $h = T(g)$ be the linear operator defined by Proposition 5.3. Then we consider the problem (9.1) written as the fixed point problem

$$h_1 \,=\, T(\,G(h_1)\,), \quad \|h\|_* \leq \mathcal{K}\alpha. \tag{9.6}$$

We have

$$\|T(\,G(h_1)\,)\|_* \leq \|T\|\,\|G(0)\|_{p,4-\frac{4}{p}} + C\alpha\|h_1\|_*\,.$$

Hence fixing $\mathcal{K} > 2C_2\|T\|$, we find that for all α sufficiently small, the operator $T\,G$ is a contraction mapping of the ball $\|h\|_* \leq \mathcal{K}\alpha$ into itself. We thus have the existence of a unique solution of the fixed problem (9.6), namely a unique solution h_1 to problem (9.1) satisfying (9.2) and the proof of Proposition 5.4 is concluded.

10. *Conclusion of the proof of Theorem 6*

We denote in what follows

$$r(x) = \sqrt{x_1^2 + x_2^2}, \quad \hat{r} = \frac{1}{r}(x_1, x_2, 0), \quad \hat{\theta} = \frac{1}{r}(-x_2, x_1, 0)\,.$$

We consider the four Jacobi fields associated to rigid motions, $z_1, \ldots, z_4$ introduced in (2.13). Let J be the number of bounded, linearly independent Jacobi fields of $\mathcal{J}$. By our assumption and the asymptotic expansion of the ends (2.11), $3 \leq J \leq 4$. (Note that when M is a catenoid, $z_4 = 0$ and $J = 3$.) Let us choose

$$\hat{z}_j = \sum_{l=1}^{4} d_{jl} z_{0l}, j = 1, ..., J \tag{10.1}$$

be normalized such that

$$\int_M q(y)\hat{z}_i\hat{z}_j = 0, \text{ for } i \neq j, \int_M q(y)\hat{z}_i^2 = 1, i, j = 1, \cdots, J\,. \tag{10.2}$$

In what follows we fix the function q as

$$q(y) \,:=\, \frac{1}{1 + r(y)^4}\,. \tag{10.3}$$

So far we have built, for certain constants $\tilde{c}_i$ a solution u of equation (5.36), namely

$$\Delta u + f(u) = \sum_{j=1}^{J} \tilde{c}_i \hat{z}_i(\alpha y) w'(t) q(\alpha y)\zeta_2$$

where u, defined in (5.35) satisfies the following properties

$$u(x) = w(t) + \phi(y,t) \tag{10.4}$$

near the manifold, meaning this $x = y + (t + h(\alpha y)\,)\,\nu(\alpha y)$ with

$$y \in M_\alpha, \quad |t| \le \frac{\delta}{\alpha} + \gamma \log(2 + r(\alpha y)).$$

The function ϕ satisfies in this region the estimate

$$|\phi| + |\nabla\phi| \le C\alpha^2 \frac{1}{1 + r^2(\alpha y)} e^{-\sigma|t|}\,. \tag{10.5}$$

Moreover, we have the validity of the global estimate

$$|\nabla u(x)| \le \frac{C}{1 + r^3(\alpha x)} e^{-\sigma\frac{\delta}{\alpha}}\ .$$

We introduce the functions

$$Z_i(x) = \partial_{x_i} u(x),\ i = 1,2,3, \quad Z_4(x) = -\alpha x_2 \partial_{x_2} u + \alpha x_1 \partial_{x_2} u\,.$$

From the expansion (10.4) we see that

$$\nabla u(x) = w'(t)\,\nabla t\ +\ \nabla\phi.$$

Now, $t = z - h(\alpha y)$ where z designates normal coordinate to M_α. Since $\nabla z = \nu = \nu(\alpha y)$ we then get

$$\nabla t = \nu(\alpha y) - \alpha \nabla h(\alpha y).$$

Let us recall that h satisfies $h = (-1)^k \beta_k \log r + O(1)$ along the k-th end, and

$$\nabla h = (-1)^k \frac{\beta_k}{r} \hat{r} + O(r^{-2})\,.$$

From estimate (10.5) we we find that

$$\nabla u(x) = w'(t)(\nu - \alpha(-1)^k \frac{\beta_k}{r_\alpha}\hat{r}) + O(\alpha r_\alpha^{-2} e^{-\sigma|t|}). \tag{10.6}$$

From here we get that near the manifold,

$$Z_i(x) = w'(t)\,(z_i(\alpha y) - \alpha(-1)^k \frac{\beta_k}{r_\alpha}\hat{r} e_i) + O(\alpha r_\alpha^{-2} e^{-\sigma|t|}), \quad i = 1,2,3, \tag{10.7}$$

$$Z_4(x) = w'(t)\, z_{04}(\alpha y) + O(\alpha r_\alpha^{-1} e^{-\sigma|t|}). \tag{10.8}$$

Using the characterization (5.36) of the solution u and barriers (in exactly the same way as in Lemma **??** below which estimates eigenfunctions of the linearized operator), we find the following estimate for $r_\alpha(x) > R_0$:

$$|\nabla u(x)| \le C \sum_{k=1}^{m} e^{-\sigma |x_3 - \alpha^{-1}(F_k(\alpha x') + \beta_j \alpha \log|\alpha x'|)\,|} \,. \tag{10.9}$$

We claim that

$$\int_{\mathbb{R}^3} (\Delta u + f(u)) Z_i(x)\, dx = 0 \quad \text{for all} \quad i = 1, \dots, 4 \tag{10.10}$$

so that

$$\sum_{j=1}^{J} \tilde{c}_j \int_{\mathbb{R}^3} q(\alpha x) \hat{z}_j(\alpha y) w'(t) Z_i(x)\, \zeta_2\, dx \;=\; 0 \quad \text{for all} \quad i = 1, \dots, 4. \tag{10.11}$$

Let us accept this fact for the moment. Let us observe that from estimates (10.7) and (10.8),

$$\alpha^2 \int_{\mathbb{R}^3} q(\alpha x) \hat{z}_j(\alpha y) w'(t) \sum_{l=1}^{4} d_{il} Z_l(x)\, \zeta_2\, dx = \int_{-\infty}^{\infty} w'(t)^2 dt \int_M q\, \hat{z}_j \hat{z}_i dV + o(1)$$

with $o(1)$ is small with α. Since the functions $\hat{z}_i$ are linearly independent on any open set because they solve an homogeneous elliptic PDE, we conclude that the matrix with the above coefficients is invertible. Hence from (10.11) and (10.2), all $\tilde{c}_i$'s are necessarily zero. We have thus found a solution to the Allen Cahn equation (2.1) with the properties required in Theorem 6.

It remains to prove identities (10.10). The idea is to use the invariance of $\Delta + f(u)$ under rigid translations and rotations. This type of Pohozaev identity argument has been used in a number of places, see for instance [31].

In order to prove that the identity (10.10) holds for $i = 3$, we consider a large number $R >> \frac{1}{\alpha}$ and the infinite cylinder

$$C_R = \{x \;/\; x_1^2 + x_2^2 < R^2\}.$$

Since in C_R the quantities involved in the integration approach zero at exponential rate as $|x_3| \to +\infty$ uniformly in (x_1, x_2), we have that

$$\int_{C_R} (\Delta u + f(u)) \partial_{x_3} u - \int_{\partial C_R} \nabla u \cdot \hat{r}\, \partial_{x_3} u = \int_{C_R} \partial_{x_3} \left(F(u) - \frac{1}{2} |\nabla u|^2 \right) = 0.$$

We claim that

$$\lim_{R\to+\infty}\int_{\partial C_R}\nabla u\cdot\hat r\ \partial_{x_3}u\ =\ 0.$$

Using estimate (10.6) we have that near the manifold,

$$\partial_{x_3}u\nabla u(x)\cdot\hat r = w'(t)^2((\nu-\alpha(-1)^k\frac{\beta_k}{r_\alpha}\hat r)\cdot\hat r)\nu_3 + O(\alpha e^{-\sigma|t|}\frac{1}{r^2}).$$

Let us consider the k-th end, which for large r is expanded as

$$x_3 = F_{k,\alpha}(x_1,x_2) = \alpha^{-1}(a_k\log\alpha r + b_k + O(r^{-1}))$$

so that

$$(-1)^k\nu = \frac{1}{\sqrt{1+|\nabla F_{k,\alpha}|^2}}(\nabla F_{k,\alpha},-1) = \frac{a_k}{\alpha}\frac{\hat r}{r} - e_3\ +\ O(r^{-2})\,. \tag{10.12}$$

Then on the portion of C_R near this end we have that

$$(\nu-\alpha(-1)^k\frac{\beta_k}{r_\alpha}\hat r)\cdot\hat r\,\nu_3 = -\alpha^{-1}\frac{a_k+\alpha\beta_k}{R} + O(R^{-2}). \tag{10.13}$$

In addition, also, for $x_1^2+x_2^2=R^2$ we have the expansion

$$t = (x_3 - F_{k,\alpha}(x_1,x_2) - \beta_k\log\alpha r + O(1))(1+O(R^{-2}))$$

with the same order valid after differentiation in x_3, uniformly in such (x_1,x_2). Let us choose $\rho=\gamma\log R$ for a large, fixed γ. Observe that on ∂C_R the distance between ends is greater than 2ρ whenever α is sufficiently small. We get,

$$\int_{F_{k,\alpha}(x_1,x_2)+\beta_k\log\alpha r-\rho}^{F_{k,\alpha}(x_1,x_2)+\beta_k\log\alpha r+\rho} w'(t)^2dx_3 = \int_{-\infty}^{\infty}w'(t)^2dt + O(R^{-2})\,.$$

Because of estimate (10.9) we conclude, fixing appropriately γ, that

$$\int_{\bigcap_k\{|x_3-F_{k,\alpha}|>\rho\}}\partial_{x_3}u\nabla u(x)\cdot\hat r\,dx_3 = O(R^{-2})\,.$$

As a conclusion

$$\int_{-\infty}^{\infty}\partial_{x_3}u\nabla u\cdot\hat r\,dx_3 = -\frac{1}{\alpha R}\sum_{k=1}^m(a_k+\alpha\beta_k)\int_{-\infty}^{\infty}w'(t)^2\,dt\ + O(R^{-2})$$

and hence

$$\int_{\partial C_R}\partial_{x_3}u\nabla u(x)\cdot\hat r = -\frac{2\pi}{\alpha}\sum_{k=1}^m(a_k+\alpha\beta_k) + O(R^{-1})\,.$$

But $\sum_{k=1}^{m} a_k = \sum_{k=1}^{m} \beta_k = 0$ and hence (10.10) for $i = 3$ follows after letting $R \to \infty$.

Let us prove the identity for $i = 2$. We need to carry out now the integration against $\partial_{x_2} u$. In this case we get

$$\int_{C_R} (\Delta u + f(u))\partial_{x_2} u = \int_{\partial C_R} \nabla u \cdot \hat{r}\, \partial_{x_2} u + \int_{C_R} \partial_{x_2} \left(F(u) - \frac{1}{2}|\nabla u|^2 \right).$$

We have that

$$\int_{C_R} \partial_{x_2} \left(F(u) - \frac{1}{2}|\nabla u|^2 \right) = \int_{\partial C_R} \left(F(u) - \frac{1}{2}|\nabla u|^2 \right) n_2$$

where $n_2 = x_2/r$. Now, near the ends estimate (10.6) yields

$$|\nabla u|^2 = |w'(t)|^2 + O(e^{-\sigma|t|}\frac{1}{r^2})$$

and arguing as before, we get

$$\int_{-\infty}^{\infty} |\nabla u|^2 dx_3 = m \int_{-\infty}^{\infty} |w'(t)|^2 dt + O(R^{-2}).$$

Hence

$$\int_{\partial C_R} |\nabla u|^2 n_2 = m \int_{-\infty}^{\infty} |w'(t)|^2 dt \int_{[r=R]} n_2 + O(R^{-1}) \, .$$

Since $\int_{[r=R]} n_2 = 0$ we conclude that

$$\lim_{R \to +\infty} \int_{\partial C_R} |\nabla u|^2 \, n_2 = 0.$$

In a similar way we get

$$\lim_{R \to +\infty} \int_{\partial C_R} F(u) \, n_2 = 0.$$

Since near the ends we have

$$\partial_{x_2} u = w'(t)(\nu_2 - \alpha(-1)^k \frac{\beta_k}{r_\alpha} \hat{r} e_2) + O(\alpha r^{-2} e^{-\sigma|t|})$$

and from (10.12) $\nu_2 = O(R^{-1})$, completing the computation as previously done yields

$$\int_{\partial C_R} \nabla u \cdot \hat{r}\, \partial_{x_2} u = O(R^{-1}).$$

As a conclusion of the previous estimates, letting $R \to +\infty$ we finally find the validity of (10.10) for $i = 2$. Of course the same argument holds for $i = 1$.

Finally, for $i = 4$ it is convenient to compute the integral over C_R using cylindrical coordinates. Let us write $u = u(r, \theta, z)$. Then

$$\int_{C_R} (\Delta u + f(u))\,(x_2 \partial_{x_1} u - x_1 \partial_{x_1} u) =$$

$$\int_0^{2\pi} \int_0^R \int_{-\infty}^{\infty} [u_{zz} + r^{-1}(r u_r)_r + f(u)]\, u_\theta\, r\, d\theta\, dr\, dz \quad =$$

$$-\frac{1}{2} \int_0^{2\pi} \int_0^R \int_{-\infty}^{\infty} \partial_\theta\, [u_z^2 + u_r^2 - 2F(u)]\, r\, d\theta\, dr\, dz + R \int_{-\infty}^{\infty} \int_0^{2\pi} u_r\, u_\theta(R, \theta, z)\, d\theta\, dz \; =$$

$$0 + \int_{\partial C_R} u_r u_\theta \; .$$

On the other hand, on the portion of ∂C_R near the ends we have

$$u_r\, u_\theta = w'(t)^2 R(\nu \cdot \hat{r})(\nu \cdot \hat{\theta}) + O(R^{-2} e^{-\sigma|t|}).$$

From (10.12) we find

$$(\nu \cdot \hat{r})(\nu \cdot \hat{\theta}) = O(R^{-3}),$$

hence

$$u_r\, u_\theta = w'(t)^2 O(R^{-2}) + O(R^{-2} e^{-\sigma|t|})$$

and finally

$$\int_{\partial C_R} u_r\, u_\theta = O(R^{-1}).$$

Letting $R \to +\infty$ we obtain relation (10.10) for $i = 4$. The proof is concluded.

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