ANALYSIS OF BLOW-UP LOCUS AND EXISTENCE OF WEAK SOLUTIONS FOR NONLINEAR SUPERCRITICAL PROBLEMS

KELEI WANG AND JUNCHENG WEI

ABSTRACT. We give a qualitative analysis of sequences of stationary solutions to the supercritical problem

$$-\Delta u = |u|^{p-1}u \ \text{ in } \Omega, \ \ p > \frac{n+2}{n-2}.$$

A consequence of the analysis is the existence of positive singular weak solutions on a convex domain when $p \in (\frac{n+2}{n-2}, \frac{n+1}{n-3})$, with only isolated singularities.

1. INTRODUCTION AND MAIN RESULTS

Of concern is the local qualitative behavior of sequences of stationary weak solutions to

$$-\Delta u = |u|^{p-1}u \text{ in } B_2 \tag{1.1}$$

where $p > \frac{n+2}{n-2}$ and B_2 denotes the open ball in $\mathbb{R}^n$ with radius 2. Throughout this paper $B_r(x)$ always denotes the open ball of radius r with center at x, and B_r is a ball centered at the origin.

For a weak solution $u \in H^1(B_2) \cap L^{p+1}(B_2)$, we say that it is stationary if for any smooth vector field Y with compact support,

$$\int \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1}\right) \text{div}Y - DY(\nabla u, \nabla u) = 0. \tag{1.2}$$

For smooth solutions this condition follows from variations of the energy functional

$$E(u) = \int \frac{1}{2}|\nabla u|^2 - \frac{|u|^{p+1}}{p+1}$$

with respect to perturbations of the parametrization of the domain, that is,

$$\frac{d}{dt}E(u(x + tY(x)))\Big|_{t=0} = 0. \tag{1.3}$$

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Formula (1.2) can also be obtained by multiplying (1.1) by $Y \cdot \nabla u$ and integrating by parts, granted that this solution has enough regularity. Such condition is classical in many works dealing with partial regularity. In this problem it was first used by Pacard in [14, 15].

Since p is assumed to be supercritical, solutions to (1.1) may not be smooth. Thus we need to define

Definition 1.1. *For a solution u of (1.1), its singular set $\mathcal{S}(u)$ consists of those points, such that in any neighborhood of this point u is not bounded.*

Roughly speaking $\mathcal{S}(u) = \{u = \infty\}$. By definition it is a closed set. Pacard's partial regularity result ([14]) says that for a stationary weak solution u, $\mathcal{S}(u)$ is a closed set satisfying

$$H^{n-2\frac{p+1}{p-1}}(\mathcal{S}(u)) = 0.$$

In particular, $\dim(\mathcal{S}(u)) \leq n - 2\frac{p+1}{p-1}$.

Let $u_i \in H^1(B_2) \cap L^{p+1}(B_2)$ be a sequence of stationary solutions to (1.1), with the energy bound

$$\sup_i \int_{B_2} |\nabla u_i|^2 + |u_i|^{1+p} := M < +\infty. \tag{1.4}$$

By this assumption, we can assume that u_i converges weakly to u in $H^1(B_2) \cap L^{p+1}(B_2)$. By the compact Sobolev embedding theorem, u_i converges strongly to u in $L^q(B_2)$ for any $q < p + 1$. In particular u_i^p converges to u^p in $L^1(B_2)$, and u is a weak solution to (1.1).

Denote the measure

$$\mu_i = \left(\frac{p-1}{2} |\nabla u_i|^2 + \frac{p-1}{p+1} |u_i|^{p+1} \right) dx.$$

There exists a positive Radon measure ν such that,

$$\mu_i \rightharpoonup \left(\frac{p-1}{2} |\nabla u|^2 + \frac{p-1}{p+1} |u|^{p+1} \right) dx + \nu := \mu \text{ weakly as measures.}$$

Note that u_i converges strongly to u in $H^1(B_1) \cap L^{p+1}(B_1)$ if and only if $\nu = 0$ in B_1. Let

$$\Sigma := \mathcal{S}(u) \cup \text{spt}(\nu),$$

which we call the blow up locus of this sequence (u_i).

The purpose of this paper is to give a qualitative characterization of the blow-up locus set. Denote $n - m$ to be the integer part of $n - 2\frac{p+1}{p-1}$. It is obvious that $m \geq 3$. With these notations we have

Theorem 1.2.
- *For any k, u_i converges to u in $C_{loc}^k(B_1 \setminus \Sigma)$.*
- *If $n - 2\frac{p+1}{p-1}$ is not an integer, $\Sigma \cap B_1 = \mathcal{S}(u)$ and $\nu = 0$. Hence $u_i \to u$ strongly in $H^1(B_1) \cap L^{p+1}(B_1)$, and u satisfies the stationary condition.*
- *If $n - 2\frac{p+1}{p-1}$ is an integer $n - m$, i.e., $p = \frac{m+2}{m-2}$, $\Sigma \cap B_1$ is $(n-m)$-countable rectifiable with $H^{n-m}(\Sigma \cap B_1) < +\infty$.*
- *If $n - 2\frac{p+1}{p-1}$ is an integer $n - m$ and $H^{n-m}(\Sigma \cap B_1) > 0$, after extracting a subsequence, a rescaled subsequence of u_i converges to a nontrivial smooth solution $\bar{v}$ to the low-dimensional Yamabe problem*

$$-\Delta \bar{v} = |\bar{v}|^{\frac{4}{m-2}}\bar{v} \quad in \ \mathbb{R}^m, \quad \int_{\mathbb{R}^m}(|\nabla \bar{v}|^2 + |\bar{v}|^{p+1}) \le C(M). \qquad (1.5)$$

Several remarks are in order: first, the above theorem implies that the only possible singularity formulation is through low-dimensional bubble (1.5). On the other hand, there are indeed sequences of bubbling solutions when p is close to $\frac{m+2}{m-2}$ (del Pino-Musso-Pacard [4]). We conjecture that when $p = \frac{m+2}{m-2}$, the blow-up locus Σ must be a minimal submanifold. (The problem is to show that the limit function u is also stationary. This may not be true in general by examples of Ding-Li-Li [5] in harmonic map theory.) Second, Theorem 1.2 is reminiscent of similar results for harmonic maps by Lin [9]. Indeed our proof is motivated by ideas of [9]. See also Lin-Riviere [11], Li-Tian [12] and Riviere [17], Tian [21]. Finally, Theorem 1.2 also covers the sign-changing case.

As an application we can improve Pacard's result when $n - 2\frac{p+1}{p-1}$ is not an integer, that is

Theorem 1.3. *Let u be a stationary solution to (1.1), then the Hausdorff dimension of $\mathcal{S}(u)$ is no more than $n - m$. Moreover, if $p < \frac{n+1}{n-3}$, $\mathcal{S}(u)$ is a discrete set.*

As another application, we consider a problem studied by Dancer [3]. Given a smooth bounded domain $\Omega \subset \mathbb{R}^n$, consider the problem

$$\begin{cases} -\Delta u = \lambda(1+u)^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (1.6)$$

where $p > \frac{n+2}{n-2}, \lambda > 0$.

In [3], Dancer proved the existence of a family of positive solutions $(\lambda(s), u(s))$ such that $\|u(s)\|_{L^\infty(\Omega)} \to +\infty$ while $\lambda(s)$ bounded (from below and above). If Ω is star-shaped, $u(s)$ are uniformly bounded in

$H^1(\Omega)$ and $\lambda(s)$ is uniformly bounded from below and also above. Furthermore Dancer showed that under the assumptions that Ω is convex and possesses n-axis of symmetries and that $\frac{n+2}{n-2} < p < \frac{n+1}{n-3}$, then the sequence $(\lambda(s), u(s))$ converges to (λ_*, u_*) in $\mathbb{R} \times C_{loc}^k(\Omega \backslash \{0\})$ where u^* is a weak solution of (1.6) with only singularity at the origin.

In the following we remove the symmetry assumption of Dancer.

Theorem 1.4. *If* $\frac{n+2}{n-2} < p < \frac{n+1}{n-3}$ *and* Ω *is convex, then given any sequence* (λ_i, u_i) *in this family,*

- *There exists a subsequence such that* $\lambda_i \to \lambda_*$, $u_i \to u_*$ *strongly in* $H^1(\Omega)$;
- u_* *is a stationary* H^1 *weak solution of* (1.6)*, and it is smooth outside finitely many points* $x_i \in \Omega$, $1 \le i \le K$;
- *For any* k, u_i *converges to* u_* *in* $C_{loc}^k(\Omega \setminus \cup_{i=1}^K \{x_i\})$.

The convexity is used to guarantee that u_i are smooth near $\partial\Omega$ (uniformly in i). This can be proved by the moving plane method. By this near boundary regularity we see the blow up locus can only appear in the interior of Ω, thus we can apply Theorem 1.2 and Theorem 1.3.

The organization of the paper is as follows: In Section 2, we collect some basic estimates including the monotonicity formula and ϵ-regularity. In Section 3 we give the basic Hausdorff measure estimate. Then in Section 4 we consider the case where $2\frac{p+1}{p-1}$ is not an integer and carry out the important dimension reduction technique to prove Theorem 1.3. The remaining part is devoted to the analysis of the case when $2\frac{p+1}{p-1}$ is an integer. We construct the bubbling sequence in Section 5. In Section 6 we give a quantization of the density function. Finally we discuss the stationary property of the blow-up locus.

2. Preliminary Analysis

We collect some preliminary analysis in this section. The basic tool used in this paper is the following monotonicity formula.

Theorem 2.1. *For any* $B_R(x) \subset B_1$ *and* $r \in (0, R)$,

$$E(r; x, u) := \frac{r^{-n+2\frac{p+1}{p-1}}}{p+3} \int_{B_r(x)} \left(\frac{p-1}{2} |\nabla u|^2 + \frac{p-1}{p+1} u^{p+1} \right)$$

$$+\frac{1}{p+3}\frac{d}{dr}\left[r^{-n+2\frac{p+1}{p-1}}\int_{\partial B_r(x)}u^2\right]$$

is nondecreasing in r. Moreover, if $E(r;x,u)\equiv const.$ in $(0,R)$, then u is homogeneous with respect to x

$$u(x+\lambda y)=\lambda^{-\frac{2}{p-1}}u(x+y),\quad y\in B_R(x),\ \lambda\in(0,1).$$

Proof. This follows directly from Pacard [14]. In fact, by the proof in [14], we have

$$\frac{d}{dr}E(r;x,u)=c(n,p)r^{2\frac{p+1}{p-1}-n}\int_{\partial B_r(x)}\left(\frac{\partial u}{\partial r}+\frac{2}{p-1}r^{-1}u\right)^2\geq 0.\quad(2.1)$$

This also gives the homogeneity of u when $E\equiv const..$ $\qquad\square$

An equivalent form is

$$E(r;x,u)=r^{-n+2\frac{p+1}{p-1}}\int_{B_r(x)}\left(\frac{|\nabla u|^2}{2}-\frac{|u|^{p+1}}{p+1}\right)+r^{-n+2\frac{p+1}{p-1}-1}\int_{\partial B_r(x)}\frac{u^2}{p-1}.$$

Next we recall the ε−regularity theorem, which was proved in [15]. (See Proposition 2 there. Although the result was only stated for positive solutions, its proof also holds for sign-changing solutions after suitable modifications.)

Theorem 2.2. *There exist two constants $\varepsilon_0, C>0$, depending only on p and n, such that if u is a stationary weak solution of (1.1) in B_{2R}, satisfying*

$$R^{2\frac{p+1}{p-1}-n}\int_{B_R}\frac{p-1}{2}|\nabla u|^2+\frac{p-1}{p+1}|u|^{p+1}\leq\varepsilon_0,$$

then

$$\sup_{B_{R/2}}\left(R^2|\nabla^2 u|+R|\nabla u|+|u|\right)\leq CR^{-\frac{2}{p-1}}.$$

In the proof if we tract the dependence of C on ε carefully, we can show that as $\varepsilon_0\to 0$, $C(\varepsilon_0)\to 0$.

The following is a technical result, which will be used in the latter part of this paper to treat the boundary term in the monotonicity formula $E(r;x,u)$.

Lemma 2.3. *If $u\in H^1(B_1)$, then for any $s\in[0,n-1)$, the set*

$$E:=\{x:\limsup_{r\to 0}r^{-s}\int_{\partial B_r(x)}u^2>0\}$$

has zero H^{s+1} measure.

Proof. Note that by the trace theorem for H^1 space, $\int_{\partial B_r(x)} u^2$ is well defined for every $\partial B_r(x) \subset B_1$.

We claim that E belongs to the set

$$\{x : \limsup_{r \to 0} r^{-s-1} \int_{B_r(x)} u^2 + |\nabla u|^2 > 0\}.$$

Indeed, if

$$\limsup_{r \to 0} r^{-s-1} \int_{B_r(x)} u^2 + |\nabla u|^2 = 0,$$

then for any r, there exists $t_0 \in (r/2, r)$ such that

$$\int_{\partial B_{t_0}(x)} u^2 = o(r^s).$$

Next by direct differentiating in r, we see

$$r^{1-n} \int_{\partial B_r(x)} u^2$$
$$\leq t_0^{1-n} \int_{\partial B_{t_0}(x)} u^2 + 2 \int_{B_r(x) \setminus B_{t_0}(x)} |y - x|^{1-n} u \frac{(y - x) \cdot \nabla u}{|y - x|}$$
$$\leq t_0^{1-n} \int_{\partial B_{t_0}(x)} u^2 + 2^n r^{1-n} \left(\int_{B_r(x) \setminus B_{t_0}(x)} u^2 \right)^{\frac{1}{2}} \left(\int_{B_r(x) \setminus B_{t_0}(x)} |\nabla u|^2 \right)^{\frac{1}{2}}$$
$$= o(r^{s+1-n}).$$

That is, for $r \to 0$,

$$\int_{\partial B_r(x)} u^2 = o(r^s).$$

This proves the claim, and then by [10, Lemma 2.1.1] we get the measure estimate. $\square$

The next result is Lemma 4 in [14].

Lemma 2.4. *There exists a constant $C > 0$ depending only on p and n, such that for a stationary solution u in B_1, for any $x \in B_{1/4}$ and $r \in (0, 1/4)$,*

$$r^{2\frac{p+1}{p-1}-n} \int_{B_r(x)} \frac{p-1}{2} |\nabla u|^2 + \frac{p-1}{p+1} |u|^{p+1} \leq C E(2r; x, u).$$

By the monotonicity formula we have

Corollary 2.5. *There exists a constant $C > 0$ depending only on p and n, such that for any stationary solution u in B_1, for any $x \in B_{1/2}$ and $r \in (0, 1/8)$,*

$$r^{2\frac{p+1}{p-1}-n} \int_{B_r(x)} |\nabla u|^2 + |u|^{p+1} \leq C \left(\int_{B_1} |\nabla u|^2 + |u|^{p+1} + \left(\int_{B_1} |u|^{p+1} \right)^{\frac{2}{p+1}} \right).$$
(2.2)

Proof. For any $x \in B_{1/2}$,

$$\begin{aligned}
E(\frac{1}{4}; x, u) &\leq 4 \int_{\frac{1}{4}}^{\frac{1}{2}} E(\rho; x, u) d\rho \\
&\leq C(n, p) \left(\int_{B_1} |\nabla u|^2 + |u|^{p+1} + \int_{B_1} u^2 \right) \\
&\leq C(n, p) \left(\int_{B_1} |\nabla u|^2 + |u|^{p+1} + \left(\int_{B_1} |u|^{p+1} \right)^{\frac{2}{p+1}} \right).
\end{aligned}$$

Then we can apply the previous lemma to get the claim. $\qquad \square$

Define

$$\mu_{1,i} = \frac{1}{2}|\nabla u_i|^2 dx \rightharpoonup \mu_1 = \frac{1}{2}|\nabla u|^2 dx + \nu_1,$$

$$\mu_{2,i} = \frac{1}{p+1}|u_i|^{p+1} dx \rightharpoonup \mu_2 = \frac{1}{p+1}|u|^{p+1} dx + \nu_2.$$

Hence we have $\mu_i = (p-1)(\mu_{1,i} + \mu_{2,i})$, $\mu = (p-1)(\mu_1 + \mu_2)$ and $\nu = (p-1)(\nu_1 + \nu_2)$.

We have the following energy partition between ν_1 and ν_2.

Lemma 2.6. $2\nu_1 = (p+1)\nu_2$.

Proof. Because $u_i \in H^1(B_2) \cap L^{p+1}(B_2)$, by testing the equation (1.1) with $u_i \eta^2$, where $\eta \in C_0^\infty(B_2)$, we get

$$\int_{B_2} |\nabla u_i|^2 \eta^2 - |u_i|^{p+1} \eta^2 = \int_{B_2} u_i^2 \Delta \frac{\eta^2}{2}.$$

By taking $i \to +\infty$, and noting that u_i converges to u strongly in $L^2(B_2)$, we get

$$\int_{B_2} \left(|\nabla u|^2 \eta^2 - |u|^{p+1} \eta^2 \right) + \int_{B_2} 2\eta^2 d\nu_1 - (p+1)\eta^2 d\nu_2 = \int_{B_2} u^2 \Delta \frac{\eta^2}{2}.$$
(2.3)

On the other hand, since $u \in H^1(B_2) \cap L^{p+1}(B_2)$ is an L^1 weak solution to (1.1), by choosing test functions in the form $u^M \eta^2$, where $u^M = \max\{\min\{u, M\}, -M\}$, and then letting $M \to +\infty$, we also have

$$\int_{B_2} |\nabla u|^2 \eta^2 - |u|^{p+1} \eta^2 = \int_{B_2} u^2 \Delta \frac{\eta^2}{2}.$$

Substituting these into (2.3), we see

$$\int_{B_2} 2\eta^2 d\nu_1 - (p+1)\eta^2 d\nu_2 = 0.$$

Since η can be chosen arbitrarily, this gives the claim. $\square$

3. ANALYSIS OF THE BLOW UP LOCUS

In this section we use notations as in Theorem 1.2. Define

$$\widetilde{\Sigma} = \cap_{0<r<1}\{x \in B_1 : \limsup_{i\to\infty} r^{-n+2\frac{p+1}{p-1}} \mu_i(B_r(x)) \geq \frac{\varepsilon_0}{2}\}.$$

Below we will show that this coincides with Σ defined in the introduction.

Lemma 3.1. $\widetilde{\Sigma}$ *is a closed set. For any domain* $\Omega \subset\subset B_1 \setminus \widetilde{\Sigma}$ *and* $k > 0$, u_i *converges to* u *in* $C^k(\Omega)$.

Proof. By definition, if x_0 does not belong to $\widetilde{\Sigma}$, there exists an $r_0 > 0$ such that, for all i large,

$$r_0^{2\frac{p+1}{p-1}-n} \mu_i(B_{r_0}(x)) \leq \varepsilon_0.$$

By Theorem 2.2,

$$\sup_{B_{r_0/2}(x_0)} |u_i| \leq C r_0^{-\frac{2}{p-1}}.$$

Then standard elliptic estimates show that u_i are uniformly bounded in $C^{k+1}(B_{r_0/4}(x_0))$ for any k. This then implies that for any $r \in (0, r_0/4)$,

$$\mu_i(B_r(x_0)) \leq C r^{2\frac{p+1}{p-1}}.$$

Then we get an $r_1 > 0$, which is independent of i, such that $B_{r_1}(x_0) \cap \widetilde{\Sigma} = \emptyset$. So $\widetilde{\Sigma}$ is relatively closed.

Since u_i converges to u in $L^2(B_{r_1}(x_0))$, u_i also converges to u in $C^k(B_{r_1}(x_0))$. $\square$

From this proof we see

Corollary 3.2. u *is smooth outside* $\widetilde{\Sigma}$. *That is,* $\mathcal{S}(u) \subset \widetilde{\Sigma}$.

Lemma 3.1 also shows that u_i converges strongly to u in $H^1_{loc} \cap L^{p+1}_{loc}$ outside $\widetilde{\Sigma}$. Hence by the definition of ν we obtain

Corollary 3.3. $spt(\nu) \subset \widetilde{\Sigma}$.

Since we will encounter many times the weak convergence of positive Radon measures. The following facts may be helpful to keep in mind.

(1) For a positive Radon measure μ defined in B_1, except a countable set of $r \in (0,1)$,

$$\mu(\partial B_r) := \lim_{h \to 0} \mu(B_{r+h} \setminus \overline{B_{r-h}}) = 0.$$

(2) Assume that a sequence of positive Radon measures μ_i converges weakly to μ. Then for any open bounded set Ω,

$$\liminf_{i \to +\infty} \mu_i(\Omega) \geq \mu(\Omega),$$

$$\limsup_{i \to +\infty} \mu_i(\overline{\Omega}) \leq \mu(\overline{\Omega}).$$

(3) Combining the previous two points, we see for a.a. $r > 0$,

$$\lim_{i \to +\infty} \mu_i(B_r) = \mu(B_r).$$

Lemma 3.4. $H^{n-2\frac{p+1}{p-1}}(\widetilde{\Sigma} \cap B_1) < +\infty$.

Proof. For any $x \in \widetilde{\Sigma}$ and $r \in (0,1)$, if $\mu(\partial B_r(x)) = 0$, then

$$\mu(B_r) = \lim_{i \to +\infty} \mu_i(B_r).$$

So by the definition of $\widetilde{\Sigma}$, we have

$$r^{-n+2\frac{p+1}{p-1}} \mu(B_r(x)) \geq c\varepsilon_0. \tag{3.1}$$

If $\mu(\partial B_r(x)) \neq 0$, we can take an increasing sequence $r_i \to r$ with $\mu(\partial B_{r_i}(x)) = 0$, so that (3.1) holds for r_i. Then by letting $i \to +\infty$, we see (3.1) also holds for such r, and hence for any $r > 0$.

The measure estimate can be proved by the Vitali covering theorem, see [8, Theorem 3.2]. $\square$

Remark 3.5. *In fact the proof shows that, for any $x \in \widetilde{\Sigma} \cap B_1$ and $r \in (0, 1/2)$,*

$$H^{n-2\frac{p+1}{p-1}}(B_r \cap \widetilde{\Sigma}) \leq C(M) r^{n-2\frac{p+1}{p-1}}.$$

Concerning the upper bound, we have

Lemma 3.6. *For any $x \in \widetilde{\Sigma}$ and $r \in (0,1)$,*

$$\mu(B_r(x)) \leq C(M) r^{n-2\frac{p+1}{p-1}}. \tag{3.2}$$

Proof. By (2.2), for any $i > 0$, $x \in B_1$ and $r \in (0,1)$,

$$r^{-n+2\frac{p+1}{p-1}}\mu_i(B_r(x)) \leq C\left(\int_{B_2}|\nabla u_i|^2 + |u_i|^{p+1} + \left(\int_{B_1}|u_i|^{p+1}\right)^{\frac{2}{p+1}}\right).$$

Then (3.2) follows frow the weak convergence of μ_i to μ. $\qquad \square$

In particular, $\mu\lfloor_\Sigma$ is absolutely continuous with respect to $H^{n-2\frac{p+1}{p-1}}\lfloor_{\widetilde{\Sigma}}$. However we can show

Lemma 3.7. $\mu\lfloor_{\widetilde{\Sigma}} = \nu$.

Proof. Because $u \in H^1(B_1) \cap L^{p+1}(B_1)$, by [10, Lemma 2.1.1], the set

$$\{x \in B_1 : \limsup_{r \to 0} r^{-n+2\frac{p+1}{p-1}}\int_{B_r(x)}|\nabla u|^2 + |u|^{p+1} > 0\}$$

has zero $n - 2(p+1)/(p-1)$ Hausdorff dimensional measure. This means, for $H^{n-2(p+1)/(p-1)}$−a.a. $x \in \widetilde{\Sigma}$,

$$\limsup_{r \to 0} r^{-n+2\frac{p+1}{p-1}}\int_{B_r(x)}|\nabla u|^2 + |u|^{p+1} = 0. \qquad (3.3)$$

Because $H^{n-2\frac{p+1}{p-1}}(\widetilde{\Sigma}) < +\infty$, the measure $(|\nabla u|^2 + |u|^{p+1})dx$ restricted to $\widetilde{\Sigma}$ is zero. So $\mu\lfloor_{\widetilde{\Sigma}} = \nu$. $\qquad \square$

This result, combined with (3.1) and (3.2), implies that ν and $H^{n-2\frac{p+1}{p-1}}\lfloor_{\widetilde{\Sigma}}$ are mutually continuous with respect to each other. Together with Lemma 3.1 and (3.1), this lemma also implies that, when $\nu \neq 0$, the support of ν is exactly $\widetilde{\Sigma}$. Since we always have $\mathcal{S}(u) \subset \widetilde{\Sigma}$, we see in this case $\Sigma = \widetilde{\Sigma}$.

If $\nu = 0$, the proof of Lemma 3.7 implies that for $H^{n-2\frac{p+1}{p-1}}$-a.a. $x \in \Sigma$,

$$\limsup_{r \to 0} r^{2\frac{p+1}{p-1}-n}\mu(B_r(x)) = 0.$$

Combining this with Lemma 3.4, we see

$$H^{n-2\frac{p+1}{p-1}}(\widetilde{\Sigma}) = 0.$$

In this case, we still have

Lemma 3.8. *If* $\nu = 0$, $\widetilde{\Sigma} = \mathcal{S}(u) = \Sigma$.

Proof. The assumption that $\nu = 0$ implies the strong convergence of u_i in $H^1_{loc} \cap L^{p+1}_{loc}$. If x_0 does not belong to $\mathcal{S}(u)$, by definition u is smooth in an open ball $B_{r_0}(x_0)$. Then there exists another $r_1 < r_0$ so that

$$r_1^{2\frac{p+1}{p-1}-n}\int_{B_{r_1}(x_0)}\frac{p-1}{2}|\nabla u|^2 + \frac{p-1}{p+1}|u|^{p+1} \leq \frac{\varepsilon_0}{4}.$$

By the strong convergence of u_i in $H^1_{loc} \cap L^{p+1}_{loc}$, for all i large,

$$r_1^{2\frac{p+1}{p-1}-n} \int_{B_{r_1}(x_0)} \frac{p-1}{2}|\nabla u_i|^2 + \frac{p-1}{p+1}|u_i|^{p+1} < \frac{\varepsilon_0}{2}.$$

Thus we can argue as in the proof of Lemma 3.1 to show that x_0 does not belong to $\widetilde{\Sigma}$. This gives $\widetilde{\Sigma} \subset \mathcal{S}(u)$, and the other direction has already been given in Corollary 3.2. $\square$

This finishes the proof of the first claim in Theorem 1.2. Next we study the structure of Σ.

Lemma 3.9. *For $H^{n-2\frac{p+1}{p-1}}-a.a.\ x \in \Sigma$,*

$$\Theta(x) := \lim_{r \to 0} r^{-n+2\frac{p+1}{p-1}} \nu(B_r(x)) \in (\frac{\varepsilon_0}{2}, C(M)),$$

exists.

Proof. Fix a point in Σ, and without loss of generality, assume it is 0.

By (2.1), for any $0 < r_1 < r_2 < 1$,

$$E(r_2; 0, u_i) \geq E(r_1; 0, u_i).$$

Here we use the second formulation of $E(r; x, u)$.

By the weak convergence of u_i in H^1_{loc} and the trace theorem, for any ball $B_r(x)$,

$$\lim_{i \to +\infty} \int_{\partial B_r(x)} u_i^2 = \int_{\partial B_r(x)} u^2.$$

For a.a. $r \in (0, 1)$, $\mu(\partial B_r(x)) = 0$. For such r, we have

$$\mu(B_r(x)) = \lim_{i \to \infty} \mu_i(B_r(x)).$$

The same claims also hold for ν, ν_1 and ν_2. If r_1, r_2 satisfy these conditions, then passing to the limit in the monotonicity formula for u_i we obtain

$$E(r_2; 0, u) + r_2^{2\frac{p+1}{p-1}-n}(\nu_1 - \nu_2)(B_{r_2}) \geq E(r_1; 0, u) + r_1^{2\frac{p+1}{p-1}-n}(\nu_1 - \nu_2)(B_{r_2}).$$
$$(3.4)$$

Note that by Lemma 2.6, $\nu_1 - \nu_2 = \frac{p-1}{2}\nu_2 = \frac{1}{p+3}\nu$.

If $\mu(\partial B_{r_1}) \neq 0$ or $\mu(\partial B_{r_2}) \neq 0$, we can take $\bar{r}_1 > r_1$, $\bar{r}_2 < r_2$, with $\bar{r}_1 < \bar{r}_2$ and $\mu(\partial B_{\bar{r}_1}) = \mu(\partial B_{\bar{r}_2}) = 0$, so that

$$E(\bar{r}_2; 0, u) + \bar{r}_2^{2\frac{p+1}{p-1}-n}(\nu_1 - \nu_2)(B_{\bar{r}_2}) \geq E(\bar{r}_1; 0, u) + \bar{r}_1^{2\frac{p+1}{p-1}-n}(\nu_1 - \nu_2)(B_{\bar{r}_1}).$$

For any $\varepsilon > 0$, we can choose $\bar{r}_2$ close to r_2 so that

$$(\nu_1 - \nu_2)(B_{\bar{r}_2}) \geq (\nu_1 - \nu_2)(B_{r_2}) - \varepsilon.$$

Then by noting that $E(r; 0, u)$ is continuous in r, and $(\nu_1 - \nu_2)(B_{\bar{r}_1}) \geq (\nu_1 - \nu_2)(B_{r_1})$, we can let $\bar{r}_1 \to r_1$, $\bar{r}_2 \to r_2$ to get (3.4).Thus (3.4) holds for any $0 < r_1 < r_2 < 1$.

By Lemma 3.6, we directly get a lower bound for

$$\bar{E}(r) := E(r; 0, u) + \frac{1}{p+3} r^{2\frac{p+1}{p-1}-n} \nu(B_r) \geq -C(M).$$

By the monotonicity of $\bar{E}(r)$, we can use the same method as in the proof of Corollary 2.5 to obtain an upper bound for $\bar{E}(r)$.

Then once again by the monotonicity, the limit

$$\lim_{r \to 0} \left(E(r; 0, u) + \frac{1}{p+3} r^{2\frac{p+1}{p-1}-n} \nu(B_r) \right)$$

exists.

Now we assume that (3.3) holds at 0, which is true $H^{n-2\frac{p+1}{p-1}}$ a.e. in Σ. By Lemma 2.3, we can also assume that at this point

$$\lim_{r \to 0} r^{2\frac{p+1}{p-1}-n-1} \int_{\partial B_r(x)} u^2 = 0. \tag{3.5}$$

With this choice, at this point

$$\lim_{r \to 0} E(r; 0, u) = 0.$$

Thus the limit

$$\Theta(0) = \lim_{r \to 0} r^{2\frac{p+1}{p-1}-n} \nu(B_r)$$

exists.

Finally, the upper bound of Θ is a direct consequence of (3.2). Concerning the lower bound, we can use (3.3) and (3.5) again to see that, as $r \to 0$,

$$\begin{aligned}
\Theta(0) &= r^{2\frac{p+1}{p-1}-n} \nu(B_r) + o(1) \\
&= r^{2\frac{p+1}{p-1}-n} \mu(B_r) + o(1) \\
&\geq \frac{\varepsilon_0}{2} + o(1).
\end{aligned}$$

Here $o(1)$ goes to 0 as $r \to 0$. $\hspace{2em}\square$

By the Radon-Nikodym theorem, we get

Corollary 3.10. $\nu = \Theta(x) H^{n-m} \lfloor_\Sigma$.

From the proof we also get

Corollary 3.11. *For any* $x \in B_1$ *and* $r \in (0, 1/2)$,

$$\bar{E}(r; x) := E(r; x, u) + \frac{1}{p+3} r^{2\frac{p+1}{p-1}-n} \nu(B_r(x))$$

is non-decreasing in r.

By Marstrand theorem ([10, Theorem 1.3.12] and [13]), if $n - 2\frac{p+1}{p-1}$ is not an integer, we must have $\nu = 0$. This then implies that

$$\left(\frac{p-1}{2}|\nabla u_i|^2 + \frac{p-1}{p+1}|u_i|^{p+1}\right) dx \rightharpoonup \left(\frac{p-1}{2}|\nabla u_i|^2 + \frac{p-1}{p+1}|u_i|^{p+1}\right) dx.$$

Because $u_i \rightharpoonup u$ weakly in $H^1_{loc} \cap L^{p+1}_{loc}$, $u_i \to u$ strongly in $H^1_{loc} \cap L^{p+1}_{loc}$. Then the stationary condition for u_i can be passed to the limit, so u also satisfies the stationary condition. This finishes the proof of the second claim of Theorem 1.2.

4. THE DIMENSION REDUCTION

In this section we consider the partial regularity for a stationary solution u to (1.1), where $n - 2\frac{p+1}{p-1}$ is not an integer. A crucial point is the fact we have just established: weak convergent solutions of (1.1) also converges strongly in $H^1_{loc} \cap L^{p+1}_{loc}$.

Pacard's partial regularity result ([14]) says that the singular set of u, $\mathcal{S}(u)$ is a closed set satisfying

$$H^{n-2\frac{p+1}{p-1}}(\mathcal{S}(u)) = 0.$$

In particular, $\dim(\mathcal{S}(u)) \le n - 2\frac{p+1}{p-1}$. We will use Federer's dimension reduction principle to reduce this dimension to $n - m$, the integer part of $n - 2\frac{p+1}{p-1}$.

By the monotonicity of $E(r; x, u)$, we can define the density function

$$\Theta(x, u) = \lim_{r \to 0^+} E(r; x, u).$$

By [14, Lemma 2 and Lemma 3], we have

Lemma 4.1. $\Theta(x, u) \ge 0$ *is an upper-continuous function.*

Next we have the following characterization of singular points.

Proposition 4.2. *There exists a constant ε_1 depending only on n and p, such that for any stationary weak solution u of (1.1), $x \in \mathcal{S}(u)$ if and only if $\Theta(x, u) \ge \varepsilon_1$.*

Proof. If x is in the regular set of u, there exists an $r_0 > 0$ such that u is smooth in $B_{r_0}(x)$. Then there exists a constant C such that for every $r < r_0$,

$$\int_{B_r(x)} |\nabla u|^2 + |u|^{p+1} \le Cr^n, \quad \int_{\partial B_r(x)} u^2 \le Cr^{n-1}.$$

Substituting this into the second formulation of $E(r; x, u)$ we get

$$E(r; x, u) \leq Cr^{2\frac{p+1}{p-1}}.$$

Thus $\Theta(x, u) = 0$.

On the other hand, for $x \in \mathcal{S}(u)$, by Theorem 2.2 and Lemma 2.4, there exists a universal constant $\varepsilon_1 > 0$ such that for any $r > 0$,

$$E(r; x, u) \geq \varepsilon_1.$$

By definition, we get $\Theta(x; u) \geq \varepsilon_1$. □

Assume $0 \in \mathcal{S}(u)$. For $\lambda \to 0$, define the blow up sequence

$$u^\lambda(x) = \lambda^{\frac{2}{p-1}} u(\lambda x).$$

By rescaling Lemma 2.4, for any $x \in B_{\lambda^{-1}/2}$ and $r \in (0, \lambda^{-1}/2)$,

$$\int_{B_r(x)} |\nabla u^\lambda|^2 + |u^\lambda|^{p+1} \leq C(M) r^{n-2\frac{p+1}{p-1}}.$$

Here $C(M)$ is a constant independent of $\lambda \to 0$.

By Theorem 1.2, we can subtract a subsequence $\lambda_i \to 0$ such that $u_i := u^{\lambda_i}$ converges strongly to a stationary solution u_∞ in $H^1_{loc}(\mathbb{R}^n) \cap L^{p+1}_{loc}(\mathbb{R}^n)$.

By the weak convergence of u_i in $H^1_{loc}(\mathbb{R}^n)$ and the trace theorem, for any $r > 0$ and $x \in \mathbb{R}^n$,

$$\int_{\partial B_r(x)} u_\infty^2 = \lim_{i \to +\infty} \int_{\partial B_r(x)} u_i^2.$$

Then we get

$$E(r; 0, u_\infty) = \lim_{i \to +\infty} E(r; 0, u_i).$$

On the other hand, a direct scaling shows

$$E(r; 0, u_i) = E(\lambda_i r; 0, u).$$

By the monotonicity of $E(r; 0, u)$, we obtain

$$E(r; 0, u_\infty) \equiv \lim_{r \to 0} E(r; 0, u) = \Theta(0, u), \ \forall r > 0.$$

By Theorem 2.1, u_∞ is homogeneous, that is, for any $\lambda > 0$,

$$u_\infty(\lambda x) = \lambda^{-\frac{2}{p-1}} u_\infty(x) \ \text{ a.e. in } \mathbb{R}^n.$$

In particular, the singular set $\mathcal{S}(u_\infty)$ is a cone, that is,

$$\lambda \mathcal{S}(u_\infty) = \mathcal{S}(u_\infty), \quad \forall \lambda > 0.$$

By Theorem 1.2 we have

Lemma 4.3. *For any $\varepsilon > 0$, if i large, $\mathcal{S}(u_i) \cap B_1$ lies in an $\varepsilon-$neighborhood of $\mathcal{S}(u_\infty) \cap B_1$.*

Proof. Because $n-2\frac{p+1}{p-1}$ is not an integer, the blow up locus $\Sigma = \mathcal{S}(u_\infty)$. Since u_i converges to u_∞ in any C^k topology away from Σ, for all i large, u_i is smooth outside the $\varepsilon-$neighborhood of $\mathcal{S}(u_\infty)$, and by this the claim can be seen. □

The following result is the key step to apply Federer's dimension reduction principle. The proof can be found in [22] (cf. Lemma 3.2 therein. The proof only uses the validation of the monotonicity formula, Theorem 2.1).

Lemma 4.4. *Given a stationary weak solution u of (1.1) on $\mathbb{R}^n$, assume that u is homogeneous, that is, $\forall \lambda > 0$,*

$$u(\lambda x) = \lambda^{-\frac{2}{p-1}} u(x).$$

Then $\forall x \in \mathbb{R}^n$, $\Theta(x, u) \le \Theta(0, u)$. Moreover, if $\Theta(x, u) = \Theta(0, u)$, then u is invariant in the direction of x, i.e. $\forall t \in \mathbb{R}$,

$$u(y + tx) = u(y), \quad a.e. \ y \in \mathbb{R}^n.$$

The last claim means u can viewed as a solution of (1.1) in $\mathbb{R}^{n-1}$. The following result shows that the stationary condition is preserved under this operation.

Lemma 4.5. *Let $u = u(x_1, \cdots, x_{n-1}) \in H^1_{loc}(\mathbb{R}^{n-1}) \cap L^{p+1}_{loc}(\mathbb{R}^{n-1})$ be a weak solution of (1.1) in $\mathbb{R}^{n-1}$. Take $\bar{u}$ to be the trivial extension of u to $\mathbb{R}^n$,*

$$\bar{u}(x_1, \cdots, x_n) = u(x_1, \cdots, x_{n-1}).$$

Then u is stationary if and only if $\bar{u}$ is stationary.

Proof. First assume $\bar{u}$ is stationary but u is not stationary. By definition there exists a vector field $Y \in C_0^\infty(\mathbb{R}^{n-1}, \mathbb{R}^{n-1})$, such that

$$\int_{\mathbb{R}^{n-1}} \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1}\right) \operatorname{div}Y - DY(\nabla u, \nabla u) = \delta > 0.$$

For any T, take a function $\eta_T \in C_0^\infty((-T-1, T+1))$ such that $\eta \equiv 1$ in $(-T, T)$, $|\eta'| \le 2$. Then

$$\bar{Y}(x_1, \cdots, x_{n-1}, x_n) = Y(x_1, \cdots, x_{n-1})\eta(x_n)$$

is a smooth vector field in $\mathbb{R}^n$ with compact support. So

$$\int_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla \bar{u}|^2 - \frac{1}{p+1}|\bar{u}|^{p+1}\right) \operatorname{div}\bar{Y} - D\bar{Y}(\nabla \bar{u}, \nabla \bar{u}) = 0.$$

However, direct calculation shows that this also equals

$$\int_{\mathbb{R}^{n-1}\times\{-T<x_n<T\}} \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1}\right) \operatorname{div}Y - DY(\nabla u, \nabla u)$$

$$+ \int_{\mathbb{R}^{n-1} \times \{T < |x_n| < T+1\}} \left(\frac{1}{2} |\nabla \bar{u}|^2 - \frac{1}{p+1} |\bar{u}|^{p+1} \right) \mathrm{div} \bar{Y} - D\bar{Y}(\nabla \bar{u}, \nabla \bar{u})$$

$$= 2T\delta + O(1).$$

Hence if we choose T large we get a contradiction. This proves the stationary condition for u.

Now assume u is stationary. For any vector field $\bar{Y} \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$, by noting that $\frac{\partial \bar{u}}{\partial x_n} = 0$ a.e., we have

$$\int_{\mathbb{R}^n} \left(\frac{1}{2} |\nabla \bar{u}|^2 - \frac{1}{p+1} |\bar{u}|^{p+1} \right) \mathrm{div} \bar{Y} - D\bar{Y}(\nabla \bar{u}, \nabla \bar{u})$$

$$= \int_{-\infty}^{+\infty} \int_{\mathbb{R}^{n-1}} \left(\frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} |u|^{p+1} \right) \sum_{1 \leq i \leq n-1} \frac{\partial \bar{Y}_i}{\partial x_i} - \sum_{1 \leq i,j \leq n-1} \frac{\partial \bar{Y}_i}{\partial x_j} \frac{\partial \bar{u}}{\partial x_i} \frac{\partial \bar{u}}{\partial x_j}$$

$$+ \int_{\mathbb{R}^{n-1}} \left(\frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} |u|^{p+1} \right) \times \int_{-\infty}^{+\infty} \frac{\partial \bar{Y}_n}{\partial x_n}$$

$$= 0.$$

This proves the stationary condition for $\bar{u}$. $\qquad \square$

When the blow up limit u_∞ has a singular point $x_1 \neq 0$, the next step in Federer's dimension reduction argument is to blow up u_∞ once again at x_1, thus obtaining another homogeneous solution $u_{\infty,1}$. By Lemma 4.1 and Lemma 4.4, we can show that $u_{\infty,1}$ is translation invariant along the direction x_1. Hence we can view it as a solution defined on $\mathbb{R}^{n-1}$, which is also stationary by Lemma 4.5. Note that this operation also decreases the Hausdorff dimension of its singular set by 1. We can repeat this step until we get a homogeneous solution defined on $\mathbb{R}^k$, which is singular only at the origin 0. Since by our assumption $p \in (\frac{m+2}{m-2}, \frac{m+1}{m-3})$ (in particular, p is subcritical in dimension $m-1$), it can be directly verified that $k \geq m$. Roughly speaking, after at most $n - m$ steps, we get a solution with singular set of dimension 0. Recall that at each step of reduction we decrease the dimension of singular sets by 1, thus the dimension of $\mathcal{S}(u)$ is at most $n - m$. This proves Theorem 1.3. For a precise treatment of this argument and also the case when $p \in (\frac{n+2}{n-2}, \frac{n+1}{n-3})$, we refer to [18, Appendix A] and [10, Section 2.3].

5. THE BUBBLE CONSTRUCTION

In this section and the following parts of this paper we consider the case where $n - 2\frac{p+1}{p-1}$ is an integer. Denote $m = 2\frac{p+1}{p-1}$, that is, p is the critical Sobolev exponent in $\mathbb{R}^m$.

We first explore the local properties of Σ near an arbitrary point, say $0 \in \Sigma$. For any $\lambda > 0$, define $\mu^\lambda := L^{\lambda,0}_\sharp \mu$, i.e.

$$\mu^\lambda(A) := \lambda^{2\frac{p+1}{p-1} - n} \mu(\lambda A) \text{ for any Borel set } A.$$

By (3.2), for any ball B_r,

$$\mu^\lambda(B_r) = \lambda^{2\frac{p+1}{p-1} - n} \mu(B_{\lambda r}) \leq C(M) r^{n - 2\frac{p+1}{p-1}}.$$

Hence we can subtract a subsequence $\lambda_j \to 0$, so that μ^{λ_j} converges weakly to a positive Radon measure μ_0 on $\mathbb{R}^n$. Note that for a.a. $r > 0$, we have $\mu_0(\partial B_r) = 0$. Then for such r, by (3.1),

$$\mu_0(B_r) = \lim_{j \to +\infty} \mu_{\lambda_j}(B_r) \geq c\varepsilon_0 r^{n - 2\frac{p+1}{p-1}}.$$

A posterior this holds for all $r > 0$ by continuity, not only those r with $\mu_0(\partial B_r) = 0$. In particular this implies that $\mu_0 \neq 0$ is nontrivial.

By a rescaling using Remark 3.5, we see for any $\lambda, r > 0$,

$$H^{n-m}(B_r \cap \Sigma^\lambda) \leq C(M) r^{n-m}.$$

where $\Sigma^\lambda := \lambda^{-1}\Sigma$. By the Blaschke Selection Theorem (cf. [10, Theorem 2.1.5]), after further subtracting a subsequence of $\lambda_j \to 0$, we can assume that Σ^{λ_j} converges to a closed set Σ^0.

By Lemma 3.9 and Preiss theorem ([16], see also [10] for a direct proof without using Preiss theorem), Σ is countably $(n-m)$-rectifiable. In particular, for H^{n-m}-a.a. $x \in \Sigma$, there exists a tangent plane T to Σ. (This can also be proved directly, see [9, Section 2].) Thus for any $\varepsilon > 0$, as $\lambda \to 0$,

$$\lambda^{-1}(\Sigma - \{x\}) \cap B_1 \text{ belongs to an } \varepsilon \text{ neighborhood of } T.$$

Recall that

$$\Theta(x) = \lim_{r \to 0} r^{m-n} \nu(B_r(x)), \ H^{n-m} - a.e. \text{ in } \Sigma,$$

is Borel measurable. We have (for a direct proof see for example [21, Lemma 3.2.2])

Lemma 5.1. $\Theta(x)$ is H^{n-m} approximate continuous at $H^{n-m}-a.a.$ $x \in \Sigma$. Here $\Theta(x)$ is H^{n-m} approximate continuous at x_0 if for any $\varepsilon > 0$,

$$\lim_{r \to 0} \frac{H^{n-m}(\{x \in \Sigma \cap B_r(x_0) : |\Theta(x) - \Theta(x_0)| > \varepsilon\})}{H^{n-m}(B_r(x_0) \cap \Sigma)} = 0.$$

If 0 is an approximate continuous point of Θ, then by the same proof of [9, Lemma 2.1], there exists a tangent plane T of Σ at 0. Without

loss of generality, assume $T = \mathbb{R}^{n-m} \times \{0\} \subset \mathbb{R}^n$. We can also assume that (3.3) holds at 0, i.e.

$$\lim_{r \to 0} r^{m-n} \int_{B_r} \left(|\nabla u|^2 + |u|^{p+1} \right) = 0.$$

In this case $\Sigma^0 = T$ and $\mu_0 = \Theta(0) H^{n-m} \lfloor_T$.

In the following we will always assume that such a base point has been chosen. Denote $x = (x', x'') \in \mathbb{R}^{n-m} \times \mathbb{R}^m$, and open balls in $\mathbb{R}^{n-m}$ (or $\mathbb{R}^m$) by $B_r'(x')$ (respectively, $B_r''(x'')$).

For each λ, the sequence $u_i^\lambda(x) := \lambda^{\frac{2}{p-1}} u_i(\lambda x)$ converges to $u^\lambda(x) := \lambda^{\frac{2}{p-1}} u(\lambda x)$ weakly in $H^1(B_{2\lambda^{-1}}) \cap L^{p+1}(B_{2\lambda^{-1}})$. As $i \to +\infty$, the measures

$$\mu_i^\lambda = L_\sharp^{\lambda,0} \mu_i \rightharpoonup \left(\frac{p-1}{2} |\nabla u^\lambda|^2 + \frac{p-1}{p+1} |u^\lambda|^{p+1} \right) dx + \nu^\lambda = \mu^\lambda.$$

For each j, we can choose an $i(j)$ and R_j large so that the Levi distance between $\mu_{i(j)}^{\lambda_j} \lfloor_{B_{R_j}}$ and $\nu^{\lambda_j} \lfloor_{B_{R_j}}$ is smaller than $1/j$. Then by a diagonal argument, we find a sequence of stationary solutions v_j to (1.1), satisfying

$$\int_{B_r(x)} |\nabla v_j|^2 + |v_j|^{p+1} \le C(M) r^{n-2\frac{p+1}{p-1}}, \quad \text{for all } x \in B_{R_j/2}, \ r \in (0, R_j/2),$$

and

$$\mu_j := \left(\frac{p-1}{2} |\nabla v_j|^2 + \frac{p-1}{p+1} |v_j|^{p+1} \right) dx \rightharpoonup \mu_0 = \Theta(0) H^{n-m} \lfloor_T. \quad (5.1)$$

First we note

Lemma 5.2. *In $L^1_{loc}(\mathbb{R}^n)$,*

$$\sum_{k=1}^{n-m} \left| \frac{\partial v_j}{\partial x_k} \right|^2 dx \to 0.$$

Proof. Because

$$\mu_j := \left(\frac{p-1}{2} |\nabla v_j|^2 + \frac{p-1}{p+1} |v_j|^{p+1} \right) dx \rightharpoonup 0 \quad \text{outside } T,$$

and T is a subspace of $\mathbb{R}^n$ with codimension $m \ge 3$, v_j must converge weakly to 0 in $H^1_{loc}(\mathbb{R}^n) \cap L^{p+1}_{loc}(\mathbb{R}^n)$. By the compact Sobolev embedding theorem, v_j converges to 0 in $L^2_{loc}(\mathbb{R}^n)$.

For a.a. $r > 0$, $\mu_0(\partial B_r(0)) = 0$. By the strong convergence of v_j in $L^2_{loc}(\mathbb{R}^n)$, after passing to a subsequence of j again, we also have for

a.a. $r > 0$,

$$\int_{\partial B_r(0)} v_j^2 \to 0.$$

For such r, we have

$$
\begin{aligned}
\mu_0(B_r(0)) &= (p+3)(\nu_1(B_r(0)) - \nu_2(B_r(0))) \\
&= \lim_{j \to +\infty} (p+3) \int_{B_r(0)} \frac{|\nabla v_j|^2}{2} - \frac{|v_j|^{p+1}}{p+1}.
\end{aligned}
$$

Since $\mu_0(B_r(0)) \equiv Cr^{n-2\frac{p+1}{p-1}}$ for some constant C, for any generic $0 < \sigma < R < +\infty$ (avoiding a countable set),

$$\lim_{j \to +\infty} E(R; 0, v_j) - E(\sigma; 0, v_j) = 0.$$

By the monotonicity formula for v_j, we see

$$\lim_{j \to +\infty} \int_{B_R(0) \setminus B_\sigma(0)} \frac{(\frac{y}{|y|} \nabla v_j + \frac{2}{p-1}|y|^{-1} v_j)^2}{|y|^{n-m}} = 0. \tag{5.2}$$

Because v_j converges to 0 in $L^2_{loc}(\mathbb{R}^n)$, after an expansion we obtain

$$\lim_{j \to +\infty} \int_{B_R(0) \setminus B_\sigma(0)} |y \cdot \nabla v_j|^2 = 0.$$

Take the standard basis ξ_k of $\mathbb{R}^{n-m}$, $1 \le k \le n-m$. If we choose R and σ suitably, the same argument above still works if we replace the center of ball by ξ_k, which gives

$$\lim_{j \to +\infty} \int_{B_R(\xi_k) \setminus B_\sigma(\xi_k)} |(y - \xi_k) \cdot \nabla v_j|^2 = 0.$$

If R is large and σ is small, in $D = \cap_k (B_R(\xi_k) \setminus B_\sigma(\xi_k)) \cap (B_R(0) \setminus B_\sigma(0))$,

$$
\begin{aligned}
&\lim_{j \to +\infty} \int_D |\xi_k \cdot \nabla v_j|^2 \\
&= \lim_{j \to +\infty} \int_D |(y - \xi_k) \cdot \nabla v_j|^2 - |y \cdot \nabla v_j|^2 + 2(y \cdot \nabla v_j)(\xi_k \cdot \nabla v_j) \\
&\le \lim_{j \to +\infty} \int_D |(y - \xi_k) \cdot \nabla v_j|^2 + 10|y \cdot \nabla v_j|^2 + \frac{1}{2}(\xi_k \cdot \nabla v_j)^2.
\end{aligned}
$$

So

$$\lim_{j \to +\infty} \int_D |\xi_k \cdot \nabla v_j|^2 = 0.$$

Then a suitable covering using translations of D gives the result. $\qquad\square$

In the proof we have used the following fact. For any $\varepsilon > 0$, there exists a constant C such that, for all j large,

$$\sup_{\{|x''|>\varepsilon\}} |v_j| \leq C. \tag{5.3}$$

In particular, v_j are smooth outside the $\varepsilon-$neighborhood of T.

By Pacard's partial regularity result [15] for stationary solutions, $H^{n-m}(\mathcal{S}(v_j)) = 0$. Since the projection π from $\mathbb{R}^n$ to T is a $1-$Lipschitz map, direct calculation using the definition of Hausdorff measures gives

$$H^{n-m}(\pi\mathcal{S}(v_j)) = 0.$$

In other words, for $H^{n-m}-$a.a. $x' \in T$, $(\{x'\} \times \mathbb{R}^m) \cap \mathcal{S}(v_j) = \emptyset$. Thus for all j large, $v_j(x', \cdot)$ are smooth functions in $\mathbb{R}^m$ for $H^{n-m}-$a.a. $x' \in T$, and these points form an open set. (Note that the regular set of v_j is open.)

Let

$$f_j(x') := \int_{B_1''} \sum_{k=1}^{n-m} \left| \frac{\partial v_j}{\partial x_k}(x', x'') \right|^2 dx''.$$

Lemma 5.2 says f_j converges to 0 in $L_{loc}^1(\mathbb{R}^{n-m})$. By the weak$-L^1$ estimate for the Hardy-Littlewood maximal function, for $H^{n-m}-$a.a. $x' \in T$,

$$Mf_j(x') := \sup_{0<r\leq 1/2} r^{m-n} \int_{B_r'(x')} f_j(y')dy' \to 0, \quad \text{as } j \to +\infty. \tag{5.4}$$

For any $\delta > 0$, we can take an open set $E_j \subset B_1'$ with

$$H^{n-m}(E_j) \geq (1-\delta)H^{n-m}(B_1'), \tag{5.5}$$

such that for any $x' \in E_j$, (5.4) holds and v_j is smooth in a neighborhood of $\{x'\} \times B_{1/2}''$.

Take an arbitrary sequence $x_j' \in E_j$. Then for all $x'' \in B_{1/2}''$,

$$\lim_{\delta\to 0} \delta^{m-n} \int_{B_\delta(x_j',x'')} \frac{p-1}{2}|\nabla v_j|^2 + \frac{p-1}{p+1}|v_j|^{p+1} = 0. \tag{5.6}$$

On the other hand, by the definition of the blow up locus, there exists a $\delta_j > 0$, which goes to 0 as $j \to +\infty$, such that

$$\delta_j^{m-n} \int_{B_{\delta_j}(x_j',0)} \frac{p-1}{2}|\nabla v_j|^2 + \frac{p-1}{p+1}|v_j|^{p+1} \geq \frac{\varepsilon_0}{2}. \tag{5.7}$$

These two imply the existence of an $r_j \in (0, \delta_j)$, such that

$$\max_{x''\in B_{1/2}''} r_j^{m-n} \int_{B_{r_j}(x_j',x'')} \frac{p-1}{2}|\nabla v_j|^2 + \frac{p-1}{p+1}|v_j|^{p+1} = c_1\varepsilon_0, \tag{5.8}$$

where we have chosen a fixed constant $c_1 \in (0, 1/2)$.

Assume this maxima is attained at x_j'', and denote $x_j = (x_j', x_j'')$. Define

$$\bar{v}_j(x) = r_j^{\frac{2}{p-1}} v_j(x_j + r_j x).$$

By Lemma 2.4, for any $r \in (0, r_j^{-1}/2)$,

$$\int_{B_r(0)} \frac{p-1}{2} |\nabla \bar{v}_j|^2 + \frac{p-1}{p+1} |\bar{v}_j|^{p+1} \leq C(M) r^{n-2\frac{p+1}{p-1}}. \qquad (5.9)$$

Without loss of generality, assume that $\bar{v}_j$ converges weakly to $\bar{v}$ in $H_{loc}^1(\mathbb{R}^n) \cap L_{loc}^{p+1}(\mathbb{R}^n)$. By the compact Sobolev embedding, $\bar{v}_j$ converges to $\bar{v}$ in $L_{loc}^q(\mathbb{R}^n))$ for any $q < p+1$. In particular, $\bar{v}$ is an H^1 weak solution of (1.1) in $\mathbb{R}^n$.

By (5.4) and Fatou lemma, for all $0 < r \leq r_j^{-1}/2$,

$$r^{m-n} \int_{B_r(0)} \sum_{k=1}^{n-m} \left| \frac{\partial \bar{v}}{\partial x_k}(x', x'') \right|^2$$

$$\leq \liminf_{j \to +\infty} r^{m-n} \int_{B_r(0)} \sum_{k=1}^{n-m} \left| \frac{\partial \bar{v}_j}{\partial x_k}(x', x'') \right|^2 \qquad (5.10)$$

$$= \liminf_{j \to +\infty} (rr_j)^{m-n} \int_{B_{rr_j}(x_j)} \sum_{k=1}^{n-m} \left| \frac{\partial v_j}{\partial x_k}(x', x'') \right|^2 = 0.$$

Hence $\bar{v}(x', x'') = \bar{v}(x'')$. In (5.9), we can replace B_r by the cylinder $B_{r/2}' \times B_{r/2}''$, which then gives (noting that $m = 2\frac{p+1}{p-1}$ and r can be arbitrarily large)

$$\int_{\mathbb{R}^m} |\nabla'' \bar{v}(x'')|^2 + |v(x'')|^{p+1} dx'' \leq C(M).$$

Because $\bar{v}$ is a solution to (1.1), with p the critical exponent in $\mathbb{R}^m$, by [19, Lemma B.3] it is a smooth solution.

We can assume the measures

$$(|\nabla \bar{v}_j|^2 + |\bar{v}_j|^{p+1}) dx \rightharpoonup (|\nabla \bar{v}|^2 + |\bar{v}|^{p+1}) dx + \tau.$$

Here τ is a positive Radon measure.

Lemma 5.3. τ is translation invariant under x' direction.

Proof. For any $\varphi \in C_0^\infty(\mathbb{R}^n)$, define

$$F_j(a) = \int_{\mathbb{R}^n} \left(\frac{1}{2} |\nabla \bar{v}_j|^2 - \frac{1}{p+1} |\bar{v}_j|^{p+1} \right)(x+a)\varphi(x) dx.$$

$F_j(a)$ are smooth functions of a. Then for $k = 1, \cdots, n - m$,

$$\frac{\partial F_j}{\partial a_k} = \int_{\mathbb{R}^n} \frac{\partial}{\partial x_k} \left(\frac{1}{2} |\nabla \bar{v}_j|^2 - \frac{1}{p+1} |\bar{v}_j|^{p+1} \right) (x + a) \varphi(x) dx.$$

Define $Y = \varphi(x) \frac{\partial}{\partial x_k}$, then div$Y = \frac{\partial \varphi}{\partial x_k}$. By the stationary condition,

$$\frac{\partial F_j}{\partial a_k} = -\sum_{l=1}^{n} \int_{\mathbb{R}^n} \frac{\partial \bar{v}_j}{\partial x_k} (x + a) \frac{\partial \bar{v}_j}{\partial x_l} (x + a) \frac{\partial \varphi}{\partial x_l} (x) dx.$$

By (5.10), this goes to 0 uniformly on any compact set of $\mathbb{R}^n$. Since the measures

$$\left(\frac{1}{2} |\nabla \bar{v}_j|^2 - \frac{1}{p+1} |\bar{v}_j|^{p+1} \right) dx \rightharpoonup \left(\frac{1}{2} |\nabla \bar{v}|^2 - \frac{1}{p+1} |\bar{v}|^{p+1} \right) dx + \frac{p-1}{4(p+1)} \tau,$$

and $\left(\frac{1}{2} |\nabla \bar{v}|^2 - \frac{1}{p+1} |\bar{v}|^{p+1} \right) dx$ is translation invariant in x_k directions, $1 \leq k \leq n - m$, τ is also translation invariant in x_k directions. $\qquad \square$

By the choice of x_j (see (5.8)), we have

$$c_1 \varepsilon_0 = \int_{B_1} |\nabla \bar{v}_j|^2 + |\bar{v}_j|^{p+1} = \max_{x'' \in B''_{r_j^{-1}/2}} \int_{B_1(0,x'')} |\nabla v_j|^2 + |v_j|^{p+1}. \quad (5.11)$$

Take two nonnegative functions $\varphi \in C_0^\infty(B_2')$, $\psi \in C_0^\infty(B_2'')$, such that $\varphi \equiv 1$ in B_1', $\psi \equiv 1$ in B_1''. For $a \in B_1' \times B_1''$, define

$$F_j(a) = \int (|\nabla \bar{v}_j|^2 + |\bar{v}_j|^{p+1})(x + a) \varphi(x') \psi(x'') dx.$$

Similar to the above discussion, F_j are uniformly bounded in $C^1(B_1' \times B_1'')$, and it converges uniformly to

$$\int \left(|\nabla \bar{v}|^2 + |\bar{v}|^{p+1} \right) \varphi(x' - a') \psi(x'' - a'') + \int \varphi(x' - a') \psi(x'' - a'') d\tau$$

$$= \int \left(|\nabla \bar{v}|^2 + |\bar{v}|^{p+1} \right) \varphi(x') \psi(x'' - a'') + \int \varphi(x') \psi(x'' - a'') d\tau,$$

which is independent of a'. Thus for any $R > 0$ fixed, if j is large enough,

$$\max_{x' \in B_2', x'' \in B_R''} \int_{B_1(x',x'')} |\nabla v_j|^2 + |v_j|^{p+1} \leq 2c_1 \varepsilon_0 < \varepsilon_0.$$

By Theorem 2.2 and standard elliptic estimates, for any k, $\bar{v}_j$ are uniformly bounded in $C_{loc}^k(B_{3/2}'(0) \times B_{R-1}''(0))$. Then we can take limit in (5.11) to get

$$c_1 \varepsilon_0 = \int_{B_1} |\nabla \bar{v}|^2 + |\bar{v}|^{p+1}. \quad (5.12)$$

In particular, $\bar{v}$ is nontrivial.

In conclusion, after two rescalings from u_i, we construct a nontrivial smooth solution $\bar{v}$ to the equation

$$-\Delta\bar{v} = |\bar{v}|^{\frac{4}{m-2}}\bar{v} \quad \text{in } \mathbb{R}^m.$$

Moreover, $\bar{v}$ satisfies

$$\int_{\mathbb{R}^m} |\nabla''\bar{v}|^2 + |\bar{v}|^{\frac{2m}{m-2}} \le C(M). \tag{5.13}$$

This proves the last part of Theorem 1.2.

Note that positive solutions (to (1.1) in $\mathbb{R}^m$) have the least energy. By [2], up to a translation and scaling, for every $1 \le i \le N$, the positive solution has the form

$$[m(m-2)]^{\frac{m-2}{4}} \left(1 + |y''|^2\right)^{-\frac{m-2}{2}}.$$

Since translations and scalings in $\mathbb{R}^m$ do not change the energy, there exists a constant $c(m)$ depending only on m, such that for any positive solution $\bar{v}$ of (1.1) on $\mathbb{R}^m$,

$$\int_{\mathbb{R}^m} \frac{p-1}{2}|\nabla\bar{v}|^2 + \frac{p-1}{p+1}|\bar{v}|^{p+1} = c(m).$$

For any $R > 0$, by a rescaling and using the smooth convergence of $\bar{v}_j$ in B_R, we have

$$\int_{B_1''} \left(\frac{p-1}{2}|\nabla v_j(x_j', x'')|^2 + \frac{p-1}{p+1}|v_j(x_j', x'')|^{p+1}\right) dx''$$

$$\ge \int_{B_{Rr_j}''(x_j'')} \left(\frac{p-1}{2}|\nabla v_j(x_j', x'')|^2 + \frac{p-1}{p+1}|v_j(x_j', x'')|^{p+1}\right) dx''$$

$$= \int_{B_R''} \left(\frac{p-1}{2}|\nabla\bar{v}_j(0, x'')|^2 + \frac{p-1}{p+1}|\bar{v}_j(0, x'')|^{p+1}\right) dx'' \tag{5.14}$$

$$\to \int_{B_R''} \left(\frac{p-1}{2}|\nabla\bar{v}(x'')|^2 + \frac{p-1}{p+1}|\bar{v}_j(x'')|^{p+1}\right) dx''$$

$$\ge c(m) - \sigma(R).$$

Here $\sigma(R)$ is defined by the following lemma.

Lemma 5.4. *There exists a positive, continuous non-increasing function $\sigma(R)$ defined on $[0, +\infty)$ with*

$$\lim_{R\to+\infty} \sigma(R) = 0,$$

such that for any solution v of (1.1) on $\mathbb{R}^m$, satisfying (5.12),

$$\int_{B_R''} \frac{p-1}{2}|\nabla'' v|^2 + \frac{p-1}{p+1}|v|^{\frac{2m}{m-2}} \geq c(m) - \sigma(R).$$

Proof. Arguing by contradiction, we can assume that there exist a constant $\sigma > 0$, a sequence of solutions v_j satisfying all of the assumptions, and $R_j \to +\infty$ such that

$$\int_{B_{R_j}''} \frac{p-1}{2}|\nabla'' v_j|^2 + \frac{p-1}{p+1}|v_j|^{\frac{2m}{m-2}} < c(m) - \sigma. \qquad (5.15)$$

With the above uniform bound we can assume that v_j converges weakly to v in $H_{loc}^1(\mathbb{R}^m) \cap L_{loc}^{2m/(m+2)}(\mathbb{R}^m)$. If v is nonzero, by Fatou lemma we get

$$\int_{\mathbb{R}^m} \frac{p-1}{2}|\nabla'' v|^2 + \frac{p-1}{p+1}|v|^{\frac{2m}{m-2}} \leq c(m) - \sigma.$$

This is a contradiction since the lowest energy is exactly $c(m)$.

If $v = 0$, by (5.12) and Struwe's global compactness theorem [19, Theorem 3.1], there must exists an blow up point $x_0 \in B_2''$ such that (at least) one bubble concentrates at x_0. More precisely, there exists $x_j'' \to x_0$ and $r_j \to 0$ such that

$$r_j^{\frac{m-2}{2}} v_j(x_j'' + r_j x'')$$

converges to a nontrivial solution of (1.1) weakly in $H_{loc}^1(\mathbb{R}^m) \cap L_{loc}^{2m/(m+2)}(\mathbb{R}^m)$. This bubble carries energy at least $c(m)$, which is concentrated in a small ball around x_0. Thus we get a contradiction once again.   □

Since (5.14) holds for any $x' \in E_j$, by noting (5.5), we get

$$\liminf_{j \to +\infty} \int_{B_1'} \int_{B_1''} \left(\frac{p-1}{2}|\nabla v_j(x', x'')|^2 + \frac{p-1}{p+1}|v_j(x', x'')|^{p+1} \right) dx'' dx'$$

$$\geq \liminf_{j \to +\infty} \int_{E_j} \int_{B_1''} \left(\frac{p-1}{2}|\nabla v_j(x', x'')|^2 + \frac{p-1}{p+1}|v_j(x', x'')|^{p+1} \right) dx'' dx'$$

$$\geq (c(m) - \sigma(R))(1 - \delta) H^{n-m}(B_1').$$

After letting $R \to +\infty$ and $\delta \to 0$, and noting (5.1), we obtain

Corollary 5.5. *For $H^{n-m}-a.a.$ $x \in \Sigma$, $\Theta(x) \geq c(m)$.*

6. Quantization of Density Function

In the previous section we have constructed a sequence of v_j such that (here we use the same notations as in the previous section)

$$\left(\frac{p-1}{2}|\nabla v_j|^2 + \frac{p-1}{p+1}|v_j|^{p+1}\right)dx \rightharpoonup \Theta(0)H^{n-m}\lfloor_{\mathbb{R}^{n-m}}. \quad (6.1)$$

In this section we prove the quantization of $\Theta(0)$, under the following assumption

$$\Delta'v_j \to 0, \quad \text{in } L_{loc}^{\frac{p+1}{p}}(\mathbb{R}^n). \quad (6.2)$$

Note that since v_j are uniformly bounded in $L_{loc}^{p+1}(\mathbb{R}^n)$, by standard interior $W^{2,\frac{p+1}{p}}$ estimates, $D^2 v_j$ are uniformly bounded in $L_{loc}^{\frac{p+1}{p}}(\mathbb{R}^n)$. In view of Lemma 5.2, it is natural to conjecture that (6.2) holds.

Theorem 6.1. *There exists at most N solutions of* (1.1) *in $\mathbb{R}^m$, w_i, $1 \leq i \leq N$, with*

$$\int_{\mathbb{R}^m}\frac{p-1}{2}|\nabla w_i|^2 + \frac{p-1}{p+1}|w_i|^{p+1} < +\infty,$$

such that

$$\Theta(0) = \sum_{i=1}^{N}\int_{\mathbb{R}^m}\frac{p-1}{2}|\nabla w_i|^2 + \frac{p-1}{p+1}|w_i|^{p+1}.$$

Here

$$N \leq \frac{C(M)}{c(m)}.$$

Before proving this theorem, we first show that the problem can be reduced to a slice.

Lemma 6.2. *For a.a. $x' \in B_1'$, on B_1'',*

$$\left(\frac{p-1}{2}|\nabla v_j(x',x'')|^2 + \frac{p-1}{p+1}|v_j(x',x'')|^{p+1}\right)dx'' \rightharpoonup \Theta(0)\delta_0, \quad (6.3)$$

where δ_0 is the Dirac measure supported at the origin $0 \in \mathbb{R}^m$.

Proof. Fix a $\varphi \in C_0^\infty(B_1'')$ such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in $B_{1/2}''$. For any smooth vector field $X \in C_0^\infty(B_1', \mathbb{R}^{n-m})$, substitute φX into the stationary condition (1.2). This leads to

$$0 = \int_{B_1'}\int_{B_1''}\left(\frac{1}{2}|\nabla v_j|^2 - \frac{1}{p+1}|v_j|^{p+1}\right)\varphi\text{div}X\,dx''dx' \quad (6.4)$$

$$- \int_{B_1'}\int_{B_1''}\left[\varphi DX(\nabla'v_j,\nabla'v_j) + (\nabla''\varphi\cdot\nabla''v_j)(X\cdot\nabla'v_j)\right]dx''dx'.$$

By using the equation (1.1) and integrating by parts, we have

$$
\int_{B_1'} \int_{B_1''} |v_j|^{p+1} \varphi \mathrm{div} X \, dx'' dx'
$$

$$
= -\int_{B_1'} \int_{B_1''} \Delta v_j v_j \varphi \mathrm{div} X \, dx'' dx' \tag{6.5}
$$

$$
= \int_{B_1'} \int_{B_1''} |\nabla v_j|^2 \varphi \mathrm{div} X \, dx'' dx'
$$

$$
+ \int_{B_1'} \int_{B_1''} \left[v_j \left(\nabla' v_j \cdot \nabla' \mathrm{div} X \right) \varphi + v_j \left(\nabla'' v_j \cdot \nabla'' \varphi \right) \mathrm{div} X \right] dx'' dx'.
$$

Integrating by parts once again,

$$
\int_{B_1'} \int_{B_1''} v_j \left(\nabla' v_j \cdot \nabla' \mathrm{div} X \right) \varphi \, dx'' dx'
$$

$$
= -\int_{B_1'} \int_{B_1''} \left(|\nabla' v_j|^2 + \Delta' v_j v_j \right) \mathrm{div} X \varphi \, dx'' dx'.
$$

By Lemma 5.2, (6.2) and the uniform $L^{p+1}(B_1' \times B_1'')$ bound on v_j,

$$
\lim_{j \to \infty} \int_{B_1'} \int_{B_1''} \left(|\nabla' v_j|^2 + |\Delta' v_j v_j| \right) dx'' dx' = 0, \tag{6.6}
$$

$$
\lim_{j \to \infty} \int_{B_1'} \int_{B_1''} \sum_{k,l=1}^{n-m} \left| \frac{\partial v_j}{\partial x_k} \right| \left| \frac{\partial v_j}{\partial x_l} \right| dx'' dx' = 0. \tag{6.7}
$$

By Lemma 5.2 and the uniform $L^2(B_1' \times B_1'')$ bound on $\nabla'' v_j$ and Cauchy inequality,

$$
\lim_{j \to \infty} \int_{B_1'} \int_{B_1''} \left| \left(\nabla'' \varphi \cdot \nabla'' v_j \right) \nabla' v_j \right| dx'' dx' = 0. \tag{6.8}
$$

Because $v_j \to 0$ in $L^2(B_1' \times B_1'')$ (note that v_j converges uniformly to 0 on any compact set in $B_1' \times (B_1'' \setminus \{0\})$), combining this with the uniform $L^2(B_1' \times B_1'')$ bound on $\nabla'' v_j$ and Cauchy inequality, we get

$$
\lim_{j \to \infty} \int_{B_1'} \int_{B_1''} \left| v_j \left(\nabla'' v_j \cdot \nabla'' \varphi \right) \right| dx'' dx' = 0. \tag{6.9}
$$

Substituting (6.5)-(6.9) into (6.4), we obtain

$$
\int_{B_1'} \left[\int_{B_1''} \left(\frac{1}{2} - \frac{1}{p+1} \right) |\nabla v_j|^2 \varphi \, dx'' \right] \mathrm{div} X \, dx' = \int_{B_1'} f_j \cdot X + A_j \cdot DX \, dx',
$$

where $f_j \in L^1(B_1', \mathbb{R}^{n-m})$ and $A_j \in L^1(B_1', \mathbb{R}^{n-m} \otimes \mathbb{R}^{n-m})$ (i.e. A_j are matrix valued), satisfying

$$\lim_{j \to \infty} \int_{B_1} |f_j| + |A_j| dx' = 0.$$

By (6.1) and Lemma 2.6,

$$\left[\int_{B_1''} \left(\frac{1}{2} - \frac{1}{p+1} \right) |\nabla v_j|^2 \varphi dx'' \right] \to \frac{\Theta(0)}{p+3}, \quad \text{weakly in } L^1(B_1').$$

Then we can apply Allard's strong constancy lemma (cf. [1, Section 1]) to conclude that

$$\int_{B_1''} \left(\frac{1}{2} - \frac{1}{p+1} \right) |\nabla v_j|^2 \varphi dx'' \to \frac{\Theta(0)}{p+3}, \quad \text{in } L^1(B_1').$$

We can also substitute (6.5) into (6.4) to eliminate $|\nabla v_j|^2$. This leads to the strong convergence of

$$\int_{B_1''} \left(\frac{1}{2} - \frac{1}{p+1} \right) |v_j|^{p+1} \varphi dx'',$$

in $L^1(B_1')$.

Now we have proved that

$$\int_{B_1''} \left(\frac{p-1}{2} |\nabla v_j|^2 + \frac{p-1}{p+1} |v_j|^{p+1} \right) \varphi dx'' \to \Theta(0)$$

in $L^1(B_1')$. Since v_j converges to 0 in $C^1 \left(B_1' \times \left(B_1'' \setminus B_{1/3}'' \right) \right.$ and $1 - \varphi \equiv 0$ in $B_{1/2}''$,

$$\lim_{j \to 0} \int_{B_1'} \int_{B_1''} \left(\frac{p-1}{2} |\nabla v_j|^2 + \frac{p-1}{p+1} |v_j|^{p+1} \right) (1 - \varphi) \, dx'' dx' = 0.$$

Thus

$$\int_{B_1''} \left(\frac{p-1}{2} |\nabla v_j(x', x'')|^2 + \frac{p-1}{p+1} |v_j(x', x'')|^{p+1} \right) dx'' \to \Theta(0)$$

in $L^1(B_1')$.

After passing to a subsequence of j, we can assume that for a.a. $x' \in B_1'$,

$$\int_{B_1''} \left(\frac{p-1}{2} |\nabla v_j(x', x'')|^2 + \frac{p-1}{p+1} |v_j(x', x'')|^{p+1} \right) dx'' \to \Theta(0).$$

Then by noting that

$$\left(\frac{p-1}{2} |\nabla v_j(x', x'')|^2 + \frac{p-1}{p+1} |v_j(x', x'')|^{p+1} \right) \to 0,$$

uniformly on any compact set of $B_1'' \setminus \{0\}$, we must have

$$\left(\frac{p-1}{2} |\nabla v_j(x', x'')|^2 + \frac{p-1}{p+1} |v_j(x', x'')|^{p+1} \right) dx'' \rightharpoonup \Theta(0)\delta_0,$$

weakly as measures. $\square$

Proof of Theorem 6.1. By (6.2) and the previous lemma, for each j, there exists a closed set $E_j \subset B_1'$ with $\lim_{j \to +\infty} H^{n-m}(E_j) = 0$, such that for any $x_j' \in B_1' \setminus E_j$, (6.3) holds and

$$\lim_{j \to \infty} \int_{B_1''} |\Delta'' v_j(x_j', x'')|^{\frac{2m}{m+2}} dx'' = 0. \tag{6.10}$$

Moreover, as in the previous section, we can also restrict E_j further so that v_j is smooth in a neighborhood of $E_j \times B_1''$.

Let $x_j' \in B_1' \setminus E_j$ be an arbitrary sequence, and

$$\tilde{v}_j(x') = v_j(x_j', x'').$$

Thanks to (6.3), $\tilde{v}_j$ are uniformly bounded in $H^1(B_1'') \cap L^{\frac{2m}{m-2}}(B_1'')$.

Since v_j is smooth in a neighborhood of $\{x_j'\} \times B_1''$,

$$\Delta'' \tilde{v}_j + |\tilde{v}_j|^{p-1} \tilde{v}_j = -\Delta' \tilde{v}_j.$$

Thus by (6.10) and the Sobolev embedding theorem in dimension m, for any $\varphi \in H_1(B_1'')$,

$$
\begin{aligned}
\int_{B_1''} \left(\Delta'' \tilde{v}_j + |\tilde{v}_j|^{p-1} \tilde{v}_j \right) \varphi &= \int_{B_1''} -\Delta' \tilde{v}_j \varphi \\
&\leq \left(\int_{B_1''} |\Delta'' v_j(x_j', x'')|^{\frac{2m}{m+2}} \right)^{\frac{m+2}{2m}} \left(\int_{B_1''} |\varphi|^{\frac{2m}{m-2}} \right)^{\frac{m-2}{2m}} \\
&= o_j(\|\varphi\|_{H^1(B_1)}).
\end{aligned}
$$

Hence $\tilde{v}_j$ is a Palais-Smale sequence.

Then by [19, Theorem 3.1] (see also [6, Section 3.1]), we get N functions $w_i(x'')$, $1 \leq i \leq N$, with

$$-\Delta'' w_i = |w_i|^{\frac{4}{m-2}} w_i, \quad \int_{\mathbb{R}^m} |\nabla'' w_i|^2 + |w_i|^{\frac{2m}{m-2}} < +\infty,$$

and N points $x_{i,j}$ and radius $R_{i,j} > 0$, $1 \le i \le N$, such that

$$\|\tilde{v}_j - \sum_{i=1}^{N} R_{i,j}^{\frac{m-2}{2}} w_i(R_{i,j}(x - x_{i,j}))\|_{H^1(B_1'')} \to 0. \qquad (6.11)$$

Their $L^{\frac{2m}{m-2}}(B_1'')$ norm also converge to 0.

By Lemma 3.9 and (6.3),

$$\lim_{j \to +\infty} \int_{B_1''} \frac{p-1}{2}|\nabla \tilde{v}_j|^2 + \frac{p-1}{p+1}|\tilde{v}_j|^{p+1} = \Theta(0) \le C(M). \qquad (6.12)$$

There also exists a constant $c_1(m)$ depending only on m such that (see Lemma 5.4)

$$\int_{\mathbb{R}^m} \frac{p-1}{2}|\nabla w_i|^2 + \frac{p-1}{p+1}|w_i|^{p+1} \ge c(m). \qquad (6.13)$$

Note that this energy is invariant under scaling and translations in $\mathbb{R}^m$. Combining (6.11)-(6.13) we get the upper bound for N. $\qquad \square$

If the sequence of solutions are positive, then $w_i \ge 0$ for every $1 \le i \le N$ (for a proof see [6, Section 3.2]). By [2], up to a translation and scaling, for every $1 \le i \le N$,

$$w_i(y'') = [m(m-2)]^{\frac{m-2}{4}} \left(1 + |y''|^2\right)^{-\frac{m-2}{2}}.$$

Note that the translation and scaling do not change the energy of w_i, so

$$\int_{\mathbb{R}^m} \frac{p-1}{2}|\nabla w_i|^2 + \frac{p-1}{p+1}|w_i|^{p+1}$$
$$= 4\alpha(m)(m-1)[m(m-2)]^{\frac{m-2}{2}} \int_0^\infty (1+r^2)^{-m} r^{m-1} dr$$
$$= c(m).$$

Together with (6.4) and (6.11), this gives

Corollary 6.3. *In the case of positive solutions, there exists a constant $c(m)$ depending only on m such that, for $H^{n-m}-a.a.$ $x \in \Sigma$, $\Theta(x)/c(m)$ is an integer.*

7. STATIONARY PROPERTY OF BLOW UP LOCUS

In this section we prove that the stationary property of the limit function u is equivalent to that of the blow-up locus.

For any smooth vector field Y with compact support, the stationary condition for u_i says

$$\int \left(\frac{1}{2}|\nabla u_i|^2 - \frac{1}{p+1}|u_i|^{p+1}\right) \mathrm{div} Y - DY(\nabla u_i, \nabla u_i) = 0.$$

Let the Radon measure

$$\gamma := \frac{m(m-2)}{4(m-1)}\nu.$$

Then we have the weak convergence

$$|\nabla u_i|^2 dx \rightharpoonup |\nabla u|^2 dx + \gamma.$$

Let

$$\tau^i_{\alpha\beta} := |\nabla u_i|^{-2} \frac{\partial u_i}{\partial x_\alpha}\frac{\partial u_i}{\partial x_\beta}$$

if $|\nabla u_i| \neq 0$ (if $|\nabla u_i| = 0$, we simply take $[\tau^i_{\alpha\beta}] = 0$). We can assume the matrix-valued measures

$$\tau^i_{\alpha\beta}|\nabla u_i|^2 dx \rightharpoonup \frac{\partial u}{\partial x_\alpha}\frac{\partial u}{\partial x_\beta} dx + \tau_{\alpha\beta} d\nu.$$

Here $[\tau_{\alpha\beta}]$ is symmetric, non-negative definite, measurable with respect to γ, and $\sum_{\alpha=1}^n \tau_{\alpha\alpha} = 1$ γ−a.e.. By passing to the limit in the stationary condition for u_i, we get

$$\begin{aligned}
0 &= \int \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1}\right)\mathrm{div} Y - DY(\nabla u, \nabla u) \quad (7.1)\\
&\quad + 2\int \left(\frac{1}{m}\mathrm{div} Y - \sum_{\alpha,\beta=1}^n D_\alpha Y^\beta \tau_{\alpha\beta}\right) d\gamma.
\end{aligned}$$

For any $1 \leq \alpha, \beta \leq n$, $\tau_{\alpha\beta}$ is approximate continuous in the following sense: given an $\varepsilon > 0$, for γ−a.a. x_0,

$$\lim_{r\to 0} \frac{\gamma(\{|\tau_{\alpha\beta}(x) - \tau_{\alpha\beta}(x_0)| > \varepsilon\} \cap B_r(x_0))}{\gamma(B_r(x_0))} = 0.$$

Assume that $0 \in \Sigma = \mathrm{spt}(\gamma)$ satisfies this condition for all $1 \leq \alpha, \beta \leq n$, and $\mu_0 = \Theta(0)H^{n-m}\lfloor_{\mathbb{R}^{n-m}}$ is defined as in Section 5. Then as $\lambda_j \to 0$,

$$\begin{aligned}
\lim_{j\to+\infty} L^{\lambda_j}_\sharp [\tau_{\alpha\beta}(x)d\gamma] &= \frac{m(m-2)}{4(m-1)}\tau_{\alpha\beta}(0)d\mu_0\\
&= \frac{m(m-2)}{4(m-1)}\Theta(0)\tau_{\alpha\beta}(0)H^{n-m}\lfloor_{\mathbb{R}^{n-m}}.
\end{aligned}$$

For the sequence v_j constructed in Section 5, similar to the discussion above,

$$\frac{\partial v_j}{\partial x_\alpha}\frac{\partial v_j}{\partial x_\beta}dx \rightharpoonup \frac{m(m-2)}{4(m-1)}\tau_{\alpha\beta}(0)d\mu_0.$$

The stationary condition for v_j can be written as

$$\int \left(\frac{1}{2}|\nabla v_j|^2 - \frac{1}{p+1}|v_j|^{p+1}\right)\text{div}Y - \sum_{\alpha,\beta=1}^{n} D_\alpha Y^\beta \frac{\partial v_j}{\partial x_\alpha}\frac{\partial v_j}{\partial x_\beta} = 0.$$

Passing to the limit, by Lemma 2.6, we get

$$\int \left(\frac{1}{2} - \frac{m-2}{2m}\right)\text{div}Y d\mu_0 - \sum_{\alpha,\beta=1}^{n} D_\alpha Y^\beta \tau_{\alpha\beta}(0)d\mu_0 = 0.$$

Since $\mu_0 = \Theta(0)H^{n-m}\lfloor_{\mathbb{R}^{n-m}}$, this can be written as

$$\int_{\mathbb{R}^{n-m}} \sum_{\alpha,\beta=1}^{n} \left(\frac{1}{m}\delta_{\alpha\beta} - \tau_{\alpha\beta}(0)\right) D_\alpha Y^\beta dH^{n-m} = 0.$$

Here $\delta_{\alpha\beta}$ is the Kronecker symbol.

First, for any $n - m + 1 \leq k \leq n$ and $\eta \in C_0^\infty(\mathbb{R}^n)$, by choosing $Y^\beta = \eta\delta_{k\beta}$, we see

$$\tau_{\alpha\beta}(0) = \frac{1}{m}\delta_{\alpha\beta}, \quad \text{if } \alpha \text{ or } \beta \geq n - m + 1.$$

Let $[A_{\alpha\beta}] = [\tau_{\alpha\beta(0)}]_{1\leq\alpha,\beta\leq n-m}$. It is symmetric, nonnegative definite, with

$$\sum_{\alpha=1}^{n-m} A_{\alpha\alpha} = \sum_{\alpha=1}^{n} \tau_{\alpha\alpha}(0) - \sum_{\alpha=n-m+1}^{n} \tau_{\alpha\alpha}(0) = 0.$$

Thus $A = 0$. This implies

$$[\tau_{\alpha\beta}] = \frac{1}{m}\left(Id - S\right),$$

where Id is the identity operator on $\mathbb{R}^n$ and S is the projection operator onto $\mathbb{R}^{n-m}$, i.e. the tangent plane of Σ at 0. (Note that this tangent plane exists uniquely H^{n-m}−a.e. on Σ, because Σ is $(n-m)$−rectifiable.)

Substituting this into (7.1), we get

$$\begin{aligned}
0 &= \int \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1}\right)\text{div}Y - DY(\nabla u, \nabla u) \quad (7.2) \\
&\quad + \frac{2}{m}\int DY \cdot S d\gamma.
\end{aligned}$$

Let G^{n-m} be the $(n-m)$-dimensional Grassmanian space of $\mathbb{R}^n$, i.e. the space of (unoriented) $(n-m)-$dimensional subspace in $\mathbb{R}^n$. Define the varifold V_Σ associated to Σ as

$$< V_\Sigma, \Phi >= \int_\Sigma \Phi(x, S(x)) d\gamma(x),$$

for any $\Phi \in C_0^0(B_1 \times G^{n-m})$. Here $S(x)$ is the weak tangent plane of Σ at x, which exists uniquely $H^{n-m}-$a.e. in Σ. We say V_Σ is stationary if for any smooth vector field Y with compact support (see [10, Section 6.2] for more discussions),

$$\int_\Sigma DY \cdot S(x) d\gamma(x) = 0.$$

By (7.2) we get

Theorem 7.1. *u is a stationary solution if and only if V_Σ is stationary.*

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K. Wang- Wuhan Institute of Physics and Mathematics, The Chinese Academy of Sciences, Wuhan 430071, China.
E-mail address: `wangkelei@wipm.ac.cn`

J. Wei - Department Of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2 Canada and Department of Mathematics, Chinese University Of Hong Kong, Shatin, Hong Kong.
E-mail address: `wei@math.cuhk.edu.hk`