# Existence and uniqueness of singular solution to stationary Schrödinger equation with supercritical nonlinearity

Fouad Hadj Selem,
Laboratoire de Mathématiques UMR 6620 - CNRS,
Université Blaise Pascal,
Campus des Cézeaux -B.P. 80026
63171 Aubière cedex, France
E-mail: fouad.hadjselem@math.univ-bpclermont.fr

Hiroaki Kikuchi,
Department of Mathematics,
Tsuda College,
2-1-1 Tsuda-machi, Kodaira-shi, Tokyo 187-8577, Japan
E-mail: hiroaki@tsuda.ac.jp

Juncheng Wei,
Department of Mathematics,
The Chinese University of Hong Kong,
Shatin, Hong-Kong, China
E-mail: wei@math.cunk.edu.hk

**Abstract**

In this paper, we study a singular solution to a stationary Schrödinger equation with the harmonic potential and the Sobolev supercritical nonlinearity in the spirit of Merle and Peletier [9]. Contrary to the situation Merle and Peletier [9] considered, our spatial domain is the whole space $\mathbb{R}^d$ and our equation is non-autonomous. For these reasons, there are several points we need to take another approach in proving the existence and the uniqueness of the singular solution.

## 1 Introduction

In this paper, we consider the following semilinear elliptic equation:

$$\begin{cases} -\Delta u + |x|^2 u - \lambda u - |u|^{p-1}u = 0, \quad x \in \mathbb{R}^d, & (1) \\ u(x) > 0, \quad x \in \mathbb{R}^d, & (2) \\ u(x) \to 0 \quad \text{as } |x| \to \infty, & (3) \end{cases}$$

where $d \geq 3, \lambda > 0$ and $p > 1$.

Hirose and Ohta [5, 6] showed that for each $\lambda > \lambda_1$, the equations (1)–(3) has a unique solution in case of $p \in (1, 2^* - 1)$, where $\lambda_1$ is the first eigenvalue of the operator $-\Delta + |x|^2$ and $2^*$ is the Sobolev critical exponent, that is, $2^* = 2d/(d-2)$. On the other hand, there is a numerical observation which suggests that contrary to the Sobolev subcritical case $1 < p < 2^* - 1$, the equations (1)–(3) has many solutions for some $\lambda \in (0, \lambda_1)$ in the Sobolev supercritical case $p > 2^* - 1$ (see Figures 10 and 11 of [3] in detail). The motivation of this study comes from the observation. We note that similar phenomena can be proved rigorously for the following semilinear elliptic equations:

$$\begin{cases} -\Delta u - \nu u - |u|^{p-1}u = 0, \quad x \in B, & (4) \\ u(x) > 0, \quad x \in B, & (5) \\ u = 0, \quad x \in \partial B, & (6) \end{cases}$$

where $\nu > 0, p > 1$ and $B$ is the unit ball in $\mathbb{R}^d$. To state it more precisely, Dolbeault and Flores [1] and Guo and Wei [2] respectively showed that there exists a unique $\nu_* \in (0, \nu_1)$ such that for any $k \in \mathbb{N}$, the equations (4)–(6) has at least $k$ solutions if $\nu$ is sufficiently close to $\nu_*$ in case of $p \in (2^* - 1, p_c)$, where $\nu_1$ is the first eigenvalue of the operator $-\Delta$ in $B$ with the Dirichlet boundary condition and $p_c$ is the so-called Joseph and Lundgren exponent introduced in [7], that is,

$$p_c := \begin{cases} \infty & \text{if } 2 \le d \le 10, \\ \frac{(d-2)^2 - 4d + 8\sqrt{d-1}}{(d-2)(d-10)} & \text{if } d \ge 11. \end{cases}$$

Guo and Wei [2] also showed that for any $\nu \in (\nu_*, \nu_1)$, (4)–(6) has exactly one solution for $\nu \in (\nu_*, \nu_1)$ and has no solution for $\nu > \nu_*$ in case of $p \ge p_c^2$, where $p_c^2 \ge p_c$. In their proofs [1, 2], the analysis at $\nu = \nu_*$ is crucial. In fact, Merle and Peletier [9] showed that the equations (4)–(6) with $\nu = \nu_*$ has a singular solution $V$ satisfying

$$V(x) = A(p,d)|x|^{-\frac{2}{p-1}} \left\{1 - B(p,d,\nu_*)|x|^2 + o(|x|^2)\right\} \quad \text{as } |x| \to 0, \tag{7}$$

where

$$A = A(p,d) = \left\{\frac{2}{p-1}\left(d - 2 - \frac{2}{p-1}\right)\right\}^{\frac{1}{p-1}}, \tag{8}$$

$$B = B(p,d,\lambda) = \lambda\left\{4\left(d - 1 - \frac{3}{p-1}\right)\right\}^{-1}. \tag{9}$$

The singular solution $V$ plays an important role in the above results [1, 2]. Therefore, in order to study the multiplicity of the solutions to (1)-(3), it seems worthwhile to investigate whether the equations (1)-(3) has a singular solution like (7). Our first result is the following:

**Theorem 1.** *Let $p > 2^* - 1$. Then, there exists a unique value $\lambda_* \in (0, \lambda_1)$ such that the equations (1)–(3) with $\lambda = \lambda_*$ has a radial solution $W$ satisfying*

$$W(x) = A(p,d)|x|^{-\frac{2}{p-1}} \left\{1 - B(p,d,\lambda_*)|x|^2 + o(|x|^2)\right\} \quad as\ |x| \to 0, \tag{10}$$

*where the constants $A(p,d)$ and $B(p,d,\lambda)$ is given by (8) and (9).*

Before stating our second result, we recall that it is shown in [4] that there exists a bifurcation branch $\mathcal{C} \subset (0, \lambda_1) \times \Sigma$ such that

$$\mathcal{C} = \{(\lambda, u) \in (0, \lambda_1) \times \Sigma \mid u \text{ is a solution to (1)–(3)}\} \tag{11}$$

satisfying

$$\sup \{\|u\|_{L^\infty} \mid (\lambda, u) \in \mathcal{C}\} = \infty,$$

where $\Sigma$ is the function space defined by

$$\Sigma = \left\{u \in H^1(\mathbb{R}^d) \mid |x|u \in L^2(\mathbb{R}^d)\right\}.$$

We are concerned with the asymptotic behavior of the solution with $\|u\|_{L^\infty} \to \infty$. Concerning this problem, we obtain the following:

**Theorem 2.** *Let $p > 2^* - 1$ and $\{(\lambda_n, u_n)\} \subset \mathcal{C}$ with $\|u_n\|_{L^\infty} \to \infty$ as $n \to \infty$, where $\mathcal{C}$ is given by* (11) *Then, we have*

$$\lambda_n \to \lambda_* \qquad \text{as } n \to \infty, \tag{12}$$

*where $\lambda_* \in (0, \lambda_1)$ is the unique value given in Theorem 1. Moreover, we have that*

$$u_n \to W_{\lambda_*} \qquad \text{in } \Sigma \text{ as } n \to \infty. \tag{13}$$

The proof of Theorem 2 is quite similar to that of Merle and Peletier [9, Theorem 1.2]. Thus, we omit it.

We prove Theorem 1 in the sprit of Merle and Peletier [9]. However, we meet several difficulty to show the existence of the singular solution $W$ and uniqueness of the value $\lambda_*$. One of the reason is that our spatial domain is whole space $\mathbb{R}^d$ while Merle and Peletier [9] considered the equations (4)–(6) on the unit ball $B$. The difference forces us to do an additional argument to prove the existence of a singular solution. Indeed, after constructing a local solution $W$ near the origin following Merle and Peletier [9], we need to extend the local solution globally. For this purpose, we shall employ the shooting method. The second difficulty comes from the fact that our equations (1)–(3) is non-autonomous. Merle and Peletier [9] obtained the existence of the singular solution $V$ and the uniqueness of the value $\nu_*$ at the same time by a scaling argument. However, we cannot apply the scaling argument because of the presence of the potential term. For this, we need to take a different approach to show the uniqueness of the value $\lambda_*$. To this end, we shall use the ideas of Wang [11] and Guo and Wei [2].

This paper is organized as follows. In Section 2, following Merle and Peletier [9], we construct a local solution to (1) near the origin for any $\lambda > 0$ and investigate the asymptotic behavior. In Section 3, we prove that there exists $\lambda_* > 0$ such that the solution to (1) obtained in Section 2 exists globally and satisfies (2) and (3). In Section 4, we shall show the uniqueness of the value $\lambda_* > 0$.

## 2 Local existence

In this section, we shall show a existence of a local solution to (1) near the origin $x = 0$ and investigate the asymptotic behavior of the solution. To this end, we transform the equation (1). We first note that from the result of Li and Ni [8], the solution to (1) becomes radially symmetric. Therefore, the equations (1)–(3) becomes the following ordinary differential equations:

$$\begin{cases} -u_{rr} - \frac{d-1}{r}u_r + r^2 u - \lambda u - |u|^{p-1}u = 0, & r > 0, \\ u(r) > 0 & r > 0, \\ u(r) \to 0 & \text{as } r \to \infty. \end{cases} \tag{14}$$

We put

$$v = A^{-1} r^{\theta} u, \tag{15}$$

where $\theta = 2/(p-1)$. In order to prove Theorem 1, we seek a solution to the following:

$$\begin{cases} -v_{rr} - \frac{k-1}{r}v_r - \frac{A^{p-1}}{r^2}\{v^p - v\} + r^2 v - \lambda v = 0, & r > 0, \\ v(r) > 0, \quad r > 0, \\ v(r) \to 1 \quad \text{as } r \to 0 \qquad \text{and} \qquad v(r) \to 0 \quad \text{as } r \to \infty, \end{cases} \tag{16}$$

where $k = d - 4/(p-1)$.

We now carry out so called Emden-Fowler transformation to make the equation autonomous except for the potential term and the term involving the parameter $\lambda$. We set

$$t = \frac{\log r}{m} - \log \frac{\beta}{2m}, \quad y(t) = v(r), \tag{17}$$

where $\beta \in \mathbb{R}$ and $m \in \mathbb{R}$ are defined by

$$\beta = \frac{\lambda}{m(d-2-\theta)}, \qquad m = A^{-\frac{p-1}{2}} = \left\{ \frac{2}{p-1}\left(d - 2 - \frac{2}{p-1}\right)\right\}^{-\frac{1}{2}} = \{\theta(d-2-\theta)\}^{-\frac{1}{2}}.$$

Then, we see that $y(t)$ satisfies the following:

$$\begin{cases} y'' + \alpha y' - y + y^p - \gamma e^{4mt} y + e^{2mt} y = 0, \quad -\infty < t < \infty, & (18) \\ y(t) \to 1 \quad \text{as } t \to -\infty, & (19) \\ v(t) \to 1 \quad \text{as } t \to -\infty \qquad \text{and} \qquad v(t) \to 0 \quad \text{as } t \to \infty, & (20) \end{cases}$$

where $\alpha = (k-2)m$ and $\gamma = 1/\lambda^2 m^2$. Here, we denote by $y'$ the derivative of $y$ with respect to the variable $t$. Then, following Merle and Peletier [9], we obtain the following proposition:

**Proposition 3.** *Let $p > 2^* - 1$. For each $\lambda > 0$, there exist $T_\lambda \in \mathbb{R}$ and a unique solution $y_\lambda \in C([-\infty, T_\lambda), \mathbb{R})$ to (18) satisfying*

$$y_\lambda(t) = 1 - \frac{\theta(d-2-\theta)}{4(d-1) - 6\theta} e^{2mt}[1 + O(e^{2mt})] \quad \text{as } t \to -\infty. \tag{21}$$

Since the proof of Proposition 3 is similar to that of Merle and Peletier [9, Lemmata 3.1 and 3.2], we omit it.

# 3 Existence of the singular solution

In this section, we show that there exists $\lambda_* > 0$ such that the local solution obtained in Proposition 3 to the equation (18) with $\lambda = \lambda_*$ exists globally and vanished at infinity. This shows that there exists a solution $W$ to (1) satisfying (10). To this end, we shall employ the shooting method. For each $\lambda > 0$, we denote by $y_\lambda$ the solution to (18). We set

$$I_+ = \{\lambda > 0 \mid \text{there exists } T \in \mathbb{R} \text{ such that } y_\lambda'(T) = 0 \text{ and } y_\lambda(t) > 0 \text{ for all } -\infty < t < \infty\},$$
$$I_- = \{\lambda > 0 \mid \text{there exists } T \in \mathbb{R} \text{ such that } y_\lambda(T) = 0 \text{ and } y_\lambda'(t) < 0 \text{ for all } -\infty < t < T\},$$
$$I_0 = \{\lambda > 0 \mid y_\lambda(t) > 0, y_\lambda'(t) < 0 \text{ for all } -\infty < t < \infty \text{ and } y_\lambda(t) \to 0 \text{ as } t \to \infty\}.$$

Concerning these sets, we obtain the following result:

**Lemma 4.** *Let the sets $I_\pm$ and $I_0$ be defined above. Then, we have*

$$(0, \infty) = I_+ \cup I_0 \cup I_-.$$

*Proof.* Obviously, $I_+ \cap I_- = \emptyset$. We claim that if $\lambda \notin I_+ \cup I_-$, we have $\lambda \in I_0$. Suppose that $\lambda \notin I_+ \cup I_-$. Then, one of the following cases occurs:

(Case 1) $y_\lambda$ meets the line $x = 0$ with zero derivative,

(Case 2) $y_\lambda$ blows up, that is, there exists $T_\lambda \in \mathbb{R}$ such that $y_\lambda'(t), y_\lambda(t) \to \infty$ as $t \to T_\lambda$,

(Case 3) $y_\lambda(t) > 0, y_\lambda'(t) < 0$ for all $t \in \mathbb{R}$.

First, we show that (Case 1) does not occur. Suppose that there exists $R \in \mathbb{R}$ such that $y_\lambda(R) = y_\lambda'(R) = 0$. This implies $y_\lambda \equiv 0$ from the uniqueness of the Cauchy problem. Thus, this is impossible.

Second, we shall eliminate the possibility that (Case 2) occurs. Since $y_\lambda(t) > 0$ for $t \in (-\infty, T_\lambda)$, we have

$$0 > y_\lambda'' + \alpha y_\lambda' - y_\lambda - \gamma e^{4mt} y_\lambda > y_\lambda'' + \alpha y_\lambda' - y_\lambda - \gamma e^{4mT_\lambda} y_\lambda. \tag{22}$$

We put

$$z_\lambda = y_\lambda' + C_\lambda y_\lambda,$$

where

$$C_\lambda = \frac{\alpha + \sqrt{\alpha^2 + 4(1 + \gamma e^{4mT_\lambda})}}{2}.$$

Then it follows (22) that

$$z_\lambda' - (C_\lambda - \alpha) z_\lambda < 0 \tag{23}$$

for $t \in (-\infty, T_\lambda)$. Multiplying (23) by $e^{-(C_\lambda - \alpha)t}$, we obtain

$$\left(e^{-(C_\lambda - \alpha)t} z_\lambda\right)' < 0,$$

for $t \in (-\infty, T_\lambda)$. Therefore, we see that

$$z_\lambda(t) < e^{(C_\lambda - \alpha)(t-s)} z_\lambda(s) \tag{24}$$

for $-\infty < s < t < T_\lambda$. The estimate (24) implies that (Case 2) does not occur.

Therefore, we see that if $\lambda \notin I_+ \cup I_-$, we have $y_\lambda(t) > 0,\ y'_\lambda(t) < 0$ for all $t \in \mathbb{R}$. Then, there exist $\{t_n\} \subset \mathbb{R}$ with $\lim_{n\to\infty} t_n = \infty$ and $l \geq 0$ such that

$$y_\lambda(t_n) \to l, \quad y'_\lambda(t_n) \to 0, \quad y''_\lambda(t_n) \to 0$$

as $n \to \infty$. Suppose that $l \neq 0$. It follows from (18) that

$$0 \leftarrow y''_\lambda + \alpha y'_\lambda = y_\lambda - y_\lambda^p + \gamma e^{4mt_n} y_\lambda - e^{2mt_n} y_\lambda \to \infty \qquad \text{as } n \to \infty, \tag{25}$$

which is a contradiction. Therefore, we obtain $l = 0$. This complete the proof. □

**Lemma 5.** *The sets $I_\pm$ are open.*

*Proof.* Openness of the set $I_-$ is clear from the continuous dependence of the solution on $\lambda$. Thus, we consider the set $I_+$. Let $\lambda_* \in I_+$. We claim that there exist a local minimum $t_* \in \mathbb{R}$, that is, $y'_{\lambda_*}(t_*) = 0$ and $y''_{\lambda_*}(t_*) > 0$. Suppose that $y'_{\lambda_*}(t) \leq 0$ for all $-\infty < t < \infty$. Then, there exists $l \geq 0$ such that $y_{\lambda_*}(t) \to l$ as $t \to \infty$. Suppose that $l > 0$. Then, we can drive a contradiction by a same argument as in (25). Thus, we have $l = 0$, which implies that $y'_\lambda(t) < 0$ for all $-\infty < t < \infty$ from the result of Li and Ni [8]. This contradicts the fact that $\lambda_* \in I_+$. Therefore, there exists $t_1 \in \mathbb{R}$ such that $y'_{\lambda_*}(t_1) > 0$. It follows from Proposition 3 that $y'_{\lambda_*}(t_2) < 0$ if $t_2 \in \mathbb{R}$ is sufficiently small. From this, we infer that there exists $t_* \in \mathbb{R}$ such that $y'_{\lambda_*}(t_*) = 0$ and $y''_{\lambda_*}(t_*) > 0$. Thus, our claim holds.

Then, there exist $t_3 < t_* < t_4$ such that $y_{\lambda_*}(t_i) > y_{\lambda_*}(t_*)$ for $i = 3$ and 4. It follows from the continuous dependence of the solution on the parameter $\lambda$ that

$$y_\lambda(t_i) > y_\lambda(t_*) \quad \text{for } i = 3 \text{ and } 4 \text{ if } |\lambda - \lambda_*| > 0 \text{ is sufficiently small.}$$

Thus, there exists $t_0 \in (t_1, t_2)$ such that $y'_\lambda(t_0) = 0$, which yields that $\lambda \in I_+$. This completes the proof. □

**Lemma 6.** *The set $I_-$ is nonempty.*

*Proof.* First, we note that from the result of Merle and Peletier [9] that there exist $T_0 \in \mathbb{R}$ and a unique solution $w_0$ to the following ordinary differential equation:

$$\begin{cases} w'' + \alpha w' - w + w^p + e^{2mt} w = 0, & -\infty < t < T_0, \\ w \to 1 & \text{as } t \to -\infty, \\ w(T_0) = 0. \end{cases} \tag{26}$$

Suppose the contrary that $\lambda \in I_0 \cup I_+$ for any $\lambda > 0$. We take $\delta > 0$ sufficiently small so that the solution $w(t)$ exists for $t \in (-\infty, T_0 + \delta)$. Then, we put $T_* = T_0 + \delta$. We first show that there exist a sufficiently large $\lambda_1 > 0$ and a constant $C > 0$, which is independent of $\lambda$, such that

$$\sup_{t\in(-\infty,T_*)} y_\lambda(t) \le C \tag{27}$$

for $\lambda > \lambda_1$. We can take $\lambda > 0$ sufficiently large so that

$$\gamma = \frac{1}{\lambda^2 m^2} < e^{-2mT_*}. \tag{28}$$

For such $\gamma > 0$, we have by (18) that

$$0 > y_\lambda'' + \alpha y_\lambda' - y_\lambda + y_\lambda^p$$

for $t \in (-\infty, T_*)$, where we have used the fact that $y_\lambda(t) > 0$ for all $-\infty < t < \infty$. This yields that

$$y_\lambda'' + \alpha y_\lambda' < y_\lambda - y_\lambda^p < \max_{s>0}\{s - s^p\} = (p-1)p^{-p/(p-1)}. \tag{29}$$

It follows from (21) that there exists a sufficiently small $\varepsilon_0$ and $T_1 \in (-\infty, T_*)$ (independent of $\lambda$) such that

$$1 - \varepsilon_0 < y_\lambda(t) < 1, \quad y_\lambda'(t) < 0 \tag{30}$$

for $t \in (-\infty, T_1]$. Integrating (29) from $T_1$ to $t$, we have

$$y_\lambda'(t) + \alpha y_\lambda(t) < (1-\varepsilon_0)\alpha + C_p(t - T_1), \tag{31}$$

where $C_p = (p-1)p^{-p/(p-1)}$. By (31), we see that (27) holds.

Next, we put

$$s = -t, \qquad \eta(s) = w(s) - 1. \tag{32}$$

Then, $\eta$ satisfies

$$\eta'' - \alpha\eta' + (p-1)\eta = f(s,\eta),$$

where

$$f(s,\eta) = -e^{-2ms}(1+\eta) - \varphi(\eta), \qquad \varphi(\eta) = (1+\eta)^p - 1 - p\eta.$$

Similarly, we put

$$\zeta_\lambda(s) = y_\lambda(s) - 1. \tag{33}$$

Then, $\zeta_\lambda$ satisfies the following:

$$\zeta_\lambda'' - \alpha\zeta + (p-1)\zeta = g_\lambda(s,\zeta),$$

where $g_\lambda(s,\zeta) = -\gamma e^{-4ms}\{1+\zeta\} + f(s,\zeta)$. We distinguish the following there cases:

$$\text{(Case 1) } p - 1 > \frac{\alpha^2}{4}, \qquad \text{(Case 2) } p - 1 = \frac{\alpha^2}{4}, \qquad \text{(Case 3) } p - 1 < \frac{\alpha^2}{4}.$$

We shall discuss (Case 1) only and the other cases can be proved similarly. We put

$$\mu = \sqrt{p - 1 - \frac{\alpha^2}{4}}.$$

Then, by using the method of variation of parameters, we see that $\eta$ and $\zeta_\lambda$ satisfy the following integral equations respectively;

$$\eta(s) = \frac{1}{\mu} e^{\frac{\alpha}{2}s} \int_s^\infty e^{-\frac{\alpha}{2}\sigma} \sin(\mu(\sigma - s)) f(\sigma, \eta) d\sigma,$$
$$\zeta_\lambda = \frac{1}{\mu} e^{\frac{\alpha}{2}s} \int_s^\infty e^{-\frac{\alpha}{2}\sigma} \sin(\mu(\sigma - s)) g_\lambda(\sigma, \zeta_\lambda) d\sigma.$$

Then, we have

$$\begin{aligned} |\eta(s) - \zeta_\lambda(s)| &\le \frac{1}{\mu} e^{\frac{\alpha}{2}s} \int_s^\infty e^{-\frac{\alpha}{2}\sigma} |\sin(\mu(\sigma - s))| |f(\sigma, \eta) - g_\lambda(\sigma, \zeta_\lambda)| d\sigma \\ &\le \frac{1}{\mu} \int_s^\infty \gamma e^{-4m\sigma} |1 + \zeta_\lambda(\sigma)| d\sigma + \frac{1}{\mu} \int_s^\infty |f(\sigma, \eta) - f(\sigma, \zeta_\lambda)| d\sigma. \end{aligned}$$

Since $f$ is Lipschitz continuous, there exists a constant $L > 0$ such that $|f(\sigma, \eta) - f(\sigma, \zeta_\lambda)| v \le L|\eta - \zeta_\lambda|$. This together with (27) gives us that

$$|\eta(s) - \zeta_\lambda(s)| \le \gamma \frac{C}{\mu} + \frac{L}{\mu} \int_s^\infty |\eta(\sigma) - \zeta_\lambda(\sigma)| d\sigma$$

for some constant $C > 0$. For any $\varepsilon > 0$, we can take $\lambda > 0$ sufficiently large so that

$$\gamma \frac{C}{\mu} = \frac{C}{\mu m^2 \lambda} < \varepsilon.$$

This yields that

$$|\eta(s) - \zeta_\lambda(s)| \le \varepsilon + C_1 \int_s^\infty |\eta - \zeta_\lambda| d\sigma$$

for some constant $C_1 > 0$. Then, the Gronwall's inequality gives us that

$$|\eta(s) - \zeta_\lambda(s)| \le \varepsilon(1 + C_1 s e^{C_1 s})$$

for all $s \in (-T_*, \infty)$. This together with (32) yields that

$$|y_\lambda(t) - w(t)| \le \varepsilon(1 + C_1|t|e^{-C_1 t}) \tag{34}$$

for all $t \in (-\infty, T_*)$. Since $w$ has a zero at $t = T_*$, (34) implies that $y_\lambda$ has a zero for sufficiently large $\lambda > 0$. Thus, we see that the set $I_-$ is nonempty. □

**Lemma 7.** *The set $I_+$ is non-empty.*

*Proof.* First, we shall show that if $\lambda > 0$ is sufficiently small, $y_\lambda$ does not have zero in $(-\infty, \infty)$. Suppose the contrary that there exists $\lambda_n \subset (0, \infty)$ with $\lim_{n\to\infty} \lambda_n = 0$ such that $y_{\lambda_n}$ have a zero at $t = T_n$. Thanks to the asymptotic (21), there exists $C \in \mathbb{R}$ (independent of $n$) such that $T_n \ge C$ for all $n \in \mathbb{N}$. Multiplying the equation (18) by $y'_{\lambda_n}$ and integrating the resulting equation from $-\infty$ to $T_n$, we have

$$\begin{aligned} &\left[\frac{1}{2}(y'_{\lambda_n})^2\right]_{-\infty}^{T_n} + \alpha \int_{-\infty}^{T_n} |y'_{\lambda_n}|^2 ds + \left[-\frac{y_{\lambda_n}^2}{2} + \frac{y_{\lambda_n}^{p+1}}{p+1} - \frac{\gamma}{2}|e^{4mt} y_{\lambda_n}^2| + \frac{|e^{2mt} y_{\lambda_n}^2|}{2}\right]_{-\infty}^{T_n} \\ &= -2\gamma \int_{-\infty}^{T_n} e^{4ms} |y_{\lambda_n}|^2 ds + m \int_{-\infty}^{T_n} e^{2ms} |y_{\lambda_n}|^2 ds. \end{aligned} \tag{35}$$

Since $y_{\lambda_n}(t) \to 1$ as $t \to -\infty$, the left hand side of (35) yields

$$\begin{aligned}
&\left[\frac{1}{2}(y'_{\lambda_n})^2\right]_{-\infty}^{T_n} + \alpha\int_{-\infty}^{T_n} |y'_{\lambda_n}|^2 ds + \left[-\frac{y_{\lambda_n}^2}{2} + \frac{y_{\lambda_n}^{p+1}}{p+1} - \frac{\gamma}{2}|e^{4mt}y_{\lambda_n}^2| + \frac{|e^{2mt}y_{\lambda_n}^2|}{2}\right]_{-\infty}^{T_n} \\
&\geq \frac{1}{2}y_{\lambda_n}^2(T_n) + \left[-\frac{y_{\lambda_n}^2}{2} + \frac{y_{\lambda_n}^{p+1}}{p+1} - \frac{\gamma}{2}|e^{4mt}y_{\lambda_n}^2| + \frac{|e^{2mt}y_{\lambda_n}^2|}{2}\right]_{-\infty}^{T_n} \\
&\geq \left[-\frac{y_{\lambda_n}^2}{2} + \frac{y_{\lambda_n}^{p+1}}{p+1} - \frac{\gamma}{2}|e^{4mt}y_{\lambda_n}^2| + \frac{|e^{2mt}y_{\lambda_n}^2|}{2}\right]_{-\infty}^{T_n} \\
&= \frac{1}{2} - \frac{1}{p+1} > 0.
\end{aligned} \tag{36}$$

On the other hand, using the asymptotic (21) of $y_{\lambda_n}$ again, there exists $\hat{T}(< T_n)$ (independent of $\lambda$) such that $1/2 < y_{\lambda_n} < 1$ for $t \in (-\infty, \hat{T})$. This together with the fact that $\gamma = 1/m^2\lambda_n^2$ yields that

$$\begin{aligned}
&-\frac{2}{m^2\lambda_n^2}\int_{-\infty}^{T_n} e^{4ms}|y_{\lambda_n}|^2 ds + m\int_{-\infty}^{T_n} e^{2ms}|y_{\lambda_n}|^2 ds \\
&= -\frac{2}{m^2\lambda_n^2}\int_{-\infty}^{\hat{T}} e^{4ms}|y_{\lambda_n}|^2 ds - \frac{2}{m^2\lambda_n^2}\int_{\hat{T}}^{T_n} e^{4ms}|y_{\lambda_n}|^2 ds + m\int_{-\infty}^{\hat{T}} e^{2ms}|y_{\lambda_n}|^2 ds \\
&\quad + m\int_{\hat{T}}^{T_n} e^{2ms}|y_{\lambda_n}|^2 ds \\
&< -\frac{1}{m^2\lambda_n^2}\int_{-\infty}^{\hat{T}} e^{4ms} ds - \frac{2}{m^2\lambda_n^2}\int_{\hat{T}}^{T_n} e^{4ms}|y_{\lambda_n}|^2 ds + m\int_{-\infty}^{\hat{T}} e^{2ms} ds + m\int_{\hat{T}}^{T_n} e^{2ms}|y_{\lambda_n}|^2 ds \\
&< -\frac{1}{m^2\lambda_n^2}\left[\frac{e^{4ms}}{4m}\right]_{-\infty}^{\hat{T}} + m\left[\frac{e^{2ms}}{2m}\right]_{-\infty}^{\hat{T}} + \int_{\hat{T}}^{T_n}\left(-\frac{2}{m^2\lambda_n^2} + me^{-2ms}\right)e^{4ms}|y_{\lambda_n}|^2 ds \\
&= \left(-\frac{1}{4m^2\lambda_n^2} + \frac{e^{-2m\hat{T}}}{2}\right)e^{4m\hat{T}} + \int_{\hat{T}}^{T_n}\left(-\frac{2}{m^2\lambda_n^2} + me^{-2m\hat{T}}\right)e^{4ms}|y_{\lambda_n}|^2 ds \\
&< 0 \quad \text{for sufficiently large } n \in \mathbb{N}.
\end{aligned}$$

This together with (35) and (36) yields a contradiction. Thus, we see that $\lambda \in I_0 \cup I_+$ for sufficiently small $\lambda > 0$.

Next, we shall show that $\lambda \in I_+$ for sufficiently small $\lambda > 0$. Suppose the contrary that there exists a sequence $\{\lambda_n\} \subset (0, \infty)$ with $\lim_{n\to\infty} \lambda_n = 0$ such that $y_{\lambda_n}$ has no critical point. Then, by Lemma 4, we see that $y'_{\lambda_n}(t) < 0$ and $y_{\lambda_n}(t) \to 0$ as $t \to \infty$. Moreover, since $y_{\lambda_n}(t) \to 1$ as $t \to -\infty$, we have

$$y_{\lambda_n}(t) < 1 \qquad \text{for all } -\infty < t < \infty. \tag{37}$$

Then, there exists $T_{1,n} \in \mathbb{R}$ such that

$$y_{\lambda_n}(t) \leq 1/4 \qquad \text{for all } t \geq T_{1,n}. \tag{38}$$

It follows from (21) that there exists $T_0 > 0$ (independent of $n$) such that $T_{1,n} \geq T_0$. We take $\lambda >$ sufficiently small so that $-\log\gamma/2m = \log(\lambda^2 m^2)/2m < T_0$. Then, integrating the equation (18) from $-\infty$ to $T_{1,n}$, we have, by (37) and (38), that

$$\begin{aligned}
y'_{\lambda_n}(T_{1,n}) &= -\alpha[y_{\lambda_n}]_{-\infty}^{T_{1,n}} + \int_{-\infty}^{T_{1,n}} \left\{ y_{\lambda_n} - y_{\lambda_n}^p + \gamma e^{4ms} y_{\lambda_n} - e^{2ms} y_{\lambda_n} \right\} ds \\
&= \frac{3}{4}\alpha + \int_{-\infty}^{-\frac{\log\gamma}{2m}} \left\{ y_{\lambda_n} - y_{\lambda_n}^p + \gamma e^{4ms} y_{\lambda_n} - e^{2ms} y_{\lambda_n} \right\} ds \\
&\quad + \int_{-\frac{\log\gamma}{2m}}^{T_{1,n}} \left\{ y_{\lambda_n} - y_{\lambda_n}^p + \gamma e^{4ms} y_{\lambda_n} - e^{2ms} y_{\lambda_n} \right\} ds \\
&\geq \frac{3}{4}\alpha + \int_{-\infty}^{-\frac{\log\gamma}{2m}} \left\{ \gamma e^{4ms} y_{\lambda_n} - e^{2ms} y_{\lambda_n} \right\} ds.
\end{aligned}$$

Taking $\lambda > 0$ sufficiently small so that $1/2 < y_{\lambda_n} < 1$ for $t \in (-\infty, -\log\gamma/2m)$, we have

$$y'_{\lambda_n}(T_{1,n}) \geq \frac{3}{4}\alpha + \frac{\gamma}{2}\int_{-\infty}^{-\frac{\log\gamma}{2m}} e^{4ms} ds - \int_{-\infty}^{-\frac{\log\gamma}{2m}} e^{2ms} ds = \frac{3}{4}\alpha + \frac{1}{8m\gamma} - \frac{1}{2m\gamma} > \frac{\alpha}{2}.$$

This contradicts with the fact that $y'_{\lambda_n}(t) < 0$ for all $-\infty < t < \infty$. Thus, we infer that $\lambda \in I_+$ for sufficiently small $\lambda > 0$. $\square$

It follows from Lemma 4 to 7 that there exists $\lambda_* \in (0, \infty)$ such that $\lambda_* \in I_0$. Therefore, $y_{\lambda_*}$ satisfies the equations (1)–(3).

# 4 Uniqueness of the singular solution

This section is devoted to the proof of Theorem 1. Since we have already shown the existence of a solution satisfying (10), it is enough to prove the uniqueness of the value $\lambda_*$. Suppose that there exist two different solutions $u$ and $v$ to the equations (1)–(3) with $\lambda = \lambda_1$ and $\lambda_2$ respectively satisfying (10). Without loss of the generality, we may assume that

$$\lambda_1 < \lambda_2. \tag{39}$$

This together with (10) implies that there exists $R_1 > 0$ such that

$$u > v \quad \text{for } r \in (0, R_1). \tag{40}$$

We rescale the solution as follows:

$$u(r) = \nu^{1/(p-1)}\widetilde{u}(\sqrt{\nu}r), \qquad v(r) = \nu^{1/(p-1)}\widetilde{v}(\sqrt{\nu}r) \tag{41}$$

for $\nu > 0$. Then, the functions $\widetilde{u}$ and $\widetilde{v}$ satisfy the following equations respectively:

$$-\widetilde{u}_{rr} - \frac{d-1}{r}\widetilde{u}_r + \frac{r^2}{\nu^2}\widetilde{u} - \frac{\lambda_1}{\nu}\widetilde{u} - \widetilde{u}^p = 0, \quad r > 0, \tag{42}$$

$$-\widetilde{v}_{rr} - \frac{d-1}{r}\widetilde{v}_r + \frac{r^2}{\nu^2}\widetilde{v} - \frac{\lambda_2}{\nu}\widetilde{v} - \widetilde{v}^p = 0, \quad r > 0. \tag{43}$$

We put

$$W = \frac{\widetilde{u}}{\widetilde{v}}. \tag{44}$$

Then, $W$ satisfies

$$W_{rr} + \left(\frac{d-1}{r} + \frac{2}{\widetilde{v}}\widetilde{v}_r\right) W_r + \frac{(\lambda_1 - \lambda_2)}{\nu} W + W(\widetilde{u}^p - \widetilde{v}^p) = 0, \qquad r > 0, \tag{45}$$

$$W(r) \to 1 \qquad \text{as } r \to 0. \tag{46}$$

Furthermore, we put

$$\rho = \log r, \qquad W(\rho) = W(r). \tag{47}$$

Then, the equation (45) is transformed into the following:

$$W_{\rho\rho} + \left(d - 2 + 2r\frac{\widetilde{v}_r}{\widetilde{v}}\right) W_\rho + \frac{(\lambda_1 - \lambda_2)}{\nu} r^2 W + r^2 \widetilde{v}^{p-1}(W^p - W) = 0, \qquad \rho \in (-\infty, \infty). \tag{48}$$

It follows from (40) that there exists $T_1 = T_1(\nu) > 0$ such that

$$W(\rho) > 1 \qquad \text{for } \rho \in (-\infty, -T_1) \tag{49}$$

By (10), we see that

$$d - 2 + \frac{2r\widetilde{v}_r}{\widetilde{v}} \to \alpha_1 \qquad \text{as } \rho \to -\infty, \tag{50}$$

$$r^2 \widetilde{v}^{p-1} \frac{W^p - W}{1 - W} \to -(p-1)\beta_1 \qquad \text{as } \rho \to -\infty, \tag{51}$$

where

$$\alpha_1 = d - 2 - \frac{4}{p-1}, \qquad \beta_1 = A(p,d)^{p-1} = \frac{2}{p-1}\left(d - 2 - \frac{2}{p-1}\right)$$

Finally, we put

$$Z = 1 - W. \tag{52}$$

Then, $Z$ satisfies the following:

$$Z_{\rho\rho} + \left(d - 2 + 2r\frac{\widetilde{v}_r}{\widetilde{v}}\right) Z_\rho - \frac{(\lambda_1 - \lambda_2)}{\nu} r^2 (1 - Z) - r^2 \widetilde{v}^{p-1} \frac{W^p - W}{1 - W} Z = 0, \qquad \rho \in (-\infty, \infty). \tag{53}$$

It follows from (10) that

$$\frac{Z}{r^2} = \frac{1 - W}{r^2} = \frac{\widetilde{v} - \widetilde{u}}{r^2 \widetilde{v}} \to (\lambda_1 - \lambda_2)\left\{4\left(d - 1 - \frac{3}{p-1}\right)\right\}^{-1} \qquad \text{as } \rho \to -\infty. \tag{54}$$

Before proving Theorem 1, we prepare the following result:

**Lemma 8.** *There exists $\nu_0 > 0$ and $T_2 > 0$ such that if we take $\nu > \nu_0$, we have that*

$$Z_\rho(\rho) \le 0 \qquad \text{for } \rho \in (-\infty, -T_2). \tag{55}$$

*Proof.* We show this by contradiction. Suppose the contrary that there exists a sequence $\{\rho_n\} \subset (-\infty, -T_1)$ with $\lim_{n\to\infty} \rho_n = -\infty$ satisfying $Z_\rho(\rho_n) > 0$. Note that $Z(\rho) < 0$ for $\rho \in (-\infty, -T_1)$ and $Z(\rho) \to 0$ as $\rho \to -\infty$. This yields that there exists $\{r_n\} \subset (-\infty, -T_1)$ with $\lim_{n\to\infty} r_n = -\infty$ such that

$$Z_\rho(r_n) = 0 \qquad \text{and} \qquad Z_{\rho\rho}(r_n) \le 0. \tag{56}$$

Namely, $r_n$ is a local maximum point of $Z$. For $\rho = r_n$, we have by (53) that

$$-\frac{(\lambda_1 - \lambda_2)}{\nu} r_n^2 (1 - Z(r_n)) - r_n^2 \widetilde{v}^{p-1}(r_n) \frac{W^p(r_n) - W(r_n)}{1 - W(r_n)} Z(r_n) \ge 0. \tag{57}$$

This together with (46), (51) and (54) gives us that

$$\begin{aligned} \frac{2}{\nu} \ge \frac{1 - Z(r_n)}{\nu} &\ge -r_n^2 \widetilde{v}^{p-1}(r_n) \frac{W^p(r_n) - W(r_n)}{1 - W(r_n)} \frac{Z(r_n)}{r_n^2(\lambda_1 - \lambda_2)} \\ &\ge \frac{(p-1)\beta_1}{2} \left\{ 4(d - 1 - \frac{3}{p-1}) \right\}^{-1}. \end{aligned} \tag{58}$$

However, we can take $\nu > 0$ sufficiently large so that

$$\frac{1}{\nu} < \frac{(p-1)\beta_1}{4} \left\{ 4(d - 1 - \frac{3}{p-1}) \right\}^{-1},$$

which contradicts with (58). Thus, (55) holds. $\square$

We are now in position to prove Theorem 1.

*Proof of Theorem 1.* We first consider the case of $2^* - 1 < p < p_c$.

It follows from (53) and (54) that there exists $T_3 = T_3(\nu) > 0$ such that for $\rho \in (-\infty, -T_3)$, we have

$$\begin{aligned} Z_{\rho\rho} + \left( d - 2 + 2r \frac{\widetilde{v}_r}{\widetilde{v}} \right) Z_\rho - r^2 \widetilde{v}^{p-1} \frac{W^p - W}{1 - W} Z &= \frac{(\lambda_1 - \lambda_2)}{\nu} r^2 (1 - Z) \\ &\ge \frac{(\lambda_1 - \lambda_2)}{\nu} r^2 \\ &\ge \frac{1}{\nu} \left\{ 4(d - 1 - \frac{3}{p-1}) \right\} Z. \end{aligned} \tag{59}$$

Thus, we obtain

$$Z_{\rho\rho} + \left( d - 2 + 2r \frac{\widetilde{v}_r}{\widetilde{v}} \right) Z_\rho - \left\{ r^2 \widetilde{v}^{p-1} \frac{W^p - W}{1 - W} + \frac{1}{\nu} \left( 4\nu(d - 1 - \frac{3}{p-1}) \right) \right\} Z \ge 0. \tag{60}$$

We set

$$g_1(\rho) := d - 2 + 2r \frac{\widetilde{v}_r}{\widetilde{v}}, \qquad g_2(\rho) := -\left\{ r^2 \widetilde{v}^{p-1} \frac{W^p - W}{1 - W} + \frac{1}{\nu} \left( 4(d - 1 - \frac{3}{p-1}) \right) \right\}. \tag{61}$$

Note that for $2^* - 1 < p < p_c$, we have

$$\alpha_1^2 - 4(p-1)\beta_1 < 0.$$

We take $\nu > 0$ sufficiently large so that

$$\alpha_1^2 - 4(p-1)\beta_1 - \frac{1}{\nu}\left(4(d-1-\frac{3}{p-1})\right) < 0$$

This together with (50) and (51) implies that there exist $T_4 = T_4(\nu)$ such that

$$[g_1(\rho)]^2 - 4g_2(\rho) < 0 \qquad \text{for } \rho \in (-\infty, -T_4). \tag{62}$$

Therefore, there exist two positive constants $b_1$ and $c_1$ such that

$$b_1^2 - 4c_1 < 0, \qquad b_1 < g_1(\rho), \qquad c_1 < g_2(\rho) \qquad \text{for } \rho \in (-\infty, -T_4). \tag{63}$$

Let $\omega$ be a non-trivial solution to the following ordinary differential equation:

$$\omega_{\rho\rho} + b_1\omega_\rho + c_1\omega = 0, \qquad \rho \in (-\infty, \infty). \tag{64}$$

From (63), the solution $\omega$ is oscillatory. Thus, there exist $a_1$ and $a_2$ with $a_2 < a_1 < -T_4$ satisfying

$$\omega(\rho) > 0 \qquad \text{for } \rho \in (a_2, a_1), \qquad \omega(a_1) = \omega(a_2) = 0. \tag{65}$$

Multiplying (60) by $\omega$ and (64) by $Z$, we have

$$Z_{\rho\rho}\omega + g_1(\rho)Z_\rho\omega + g_2(\rho)Z\omega \geq 0, \tag{66}$$

$$\omega_{\rho\rho}Z + b_1\omega_\rho Z + c_1 Z\omega = 0. \tag{67}$$

Subtracting (67) from (66), we obtain

$$(Z_\rho\omega - \omega_\rho Z)_\rho + g_1(\rho)Z_\rho\omega - b_1\omega_\rho Z + (g_2(\rho) - c_1)\omega Z \geq 0.$$

This together with (63) and (65) implies that

$$\begin{aligned}
\left\{e^{b_1\rho}(Z_\rho\omega - \omega_\rho Z)\right\}_\rho &= e^{b_1\rho}\left\{(Z_\rho\omega - \omega_\rho Z)_\rho + b_1(Z_\rho\omega - \omega_\rho Z)\right\} \\
&\geq \{-g_1(\rho)Z_\rho\omega + b_1\omega_\rho Z - (g_2(\rho) - c_1)\omega Z + b_1(Z_\rho\omega - \omega_\rho Z)\} \\
&\geq \{-(g_1(\rho) - b_1)Z_\rho\omega - (g_2(\rho) - c_1)\omega Z\} \\
&\geq 0.
\end{aligned}$$

Integrating the above from $a_2$ to $a_1$, we obtain

$$0 < -e^{b_1 a_2}\omega_\rho(a_2)Z(a_2) \leq -e^{b_1 a_1}\omega_\rho(a_1)Z(a_1) < 0$$

since $\omega_\rho(a_2) > 0$ and $\omega_\rho(a_1) < 0$. This is a contradiction. Thus, we obtain the desired result.

Next, we consider the case of $p \geq p_{\mathrm{c}}$. We put $Z = e^{\tau_1 \rho}\varphi$, where

$$\tau_1 = -\frac{\alpha_1}{2} + \frac{1}{2}\sqrt{\alpha_1^2 - 4(p-1)\beta_1}.$$

We note that $\alpha_1^2 - 4(p-1)\beta_1 \geq 0$ and $\tau_1 < -2$ for $p \geq p_{\mathrm{c}}$. Then, (49), (52) and (54) gives us that there exists $T_5 > 0$ such that

$$\varphi(\rho) < 0 \qquad \text{for } \rho \in (-\infty, -T_5) \tag{68}$$

and

$$|\varphi(\rho)| \leq Ce^{(2-\tau_1)\rho} \qquad \text{for } \rho \in (-\infty, -T_5). \tag{69}$$

Since $Z_\rho = \tau_1 e^{\tau_1\rho}\varphi + e^{\tau_1\rho}\varphi_\rho < 0, \tau_1 < 0$ and $\varphi(\rho) < 0$ for $\rho \in (-\infty, -T_5)$, we have

$$\varphi_\rho \leq 0, \qquad \tau_1\varphi \leq -\varphi_\rho \qquad \text{for } \rho \in (-\infty, -T_5). \tag{70}$$

It follows from (53) and (54) that

$$\begin{aligned}
0 &= e^{\tau_1\rho}\varphi_{\rho\rho} + 2\tau_1 e^{\tau_1\rho}\varphi_\rho + \tau_1^2 e^{\tau_1\rho}\varphi + \left(d - 2 + \frac{2r\widetilde{v}_r}{\widetilde{v}}\right)(\tau_1 e^{\tau_1\rho}\varphi + e^{\tau_1\rho}\varphi_\rho) \\
&\quad - \frac{(\lambda_1 - \lambda_2)}{\nu} r^2(1 - e^{\tau_1\rho}\varphi) - r^2\widetilde{v}^{p-1}\frac{W^p - W}{1 - W}e^{\tau_1\rho}\varphi \\
&\leq e^{\tau_1\rho}\varphi_{\rho\rho} + (2\tau_1 + \alpha_1)e^{\tau_1\rho}\varphi_\rho + \left(d - 2 + \frac{2r\widetilde{v}_r}{\widetilde{v}} - \alpha_1\right)e^{\tau_1\rho}\varphi_\rho \\
&\quad + \left\{\tau_1^2 + \left(d - 2 + \frac{2r\widetilde{v}_r}{\widetilde{v}}\right)\tau_1 - r^2\widetilde{v}^{p-1}\frac{W^p - W}{1 - W}\right\}e^{\tau_1\rho}\varphi - \frac{(\lambda_1 - \lambda_2)}{\nu}r^2(1 - e^{\tau_1\rho}\varphi).
\end{aligned} \tag{71}$$

It follows from (50) that for any $\varepsilon > 0$, there exists $T_6 > 0$ such that

$$\left| d - 2 + \frac{2r\widetilde{v}_r}{\widetilde{v}} - \alpha_1 \right| < \frac{\varepsilon}{4}.$$

Moreover, (50), (51) and the definition of $\tau_1$ yields that there exists $T_7 > T_6$ such that

$$\left| \tau_1^2 + \left(d - 2 + \frac{2r\widetilde{v}_r}{\widetilde{v}}\right) - r^2\widetilde{v}^{p-1}\frac{W^p - W}{1 - W} \right| < -\tau_1\frac{\varepsilon}{4}.$$

By (54), there exists $T_8 > T_7$ and $\nu_* > 0$ such that for $\nu > \nu_*$, we have

$$\left| \frac{(\lambda_1 - \lambda_2)}{\nu} r^2(1 - e^{\tau_1\rho}\varphi) \right| < \frac{\varepsilon}{2}\tau_1\varphi$$

for $\rho \in (-\infty, -T_8)$. These together with (71) imply that

$$\begin{aligned}
0 &< \varphi_{\rho\rho} + (2\tau_1 + \alpha_1)\varphi_\rho + \frac{\varepsilon}{4}|\varphi_\rho| + \frac{\varepsilon}{4}\tau_1\varphi + \frac{\varepsilon}{4}\tau_1\varphi \\
&< \varphi_{\rho\rho} + (2\tau_1 + \alpha_1)\varphi_\rho + \frac{\varepsilon}{2}|\varphi_\rho| - \frac{\varepsilon}{2}\varphi_\rho \\
&< \varphi_{\rho\rho} + (2\tau_1 + \alpha_1 - \varepsilon)\varphi_\rho.
\end{aligned}$$

for $\rho \in (-\infty, -T_8)$.

(54) implies that $\lim_{\rho\to-\infty} Z_\rho(\rho) = 0$. This together with (69) gives us that

$$\lim_{\rho\to-\infty} \varphi(\rho) = \lim_{\rho\to-\infty} \varphi_\rho(\rho) = 0.$$

Then, integrating from $-\infty$ to $\rho$ yields that

$$0 < \varphi_\rho + (2\tau_1 + \alpha_1 - \varepsilon)\varphi. \tag{72}$$

Multiplying (72) by $e^{(2\tau_1+\alpha_1-\varepsilon)\rho}$, we obtain

$$0 < \left\{e^{(2\tau_1+\alpha_1-\varepsilon)\rho}\varphi\right\}_\rho, \qquad \text{for } \rho \in (-\infty, -T_8). \tag{73}$$

On the other hands, (69) shows that

$$|\varphi(\rho)e^{(2\tau_1+\alpha_1-\epsilon)\rho}| \le Ce^{(2+\tau_1+\alpha_1-\epsilon)\rho} = Ce^{(2+\frac{\alpha_1}{2}+\frac{1}{2}\sqrt{\alpha^2-4(p-1)\beta}-\varepsilon)\rho} \to 0 \qquad \text{as } \rho \to -\infty.$$

Then, integrating (73) from $-\infty$ to $\rho$ $(< -T_8)$, we have

$$0 < e^{(2\tau_1+\alpha_1-\varepsilon)\rho}\varphi(\rho),$$

which yields that $\varphi(\rho) > 0$ for $\rho \in (-\infty, -T_8)$. This contradicts with (68). Thus, we obtain the desired result. □

# Acknowledgements

H.K. is partially supported by the Grant-in Aid for Research Activity Start-up # 23840037 of JSPS. J.W. is partially supported by a General Research Grant entitled "Traveling Wave Solutions and Parabolic De Giorgi Conjecture".

# References

[1] J. Dolbeault and I. Flores, *Geometry of phase space and solutions of semilinear elliptic equations in a ball*, Trans. Amer. Math. Soc., **359** (2007), 4073–4087.

[2] Z. Guo and J. Wei, *Global solution branch and Morse index estimates of a semilinear elliptic equation with super-critical exponent*, Trans. Amer. Math. Soc., **363** (2011), 4777-4799.

[3] F. HadjSelem, *Radial solutions with prescribed numbers of zeros for the nonlinear Schr$\ddot{0}$dinger equation with harmonic potential*, Nonlinearity, **24** (2011), 1795-1819.

[4] F. HadjSelem and H. Kikuchi, *Existence and non-existence of solution for semilinear elliptic equation with harmonic potential and Sobolev critical/supercritical nonlinearities*, J. Math. Anal. Appl., **387** (2012), 746-754.

[5] M. Hirose and M. Ohta, *Structure of positive radial solutions to scalar field equations with harmonic potential*, J. Differential Equations, **178** (2002), 519-540.

[6] M. Hirose and M. Ohta, *Uniqueness of positive solutions to scalar field equations with harmonic potential*, Funkcial. Ekvac., **50** (2007), 67-100.

[7] D. D. Joseph and T. S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Rational Mech. Anal., **49** (1972/73), 241-269.

[8] Y. Li and W.-M. Ni, *Radial symmetry of positive solution of nonlinear elliptic equations in* $\mathbb{R}^N$, Comm. Partial Differential Equations, **18** (1993), 1043–1054.

[9] F. Merle and L. A. Peletier, *Positive solutions of elliptic equations involving supercritical growth*, Proceeding of the Royal Society of Edingburgh, **118A** (1991), 49–62.

[10] W.-M. Ni and J. Serrin, *Existence and nonexistence theorems for ground states of quasilinear partial differential equations. The aumalous case*, Atti dei Convegni Lincei **77**, Roma, 6-9 maggio 1985.

[11] X. Wang, *On the Cauchy problem for reaction-diffusion equations*, Trans. Amer. Math. Soc., **337** (1993), 549–590.