The Existence and Stability of Spike Equilibria in the One-Dimensional Gray-Scott Model: The Low Feed-Rate Regime

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Abstract

In a singularly perturbed limit of small diffusivity ε of one of the two chemical species, equilibrium spike solutions to the Gray-Scott model on a bounded one-dimensional domain are constructed asymptotically using the method of matched asymptotic expansions. The equilibria that are constructed are symmetric k-spike patterns where the spikes have equal heights. Two distinguished limits are considered: the low feed-rate regime and the intermediate regime. In the low feed-rate regime, the solution branches of k-spike equilibria are found to have a saddle-node bifurcation structure. The stability properties of these branches of solutions are analyzed with respect to the large eigenvalues λ in the spectrum of the linearization. These eigenvalues, which have the property that $\lambda = O(1)$ as $\varepsilon \to 0$, govern the stability of the solution on an $O(1)$ time-scale. Precise conditions, in terms of the non-dimensional parameters, for the stability of symmetric k-spike equilibrium solutions with respect to this class of eigenvalues are obtained. In the low feed-rate regime, it is shown that a large eigenvalue instability leads either to a competition instability, whereby certain spikes in a sequence are annihilated, or an oscillatory instability (typically synchronous) of the spike amplitudes as a result of a Hopf bifurcation. In the intermediate regime, it is shown that only oscillatory instabilities are possible, and a scaling-law determining the onset of such instabilities is derived. Detailed numerical simulations are performed to confirm the results of the stability theory. It is shown that there is an equivalence principle between spectral properties of the Gray-Scott model in the low feed-rate regime and the Gierer-Meinhardt model of morphogenesis. Finally, our results are compared with previous analytical work on the Gray-Scott model.

1 Introduction

We study the existence and stability of equilibrium spike patterns in the one-dimensional Gray-Scott model in the low feed-rate regime. The Gray-Scott system, introduced in [8], models an irreversible reaction involving two reactants in a gel reactor, where the reactor is maintained in contact with a

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reservoir of one of the two chemicals in the reaction. In nondimensional variables, this system is

$$V_T = D_v V_{XX} - (F+k)V + UV^2\,, \qquad 0 < X < L\,, \quad T > 0\,, \tag{1.1a}$$

$$U_T = D_u U_{XX} + F(1-U) - UV^2\,, \qquad 0 < X < L\,, \quad T > 0\,, \tag{1.1b}$$

$$U_X = V_X = 0\,, \qquad X = 0, L\,. \tag{1.1c}$$

Here $D_u > 0$, $D_v > 0$ are the constant diffusivities, $F > 0$ is the feed rate, and $k > 0$ is a reaction-time constant. For various ranges of these parameters, (1.1) and its two-dimensional counterpart, are known to posses a rich solution structure including the existence of stable standing pulses, the propagation of traveling waves, pulse-replication behavior, and spatio-temporal chaos (cf. [2]–[5], [6], [12], [15], [16], [20], [21], [22], [24], [26], [27], [28], [29], [30], [31], and [34]).

We will analyze (1.1) in the singularly perturbed limit where D_v/D_u is asymptotically small. In our formulation, it is convenient to introduce the change of variables as used in [22],

$$v = V/\sqrt{F}\,, \qquad x = -1 + 2X/L\,, \qquad t = (F+k)T\,. \tag{1.2}$$

This leads to the alternative dimensionless system

$$v_t = \varepsilon^2 v_{xx} - v + Auv^2\,, \qquad -1 < x < 1\,, \quad t > 0\,, \tag{1.3a}$$

$$\tau u_t = D u_{xx} + (1-u) - uv^2 \qquad -1 < x < 1\,, \quad t > 0\,, \tag{1.3b}$$

$$v_x(\pm 1, t) = u_x(\pm 1, t) = 0\,; \qquad v(x,0) = v_0(x)\,, \qquad u(x,0) = u_0(x)\,. \tag{1.3c}$$

Here $A > 0$, D, $\tau > 1$, and $\varepsilon \ll 1$, are defined in terms of D_u, D_v, L, F, and k, by

$$D \equiv \frac{4D_u}{FL^2}\,, \qquad \varepsilon^2 \equiv \frac{4D_v}{L^2(F+k)}\,, \qquad \tau \equiv \frac{F+k}{F}\,, \qquad A \equiv \frac{\sqrt{F}}{F+k}\,. \tag{1.4}$$

The non-dimensional system (1.3) is particularly convenient in that it shows that, for $\varepsilon \ll 1$, the construction of equilibrium solutions depends only on the two parameters A and D, while the reaction-time constant $\tau > 1$ only influences the stability of these solutions. The parameter D measures the effect of the finite domain and the strength of the inter-spike interactions. For a k-spike pattern, the finite domain and the inter-spike interactions are only significant when $k\sqrt{D} = O(1)$. When $k\sqrt{D} \ll 1$, an equilibrium k-spike pattern for (1.3) is composed, to leading order, of k identical copies of a one-spike solution for the infinite-line problem, where (1.3) is defined on $-\infty < x < \infty$.

We will analyze the existence and stability of equilibrium k-spike patterns for (1.3) in the limit $\varepsilon \to 0$ and for $D = O(1)$, where the finite domain and inter-spike coupling cannot be neglected. There are three regimes for A where different behaviors are obtained. For the parameter range $A = O(\varepsilon^{1/2})$, referred to here as the low feed-rate regime, there is a saddle-node bifurcation structure of equilibrium k-spike patterns, and we find that the stability of these solutions depends intricately on A, D, and τ. For the intermediate regime, where $O(\varepsilon^{1/2}) \ll A \ll O(1)$, there are certain scaling laws in terms of a universal nonlocal eigenvalue problem that determine the stability of equilibrium spike patterns. In this regime, the finite domain and inter-spike coupling do not play a central role. Finally, in the regime where $A = O(1)$, the equilibrium spike patterns again exhibit a saddle-node bifurcation structure, and this regime is intimately connected with a pulse-splitting behavior of spike patterns. In this regime, which is studied in the companion paper [13], the effect of the finite domain is crucial in the analysis.

We now summarize some of our findings for the low feed-rate regime. In the low feed-rate regime, we introduce new variables $\mathcal{A}$ and ν defined by

$$A = \varepsilon^{1/2}\mathcal{A}\,, \qquad v = \varepsilon^{-1/2}\nu\,. \tag{1.5}$$

In terms of (1.5), (1.3) is transformed to

$$\nu_t = \varepsilon^2 \nu_{xx} - \nu + \mathcal{A}u\nu^2\,, \qquad -1 < x < 1\,, \quad t > 0\,, \tag{1.6a}$$

$$\tau u_t = D u_{xx} + (1-u) - \varepsilon^{-1}u\nu^2 \qquad -1 < x < 1\,, \quad t > 0\,, \tag{1.6b}$$

$$\nu_x(\pm 1, t) = u_x(\pm 1, t) = 0\,; \qquad \nu(x,0) = \varepsilon^{1/2}v_0(x)\,, \qquad u(x,0) = u_0(x)\,. \tag{1.6c}$$

We construct k-spike equilibrium patterns where the spikes in ν have a common amplitude. For these symmetric k-spike equilibria, we show that for each $k = 1, 2\ldots$ there are two branches of such solutions for (1.6) when $\mathcal{A} > \mathcal{A}_{ke}$. These branches are referred to as the small and large solution branches. They meet at the saddle-node bifurcation value

$$\mathcal{A}_{ke} \equiv \sqrt{\frac{12\theta_0}{\tanh(\theta_0/k)}}\,, \qquad \theta_0 \equiv D^{-1/2}\,. \tag{1.7}$$

These solution branches are conveniently parameterized in terms of a parameter s, with $0 < s < \infty$, defined by

$$s = \frac{1 - U_\pm}{U_\pm}\,, \qquad U_\pm = \frac{1}{2}\left[1 \pm \sqrt{1 - \frac{\mathcal{A}_{ke}^2}{\mathcal{A}^2}}\right]\,. \tag{1.8}$$

In Fig. 1, for $D = 0.75$, we plot a bifurcation diagram of the norm of ν versus $\mathcal{A}$ for the symmetric k-spike solution branches for $k = 1, \ldots, 4$ (see (2.13) below for the definition of the norm). The asymptotic construction of these k-spike equilibrium patterns is given in §2.

In §3 we derive a nonlocal eigenvalue problem that governs the stability of the symmetric k-spike equilibrium solutions constructed in §2 with respect to the eigenvalues of order $O(1)$ in the spectrum of the linearization. These eigenvalues, referred to here as the large eigenvalues, are associated with the initiation of profile instabilities, whereby the spike amplitudes will either oscillate, typically with a common frequency and phase, or else undergo a competition instability leading to the monotonic annihilation of spikes in a spike sequence. From this nonlocal eigenvalue problem, we prove in Proposition 3.10 that the large solution branch is unconditionally unstable for any $\tau > 0$ and $D > 0$. The stability properties of the small solution branch is significantly more intricate. In particular, for each $k = 1, 2\ldots$ and $D > 0$ fixed, we prove that there exists a threshold value $\mathcal{A}_{kL}$, with $\mathcal{A}_{kL} > \mathcal{A}_{ke}$, such that the small solution branch is stable with respect to profile instabilities for $\mathcal{A} > \mathcal{A}_{kL}$ provided that $\tau < \tau_{hL}$. An explicit formula for $\mathcal{A}_{kL}$ is given below in (3.27b). For the range $\mathcal{A} > \mathcal{A}_{kL}$, there is a Hopf bifurcation as τ

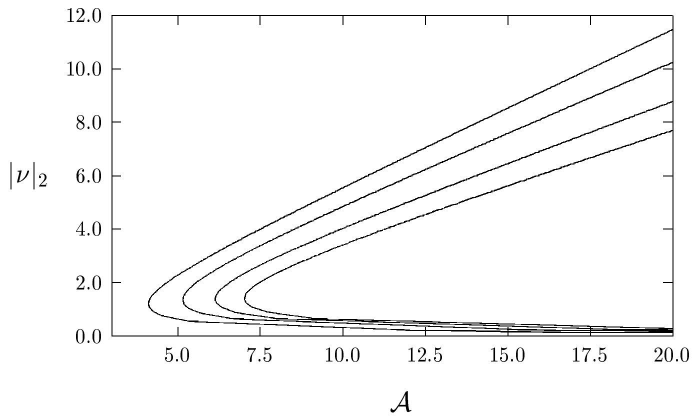
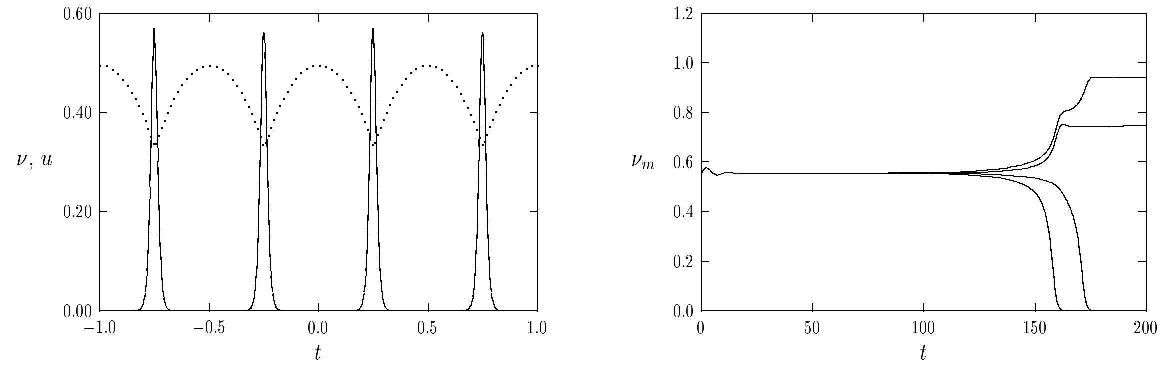


Figure 1: Bifurcation diagram of symmetric k-spike patterns in the low feed-rate regime $A = \mathcal{A}\varepsilon^{1/2}$ for $D = 0.75$ and $k = 1, 2, 3, 4$. The saddle-node values $\mathcal{A}_{ke}$ increase with k.

exceeds some critical value τ_{hL}. This bifurcation typically leads to a synchronous oscillatory instability in the spike amplitudes. The precise results are given below in Propositions 3.11 and 3.13–3.15. On the range $\mathcal{A}_{ke} < \mathcal{A} < \mathcal{A}_{kL}$ for the small solution branch, the spectrum of the linearization of (1.6) around a symmetric k-spike equilibrium solution contains at least one (unstable) real and positive eigenvalue. The existence of real positive eigenvalues in this parameter range is the mechanism for the initiation of competition instabilities whereby certain spikes in a spike sequence are annihilated. Precise results for the spectrum of the linearization for this range of $\mathcal{A}$ is given below in Proposition 3.12.

We now illustrate these instabilities for (1.6) for the parameter set $k = 4$, $D = 0.1$, and $\varepsilon = 0.01$ (this is Example 3 of §3.3). For this example, our theory yields $\mathcal{A}_{kL} = 8.127$. For the value $\mathcal{A} = 8.0$, in Fig. 2(a) we show a 1% perturbation in the equilibrium pattern, which we use as the initial condition for (1.6). Since $\mathcal{A} < \mathcal{A}_{kL}$, our theory predicts the initiation of a spike competition process. In Fig. 2(b) we plot the spike amplitudes, defined as the values of ν at its local maxima, versus time showing a spike competition process leading to the annihilation of two spikes. In Fig. 3(a) where $\mathcal{A} = 8.302 > \mathcal{A}_{kL}$ we show a synchronous decaying oscillation in the spike amplitudes when $\tau = 3.8$. In Fig. 3(b), where $\tau = 4.1$ exceeds the Hopf bifurcation value, we show a synchronous oscillatory instability leading to the simultaneous collapse of the four spikes. A mathematical theory to characterize the initiation of these fast instabilities is given in §3. Competition and synchronous oscillatory instabilities in the low feed-rate regime, which occur as a result of the finite domain and a strong inter-spike coupling mediated by the concentration field u, have not been previously reported for the Gray-Scott model (1.6).

Although it is difficult to translate detailed spectral results into precise physical mechanisms for instability, a qualitative mechanism for the two types of instabilities is as follows. For a given value of $\mathcal{A}$ and for a fixed small value of τ, a competition instability occurs when either D is too large, or equivalently when the spikes are too close together. For D large, the inhibitor u diffuses over a large



(a) The initial condition

(b) ν_m versus t

Figure 2: The parameters are $k = 4$, $D = 0.1$, $\mathcal{A} = 8.0$, $\varepsilon = 0.01$, and $\tau = 2.0$. Left figure: the initial condition for ν (solid curve) and u (dashed curve). Right figure: The spike amplitudes ν_m. The second and fourth spikes are annihilated by a spike competition. Of the two remaining spikes, the third spike has the largest amplitude.

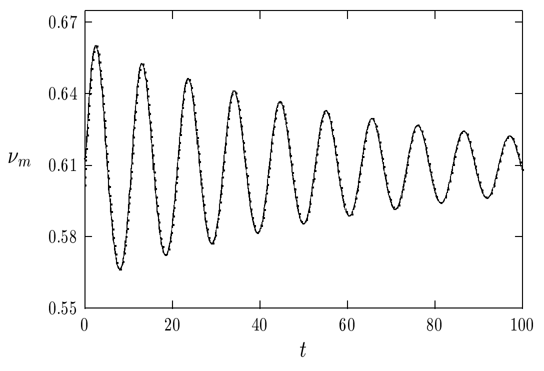
spatial extent thereby preventing the occurrence of another spike. The instability for closely spaced spikes is probably a counterpart of the spike *over-crowding* instability observed numerically in the weak-interaction regime for the two-dimensional Gray-Scott model in [28], and observed experimentally in [15] in the ferrocyanide-iodate-sulphite reaction. The positive real eigenvalue responsible for competition instabilities is the last real eigenvalue that remains in the right half-plane as D is decreased. Recall that for the shadow Gray-Scott problem where $D = \infty$, and for τ small, it is well-known that a k-spike solution is unstable due to exactly $k-1$ real positive eigenvalues. This is discussed in Remark 3.1 below. Alternatively, for a fixed D, oscillatory instabilities occur when τ is sufficiently large. For τ large, the inhibitor field u responds sluggishly to small temporal changes in the spike pattern. This leads to an oscillatory feedback loop typical in delay-type differential equations.

Similar types of competition and synchronous oscillatory instabilities of equilibrium spike patterns have been analyzed in [35] for the Gierer-Meinhardt (GM) model of morphogenesis introduced in [7]. This model, which has been widely used to model localization processes in nature, such as cell differentiation and morphogenesis (cf. [9]), biological pattern formation (cf. [18]), and the formation of sea-shell patterns (cf. [19]), can be written in dimensionless form for an activator a and an inhibitor h concentration field as

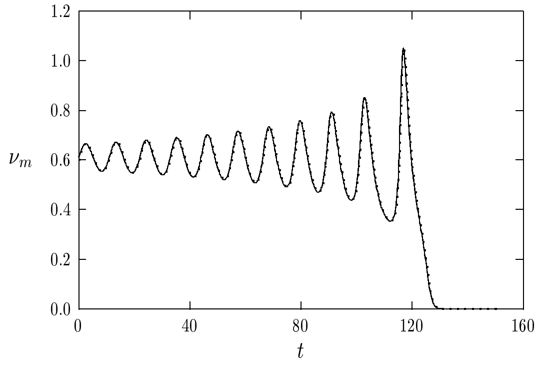
$$a_t = \varepsilon^2 a_{xx} - a + \frac{a^p}{h^q}, \qquad -1 < x < 1, \quad t > 0, \tag{1.9a}$$

$$\tau h_t = D h_{xx} - h + \varepsilon^{-1} \frac{a^m}{h^s}, \qquad -1 < x < 1, \quad t > 0, \tag{1.9b}$$

$$a_x(\pm 1, t) = h_x(\pm 1, t) = 0; \qquad a(x,0) = a_0(x), \qquad h(x,0) = h_0(x). \tag{1.9c}$$



(a) ν_m versus t



(b) ν_m versus t

Figure 3: Spike amplitudes for $k = 4$, $D = 0.1$, $\mathcal{A} = 8.302$, and $\varepsilon = 0.01$. Left figure: synchronous decaying oscillation in the spike amplitudes for $\tau = 3.8$. Right figure: synchronous oscillatory instability for $\tau = 4.1$, leading to a collapse of the four spikes. In these figures, the amplitudes of the spikes trace out identical trajectories.

Here $0 < \varepsilon^2 \ll 1$, $D > 0$, and $\tau \geq 0$, represent the activator diffusivity, inhibitor diffusivity, and reaction-time constant, respectively. The usual assumption on the exponents (p, q, m, s) of the GM model (cf. [7]) are that they satisfy

$$p > 1\,, \qquad q > 0\,, \qquad m > 1\,, \qquad s \geq 0\,, \qquad \text{with} \qquad \zeta \equiv \frac{qm}{(p-1)} - (s+1) > 0\,. \tag{1.10}$$

The relationship between oscillatory and competition instabilities in the Gray-Scott and Gierer-Meinhardt models is made precise in Proposition 3.3 below where we prove that the nonlocal eigenvalue problem for instabilities in the Gray-Scott model in the low feed-rate regime is identical to the nonlocal eigenvalue problem for the Gierer-Meinhardt model with exponent set $(p, q, m, s) = (2, s, 2, s)$, where s is defined in (1.8). This spectral equivalence principle is a new result, and allows us to use many of the detailed spectral results derived in [35] for the GM model.

The *intermediate regime*, defined by $O(1) \ll \mathcal{A} \ll O(\varepsilon^{-1/2})$, is analyzed in §4. In this regime, there are no competition instabilities, and we show in Proposition 4.2 that there is a universal nonlocal eigenvalue problem, independent of D and k, that governs the stability of a symmetric k-spike equilibrium solution with respect to oscillatory instabilities. In terms of a certain critical value of this universal nonlocal eigenvalue problem, a scaling law for the Hopf bifurcation value of τ is derived in Proposition 4.3 that yields the Hopf bifurcation value in terms of D and k. In this regime for $\mathcal{A}$, the Hopf bifurcation value is $\tau = O(\mathcal{A}^4)$, and so there are no oscillatory instabilities when $\tau = O(1)$.

We emphasize that, since we only consider eigenvalues for which $\lambda = O(1)$ as $\varepsilon \to 0$, our stability results are valid only for time intervals of $O(1)$ as $\varepsilon \to 0$. To obtain stability results for unbounded time intervals, one must consider the stability of the symmetric k-spike pattern with respect to the small eigenvalues of order $\lambda = O(\varepsilon^2)$ in the spectrum of the linearization. Additional stability thresholds, with respect to the dimensionless parameters, are obtained for this class of eigenvalues. These eigenvalues are closely related to the existence of asymmetric k-spike patterns. This problem is studied in [14].

The previous equilibrium and spectral results for the Gray-Scott model in [4], [5], [6], [2], and [3], have been based on a different nondimensionalization of the Gray-Scott model. This alternative formulation is presented in detail in §5, where we give a precise discussion of the relationship between our results and theirs. In §4 and §3.4 we also relate our results to those in [21]. The main point is that our results have a clear overlap with these previous results only in the intermediate parameter regime for $\mathcal{A}$ and for the infinite-line problem. Our analysis of synchronous spike oscillations and competition instabilities in the low feed-rate regime, which occur on a finite domain with $k\sqrt{D} \ll 1$, is new. In §5 we also list a few open problems.

2 Symmetric k-Spike Equilibria: $\mathcal{A} = O(1)$

For $\varepsilon \to 0$, and with $\mathcal{A} = O(1)$ and $D = O(1)$, we construct a symmetric k-spike equilibrium solution to (1.6) using the method of matched asymptotic expansions. The spike locations x_j, for $j = 1, \dots, k$, for this pattern satisfy

$$x_j = -1 + \frac{(2j-1)}{k}\,, \qquad j = 1, \dots, k\,. \tag{2.1}$$

For a symmetric spike pattern the spikes have equal height. Hence, we have $u(x_j) = U$, where U is independent of j.

Since the asymptotic construction of a symmetric k-spike pattern to (1.6) for $\varepsilon \ll 1$ is similar to that done in [11] for the Gierer-Meinhardt model, we will only sketch the derivation of the result. In §4, where we consider the intermediate regime $O(1) \ll \mathcal{A} \ll O(\varepsilon^{-1/2})$ we will give a detailed derivation of the equilibrium solutions for (1.3) and (1.6) and include formal error estimates associated with the inner and outer expansions.

In the inner region near the j^{th} spike, we let $y = \varepsilon^{-1}(x - x_j)$. In each inner region, we obtain that $u \sim U + O(\varepsilon)$. Therefore, from (1.6a), the leading-order inner solution for ν is

$$\nu \sim \frac{1}{\mathcal{A}U} w(y)\,, \tag{2.2}$$

where $w(y) = \frac{3}{2}\text{sech}^2\,(y/2)$ is the homoclinic solution to

$$w'' - w + w^2 = 0\,, \qquad -\infty < y < \infty\,, \tag{2.3a}$$

$$w \to 0 \quad \text{as} \quad |y| \to \infty\,; \qquad w'(0) = 0\,, \quad w(0) > 0\,. \tag{2.3b}$$

In the outer region, defined away from $O(\varepsilon)$ regions near the union of the x_j, for $j = 1, \dots, k$, we obtain that ν is exponentially small and that the term $\varepsilon^{-1}u\nu^2$ in (1.6b) can be approximated by a delta function. Consequently, the outer solution for u satisfies

$$Du_{xx} + (1-u) - \frac{6}{\mathcal{A}^2 U} \sum_{j=1}^{k} \delta(x - x_j) = 0\,, \tag{2.4a}$$

$$u_x(-1) = u_x(1) = 0\,. \tag{2.4b}$$

In obtaining (2.4a), we used $\int_{-\infty}^{\infty} w^2\, dy = 6$. The solution to (2.4) is

$$u(x) = 1 - \frac{6}{\mathcal{A}^2 U} \sum_{j=1}^{k} G(x; x_j)\,, \tag{2.5}$$

where $G(x; x_j)$ is the Green's function, satisfying

$$DG_{xx} - G = -\delta(x - x_j)\,, \qquad -1 < x < 1\,; \qquad G_x(\pm 1; x_j) = 0\,. \tag{2.6}$$

A simple calculation gives,

$$G(x; x_j) = \begin{cases} g_j \cosh\left[\theta_0(1+x)\right] / \cosh\left[\theta_0(1+x_j)\right], & -1 < x < x_j\,, \\ g_j \cosh\left[\theta_0(1-x)\right] / \cosh\left[\theta_0(1-x_j)\right], & x_j < x < 1\,, \end{cases} \tag{2.7a}$$

where

$$g_j \equiv \frac{1}{\sqrt{D}} \left(\tanh\left[\theta_0(1-x_j)\right] + \tanh\left[\theta_0(1+x_j)\right]\right)^{-1}\,, \qquad \theta_0 \equiv D^{-1/2}\,. \tag{2.7b}$$

We define $a_g \equiv \sum_{i=1}^{k} G(x_j; x_i)$, where the spike locations satisfy (2.1). A simple calculation using (2.7) shows that a_g is independent of j, and that

$$a_g \equiv \sum_{i=1}^{k} G(x_j; x_i) = \left[2\sqrt{D}\tanh\left(\theta_0/k\right)\right]^{-1}\,. \tag{2.8}$$

Evaluating (2.5) at $x = x_j$, where $u(x_j) = U$, we obtain a quadratic equation for U

$$U(U-1) = -\frac{6a_g}{\mathcal{A}^2}\,. \tag{2.9}$$

In this way, we obtain the following result for symmetric k-spike equilibrium solutions to (1.6):

<u>**Proposition 2.1**</u>: *Let $\varepsilon \to 0$, with $\mathcal{A} = O(1)$ and $D = O(1)$ in (1.6). Then, when $\mathcal{A} > \mathcal{A}_{ke}$, there are two symmetric k-spike equilibrium solutions to (1.6) given asymptotically by*

$$\nu_{\pm}(x) \sim \frac{1}{\mathcal{A}U_{\pm}} \sum_{j=1}^{k} w\left[\varepsilon^{-1}(x - x_j)\right]\,, \qquad u_{\pm}(x) \sim 1 - \frac{(1 - U_{\pm})}{a_g} \sum_{j=1}^{k} G(x; x_j)\,. \tag{2.10}$$

We label u_+, ν_+ and u_-, ν_- as the large and small solution, respectively. In (2.10), w and G satisfy (2.3) and (2.6), respectively. In addition, $U_{\pm}$ are the roots of (2.9) given by

$$U_{\pm} = \frac{1}{2}\left[1 \pm \sqrt{1 - \frac{\mathcal{A}_{ke}^2}{\mathcal{A}^2}}\,\right]\,, \tag{2.11}$$

where the existence threshold is $\mathcal{A}_{ke} = \sqrt{24a_g}$, *so that*

$$\mathcal{A}_{ke} \equiv \sqrt{\frac{12\theta_0}{\tanh(\theta_0/k)}}\,, \qquad \theta_0 \equiv D^{-1/2}\,. \tag{2.12}$$

The existence threshold (2.12) for a finite domain with k spikes is a new result. It shows that we have a saddle-node bifurcation of symmetric k-spike equilibria on the finite line when $\mathcal{A} = \mathcal{A}_{ke}$. These existence thresholds correspond to the fold points in Fig. 1 separating the upper and lower branches of symmetric k-spike equilibria (solid curves in this figure). As a remark, we notice that $\mathcal{A}_{ke}$ is an increasing function of k. For $D \ll 1$ and with $\sqrt{D}k \ll 1$, we get that $\mathcal{A}_{ke} \sim \sqrt{12}D^{-1/4}$. Thus, when $D \ll 1$, the saddle-node bifurcation value $\mathcal{A}_{ke}$ for symmetric k-spike equilibria is roughly independent of k provided that $\sqrt{D}k \ll 1$.

To display our results graphically, it is convenient to define a norm of ν by

$$|\nu|_2 \equiv \left(\varepsilon^{-1}\int_{-1}^{1} \nu^2\,dx\right)^{1/2}\,. \tag{2.13}$$

Using (2.10) and (2.11), we calculate

$$|\nu|_2 \sim \frac{2\sqrt{6k}}{\mathcal{A}}\left[1 \pm \sqrt{1 - \frac{\mathcal{A}_{ke}^2}{\mathcal{A}^2}}\right]^{-1}\,. \tag{2.14}$$

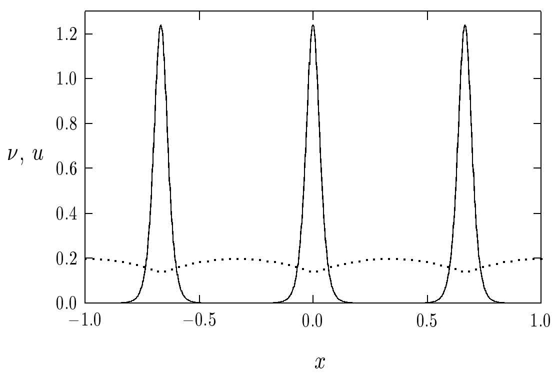
For $\mathcal{A} = 9.0$ and $\varepsilon = 0.02$, in Fig. 4(a) and Fig. 4(b) we plot the small solution when $D = 0.75$ and $D = 0.1$, respectively. Notice that as D decreases, the Green's function in (2.10) decays more rapidly away from the spike locations. Hence, for D small, u should approach the asymptotic value $u = 1$ in the outer regions. This asymptotic value $u = 1$ in the outer region is also relevant to the infinite-line problem, where we seek a one-spike solution to (1.6) on $-\infty < x < \infty$. However, as seen from Fig. 4(b), even with $D = 0.1$, u is not so close to its asymptotic value in the outer regions. Therefore, this suggests that there is range of values of D for which boundary effects due to the finite domain will be important in the analysis.

The classification of large and small solution refers to high and low concentrations of u in the core of the spike. Smaller concentrations of u in the core of the spike generate larger amplitudes for ν. Hence, each upper branch (upper solid curve) in Fig. 1 corresponds to the small solution, while each lower branch corresponds to the large solution. As shown in §3–§4, a particularly convenient way to parameterize these symmetric k-spike solution branches is to introduce a parameter s defined by

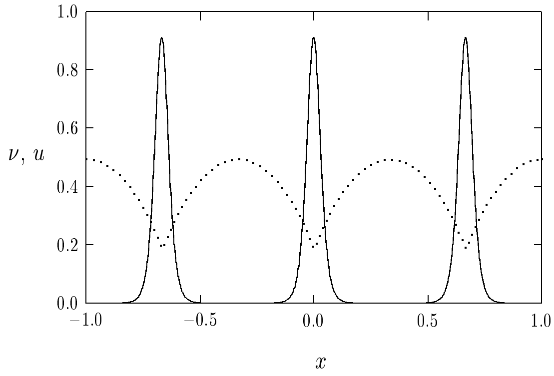
$$s = \frac{1 - U_{\pm}}{U_{\pm}}\,, \qquad 0 < s < \infty\,. \tag{2.15a}$$

Using (2.11), we can write $\mathcal{A}$ in terms of s as

$$\frac{\mathcal{A}}{\mathcal{A}_{ke}} = \frac{(1+s)}{2\sqrt{s}}\,. \tag{2.15b}$$



(a) $D = 0.75$



(b) $D = 0.1$

Figure 4: Left figure: three-spike small solution when $D = 0.75$, $\mathcal{A} = 9.0$, and $\varepsilon = 0.02$. Right figure: three-spike small solution when $D = 0.1$, $\mathcal{A} = 9.0$, and $\varepsilon = 0.02$.

Hence the small solution u_-, ν_- corresponds to the range $1 < s < \infty$, while the large solution corresponds to $0 < s < 1$. The existence threshold $\mathcal{A}_{ke}$ corresponds to $s = 1$.

To analyze the stability of symmetric k-spike equilibrium solutions we let

$$u(x,t) = u_\pm(x) + e^{\lambda t}\eta(x)\,, \qquad \nu(x,t) = \nu_\pm(x) + e^{\lambda t}\phi(x)\,, \tag{2.16}$$

where $\eta \ll 1$ and $\phi \ll 1$. Substituting (2.16) into (1.6) and linearizing, we obtain the eigenvalue problem

$$\varepsilon^2\phi_{xx} - \phi + 2\mathcal{A}u_\pm\nu_\pm\phi + \mathcal{A}\eta\nu_\pm^2 = \lambda\phi\,, \qquad -1 < x < 1\,, \tag{2.17a}$$

$$D\eta_{xx} - \eta - \varepsilon^{-1}\nu_\pm^2\eta - 2\varepsilon^{-1}u_\pm\nu_\pm\phi = \tau\lambda\eta\,, \qquad -1 < x < 1\,, \tag{2.17b}$$

$$\phi_x(\pm 1) = \eta_x(\pm 1) = 0\,. \tag{2.17c}$$

In §3 we analyze the spectrum of (2.17) corresponding to those eigenfunctions that are not locally odd functions near each spike. The corresponding eigenvalues, which determine the stability of the symmetric k-spike profile with respect to instabilities occurring on a fast $O(1)$ time-scale, are referred to as the large eigenvalues. We will show that the large solution u_+ and ν_+ is always unstable. For the small solution, we show that there are two different types of instabilities that can occur depending on the parameter ranges of D, $\mathcal{A}$, τ, and k. A competition instability, whereby spikes in a spike sequence are destroyed, can occur only for $k > 1$ when $\mathcal{A}$ is close to $\mathcal{A}_{ke}$. This instability results from a certain eigenvalue on the positive real axis $\mathrm{Re}(\lambda) > 0$. On the other hand, oscillatory instabilities in the amplitudes of the spikes as a result of a Hopf bifurcation can occur for any $\mathcal{A} > \mathcal{A}_{ke}$ when τ is sufficiently large.

3 Large Eigenvalues: Fast Profile Instabilities for $\mathcal{A} = O(1)$

In this section we analyze the stability of the symmetric k-spike equilibrium solutions of Proposition 2.1 with respect to the large eigenvalues.

3.1 The Nonlocal Eigenvalue Problem

We begin by deriving a nonlocal eigenvalue problem that is central to the stability analysis. This derivation is similar to that done in [35] for the GM model (1.9), and was originally motivated by the approach used in [25] to analyze pulses in the Fitzhugh-Nagumo system. As in [35], we look for a localized eigenfunction for ϕ in (2.17) in the form

$$\phi(x) \sim \sum_{j=1}^{k} c_j \Phi\left[\varepsilon^{-1}(x - x_j)\right] , \tag{3.1}$$

for some coefficients c_j. These coefficients are determined below to be related to eigenvectors of a certain matrix eigenvalue problem. The large eigenvalues are characterized by eigenfunctions $\Phi(y)$ for which $\int_{-\infty}^{\infty} w(y)\Phi(y)\, dy \neq 0$.

Since ϕ is localized near each x_j, the coefficients in (2.17b) can be approximated by certain Dirac masses. In particular, using (3.1) and (2.10), we obtain near $x = x_j$ that

$$\varepsilon^{-1}\nu_{\pm}^{2} \sim \frac{6}{\mathcal{A}^2 U_{\pm}^2}\delta(x - x_j) , \tag{3.2a}$$

$$2\varepsilon^{-1}u_{\pm}\nu_{\pm}\phi \sim \frac{2c_j}{\mathcal{A}}\left(\int_{-\infty}^{\infty} w(y)\Phi(y)\, dy\right) \delta(x - x_j) . \tag{3.2b}$$

Here $\delta(x)$ is the delta function, and we have used $\int_{-\infty}^{\infty} w^2\, dy = 6$. Substituting (3.2) into (2.17b), we obtain that η satisfies

$$D\eta_{xx} - \left(1 + \tau\lambda + \frac{6}{\mathcal{A}^2 U_{\pm}^2}\sum_{i=1}^{k}\delta(x - x_i)\right)\eta = \frac{2}{\mathcal{A}}\left(\int_{-\infty}^{\infty} w\Phi\, dy\right)\sum_{i=1}^{k} c_i\delta(x - x_i) , \quad |x| < 1 , \tag{3.3a}$$

$$\eta_x(-1) = \eta_x(1) = 0 . \tag{3.3b}$$

This problem for η is equivalent to

$$D\eta_{xx} - (1 + \tau\lambda)\,\eta = 0 , \qquad \eta_x(\pm 1) = 0 , \tag{3.4a}$$

$$[\eta]_i = 0 , \qquad i = 1, .., k , \tag{3.4b}$$

$$[D\eta_x]_i = -\omega_i + \frac{6}{\mathcal{A}^2 U_{\pm}^2}\eta(x_i) , \qquad i = 1, .., k . \tag{3.4c}$$

In (3.4), we have defined $[\xi]_i \equiv \xi(x_{i+}) - \xi(x_{i-})$, and ω_i by

$$\omega_i \equiv -\frac{2c_i}{\mathcal{A}}\int_{-\infty}^{\infty} w(y)\Phi(y)\, dy . \tag{3.4d}$$

To determine the eigenvalue problem for λ, we first need to compute $\eta(x_i)$ from (3.4). To do so, we solve (3.4a) on each subinterval and use the jump conditions (3.4b) and (3.4c) to patch the solution together across each subinterval. This calculation results in the matrix problem

$$\mathcal{B}\boldsymbol{\eta} = \left[(1+\tau\lambda)D\right]^{-1/2}\boldsymbol{\omega}\,, \qquad \boldsymbol{\omega} = -\frac{2\boldsymbol{c}}{\mathcal{A}}\int_{-\infty}^{\infty} w(y)\Phi(y)\,dy\,. \tag{3.5}$$

Here we have defined the vectors $\boldsymbol{\omega}$, $\boldsymbol{c}$, and $\boldsymbol{\eta}$, by $\boldsymbol{\omega}^t = (\omega_1, \ldots, \omega_k)$, $\boldsymbol{c}^t = (c_1, \ldots, c_k)$, and $\boldsymbol{\eta}^t = (\eta_1, \ldots, \eta_k)$, where t denotes transpose. The matrix $\mathcal{B}$ in (3.5) is given in terms of a tridiagonal matrix $\mathcal{B}_0$ by

$$\mathcal{B} = \mathcal{B}_0 + \frac{6}{\mathcal{A}^2 U_\pm^2\sqrt{(1+\tau\lambda)D}}I\,. \tag{3.6}$$

Here I is the $k\times k$ identity matrix, and $\mathcal{B}_0$ has the form

$$\mathcal{B}_0 \equiv \begin{pmatrix} d_\lambda & f_\lambda & 0 & \cdots & 0 & 0 & 0 \\ f_\lambda & e_\lambda & f_\lambda & \cdots & 0 & 0 & 0 \\ 0 & f_\lambda & e_\lambda & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & e_\lambda & f_\lambda & 0 \\ 0 & 0 & 0 & \cdots & f_\lambda & e_\lambda & f_\lambda \\ 0 & 0 & 0 & \cdots & 0 & f_\lambda & d_\lambda \end{pmatrix}, \tag{3.7a}$$

with matrix entries

$$d_\lambda \equiv \coth\left(\frac{2\theta_\lambda}{k}\right) + \tanh\left(\frac{\theta_\lambda}{k}\right)\,; \qquad e_\lambda \equiv 2\coth\left(\frac{2\theta_\lambda}{k}\right)\,; \qquad f_\lambda \equiv -\operatorname{csch}\left(\frac{2\theta_\lambda}{k}\right)\,. \tag{3.7b}$$

In (3.7b), θ_λ is the principal branch of the square root function defined by

$$\theta_\lambda \equiv \theta_0\sqrt{1+\tau\lambda}\,, \qquad \theta_0 \equiv D^{-1/2}\,. \tag{3.7c}$$

Next, we substitute (3.1) and (3.6) into (2.17a). This yields the nonlocal eigenvalue problem for $\Phi(y)$, for $i = 1, .., k$,

$$c_i\left(\Phi'' - \Phi + 2w\Phi\right) - \frac{12w^2}{\mathcal{A}^2U_\pm^2}\left[(1+\tau\lambda)\,D\right]^{-1/2}\left(\mathcal{B}^{-1}\boldsymbol{c}\right)_i\left(\frac{\int_{-\infty}^{\infty} w\Phi\,dy}{\int_{-\infty}^{\infty} w^2\,dy}\right) = \lambda c_i\Phi\,, \tag{3.8}$$

with $\Phi(y)\to 0$ as $|y|\to\infty$. Therefore, we must calculate the spectrum of the matrix eigenvalue problem

$$\mathcal{B}\boldsymbol{c} = \kappa\boldsymbol{c}\,. \tag{3.9}$$

From (3.6), we have

$$\kappa = \kappa_0 + \frac{6}{\mathcal{A}^2U_\pm^2}\left[(1+\tau\lambda)\,D\right]^{-1/2}\,, \tag{3.10}$$

where κ_0 and $\boldsymbol{c}$ is an eigenpair of $\mathcal{B}_0$. The eigenpairs of $\mathcal{B}_0$ were calculated explicitly in Proposition 2 of [11], where the following result was obtained:

Lemma 3.1: *The eigenvalues κ_{0j}, ordered as $0 < \kappa_{01} < \ldots < \kappa_{0k}$, and the normalized eigenvectors $\mathbf{c}_j$ of $\mathcal{B}_0$ are*

$$\kappa_{0j} = 2\tanh(\theta_\lambda/k) + 2\left[1 - \cos\left(\frac{\pi(j-1)}{k}\right)\right]\operatorname{csch}(2\theta_\lambda/k)\,, \qquad j = 1, \ldots, k\,, \tag{3.11a}$$

$$\mathbf{c}_1^t = \frac{1}{\sqrt{k}}(1, \ldots, 1)\;; \quad c_{l,j} = \sqrt{\frac{2}{k}}\cos\left(\frac{\pi(j-1)}{k}(l - 1/2)\right), \quad j = 2, \ldots, k\,. \tag{3.11b}$$

Here $\mathbf{c}^t$ denotes transpose and $\mathbf{c}_j^t = (c_{1,j}, \ldots, c_{k,j})$.

Substituting (3.9), (3.10), and (3.11), into (3.8), we obtain the following spectral problem for the large $O(1)$ eigenvalues of (2.17):

Proposition 3.2: *Assume that $0 < \varepsilon \ll 1$. Then, with $\Phi = \Phi(y)$, the $O(1)$ eigenvalues of (2.17) satisfy the nonlocal eigenvalue problem*

$$L_0\Phi - \chi w^2\left(\frac{\int_{-\infty}^{\infty} w\Phi\, dy}{\int_{-\infty}^{\infty} w\, dy}\right) = \lambda\Phi\,, \qquad -\infty < y < \infty\,, \tag{3.12a}$$

$$\Phi \to 0\,, \qquad \text{as} \quad |y| \to \infty\,. \tag{3.12b}$$

Here the operator L_0, referred to as the local operator, is defined by

$$L_0\Phi \equiv \Phi'' - \Phi + 2w\Phi\,. \tag{3.12c}$$

The multiplier $\chi = \chi(z;j)$ in (3.12) is given explicitly by

$$\chi = \chi(z;j) \equiv 2s\left(s + \frac{\sqrt{1+z}}{\tanh(\theta_0/k)}\left[\tanh(\theta_\lambda/k) + \frac{(1 - \cos[\pi(j-1)/k])}{\sinh(2\theta_\lambda/k)}\right]\right)^{-1}, \tag{3.13a}$$

where

$$z \equiv \tau\lambda\,, \qquad \theta_\lambda \equiv \theta_0\sqrt{1+z}\,, \qquad \theta_0 \equiv D^{-1/2}\,, \tag{3.13b}$$

and s is defined by

$$s \equiv \frac{1 - U_\pm}{U_\pm}\,. \tag{3.14}$$

Here $U_\pm$ is determined in terms of $\mathcal{A}/\mathcal{A}_{ke}$ by (2.11). The global eigenfunction $\phi(x)$ is given by (3.1), where the coefficients $\mathbf{c}^t = (c_1, \ldots, c_k)$ are the eigenvectors of $\mathcal{B}_0$ given in (3.11b).

We now establish an important spectral equivalence principle between the nonlocal eigenvalue problem (3.12) and a corresponding nonlocal eigenvalue problem derived in Proposition 2.3 of [35] for the GM model (1.9) with exponent set (p, q, m, s). For this general exponent set, Proposition 2.3 of [35] shows that the nonlocal eigenvalue problem for the GM model with exponent set $(2, q, 2, s)$ has exactly the same form as in (3.12), except that χ in (3.13) is to be replaced with

$$\chi = \chi(z;j) \equiv 2q\left(s + \frac{\sqrt{1+z}}{\tanh(\theta_0/k)}\left[\tanh(\theta_\lambda/k) + \frac{(1 - \cos[\pi(j-1)/k])}{\sinh(2\theta_\lambda/k)}\right]\right)^{-1}. \tag{3.15}$$

Therefore, we conclude that the nonlocal eigenvalue problems for the GM model and the Gray-Scott model are identical if we take $q = s$, to get the GM exponent set $(p, q, m, s) = (2, s, 2, s)$, where s is given in (3.14). However, in the GM model, the usual assumption on the exponents (p, q, m, s) are that they satisfy $\zeta \equiv qm/(p-1) - (1+s) > 0$. The spectral results in [35] for the GM model were obtained under this condition. With the exponent set $(2, s, 2, s)$, we calculate $\zeta = s - 1$. Since $0 < s < 1$ corresponds to the large solution, while $s > 1$ corresponds to the small solution, we obtain the following spectral equivalence principle between the Gray-Scott and the GM models:

Proposition 3.3: *In the limit $\varepsilon \ll 1$, consider the large eigenvalues of (2.17). The nonlocal eigenvalue problem for the stability of the small k-spike symmetric equilibrium solution u_-, ν_- of the Gray-Scott problem (1.6) is identical to the related nonlocal eigenvalue problem for the GM model with exponent set $(p, q, m, s) = (2, s, 2, s)$, where $s > 1$ is given in (3.14). The spectral problem for the large solution of the Gray-Scott model is also equivalent to that for a GM model with exponents $(2, s, 2, s)$, except that $\zeta < 0$ in (1.10).*

In the context of the Gray-Scott model, the eigenvalue problem (3.12) is new. The equivalence principle of Proposition 3.3 allows us to immediately analyze the stability of the small solution of the Gray-Scott model by directly appealing to the results of [35] obtained for the GM model with arbitrary exponent set (p, q, m, s) satisfying (1.10). For the large solution u_+, ν_+, we have to extend the analysis in [35] to allow for a GM model with exponent set $(2, s, 2, s)$, but with $\zeta = s - 1 < 0$.

Next, we reformulate (3.12) into a form more amenable to analysis. Let $\psi(y)$ be the solution to

$$L_0 \psi \equiv \psi'' - \psi + 2w\psi = \lambda\psi + w^2 \, ; \qquad \psi \to 0 \quad \text{as} \quad |y| \to \infty \, . \tag{3.16}$$

Then, the eigenfunctions of (3.12) can be written as

$$\Phi = \chi(\tau\lambda; j)\psi J \, , \qquad J \equiv \frac{\int_{-\infty}^{\infty} w\Phi \, dy}{\int_{-\infty}^{\infty} w^2 \, dy} \, . \tag{3.17}$$

We then substitute the formula for Φ into the definition of J. Assuming, as mentioned earlier, that $\int_{-\infty}^{\infty} w\Phi \, dy \neq 0$, we then obtain that the eigenvalues of (3.12) are the union of the zeros of the functions $g_j(\lambda) = 0$ for $j = 1, .., k$, where

$$g_j(\lambda) \equiv C_j(\lambda) - f(\lambda) \, , \qquad f(\lambda) \equiv \frac{\int_{-\infty}^{\infty} w \left(L_0 - \lambda\right)^{-1} w^2 \, dy}{\int_{-\infty}^{\infty} w^2 \, dy} \, . \tag{3.18}$$

Here we have defined $C_j(\lambda) \equiv [\chi(\tau\lambda; j)]^{-1}$, so that from (3.13a) we get

$$C_j(\lambda) = \frac{1}{2} + \frac{\sqrt{1+\tau\lambda}}{2s\tanh(\theta_0/k)} \left[\tanh(\theta_\lambda/k) + \frac{(1 - \cos[\pi(j-1)/k])}{\sinh(2\theta_\lambda/k)} \right] \, . \tag{3.19}$$

After establishing the theoretical results in §3.2, in §3.3 we numerically determine the roots of (3.18) by following a similar approach as in [35]. We use a combination of Newton's method coupled to the numerical solution to the boundary value problem (3.16) obtained by COLSYS [1]. Although (3.16) can be solved explicitly in terms of hypergeometric functions [4], we avoid this approach since with these special functions it is then difficult to prove rigorous results for the spectrum of (3.12).

3.2 Theoretical Results on the Spectrum: Large Eigenvalues

We begin by looking for roots of (3.18) on the non-negative real axis $\lambda = \lambda_R \geq 0$. The first observation is that $f(\lambda_R)$ has a singularity on the positive real axis as a consequence of Theorem 2.12 of [17].

Lemma 3.4: (From [17]): *Consider the local eigenvalue problem $L_0 \phi_l = \sigma \phi_l$ for $\phi_l \in \mathcal{H}^1(R)$. This problem admits the eigenvalues $\sigma_0 > 0$, $\sigma_1 = 0$, and $\sigma_j < 0$ for $j > 1$. The eigenvalue σ_0 is simple, and the corresponding eigenfunction ϕ_{l0} has one sign.*

A simple calculation yields $\sigma_0 = 5/4$. Therefore, $f(\lambda_R) \to +\infty$ as $\lambda_R \to \sigma_0^-$. In [35] a detailed study of the behavior of $f(\lambda_R)$ on the positive real axis was given. The following rigorous result is a consequence of Proposition 3.5 of [35]:

Proposition 3.5: (From [35]): *For $\lambda_R \geq 0$, the function $f(\lambda_R)$ in (3.18) has the local behavior,*

$$f(\lambda_R) \sim 1 + \frac{3\lambda_R}{4} + \kappa_c \lambda_R^2 + O(\lambda_R^3), \quad \text{as} \quad \lambda_R \to 0; \qquad \kappa_c \equiv \frac{\int_{-\infty}^{\infty} \left(L_0^{-1} w\right)^2 dy}{\int_{-\infty}^{\infty} w^2 \, dy} > 0. \tag{3.20a}$$

In addition, we have the global behavior

$$f'(\lambda_R) > 0 \quad \text{and} \quad f''(\lambda_R) > 0, \quad \text{for} \quad 0 < \lambda_R < \sigma_0. \tag{3.20b}$$

Moreover, $f(\lambda_R) \to +\infty$ as $\lambda_R \to \sigma_0^-$. On the other side of the singularity we have

$$f(\lambda_R) < 0, \quad \text{for} \quad \lambda_R > \sigma_0. \tag{3.20c}$$

Proof: This result is simply Proposition 3.5 of [35] for a GM model with exponents $p = m = 2$. ■

Next, we summarize the key properties of $C_j(\lambda)$ when $\lambda = \lambda_R \geq 0$ is real.

Proposition 3.6: *For any fixed $\tau > 0$, we have a monotonicity result for $\lambda_R \geq 0$ that*

$$C_k(\lambda_R) > C_{k-1}(\lambda_R) > \ldots > C_1(\lambda_R) > 0, \qquad C_k'(\lambda_R) < C_{k-1}'(\lambda_R) < \ldots < C_1'(\lambda_R). \tag{3.21a}$$

In addition, for $\tau > 0$, and for each $j = 1, \ldots, k$, we have for $\lambda_R > 0$ that

$$C_j'(\lambda_R) > 0, \qquad C_j''(\lambda_R) < 0, \qquad C_j'(\lambda_R) = O(\tau^{1/2}), \quad \text{as} \quad \tau \to +\infty. \tag{3.21b}$$

Define the coefficients B_j by $B_j = C_j(0)$ for $j = 1, .., k$. These coefficients are independent of τ and have the following properties:

$$B_1 < B_2 < \ldots < B_k, \qquad B_1 = \frac{(s+1)}{2s}, \tag{3.21c}$$

$$\frac{dB_j}{dD} > 0, \quad \text{for} \quad j = 2, \ldots, k; \qquad \frac{dB_j}{ds} < 0, \quad \text{for} \quad j = 1, \ldots, k. \tag{3.21d}$$

Proof: This is Proposition 5.1 of [35] for a GM model with exponents $(p, q, m, s) = (2, s, 2, s)$. The monotonicity result for B_j with respect to s is immediate from (3.19). ■

As shown below, the critical values of D and $\mathcal{A}$ where $B_j = 1$ for $j = 2, \ldots, k$ play a central role in the analysis. We now calculate these values. The first observation is that for the large solution branch where $0 < s < 1$, we have from (3.21c) that $B_j > 1$ for $j = 1, \ldots, k$. Therefore,

$$1 < C_1(0) < C_2(0) < \ldots < C_k(0) \,, \qquad \text{for} \quad 0 < s < 1 \,. \tag{3.22}$$

For the small solution branch where $s > 1$ we calculate the values $D = D_j$ for which $B_j = 1$ for $j = 2, \ldots, k$. Note that since $B_1 < 1$ for any D, there is no threshold value D_1. A simple calculation using (3.19) and $B_j = C_j(0)$, yields that

$$B_j = \frac{1}{2}\left(1 + \frac{1}{s}\right) + \frac{\gamma_j}{4s\sinh^2\left(\theta_0/k\right)} \,, \qquad \gamma_j \equiv 1 - \cos\left(\frac{\pi(j-1)}{k}\right) \,. \tag{3.23}$$

Setting $B_j = 1$, for $j = 2, \ldots, k$, and solving for the thresholds D_j, we get

$$D_j \equiv \frac{4}{k^2\left[\ln\left(r_j + \sqrt{r_j^2 - 1}\right)\right]^2} \,, \quad j = 2, \ldots, k \,; \qquad r_j \equiv \frac{\gamma_j}{s-1} + 1 \,, \qquad j = 2, \ldots, k \,. \tag{3.24}$$

Alternatively, for a fixed D, we can calculate thresholds $\mathcal{A} = \mathcal{A}_j$ for which $B_j = 1$. Using (2.15b), which relates s in terms of $\mathcal{A}$, we set $B_j = 1$ in (3.23) to get

$$\mathcal{A}_j = \mathcal{A}_{ke} \frac{\left[(\gamma_j/2) + 2\sinh^2\left(\theta_0/k\right)\right]}{\left(\left[(\gamma_j/2) + 2\sinh^2\left(\theta_0/k\right)\right]^2 - (\gamma_j/2)^2\right)^{1/2}} \,, \qquad j = 2, \ldots, k \,. \tag{3.25}$$

Here $\mathcal{A}_{ke}$ are the existence thresholds for symmetric k-spike patterns given in (2.12). Since $\mathcal{A}$ is an increasing function of s for $s > 1$, it follows from (3.21c), (3.21d), (3.24), and (3.25), that

$$\mathcal{A}_2 < \mathcal{A}_3 < \ldots < \mathcal{A}_k \,, \qquad D_k < D_{k-1} < \ldots < D_2 \,. \tag{3.26}$$

Therefore, when $D < D_k$, or when $\mathcal{A} > \mathcal{A}_k$, we have that $B_j < 1$ for $j = 2, \ldots, k$. We label these important critical values as

$$D_k \equiv D_{kL} \equiv \frac{4}{k^2\left[\ln\left(r_k + \sqrt{r_k^2 - 1}\right)\right]^2} \,, \qquad r_k \equiv \frac{\gamma_k}{s-1} + 1 \,, \quad \gamma_k \equiv 1 + \cos\left(\frac{\pi}{k}\right) \,, \tag{3.27a}$$

$$\mathcal{A}_k \equiv \mathcal{A}_{kL} \equiv \mathcal{A}_{ke} \frac{\left((\gamma_k/2) + 2\sinh^2\left(\theta_0/k\right)\right)}{\left(\left[(\gamma_k/2) + 2\sinh^2\left(\theta_0/k\right)\right]^2 - (\gamma_k/2)^2\right)^{1/2}} \,. \tag{3.27b}$$

Next, we look for roots of (3.18), or equivalently eigenvalues of (3.12), on the non-negative imaginary axis $\lambda = i\lambda_I$, with $\lambda_I > 0$. Substituting $\lambda = i\lambda_I$ into (3.16) and (3.18), and extracting real and imaginary parts, we obtain that the eigenvalues of (3.12) with $\lambda = i\lambda_I$ and $\lambda_I \geq 0$ are the roots of the coupled system $g_{Rj} = g_{Ij} = 0$, where

$$g_{Rj}(\lambda_I) \equiv C_{Rj}\left(\lambda_I\right) - f_R\left(\lambda_I\right) \,, \qquad g_{Ij}(\lambda_I) \equiv C_{Ij}\left(\lambda_I\right) - f_I\left(\lambda_I\right) \,, \qquad j = 1, \ldots, k \,, \tag{3.28a}$$

and

$$f_R(\lambda_I) \equiv \frac{\int_{-\infty}^{\infty} w L_0 \left[L_0^2 + \lambda_I^2\right]^{-1} w^2 \, dy}{\int_{-\infty}^{\infty} w^2 \, dy}, \qquad f_I(\lambda_I) \equiv \frac{\lambda_I \int_{-\infty}^{\infty} w \left[L_0^2 + \lambda_I^2\right]^{-1} w^2 \, dy}{\int_{-\infty}^{\infty} w^2 \, dy}. \tag{3.28b}$$

Here we have defined $g_{Rj}(\lambda_I) \equiv \mathrm{Re}\,[g_j(i\lambda_I)]$, $g_{Ij}(\lambda_I) \equiv \mathrm{Im}\,[g_j(i\lambda_I)]$, and

$$C_{Rj}(\lambda_I) \equiv \mathrm{Re}\,[C_j(i\lambda_I)] \,, \qquad C_{Ij}(\lambda_I) \equiv \mathrm{Im}\,[C_j(i\lambda_I)] \,. \tag{3.28c}$$

To analyze the spectrum of (3.12) on the imaginary axis, we need some qualitative properties of the functions f_R, f_I, C_{Rj}, and C_{Ij}. The key properties are summarized as follows.

Proposition 3.7: *The functions f_R and f_I in (3.28b) have the asymptotic behavior*

$$f_R(\lambda_I) \sim 1 - \kappa_c \lambda_I^2 + O(\lambda_I^4) \,, \quad \text{as} \quad \lambda_I \to 0 \,; \qquad f_R(\lambda_I) = 0\left(\lambda_I^{-2}\right) \,, \quad \text{as} \quad \lambda_I \to \infty \,, \tag{3.29a}$$

$$f_I(\lambda_I) \sim \frac{3\lambda_I}{4} + O(\lambda_I^3) \,, \quad \text{as} \quad \lambda_I \to 0 \,; \qquad f_I(\lambda_I) = 0\left(\lambda_I^{-1}\right) \,, \quad \text{as} \quad \lambda_I \to \infty \,. \tag{3.29b}$$

Here κ_c is given in (3.20a). Moreover, the functions $f_R(\lambda_I)$ and $f_I(\lambda_I)$ have the global behavior

$$f_R'(\lambda_I) < 0 \,, \qquad f_I(\lambda_I) > 0 \,, \quad \text{for} \quad \lambda_I > 0 \,. \tag{3.29c}$$

For $\lambda_I > 0$ and $\tau > 0$, the functions C_{Rj} and C_{Ij} satisfy

$$C_{Rj}(0) = B_j \,, \quad C_{Rj}'(\lambda_I) > 0 \,; \qquad C_{Ij}(0) = 0 \,, \qquad C_{Ij}'(\lambda_I) > 0 \,, \tag{3.30a}$$

$$C_{Rj}'(\lambda_I) = O(\tau^{1/2}) \,, \qquad C_{Ij}'(\lambda_I) = O(\tau^{1/2}) \,, \quad \text{as} \quad \tau \to \infty \,, \tag{3.30b}$$

$$C_{Rj}(\lambda_I) = C_{Rj}(0) + O(\tau\lambda_I) \,, \qquad C_{Ij}(\lambda_I) = O(\tau\lambda_I) \,, \quad \text{as} \quad \tau \to 0 \,. \tag{3.30c}$$

Here B_j are the values $C_j(0) = B_j$, whose properties were given in (3.21c) and (3.21d).

Proof: The proof of the results in (3.29) are a special case of Propositions 3.1 and 3.2 of [35] corresponding to setting $p = m = 2$ in the GM model (1.9). The proof of (3.30) is immediate from setting $\lambda = i\lambda_I$ in the definition of $C_j(\lambda)$ in (3.19). ∎

A critical ingredient for determining the number of eigenvalues in the right half-plane $\mathrm{Re}(\lambda) > 0$ is to derive a winding number criterion. Specifically, we calculate the winding number of $g_j(\lambda)$, with $\lambda = \lambda_R + i\lambda_I$ in (3.18) over the counterclockwise contour composed of the imaginary axis $-iR \leq \mathrm{Im}\lambda \leq iR$ and the semi-circle Γ_R, given by $|\lambda| = R > 0$, for $-\pi/2 \leq \arg\lambda \leq \pi/2$. Assuming that there are no zeros of $g_j(\lambda)$ on the imaginary axis, we let $R \to \infty$ and use the argument principle to determine the number of zeros of $g_j(\lambda)$ in the right half-plane. The function $g_j(\lambda)$ in (3.18) is analytic in the right half-plane, except at the simple pole $\lambda = \sigma_0 > 0$, where $\sigma_0 = 5/4$ is the unique positive eigenvalue of the operator L_0 (see Lemma 3.4 above). For any $\tau > 0$, we have from (3.19) that $C_j(\lambda) \sim O(\sqrt{\lambda})$ as $|\lambda| \to \infty$ in the right half-plane. Moreover, $f(\lambda) \to 0$ as $|\lambda| \to \infty$. Thus, for any $\tau > 0$, the change in the argument of $g_j(\lambda)$ over Γ_R as $R \to \infty$ is $\pi/2$. By using the argument principle, and $g_j(\overline{\lambda}) = \overline{g_j(\lambda)}$, we then obtain the following winding number criterion:

Proposition 3.8: *Let $\tau > 0$ and assume that there are no zeros of $g_j(\lambda)$ on the imaginary axis for $j = 1, \ldots, k$. Then, the number of eigenvalues M of (3.12) in $Re(\lambda) > 0$ satisfies*

$$M = \frac{5k}{4} + \frac{1}{\pi} \sum_{j=1}^{k} [\arg g_j]_{\Gamma_I} \,. \tag{3.31}$$

Here $[\arg g_j]_{\Gamma_I}$ denotes the change in the argument of $g_j(\lambda)$ along the semi-infinite imaginary axis $\Gamma_I = i\lambda_I$, $0 \leq \lambda_I < \infty$, traversed in the downwards direction.

Using the properties given in Propositions 3.6 and 3.7, there are only a few possible values for $[\arg g_j]_{\Gamma_I}$. This leads to a more specific winding number criterion.

Proposition 3.9: *Let $\tau > 0$. Then, there are three distinct possibilities:*

$$\text{(1): if } \; g_{Ij} < 0 \; \text{ when } \; g_{Rj} = 0 \,, \qquad \text{then } \; [\arg g_j]_{\Gamma_I} = -5\pi/4 \,, \tag{3.32a}$$

$$\text{(2): if } \; g_{Ij} > 0 \; \text{ when } \; g_{Rj} = 0 \,, \qquad \text{then } \; [\arg g_j]_{\Gamma_I} = 3\pi/4 \,, \tag{3.32b}$$

$$\text{(3): if } \; g_{Rj} > 0 \; \text{ for all } \; \lambda_I \geq 0 \,, \qquad \text{then } \; [\arg g_j]_{\Gamma_I} = -\pi/4 \,. \tag{3.32c}$$

Proof: Let $\tau > 0$. From (3.30) and (3.29) we have that $C_{Rj}(\lambda_I)$ is a positive monotone increasing function, whereas $f_R(\lambda_I)$ is a positive monotone decreasing function. Therefore, $g_{Rj} = 0$ has at most one root. First suppose that $C_{Rj}(0) < f_R(0) = 1$ so that $g_{Rj} = 0$ has a root. Then, using (3.30) and (3.29) we have that $g_{Rj} \sim b\sqrt{\lambda_I}$ and $g_{Ij} \sim b\sqrt{\lambda_I}$ as $\lambda_I \to +\infty$ for some $b > 0$. In addition, from (3.29b) and the condition $C_{Ij}(0) = 0$, we have that $g_{Rj} < 0$ and $g_{Ij} = 0$ at $\lambda_I = 0$. This implies that $\arg g_j = \pi/4$ as $\lambda_I \to +\infty$, and $\arg g_j = -\pi$ at $\lambda_I = 0$. Since the root to $g_{Rj} = 0$ is unique, this shows that the change in the argument, $[\arg g_j]_{\Gamma_I}$, is either $5\pi/4$ or $-3\pi/4$ depending on the sign of g_{Ij} at the unique root of $g_{Rj} = 0$. This proves (3.32a) and (3.32b). Next, suppose that $C_{Rj}(0) > f_R(0) = 1$. Then, since $g_{Rj}(0) > 0$ and $g'_{Rj}(\lambda_I) > 0$ from (3.29c) and (3.30a), we conclude that $g_{Rj} > 0$ for $\lambda_I \geq 0$. In this case, $\arg g_j = 0$ at $\lambda_I = 0$. Since $g_{Rj} > 0$ for all $\lambda_I > 0$, the result (3.32c) follows. ■

There are many consequences of Propositions 3.5–3.9. The first concerns the stability of the large solution u_+, ν_+.

Proposition 3.10: *For any $\tau > 0$, $D > 0$, and $0 < s < 1$, the large solution u_+, ν_+ is unstable. In this case there are exactly k eigenvalues of (3.12) in the right half-plane, and they are all located on the positive real axis in the interval $0 < \lambda_R < \sigma_0 = 5/4$. These k real eigenvalues have a common leading-order asymptotic behavior*

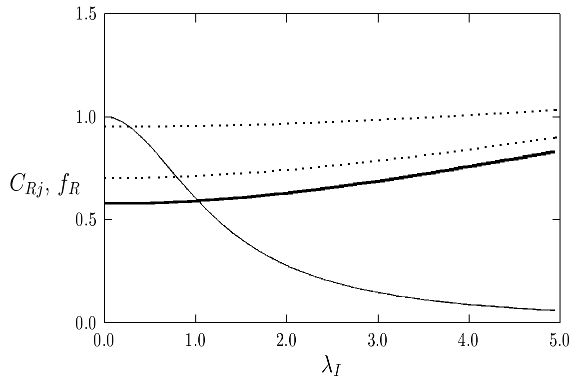
$$\lambda_R \sim \frac{5}{4} + \delta_1/\tau^{1/2} + \cdots \,, \qquad \text{as} \quad \tau \to \infty \,, \tag{3.33a}$$

where δ_1 is defined by

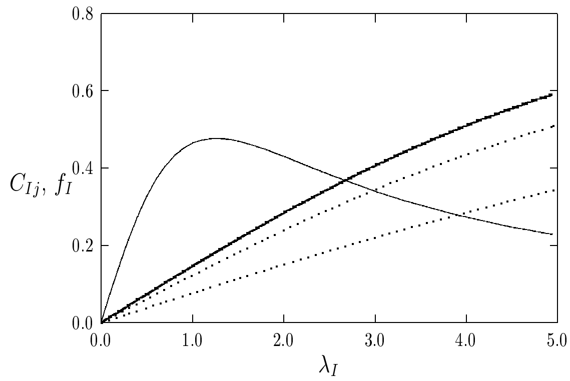
$$\delta_1 \equiv -\frac{\left(\int_{-\infty}^{\infty} w\phi_{l0}\, dy\right)\left(\int_{-\infty}^{\infty} w^2\phi_{l0}\, dy\right)}{c_\infty \int_{-\infty}^{\infty} w^2\, dy} \,, \qquad c_\infty \equiv \frac{\sqrt{5}}{4s} \left[\tanh\left(\theta_0/k\right)\right]^{-1} \,. \tag{3.33b}$$

Here ϕ_{l0} is the principal eigenfunction of L_0 (see Lemma 3.4), normalized so that $\int_{-\infty}^{\infty} \phi_{l0}^2 \, dy = 1$.

Proof: Let $\tau > 0$ and $0 < s < 1$. Since $C_{Rj}(0) = B_j > 1$ for $j = 1, \ldots, k$ from (3.22), we have $g_{Rj}(0) > 0$, and consequently $g_{Rj}(\lambda_I) > 0$ for $\lambda_I \geq 0$. Therefore, for $j = 1, \ldots, k$, condition (3) of Proposition 3.9 applies. From (3.31) this yields that there are $M = k$ eigenvalues in the right half-plane for any $\tau > 0$ and $0 < s < 1$. Next, we show that these eigenvalues are real and positive. For any $0 < s < 1$, (3.21c) and (3.21d) yield that $1 < B_1 < B_2 < \ldots < B_k$, and consequently $1 = f(0) < C_1(0) < \ldots < C_k(0)$. Proposition 3.6 proves that $C_j(\lambda_R)$ is a positive increasing and concave function, while $f(\lambda_R)$ is a positive increasing convex function on $0 \leq \lambda_R < \sigma_0$. Hence, it follows that $g_j(\lambda_R) = 0$ has exactly one root for each $j = 1, \ldots, k$ on the interval $0 \leq \lambda_R < \sigma_0$. Since $f(\lambda_R) < 0$ for $\lambda_R > \sigma_0$ from (3.20c), there can be no roots on $\lambda_R > \sigma_0$. Hence, we have k real positive eigenvalues for (3.12) on the interval $0 < \lambda_R < \sigma_0 = 5/4$. The asymptotic behavior (3.33) is obtained by expanding λ_R and ψ in fractional powers of τ similar to that done in Proposition 3.9 of [35] for the GM model, and noting from (3.19) that $C_j(\lambda_R) \sim c_\infty \tau^{1/2}$ as $\tau \to \infty$. We leave these details to the reader. ■



(a) $g_{Rj} = 0$



(b) $g_{Ij} = 0$

Figure 5: Parameter values are $D = 0.75$, $k = 3$, $\mathcal{A} = 9.023$, and $\tau = 2.0$. Left figure: $C_{Rj}(\lambda_I)$, for $j = 1, \ldots, 3$, and $f_R(\lambda_I)$ (solid curve). Right figure: $C_{Ij}(\lambda_I)$ and $f_I(\lambda_I)$ (solid curve). In these figures C_{R1} and C_{I1} are the heavy solid curves.

The next result gives a criterion for the stability of a one-spike small solution u_-, ν_-, and for a k-spike small solution when $D < D_{kL}$, or $\mathcal{A} > \mathcal{A}_{kL}$. Here D_{kL} and $\mathcal{A}_{kL}$ are the thresholds given in (3.27).

Proposition 3.11 *Let $\tau > 0$, $k = 1$, and consider the small solution u_-, ν_-, where $s > 1$. For such a solution, we have $M = 0$ when $\tau \ll 1$ and $M = 2$ when τ is sufficiently large. Moreover, there exists a value $\tau = \tau_{h1}$, depending on $\mathcal{A}$ and D, such that there is a pair of complex conjugate eigenvalues on the imaginary axis. For a multi-spike solution where $k > 1$, suppose that $D < D_{kL}$ or $\mathcal{A} > \mathcal{A}_{kL}$. Then,*

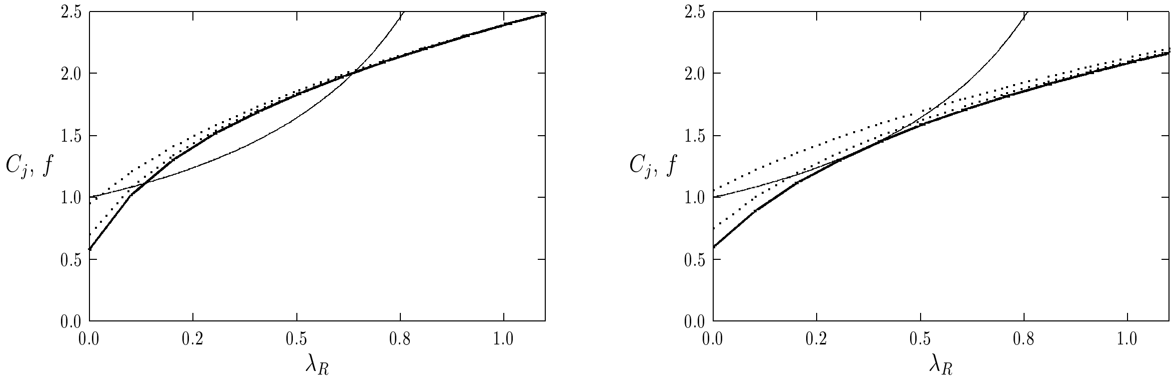
$M=0$ when $\tau \ll 1$ and $M=2k$ when $\tau \gg 1$. For $k \geq 1$, and for τ sufficiently large, these eigenvalues are real and are on the interval $0<\lambda_R<\sigma_0=5/4$. There are k eigenvalues λ_{Bj} that tend to $\sigma_0=5/4$ from below as $\tau \to \infty$, and k eigenvalues λ_{Sj} that tend to zero as $\tau \to \infty$. The λ_{Bj} for $j=1,\ldots,k$ have the common asymptotic behavior (3.33), while the asymptotic behavior of λ_{Sj} is given by

$$\lambda_{Sj} \sim \frac{\omega_j}{\tau} + O(\tau^{-2}), \quad j=1,\ldots,k. \tag{3.34}$$

Here ω_j is the unique positive root of $C_j(\omega_j \tau^{-1})=1$.

Proof: Let $\tau>0$, $s>1$, and $k \geq 1$. Assume for $k>1$ that $D<D_{kL}$, or equivalently $\mathcal{A}>\mathcal{A}_{kL}$. We first show that $M=0$ when $\tau \ll 1$ and $M=2k$ when $\tau \gg 1$. In this case, we have $C_{Rj}(0)=B_j<f_R(0)=1$ for $j=1,\ldots,k$. This, implies that $g_{Rj}(0)<0$, and so by the monotonicity of g_{Rj}, we have that $g_{Rj}=0$ has a unique root for each $j=1,\ldots,k$. Moreover, since C_{Rj} is monotone increasing in λ_I for $\tau>0$, these roots must lie in the interval $(0,\lambda_{Ic})$, where λ_{Ic} is the unique root of $C_{R1}(0)=f_R(\lambda_I)$. Hence, λ_{Ic} is independent of τ. Since $C_{Ij} \to 0$ uniformly as $\tau \to 0$ for any fixed λ_I from (3.30c), it follows that for $\tau \ll 1$ we have $g_{Ij}<0$ whenever $g_{Rj}=0$. Hence, for $\tau \ll 1$ condition (1) in Proposition 3.9 applies. Then, from the winding number criterion (3.31) we get $M=0$. Alternatively, since at each fixed λ_I, C_{Ij} increases without bound as $\tau \to \infty$ from (3.30b), we have that $g_{Ij}>0$ at the unique root of $g_{Rj}=0$. Therefore, for $\tau \gg 1$, condition (2) in Proposition 3.9 holds, and we get $M=2k$ from (3.31). Next, we show that for $k \geq 1$, and $\tau \gg 1$ that there are $2k$ eigenvalues on the real axis in $0<\lambda_R<\sigma_0$. Along the real axis, we have from Proposition 3.5 that $f(0)=1$, and $f(\lambda_R)$ is an increasing, convex, function on $0<\lambda_R<\sigma_0$. Moreover, $f(\lambda_R) \to +\infty$ as $\lambda_R \to \sigma_0^-$, and $f(\lambda_R)<0$ for $\lambda_R>\sigma_0$. In contrast, from (3.21b) we have that $C_j(\lambda_R)$ is a monotone increasing concave function, with an unbounded derivative for $\tau \gg 1$. When $D<D_{kL}$, or when $\mathcal{A}>\mathcal{A}_{kL}$, we have that $C_j(0)=B_j<f(0)=1$ for $j=1,\ldots,k$. Therefore, as τ is increased, there must be a critical value $\tau_{mj}>0$ where $C_j(\lambda_R)$ and $f(\lambda_R)$ first intersect tangentially. For $\tau>\tau_{mj}$ there must be two roots to $g_j(\lambda)=0$. The values $\tau=\tau_{mj}$ for $j=1,\ldots,k$ are the values where complex conjugate eigenvalues merge onto the real axis. Clearly, as $\tau \to \infty$, one root of each $g_j(\lambda_R)=0$ tends to σ_0^- while the other root tends to zero. The precise asymptotic behavior of these roots as $\tau \to \infty$ is obtained in a similar way as in Propositions 3.8 and 3.9 of [35]. We leave these details to the reader. Since for $k=1$ we have $M=0$ for $\tau \ll 1$ and $M=2$ for $\tau \gg 1$, the existence of a Hopf bifurcation value τ_{h1} (possibly non-unique) follows by continuity. ■

To illustrate this result, we take the parameter values $D=0.75$, $k=3$, $\mathcal{A}=9.023$, and $\tau=2.0$. For these values, we calculate from (3.27b) that $\mathcal{A}>\mathcal{A}_{3L}=8.686$, so that Proposition 3.11 applies. In Fig. 5(a) we plot the numerically computed $f_R(\lambda_I)$ and $C_{Rj}(\lambda_I)$ for $j=1,\ldots,3$. A similar plot of $f_I(\lambda_I)$ and $C_{Ij}(\lambda_I)$, for $j=1,\ldots,3$, is shown in Fig. 5(b). For this small value $\tau=2.0$ it is clear from these figures that whenever $g_{R_j}=0$ we have $g_{Ij}<0$. Since condition (1) of Proposition 3.9 holds for each $j=1,2,3$, we get from (3.31) that $M=0$. Hence, there are no eigenvalues of (3.12) in the right half-plane. In Fig. 6(a) we plot $C_j(\lambda_R)$ and $f(\lambda_R)$ with $\tau=80.0$, and for the same parameter values given above. For this large value of τ, each $C_j(\lambda_R)$ intersects $f_R(\lambda_R)$ exactly twice. Hence, we have six real positive eigenvalues on the interval $0<\lambda_R<5/4$.



(a) $g_j = 0$ (b) $g_j = 0$

Figure 6: Plots of $C_j(\lambda_R)$, for $j = 1, 2, 3$, and $f(\lambda_R)$ (solid curve). The heavy solid curve is $C_1(\lambda_R)$. Left figure: parameter values are $D = 0.75$, $k = 3$, $\mathcal{A} = 9.023$, and $\tau = 80.0$. Right figure: parameter values are $D = 0.75$, $k = 3$, $\mathcal{A} = 8.357$, and $\tau = 36.93$. For these values, $C_1(\lambda_R)$ intersects $f(\lambda_R)$ tangentially. For this data we have $\mathcal{A}_2 < \mathcal{A} < \mathcal{A}_{3L}$.

For a one-spike solution, the main limitation of Proposition 3.11 is that we are not able to prove that the Hopf bifurcation value τ_{h1} is unique. In particular, we cannot theoretically rule out the case that there are N, with $N > 1$ and odd, values of τ_{h1} where there are complex conjugate eigenvalues on the imaginary axis. To show that $N = 1$, and hence that τ_{h1} is unique, one would have to prove a strict transversal crossing condition that states that whenever $\tau > \tau_{h1}$ there are two eigenvalues in the right half-plane, so that $M = 2$ for all $\tau > \tau_{h1}$. From all of the numerical experiments that we have performed it appears that such a crossing condition does in fact hold. However, Proposition 3.11 does guarantee that there will be two real positive eigenvalues for all $\tau > \tau_{m1}$.

For the small solution branch, the qualitative difference between the spectrum of (3.12) for a one-spike and for a multi-spike solution is that only for a multi-spike solution can eigenvalues cross through the origin along the real axis $\text{Im}(\lambda) = 0$ as D or $\mathcal{A}$ is varied. However, when $D < D_{kL}$ or $\mathcal{A} > \mathcal{A}_{kL}$, there are no real positive eigenvalues when $\tau \ll 1$. The real eigenvalues that exist when $\tau \gg 1$ have occurred from the merging of complex conjugate pairs of eigenvalues onto the real axis. These complex conjugate eigenvalues arise from a Hopf bifurcation. Notice that since $C_j(0)$ is independent of τ, the eigenvalues of (3.12) can never cross through the origin along the real axis as τ is increased. Therefore, instabilities as τ is increased can only occur from Hopf bifurcations, whereas instabilities that occur as D is increased or $\mathcal{A}$ is decreased occur from real eigenvalues entering the right half-plane.

The next result characterizes the number of eigenvalues in the right half-plane for a multi-spike solution for other ranges of D and $\mathcal{A}$, where there may be positive real eigenvalues when $\tau \ll 1$.

Proposition 3.12: *Let $\tau > 0$, $k > 1$, and consider the multi-spike small solution u_-, ν_-, where $s > 1$.*

Suppose that there exists a j^* with $2 \leq j^* \leq k$ such that either $D_{j^*} < D < D_{j^*-1}$, or $\mathcal{A}_{j^*-1} < \mathcal{A} < \mathcal{A}_{j^*}$. Here D_j and $\mathcal{A}_j$ for $j = 2, \ldots, k$ are given in (3.24) and (3.25), respectively. Since there are no thresholds for $j = 1$, we conveniently label $D_1 = \infty$ and $\mathcal{A}_1 = \mathcal{A}_{1e}$, where $\mathcal{A}_{1e}$ is the existence threshold of (2.12). Then, for any $\tau > 0$, the number of eigenvalues M of (3.12) in the right half-plane satisfies the bounds

$$1 + k - j^* \leq M \leq k - 1 + j^* \,. \tag{3.35}$$

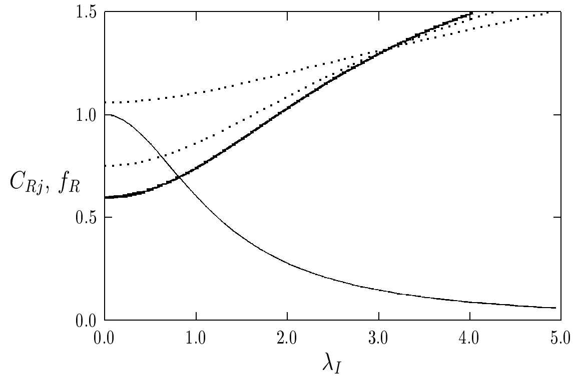
Moreover, there are at least $M_R = 1 + k - j^*$ eigenvalues on the positive real axis in the interval $0 < \lambda_R < \sigma_0 = 5/4$ for any $\tau > 0$.

Proof: The proof is simple. For the range of D and $\mathcal{A}$ stated above, we have $C_{Rj}(0) > 1$ for $j = j^*, \ldots, k$, and $C_{Rj}(0) < 1$ for $j = 1, \ldots, j^* - 1$. Hence, $g_{Rj}(\lambda_I) > 0$ for $j = j^*, \ldots, k$, and condition (3) of Proposition 3.9 holds. This yields $[\arg g_j]_{\Gamma_I} = -\pi/4$ for $j = j^*, \ldots, k$ and any $\tau > 0$. For the other indices, we proceed as in the proof of Proposition 3.11, to get $[\arg g_j]_{\Gamma_I} = -5\pi/4$ for $j = 1, \ldots, j^* - 1$ when τ is sufficiently small, and $[\arg g_j]_{\Gamma_I} = 3\pi/4$ for $j = 1, \ldots, j^* - 1$ when τ is sufficiently large. Substituting these values into the winding number criterion (3.31) we obtain (3.35). It is clear that there are at least $M_R = 1 + k - j^*$ eigenvalues on the real axis when $\tau \ll 1$. This follows since $C_j(0) = B_j > 1 = f(0)$ for $j = j^*, \ldots, k$, and the fact that for any $\tau > 0$ the curve $C_j(\lambda_R)$ must intersect $f(\lambda_R)$ exactly once for each $j = j^*, \ldots, k$. ∎

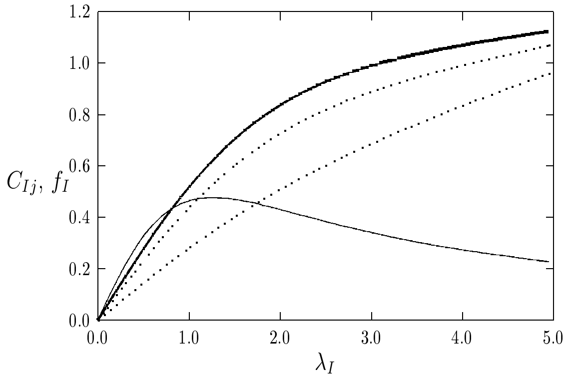
Remark 3.1: As a special case of this result, we set $j^* = 2$ to obtain $k - 1 < M < k + 1$. Therefore, when $D > D_2$, or when $\mathcal{A} < \mathcal{A}_2$, there will be at least $k - 1$ eigenvalues on the positive real axis. Here D_2 and $\mathcal{A}_2$ are given in (3.24) and (3.25), respectively. This range of the parameters is the near-shadow limit, since we know that a k-spike solution for the shadow problem, obtained by letting $D \to \infty$ in (1.6), will have $k - 1$ eigenvalues on the positive real axis when $\tau = 0$. Hence, this qualitative feature of the spectrum is preserved for finite values of D up until D crosses below D_2.

To illustrate Proposition 3.12, we take the parameter values $D = 0.75$, $k = 3$, $\mathcal{A} = 8.357$ and $\tau = 36.93$. For these values, we calculate from (3.25) and (3.27b) that $\mathcal{A}_2 = 6.86$ and $\mathcal{A}_{3L} = 8.686$, so that $\mathcal{A}_2 < \mathcal{A} < \mathcal{A}_{3L}$. Therefore, this corresponds to setting $j^* = k = 3$ in Proposition 3.12 to get $1 \leq M \leq 5$. In Fig. 6(b) we show the graphical determination of the eigenvalues of (3.12) on the positive real axis by plotting $C_j(\lambda_R)$, for $j = 1, \ldots, 3$, and $f(\lambda_R)$. Since $C_3(0) > 1$, we only get one root of $g_3 = 0$ for any $\tau > 0$. For the value $\tau = 36.93$, we have that C_1 intersects f_R tangentially. For $\tau \gg 1$, it is clear that we will have $M = 5$. For the smaller value $\tau = 6.183$, in Fig. 7(a) we plot $f_R(\lambda_I)$ and $C_{Rj}(\lambda_I)$ for $j = 1, \ldots, 3$. A similar plot of $f_I(\lambda_I)$ and $C_{Ij}(\lambda_I)$, for $j = 1, \ldots, 3$, is shown in Fig. 7(b). For this value of τ, we have $g_{R1} = g_{I1} = 0$. In addition, we have $g_{I2} < 0$ when $g_{R2} = 0$, and that $g_{R3}(\lambda_R) > 0$ for all $\lambda_R > 0$. Therefore, condition (1) and condition (3) of Proposition 3.9 holds for $j = 2$ and $j = 3$, respectively. Then, from (3.31), it is clear that this value of τ corresponds to where (3.12) has complex conjugate eigenvalues on the imaginary axis, together with one positive real eigenvalue.

The method of proof of Proposition 3.12 shows that there exists a value $\tau = \tau_{hj} > 0$ (possibly non-unique) such that $g_{Rj} = g_{Ij} = 0$. At this value of τ, there is a complex conjugate pair of eigenvalues on the imaginary axis. The minimum τ_{hL} of these values determines the stability threshold. Therefore,



(a) $g_{Rj} = 0$



(b) $g_{Ij} = 0$

Figure 7: Parameter values $D = 0.75$, $k = 3$, $\mathcal{A} = 8.357$, and $\tau = 6.183$. Left figure: $C_{Rj}(\lambda_I)$, for $j = 1, \ldots, 3$, and $f_R(\lambda_I)$ (solid curve). Right figure: $C_{Ij}(\lambda_I)$ and $f_I(\lambda_I)$ (solid curve). In these figures C_{R1} and C_{I1} are the heavy solid curves.

we define

$$\tau_{hL} \equiv \text{Min}\left(\tau_{hj}\,;\, j = 1, .., k\right) . \tag{3.36}$$

This critical value depends on k, D, and $\mathcal{A}$. The main stability results obtained in this section can now be summarized succinctly as follows.

Proposition 3.13: *Let $\tau > 0$, $k > 1$, and consider the multi-spike small solution u_-, ν_-, where $s > 1$. For $D < D_{kL}$, or $\mathcal{A} > \mathcal{A}_{kL}$, the solution will be stable with respect to the large eigenvalues when $0 < \tau < \tau_{hL}$. Alternatively, suppose that $D > D_{kL}$, or $\mathcal{A}_{ke} < \mathcal{A} < \mathcal{A}_{kL}$, then the solution is unstable for any $\tau > 0$.*

Although we have not been able to prove a strict transversal crossing condition, all of the numerical experiments that we have performed below indicate that the values τ_{hj} are uniquely defined. Specifically, when τ increases past a particular τ_{hj} an additional pair of complex conjugate eigenvalues enters into the right half-plane and remains in this plane for all $\tau > \tau_{hj}$. Therefore, we believe that when $D < D_{kL}$, or when $\mathcal{A} > \mathcal{A}_{kL}$, the multi-spike solution will be unstable for any τ with $\tau > \tau_{hL}$. When $D > D_{kL}$, or when $\mathcal{A} < \mathcal{A}_{kL}$, there is at least one positive real eigenvalue for any $\tau > 0$. In this range of D or $\mathcal{A}$, there may be additional complex conjugate pairs of eigenvalues in the right half-plane if τ is large enough.

Next, we discuss the two types of instabilities that can occur for multi-spike solutions to (1.6). We first discuss **competition instabilities** that occur as a result of eigenvalues on the positive real axis. Suppose that D satisfies $D_{kL} < D < D_{k-1}$, or equivalently $\mathcal{A}_{k-1} < \mathcal{A} < \mathcal{A}_{kL}$. Then, following the idea of the proof of Proposition 3.12, there will be exactly one eigenvalue in the right half-plane for

$0 < \tau < \tau_{hL}$, and it is located along the real axis. Therefore, on this parameter range, the $j = k$ mode governs the instability. From Proposition 3.2, the unstable eigenvector is $\boldsymbol{c}_k$ given in (3.11b). This implies that the initial instability of the small k-spike pattern has the form

$$\nu = \nu_- + \delta e^{\lambda_{Rk} t} \phi \,, \qquad \phi(x) = \sum_{n=1}^{k} c_n \Phi\left[\varepsilon^{-1}(x - x_n)\right] \,. \tag{3.37a}$$

Here $\delta \ll 1$, $\lambda_{Rk} > 0$ is the unique root of $g_k(\lambda_R) = 0$, and

$$c_n = \cos\left(\frac{\pi(k-1)}{k}(n - 1/2)\right) \,, \quad n = 1, \ldots, k \,. \tag{3.37b}$$

Since $\sum_{n=1}^{k} c_n = 0$, this instability has the effect of locally preserving the sum of the heights of the spikes. Hence, we refer to it as a **competition instability**. As shown in the numerical experiments below, this mode initiates a spike competition process, decreasing the amplitudes of some spikes while increasing the amplitudes of others. Numerically, it is found that this type of instability has the ultimate effect of annihilating spikes.

Next, we discuss the type of oscillatory instability that occurs when $D < D_{kL}$, or $\mathcal{A} > \mathcal{A}_{kL}$, as τ increases past τ_{hL}. The value of j for which the minimum in (3.36) is obtained determines the unstable eigenvector $\boldsymbol{c}_j$ in Proposition 3.2. We now develop a criterion to determine which mode goes unstable first as τ is increased.

To do so, we first try to develop an ordering principle for C_{Rj} and C_{Ij}. From (3.19) and (3.28c), we readily calculate that for $j = 1, .., k-1$ that

$$C_{Rj+1} - C_{Rj} = \beta_j \mathrm{Re}\left[E(\xi)\right] \,, \qquad C_{Ij+1} - C_{Ij} = \beta_j \mathrm{Im}\left[E(\xi)\right] \,. \tag{3.38a}$$

Here $\beta_j > 0$ and $E(\xi)$ are defined by

$$\beta_j \equiv \left(\frac{k}{2\theta_0 s}\right)\left(\frac{\sin\left[\pi(j-1/2)/k\right]\sin\left[\pi/(2k)\right]}{\tanh\left(\theta_0/k\right)}\right) > 0 \,, \qquad E(\xi) \equiv \frac{\xi}{\sinh \xi} \,, \tag{3.38b}$$

where

$$\xi \equiv \frac{2\theta_0}{k}\sqrt{1 + i\tau\lambda_I} \,. \tag{3.38c}$$

This leads to the following ordering principle:

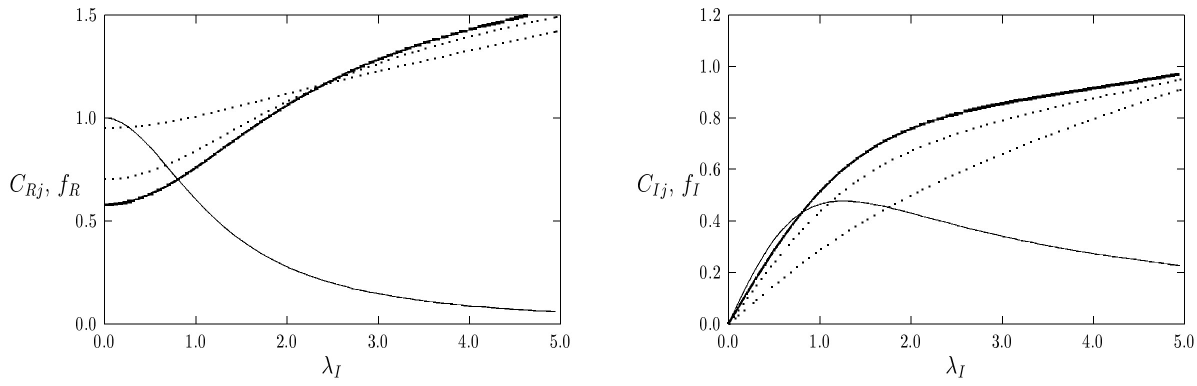
<u>Lemma 3.14</u>: *Suppose that Re*$\left[E(\xi)\right] > 0$ *and Im*$\left[E(\xi)\right] < 0$ *at each point on some interval* $0 < \lambda_I < \lambda_*$*. Then, we have the following ordering principle on* $0 < \lambda_I < \lambda_*$*:*

$$C_{R1}(\lambda_I) < \ldots < C_{Rk}(\lambda_I) \,, \qquad C_{I1}(\lambda_I) > \ldots > C_{Ik}(\lambda_I) \,. \tag{3.39}$$

When this ordering principle holds, the mode that goes unstable first must be the $j = 1$ mode. We state the result as follows.

<u>Proposition 3.15</u>: *Let* $\varepsilon \to 0$, $k > 1$ *and consider the small solution* u_-, ν_-*. Let* τ_{h1} *correspond to the minimum value of* τ *for which* $g_{R1} = g_{I1} = 0$*. Label the corresponding root by* $\lambda_I = \lambda_{h1}$*. Suppose* *that for the value $\tau = \tau_{h1}$, the ordering principle of Lemma 3.14 holds at each point on the interval $0 < \lambda_I < \lambda_{h1}$. Then, there are either zero or two eigenvalues in the right half-plane for τ in a sufficiently small neighborhood of $\tau = \tau_{h1}$.*

Proof: The proof is simple. Since the ordering principle (3.39) holds when $\tau = \tau_{h1}$, it follows by continuity that it holds in a sufficiently small neighborhood of τ_{h1}. Therefore, we will have $g_{Ij} < 0$ whenever $g_{Rj} = 0$ for $j = 2, .., k-1$. This implies that condition (1) of Proposition 3.9 holds, and so we get $[\arg g_j]_{\Gamma_I} = -5\pi/4$ for $j = 2, \ldots, k$. From (3.31) we obtain $M = 0$ or $M = 2$ depending on whether $[\arg g_1]_{\Gamma_I} = -5\pi/4$ or $[\arg g_1]_{\Gamma_I} = 3\pi/4$. ■



(a) $g_{Rj} = 0$

(b) $g_{Ij} = 0$

Figure 8: Parameter values $D = 0.75$, $k = 3$, $\mathcal{A} = 9.023$, and $\tau = 8.049$. Left figure: $C_{Rj}(\lambda_I)$, for $j = 1, \ldots, 3$, and $f_R(\lambda_I)$ (solid curve). Right figure: $C_{Ij}(\lambda_I)$ and $f_I(\lambda_I)$ (solid curve). In these figures C_{R1} and C_{I1} are the heavy solid curves.

To illustrate Proposition 3.15, we consider the parameter values of Fig. 5 where $D = 0.75$, $k = 3$, and $\mathcal{A} = 9.023 > \mathcal{A}_{3L}$. For $\tau = 8.049$, in Fig. 8(a) we plot the numerically computed $f_R(\lambda_I)$ and $C_{Rj}(\lambda_I)$ for $j = 1, \ldots, 3$. A similar plot of $f_I(\lambda_I)$ and $C_{Ij}(\lambda_I)$, for $j = 1, \ldots, 3$, is shown in Fig. 8(b). For this value of τ it is clear from these figures that the ordering principle (3.39) holds, and so the $j = 1$ sets the stability threshold. Therefore, for $\tau = 8.049$, it follows that (3.12) has no eigenvalues in the right half-plane, but has a pair of complex conjugate eigenvalues on the imaginary axis determined by the $j = 1$ mode. This is the Hopf bifurcation value.

Under the conditions of Proposition 3.15, it is guaranteed that the $j = 1$ mode goes unstable first. The corresponding eigenvector $\boldsymbol{c}_1$ is given in (3.11b) as $\boldsymbol{c}_1^t = (1, \ldots, 1)$. In this way, we obtain that at $\tau = \tau_{h1}$ the instability takes the form

$$\nu = \nu_- + \delta e^{i\lambda_{h1} t}\phi + \text{c.c}\,, \quad \phi(x) = \sum_{n=1}^{k} c_n \Phi\left[\varepsilon^{-1}(x - x_n)\right]\,, \quad c_n = 1\,, \quad n = 1, .., k\,. \tag{3.40}$$

Here c.c denotes complex conjugate, $\delta << 1$, and λ_{h1} is the root of $g_{R1} = g_{I1} = 0$, which occurs when $\tau = \tau_{h1}$. The key observation is that since $c_n = 1$, for $n = 1, .., k$, the initial form of the instability is to synchronize the amplitudes of the spikes. We refer to this as a **synchronous oscillatory instability**. Therefore, if the ordering principle in Lemma 3.14 holds, the $j = 1$ mode goes unstable first, and the effect is to synchronize the small-scale oscillations in the spike amplitudes.

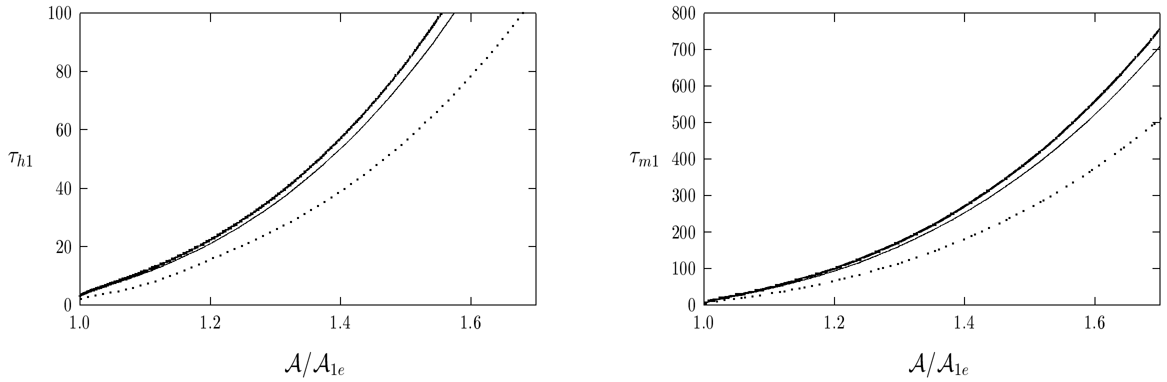
The critical condition in Lemma 3.14 is to determine the signs of $\text{Re}\,[E(\xi)]$ and $\text{Im}\,[E(\xi)]$. Writing $\xi = \xi_R + i\xi_I$, a simple calculation shows that $\text{Re}\,[E(\xi)] > 0$ if and only if

$$\tanh \xi_R \cos \xi_I + \frac{\xi_I}{\xi_R} \sin \xi_I > 0 \,. \tag{3.41a}$$

Similarly, $\text{Im}\,[E(\xi)] < 0$ if and only if

$$\sin \xi_I - \frac{\xi_I}{\xi_R} \tanh \xi_R \cos \xi_I > 0 \,. \tag{3.41b}$$

From the definition of ξ in (3.38c), it is clear that both inequalities in (3.41) will hold when $\tau\lambda_I$ is small enough. Therefore, it is natural to look for a condition for which the $j = 1$ governs the instability. The precise condition was stated in Proposition 3.15.



(a) τ_{h1}

(b) τ_{m1}

Figure 9: Left figure: the Hopf bifurcation value τ_{h1} versus $\mathcal{A}/\mathcal{A}_{1e}$ for a one-spike solution. Right figure: τ_{m1} versus $\mathcal{A}/\mathcal{A}_{1e}$ where the complex pair of eigenvalues merge onto the positive real axis. The labels are $D = 0.75$ (dashed curve), $D = 0.25$ (solid curve), and $D = 0.1$ (heavy solid curve).

Finally, we remark on the possibility of having an asynchronous small-scale oscillation near the Hopf bifurcation point τ_{hL}. Such an asynchronous instability is governed by the $j = k$ mode which, as discussed above with respect to competition instabilities, has the effect of conserving the amplitude of spikes. The $j = k$ mode will govern the initial instability when both $\text{Re}\,[E(\xi)]$ and $\text{Im}\,[E(\xi)]$ have exactly one zero-crossing on the interval $0 < \lambda_I < \lambda_{hk}$, where $\lambda_I = \lambda_{hk}$ and $\tau = \tau_{hk}$ is the root of $g_{Rk} = g_{Ik} = 0$. Although we show in Example 4 of §3.3 that asynchronous oscillatory instabilities are theoretically possible in a narrow parameter range, we have not been able to realize such oscillations in the Gray-Scott model in the low or intermediate feed-rate regime.

3.3 Numerical Results: Oscillatory and Competition Instabilities

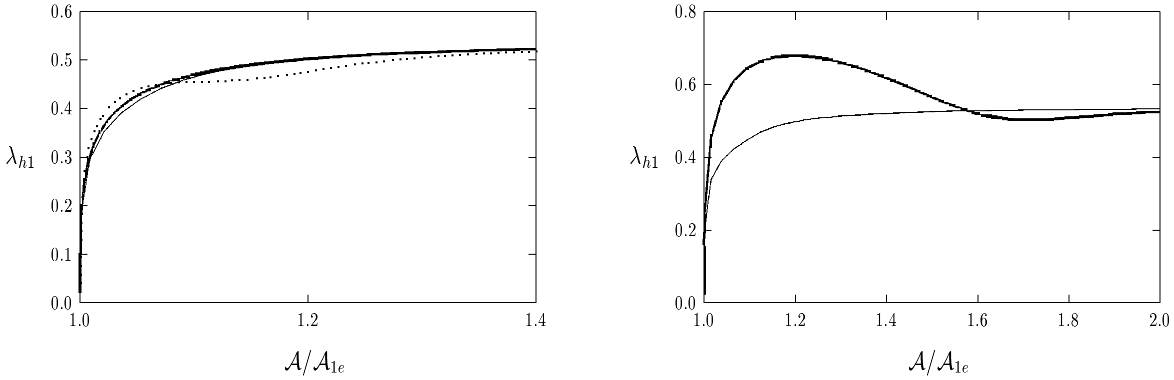
We now give some numerical results for the stability thresholds studied rigorously in §3.2. Here we will only consider instabilities of the small solution. To illustrate our results, we take $D = 0.75$ and $D = 0.1$. For $D = 0.75$, the spike interaction is strong and oscillatory instabilities occur for small values of τ. Moreover, for $D = 0.75$, spike competition instabilities due to positive real eigenvalues can also occur far from the existence threshold $\mathcal{A}_{ke}$. Alternatively, for $D = 0.1$, the inter-spike interaction is relatively weak and the finite domain does not play a very significant role in generating instabilities, unless we have many spikes. More specifically, for $k\sqrt{D} \ll 1$, instabilities due to positive real eigenvalues only occur very close to the existence threshold $\mathcal{A}_{ke}$. Recall also from (1.4), that our formulation of the Gray-Scott model required that $\tau > 1$. All of the stability thresholds for a Hopf bifurcation computed below for $D = 0.75$ and $D = 0.1$ satisfy this requirement.

For a one-spike solution, in Fig. 9(a) we plot the Hopf bifurcation threshold τ_{h1} versus $\mathcal{A}/\mathcal{A}_{1e}$ for $D = 0.75$, $D = 0.25$, and $D = 0.1$. These results suggest that τ_{h1} is an increasing function of $\mathcal{A}/\mathcal{A}_{1e}$, and that the stability threshold τ_{h1} is larger for smaller values of D. This is intuitive since for smaller values of D, the interaction of the spike with the boundaries of the domain is weaker. In Fig. 9(b) we plot the values τ_{m1} where complex conjugate eigenvalues first merge onto the real axis. In Fig. 10(a) we plot the Hopf bifurcation frequency λ_{h1} versus $\mathcal{A}/\mathcal{A}_{1e}$. This frequency tends to a limiting value $\lambda_{h1} \approx 0.53$ for $\mathcal{A}/\mathcal{A}_{1e} \gg 1$ (see §4).

Next we consider two-spike solutions for $D = 0.75$ and $D = 0.1$. From (2.12) and (3.27b), we calculate the critical values $\mathcal{A}_{2e}$ and $\mathcal{A}_{2L}$ as

$$\mathcal{A}_{2e} = 5.158\,, \quad \mathcal{A}_{2L} = 5.633\,, \quad \text{for} \quad D = 0.75\,; \qquad \mathcal{A}_{2e} = 6.427\,, \quad \mathcal{A}_{2L} = 6.433\,, \quad \text{for} \quad D = 0.1\,. \tag{3.42}$$

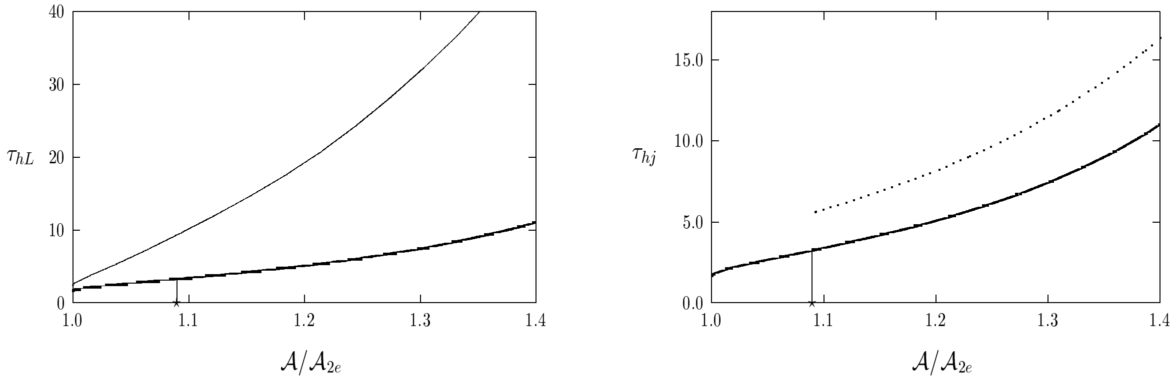
In Fig. 11(a) we plot the Hopf bifurcation value τ_{hL} as a function of $\mathcal{A}/\mathcal{A}_{2e}$ for both values of D. This threshold is set by the $j = 1$ mode. In this diagram we have indicated, by a vertical line, the value $\mathcal{A}_{2L}/\mathcal{A}_{2e}$. Recall that below this value there are eigenvalues on the positive real axis for any $\tau > 0$. Since $\mathcal{A}_{2L}$ is very close to $\mathcal{A}_{2e}$ when $D = 0.1$, a competition instability for this value of D occurs only in a narrow parameter regime. The corresponding frequency λ_{h1} is plotted in Fig. 10(b). In Fig. 11(b) we plot a more detailed stability threshold for $D = 0.75$. In this plot we have indicated by a dashed line the threshold τ_{h2} at which an additional pair of complex conjugate eigenvalues enter the right half-plane. Hence, for $\mathcal{A} > \mathcal{A}_{2L}$ and $\tau > \tau_{h2}$ there are four eigenvalues in the right half-plane. From the theory developed earlier, the curve τ_{h2} versus $\mathcal{A}/\mathcal{A}_{2e}$ terminates when $\mathcal{A} \to \mathcal{A}_{2L}^{+}$ since the complex conjugate pair for the $j = 2$ mode coalesce at the origin $\lambda = 0$. One eigenvalue then moves along the positive real axis, while the other moves along the negative real axis as $\mathcal{A}$ is decreased below $\mathcal{A}_{2L}$.



(a) λ_{h1}

(b) λ_{h1}

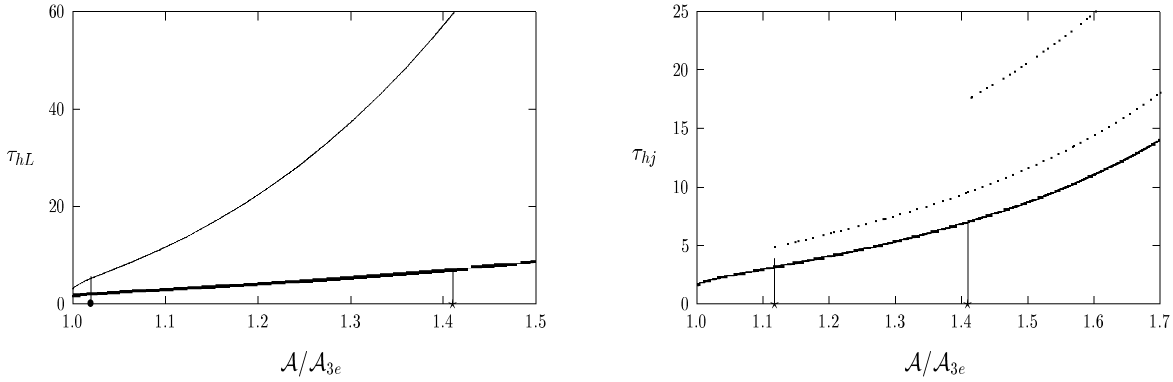
Figure 10: Left figure: the Hopf bifurcation frequency λ_{h1} versus $\mathcal{A}/\mathcal{A}_{1e}$ for a one-spike solution with $D = 0.75$ (dashed curve), $D = 0.25$ (solid curve), and $D = 0.1$ (heavy solid curve). Right figure λ_{h1} for a two-spike solution with $D = 0.75$ (heavy solid curve) and $D = 0.1$ (solid curve).



(a) τ_{hL}

(b) τ_{hj}

Figure 11: Left figure: the Hopf bifurcation value τ_{hL} versus $\mathcal{A}/\mathcal{A}_{2e}$ for a two-spike solution with $D = 0.75$ (heavy solid curve) and $D = 0.1$ (solid curve). Right figure: τ_{hL} (heavy solid curve) and τ_{h2} (dashed curve) for a two-spike solution with $D = 0.75$.



(a) τ_{hL}

(b) τ_{hj}

Figure 12: Left figure: the Hopf bifurcation value τ_{hL} versus $\mathcal{A}/\mathcal{A}_{3e}$ for a three-spike solution with $D = 0.75$ (heavy solid curve) and $D = 0.1$ (solid curve). Right figure: τ_{hL} (heavy solid curve) and τ_{hj} (dashed curves) for $j = 2, 3$ for a three-spike solution with $D = 0.75$.

For three-spike solutions with $D = 0.75$ and $D = 0.1$, we calculate from (2.12), (3.25), and (3.27b), that

$$\mathcal{A}_{3e} = 6.145\,, \quad \mathcal{A}_2 = 6.864\,, \quad \mathcal{A}_{3L} = 8.6857\,, \quad \text{for} \quad D = 0.75\,, \tag{3.43a}$$

$$\mathcal{A}_{3e} = 6.960\,, \quad \mathcal{A}_2 = 6.978\,, \quad \mathcal{A}_{3L} = 7.0904\,, \quad \text{for} \quad D = 0.1\,. \tag{3.43b}$$

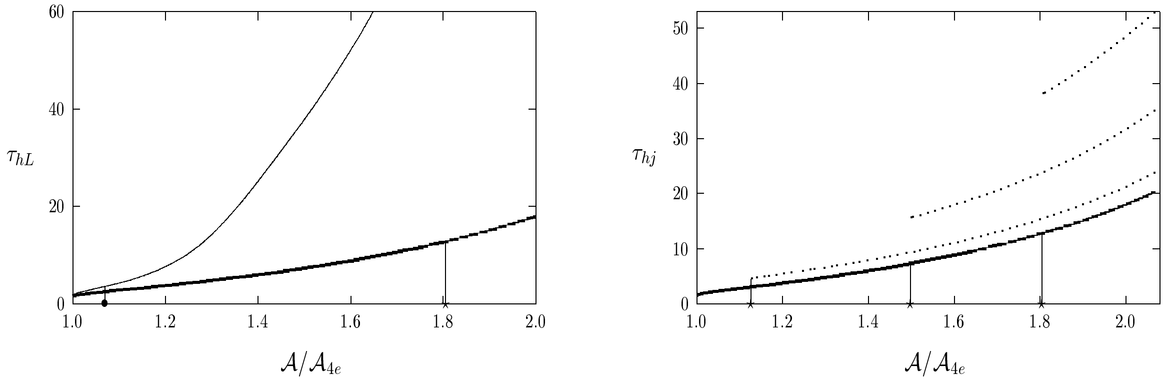
Likewise, for four-spike solutions we obtain the thresholds

$$\mathcal{A}_{4e} = 7.023\,, \quad \mathcal{A}_2 = 7.914\,, \quad \mathcal{A}_3 = 10.52\,, \quad \mathcal{A}_{4L} = 12.69\,, \quad \text{for} \quad D = 0.75\,, \tag{3.44a}$$

$$\mathcal{A}_{4e} = 7.589\,, \quad \mathcal{A}_2 = 7.619\,, \quad \mathcal{A}_3 = 7.830\,, \quad \mathcal{A}_{4L} = 8.127\,, \quad \text{for} \quad D = 0.1\,. \tag{3.44b}$$

In Fig. 12(a) and Fig. 13(a) we plot τ_{hL} versus $\mathcal{A}/\mathcal{A}_{ke}$ for three and four-spike solutions when $D = 0.75$ and $D = 0.1$. For the larger value $D = 0.75$, in Fig. 12(b) and Fig. 13(b) we show the detailed stability diagram by plotting τ_{hj} for $j = 2, \ldots, k$. In each case, the stability threshold τ_{hL} is set by the $j = 1$ mode representing synchronous oscillations. The vertical lines in these figures indicate where positive real eigenvalues enter the right half-plane. The information in Proposition 3.12 is succinctly contained in these detailed stability diagrams. For instance, from Fig. 13(b) for $D = 0.75$, we see that if $\mathcal{A}$ satisfies $7.914 < \mathcal{A} < 10.52$, then there are two real eigenvalues in the right half-plane when $\tau < \tau_{hL}$ and six eigenvalues (at least two real) in the right half-plane when $\tau > \tau_{h2}$.

We now consider some examples of the theory, and we compare the results with full-scale numerical simulations of (1.6). The solution to (1.6) is computed using the routine D03PCF of the NAG library [23] with 1500 uniformly spaced meshpoints. In each of the experiments below, we have taken the initial



(a) τ_{hL}

(b) τ_{hj}

Figure 13: Left figure: the Hopf bifurcation value τ_{hL} versus $\mathcal{A}/\mathcal{A}_{4e}$ for a four-spike solution with $D = 0.75$ (heavy solid curve) and $D = 0.1$ (solid curve). Right figure: τ_{hL} (heavy solid curve) and τ_{hj} (dashed curves) for $j = 2, 3, 4$ for a four-spike solution with $D = 0.75$.

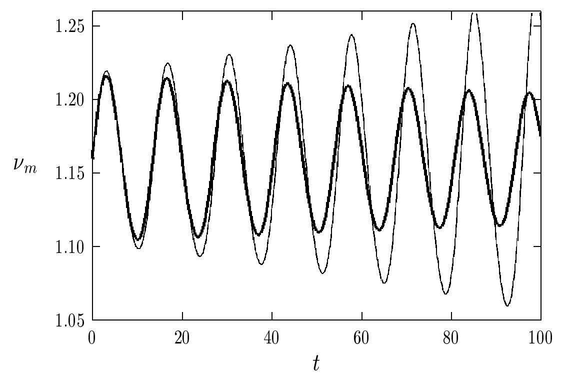
condition for (1.6) in the form

$$\nu(x,0) = \nu_-(x) \left[1 + 0.01 \sum_{j=1}^{k} b_j \cos\left(\frac{\pi(x - x_j)}{\varepsilon} \right) e^{-(x-x_j)^2/(2\varepsilon^2)} \right], \qquad u(x,0) = u_-(x). \tag{3.45}$$

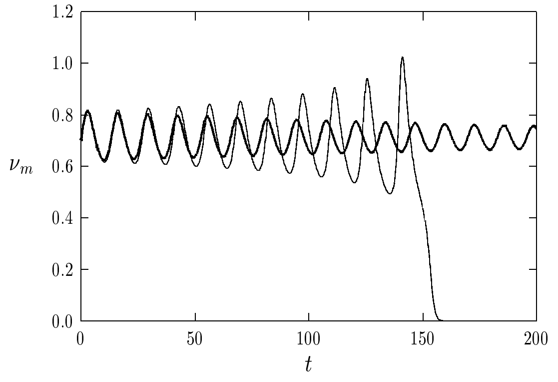
Here $b_j \equiv 1$ if j is odd and $b_j \equiv -1$ if j is even. The k-spike equilibrium solution, ν_-, and u_-, is given in Proposition 2.1.

Example 1: We first consider one-spike solutions to (1.6). In Fig. 14(a) we plot the spike amplitude $\nu_m \equiv \nu(0,t)$ versus t for the parameter values $D = 0.75$, $\mathcal{A} = 4.563$, and $\varepsilon = 0.01$. From this figure, we see that when $\tau = 7.5$ we get decaying oscillations, whereas the oscillations grow when $\tau = 7.8$. From the data used to generate Fig. 9(a), we get the Hopf bifurcation value $\tau_{hL} \approx 7.7$. In Fig. 14(b) we plot the spike amplitude $\nu_m \equiv \nu(0)$ versus t for the parameter values $D = 0.1$, $\mathcal{A} = 6.59$, and $\varepsilon = 0.01$. The oscillations are found to decay when $\tau = 8.6$, and they grow when $\tau = 8.8$. The Hopf bifurcation value from Fig. 9(a) is $\tau_{hL} \approx 8.7$. Although our theory correctly predicts the Hopf bifurcation value, it does not explain the large-scale oscillations seen in Fig. 14(b) whereby the instability ultimately leads to the annihilation of the spike.

Example 2: Next, we consider a three-spike solution to (1.6) with $D = 0.75$ and $\varepsilon = 0.1$. We first take the parameter values $\mathcal{A} = 8.6$ and $\tau = 2.0$. Since $\mathcal{A}$ satisfies $\mathcal{A}_2 < \mathcal{A} < \mathcal{A}_{3L}$ from (3.43a), and τ is below the Hopf bifurcation value, we expect a competition instability. In Fig. 15(a) we plot the initial condition (3.45) used for the numerical solution of (1.6). In Fig. 15(b) we show the competition instability that occurs from the unique real positive eigenvalue in the right half-plane. Although we



(a) ν_m versus t



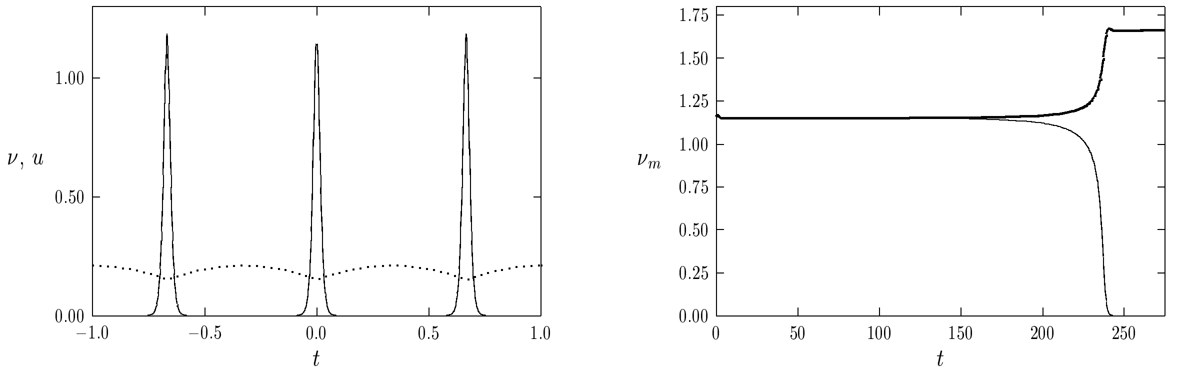
(b) ν_m versus t

Figure 14: Example 1: Plots of ν_m versus t for $k = 1$ with $\varepsilon = 0.01$. Left figure: $D = 0.75$ and $\mathcal{A} = 4.563$. The labels are $\tau = 7.5$ (heavy solid curve) and $\tau = 7.8$ (solid curve). Right figure: $D = 0.1$ and $\mathcal{A} = 6.59$. The labels are $\tau = 8.6$ (heavy solid curve) and $\tau = 8.8$ (solid curve).

can correctly predict the onset of the competition instability, the nonlinear mechanisms leading to the annihilation of the second spike as seen in Fig. 15(b) is an open problem. Next, we take the slightly larger value $\mathcal{A} = 8.86$, so that now $\mathcal{A} > \mathcal{A}_{3L} = 8.6857$. For this value, the data used to generate Fig. 12(a) yields the Hopf bifurcation value $\tau_{hL} \approx 7.5$. For $\tau = 7.25$, in Fig. 16(a) we show a synchronous decaying oscillation in the spike amplitudes. For the slightly larger value $\tau = 7.6$, in Fig. 16(b) we show both the onset of a synchronous oscillatory instability, and the ultimate simultaneous annihilation of the three spikes.

Example 3: This is the four-spike example for $D = 0.1$ and $\varepsilon = 0.01$ shown in Fig. 2 and Fig. 3 of the introduction §1. Since $\mathcal{A}_3 = 7.83$ and $\mathcal{A}_{4L} = 8.127$ from (3.44b), we will get a competition instability when $\mathcal{A} = 8.0$ and τ is below the Hopf bifurcation value. This is precisely what is seen in Fig. 2(b) where $\tau = 2.0$. For this value of $\mathcal{A}$, from the data used to generate Fig. 13(a) we will have a Hopf bifurcation when $\tau = 3.4$. At this value of τ there is a pair of complex conjugate eigenvalues on the imaginary axis, together with a positive real eigenvalue. Therefore, for $\tau = 3.2$, the instability should be a superposition of a competition instability and a synchronous decaying oscillation in the spike amplitudes. This is shown in Fig. 17. These two types of instabilities lead to an initial synchronization of the oscillations of the spike amplitudes, followed by a spike competition.

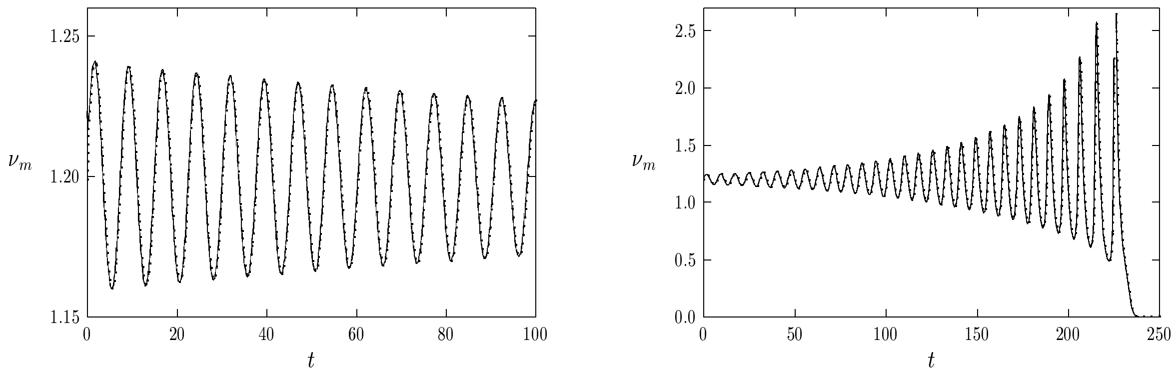
Alternatively, suppose we take $\mathcal{A} = 8.3$ so that $\mathcal{A} = 8.3 > \mathcal{A}_{4L}$. From the data used to plot Fig. 13(a) we calculate, for this value of $\mathcal{A}$, that the Hopf bifurcation occurs when $\tau \approx 4.0$. In Fig. 3(a) we show a synchronous decaying oscillation in the spike amplitudes when $\tau = 3.8$. In Fig. 3(b) we show a synchronous oscillatory instability in the spike amplitudes when $\tau = 4.1$ leading to simultaneous spike annihilation. Since D is small for this example, we might intuitively expect that the inter-spike



(a) The initial condition

(b) ν_m versus t

Figure 15: Example 2: Here $k = 3$, $D = 0.75$, $\mathcal{A} = 8.6$, $\varepsilon = 0.01$, and $\tau = 2.0$. Left figure: the initial condition for ν (solid curve) and u (dashed curve). Right figure: The spike amplitudes ν_m. The middle spike is annihilated, and the other two spikes have a common amplitude.



(a) ν_m versus t

(b) ν_m versus t

Figure 16: Example 2: The parameters are $k = 3$, $D = 0.75$, $\mathcal{A} = 8.86$, and $\varepsilon = 0.01$. Left figure: synchronous decaying amplitude oscillations when $\tau = 7.25$. Right figure: synchronous oscillatory instability for $\tau = 7.6$, leading to a collapse of the three spikes. The amplitudes of the spikes trace out identical trajectories.

interaction is very weak. However, for four spikes, the interaction is sufficiently strong so that these instabilities occur rather far from the existence threshold $\mathcal{A}_{4e} = 7.589$.

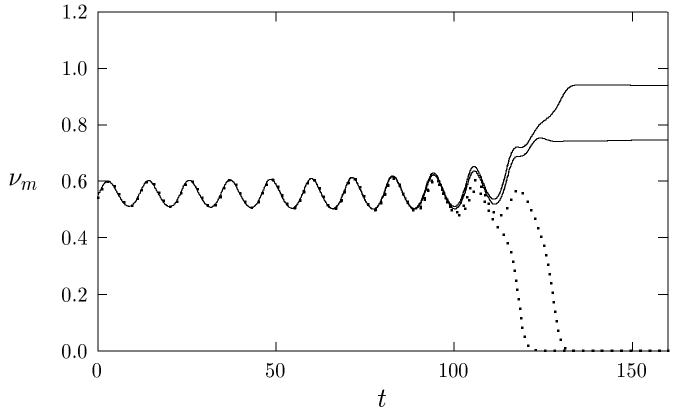
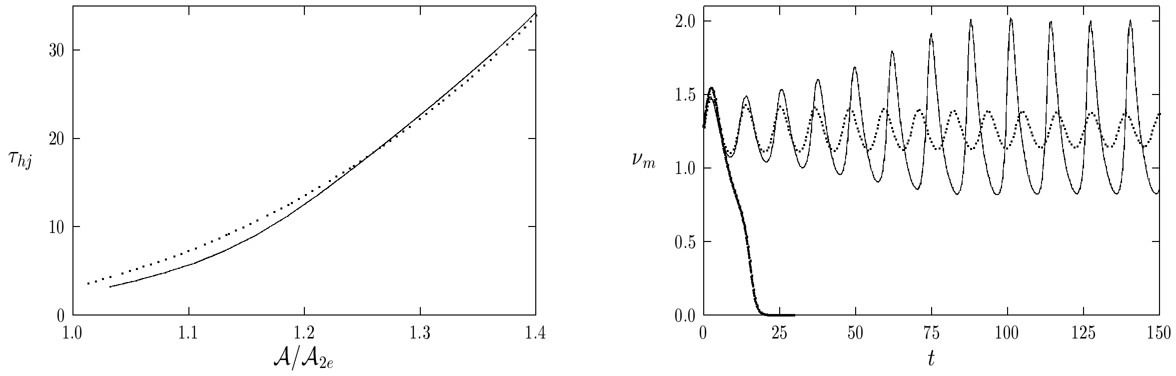


Figure 17: Example 3: The spike amplitudes when $k = 4$, $D = 0.1$, $\varepsilon = 0.01$, $\mathcal{A} = 8.0$ and $\tau = 3.2$. The synchronized oscillations of the spike amplitudes is followed by a competition leading to the annihilation of the second spike and then the fourth spike. Of the two remaining spikes, the third spike has the largest amplitude.

Example 4: In this example we try to determine asynchronous oscillations of a two-spike solution. We let $D = 0.25$ and we compute the critical values τ_{h1} and τ_{h2} versus $\mathcal{A}/\mathcal{A}_{2e}$ where complex conjugate pairs of eigenvalues enter the right half-plane. The plot is shown in Fig. 18(a). The Hopf bifurcation value τ_{hL} is the minimum of these two values. From this figure we notice that the two curves τ_{h1} and τ_{h2} cross exactly once at some value of $\mathcal{A}/\mathcal{A}_{2e}$, and that they both asymptote to a common limiting behavior as $\mathcal{A}/\mathcal{A}_{2e} \to \infty$. This limiting behavior is analyzed in §5 when we study the intermediate regime. On the range where $\tau_{h1} < \tau_{h2}$ in Fig. 18(a) we expect synchronous oscillations as τ is increased beyond τ_{h1}. Asynchronous oscillations should occur on the narrow range where $\tau_{h2} < \tau_{h1}$ as τ is increased. As a remark, when $D = 0.25$ we compute that $\mathcal{A}_{2e} = 5.614$ and $\mathcal{A}_{2L} = 5.681$. Hence, there will be no competition instabilities due to real eigenvalues crossing into the right half-plane when $\mathcal{A} > 5.681$.

To illustrate the resulting spike dynamics we take $\mathcal{A} = 7.9377$ and $\varepsilon = 0.01$. We compute that $\tau_{h1} = 36.1$ and $\tau_{h2} = 35.6$. Hence, the complex eigenvalues that first enter the right half-plane as τ is increased corresponds to the asynchronous mode. In Fig. 18(b) we plot the numerically computed spike amplitudes for three values of τ starting from the asynchronous initial condition (3.45). For each of these values of τ, the spike amplitudes are found to synchronize very quickly in time. For $\tau = 40$, the spike amplitudes both collapse, for $\tau = 34$ they exhibit a large-scale periodic oscillation, and for $\tau = 30$ the oscillations relax back to the equilibrium state as t increases. We have done many other examples to try to determine asynchronous oscillations in the low feed-rate regime without success. However, asynchronous oscillations for the spike amplitudes have been observed in the pulse-splitting



(a) τ_{h1} and τ_{h2} versus $\mathcal{A}/\mathcal{A}_{2e}$

(b) ν_m versus t

Figure 18: Example 4: Left figure: The critical values τ_{h1} (solid curve) and τ_{h2} (dashed curve) versus $\mathcal{A}/\mathcal{A}_{2e}$ for a two-spike solution with $D = 0.25$. Right figure: Spike amplitudes for $D = 0.25$, $\varepsilon = 0.01$, with $\tau = 30$ (dashed curve), $\tau = 34$ (solid curve), and $\tau = 40$ (heavy solid curve).

parameter regime where $\mathcal{A} = O(\varepsilon^{-1/2})$, or equivalently where $A = O(1)$ (see [13]). There are two possibilities for not observing such oscillations in the low feed-rate regime. One possibility is that the asynchronous mode is unstable in the weakly nonlinear regime. The second possibility results from our observation that whenever $\tau_{hL} = \tau_{hk}$, so that stability is set by the asynchronous mode, the threshold for the synchronous mode τ_{h1} has a numerical value that is typically very close to τ_{hk}. Hence, whenever asynchronous oscillations are theoretically possible as τ is increased, we have typically found that the more dominant synchronous mode is essentially also present.

3.4 The Infinite-Line Problem

Next, we determine the stability threshold for the infinite-line problem where (1.6) is posed on $-\infty < x < \infty$. We will only consider the stability of a one-spike solution centered at $x = 0$. Results for the infinite line problem were obtained using dynamical systems techniques in [4] and [5], and using a formal asymptotic analysis in [21].

The only main modification needed to the analysis leading to Propositions 2.1 and 3.1 is that the Green's function $G(x; x_j)$ in (2.6) must be replaced with $G(x; 0)$, where

$$DG_{xx} - G = -\delta(x) \,, \qquad -\infty < x < \infty \,; \qquad G \to 0 \quad \text{as} \quad |x| \to \infty \,. \tag{3.46}$$

The solution to (3.46) is

$$G(x; 0) = \left(\frac{\theta_0}{2}\right) e^{-\theta_0 |x|} \,, \qquad \theta_0 = D^{-1/2} \,. \tag{3.47}$$

Since the asymptotic construction of the one-spike equilibrium solution parallels that in §2.1, we only give the result.

Proposition 3.16: *Let $\varepsilon \to 0$, with $\mathcal{A} = O(1)$ and $D = O(1)$ in (1.6) defined now on the infinite line $-\infty < x < \infty$. Then, when $\mathcal{A} > \mathcal{A}_{1e}$, there are two one-spike solutions given asymptotically by*

$$\nu_{\pm}(x) \sim \frac{1}{\mathcal{A}U_{\pm}} w\left[\varepsilon^{-1}x\right] \,; \qquad u_{\pm}(x) \sim 1 - \frac{6}{\mathcal{A}^2 U_{\pm}} G(x;0)\,. \tag{3.48}$$

We label u_+, ν_+ and u_-, ν_- as the large and small solutions, respectively. In (3.48), w and G satisfy (2.3) and (3.47), respectively. Moreover, $U_{\pm}$ are given by

$$U_{\pm} \equiv \frac{1}{2}\left[1 \pm \sqrt{1 - \frac{\mathcal{A}_{1e}^2}{\mathcal{A}^2}}\right], \qquad \mathcal{A}_{1e} \equiv \sqrt{12} D^{-1/4}\,. \tag{3.49}$$

Since the derivation of the nonlocal eigenvalue problem is similar to that done in §3 for a multi-spike solution on the finite interval, we only give the highlights of the derivation. We introduce Φ and η by

$$\nu(x) = \nu_{\pm}(x) + e^{\lambda t}\Phi\left[\varepsilon^{-1}x\right], \qquad u(x) = u_{\pm}(x) + e^{\lambda t}\eta(x)\,. \tag{3.50}$$

Substituting (3.50) and (3.48) into (2.17b), we obtain in place of (3.4) that η satisfies

$$D\eta_{xx} - (1+\tau\lambda)\,\eta = 0\,, \quad -\infty < x < \infty\,; \qquad \eta \to 0\,, \quad \text{as} \quad |x| \to \infty\,, \tag{3.51a}$$

$$[\eta]_0 = 0\,, \qquad [D\eta_x]_0 = \frac{6}{\mathcal{A}^2 U_{\pm}^2}\eta(0) + \frac{2}{\mathcal{A}}\int_{-\infty}^{\infty} w(y)\Phi(y)\,dy\,. \tag{3.51b}$$

Substituting (3.50) and (3.48) into (2.17a), we obtain in place of (3.8) that $\Phi(y)$ satisfies

$$\Phi'' - \Phi + 2w\Phi + \frac{\eta(0)}{\mathcal{A}U_{\pm}^2} w^2 = \lambda\Phi\,, \qquad -\infty < y < \infty\,, \tag{3.52}$$

with $\Phi(y) \to 0$ as $|y| \to \infty$. The nonlocal eigenvalue problem analogous to (3.12) is obtained by solving (3.51) for $\eta(0)$, substituting the resulting expression into (3.52), and then using (3.49) for $U_{\pm}$. In this way, we obtain that $\Phi(y)$ satisfies the conditions of Proposition 3.2, except that now χ in (3.13) is to be replaced with

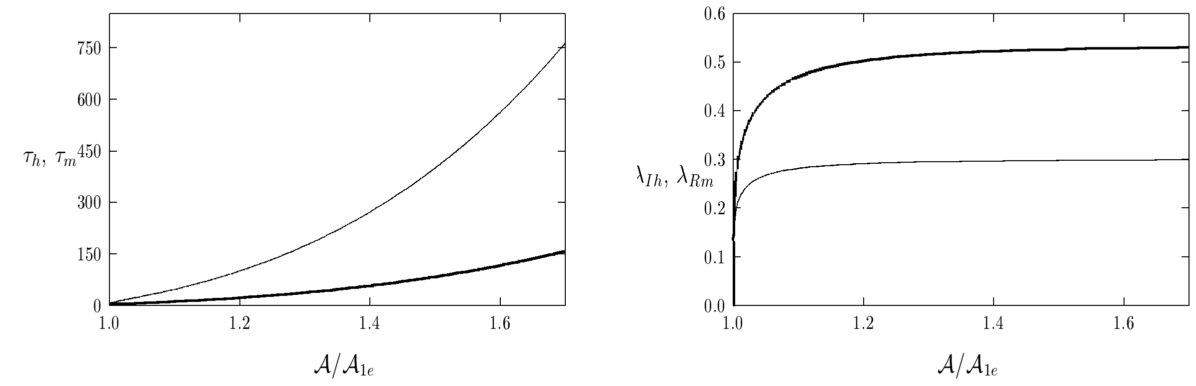
$$\chi \equiv \frac{2s}{s + \sqrt{1+\tau\lambda}}\,, \qquad s \equiv \frac{1-U_{\pm}}{U_{\pm}}\,. \tag{3.53}$$

Notice that the multiplier χ in (3.53) can be obtained by taking the limit $D \to 0$ in the multiplier in (3.13) corresponding to a k-spike symmetric pattern on the finite line. In place of (3.18) and (3.19), we then obtain that the eigenvalues λ are the roots of $g(\lambda) = 0$, where

$$g(\lambda) \equiv C(\lambda) - f(\lambda)\,, \qquad f(\lambda) \equiv \frac{\int_{-\infty}^{\infty} w\,(L_0 - \lambda)^{-1} w^2\,dy}{\int_{-\infty}^{\infty} w^2\,dy}\,, \qquad C(\lambda) = \frac{1}{2} + \frac{\sqrt{1+\tau\lambda}}{2s}\,. \tag{3.54}$$

The rigorous spectral approach used in §3.2 can be applied directly to this problem. In this way, we obtain the following result:

Proposition 3.17: *Let $\varepsilon \to 0$ and consider a one-spike solution to (1.6) on the infinite line. The large solution u_+, ν_+, where $0 < s < 1$, is unstable for any $\tau > 0$ as a result of a positive eigenvalue on the real axis. The small solution u_-, ν_-, where $s > 1$, is stable with respect to the large eigenvalues for τ sufficiently small, and is unstable with respect to the large eigenvalues for τ sufficiently large. There exists a value $\tau_h = \tau_h(s)$ (possibly non-unique) where there are complex conjugate eigenvalues on the imaginary axis.*



(a) τ_h and τ_m

(b) λ_{Ih} and λ_{Rm}

Figure 19: Left figure: the Hopf bifurcation value τ_h (heavy solid curve) versus $\mathcal{A}/\mathcal{A}_{1e}$. The complex eigenvalues merge onto the positive real axis when $\tau = \tau_m$ (solid curve). Right figure: The corresponding critical values λ_{Ih} (heavy solid curve) and λ_{Rm} (solid curve).

Using the numerical method described briefly in §3.3, we compute $\tau_h = \tau_h(s)$ numerically. Using (3.49) and $s = (1 - U_\pm)/U_\pm$, we get τ_h as a function of $\mathcal{A}/\mathcal{A}_{1e} = (1+s)\left[2\sqrt{s}\right]^{-1}$. In addition, we can compute the value of τ, labeled by $\tau_m(s)$, where the complex conjugate eigenvalues merge onto the real axis. A simple calculation using (3.53) and the local properties of $f(\lambda)$ as $\lambda \to 0$ given in Propositions 3.5 and 3.7 above, shows that when $s = 1$ we have a double root to $g(\lambda) = 0$ at $\lambda = 0$ and $\tau = 3$. The numerical results for τ_h and τ_m as a function of $\mathcal{A}/\mathcal{A}_{1e}$ are shown in Fig. 19(a). Although we cannot prove that τ_h is unique, numerical evidence suggests that when $\tau > \tau_h$ there are always two eigenvalues in the right half-plane. In Fig. 19(b) we plot the function λ_{Ih} versus $\mathcal{A}/\mathcal{A}_{1e}$ where we have a Hopf bifurcation. In this figure we also plot the function λ_{Rm} versus $\mathcal{A}/\mathcal{A}_{1e}$ where complex conjugate eigenvalues in the right half-plane first merge onto the real axis. From this figure, we notice that λ_{Ih} and λ_{Rm} have limiting behaviors for $s \gg 1$. This limiting behavior is analyzed in a more general context in §4.

The critical value τ_h for this infinite-line problem can be obtained by taking the limiting $\sqrt{D}k \ll 1$ in all of the results of §3.2 for the finite domain problem (1.6). Therefore, the results for $D = 0.1$ and

$k = 1, \ldots, 3$ computed above in §3.3 should closely approximate those for the infinite-line problem.

Finally, we relate our results for the infinite-line problem with those of [21]. To show the equivalence of our results with those in [21], we integrate (3.12) over $-\infty < y < \infty$ to obtain

$$(2-\chi) \int_{-\infty}^{\infty} w\Phi \, dy = (\lambda + 1) \int_{-\infty}^{\infty} \Phi \, dy \, . \tag{3.55}$$

Solving (3.55) for $\int_{-\infty}^{\infty} \Phi \, dy$, we substitute the resulting expression into (3.12) and use $\int_{-\infty}^{\infty} w^2 \, dy = 6$ to get

$$L_0 \Phi - \frac{\chi(\lambda+1)}{6(2-\chi)} w^2 \int_{-\infty}^{\infty} \Phi \, dy = \lambda \Phi \, . \tag{3.56}$$

Finally, using the definition of χ in (3.53) together with $w(y) = \frac{3}{2}\text{sech}^2(y/2)$, we can write (3.56) for the small solution as

$$\Phi'' - \Phi + 3\,\text{sech}^2\left(\frac{y}{2}\right)\Phi - \alpha \, \text{sech}^4\left(\frac{y}{2}\right) \int_{-\infty}^{\infty} \Phi \, dy = \lambda \Phi \, , \tag{3.57a}$$

where

$$\alpha = \frac{3\mathcal{A}^2}{8\mathcal{A}_{1e}^2} \frac{(1+\lambda)}{\sqrt{1+\tau\lambda}} \left(1 + \sqrt{1 - \frac{\mathcal{A}_{1e}^2}{\mathcal{A}^2}} \right)^2 \, . \tag{3.57b}$$

The nonlocal eigenvalue problem (3.57) is given in equation (2.13) of [21]. Our plots of τ_h and λ_{Ih} are equivalent to those in Fig. 2.4c and Fig. 2.4d of [21]. In Appendix B of [21], some properties of the spectrum for (3.57) were obtained. Our rigorous approach to the nonlocal eigenvalue problem, with results summarized in Proposition 3.17, provides an alternative proof of the stability properties.

4 The Intermediate Regime: $O(1) \ll \mathcal{A} \ll O(\varepsilon^{-1/2})$

The derivations of the spectral problem in Proposition 3.2 was based on linearizing (1.6) around the solution constructed in Proposition 2.1. A crucial feature of this solution is that the leading-order inner problems for $u_\pm$ and $\nu_\pm$ decouple near each spike. In particular, this implies that $u_\pm = U_\pm + O(\varepsilon)$ in the core of each spike. The eigenvalue problem in Proposition 3.2 is valid for the range of $\mathcal{A}$ where such a leading-order decoupling of the inner problems for $u_\pm$ and $\nu_\pm$ can be made. We now show formally that, when $D = O(1)$, this decoupling property holds for the range $O(1) \leq \mathcal{A} \ll O(\varepsilon^{-1/2})$. For the intermediate regime, which we define by $O(1) \ll \mathcal{A} \ll O(\varepsilon^{-1/2})$, we then derive certain scaling laws for the stability thresholds calculated in §3.

To construct a k-spike pattern, we first construct a symmetric one-spike equilibrium solution to (1.6) on the interval $-l < x < l$. Then, by setting $l = 1/k$, we obtain the result for a k-spike pattern for (1.6) on $-1 < x < 1$. In the inner region near $x = 0$, we let $y = \varepsilon^{-1}x$, $\nu_i(y) = \nu(\varepsilon y)$, $u_i(y) = u(\varepsilon y)$, and we expand the inner equilibrium solution for (1.6) on $-l < x < l$ as

$$\nu_i(y) = \nu_{i0}(y) + \varepsilon \nu_{i1}(y) + \cdots \, , \qquad u_i(y) = u_{i0}(y) + \varepsilon u_{i1}(y) + \cdots \, . \tag{4.1}$$

Substituting (4.1) into the equilibrium problem for (1.6), we collect powers of ε to get

$$\nu_{i0}'' - \nu_{i0} + \mathcal{A} u_{i0} \nu_{i0}^2 = 0\,; \qquad u_{i0}'' = 0\,, \qquad -\infty < y < \infty\,, \tag{4.2a}$$
$$\nu_{i1}'' - \nu_{i1} + 2\mathcal{A} u_{i0}\nu_{i0}\nu_{i1} = -\mathcal{A} u_{i1}\nu_{i0}^2\,; \qquad D u_{i1}'' = u_{i0}\nu_{i0}^2\,, \qquad -\infty < y < \infty\,. \tag{4.2b}$$

In terms of the solution w to (2.3), the solution to (4.2a) is

$$\nu_{i0}(y) = \frac{1}{\mathcal{A}U} w(y)\,, \qquad u_{i0}(y) = U\,. \tag{4.3}$$

In the outer region, ν is exponentially small and $\varepsilon^{-1} u \nu^2$ can be represented as a Dirac mass, which can be calculated using the leading-order inner solutions u_{i0} and ν_{i0}. Therefore, the outer solution u_0 satisfies,

$$D u_0'' + (1 - u_0) = \frac{6}{\mathcal{A}^2 U}\delta(x) \qquad -l < x < l\,; \qquad u_{0x}(\pm l) = 0\,. \tag{4.4}$$

The solution to (4.4) can be written in terms of the Green's function on $-l < x < l$, which satisfies

$$D G_{lxx} - G_l = -\delta(x)\,, \quad -l < x < l\,; \qquad G_{lx}(\pm l; 0) = 0\,. \tag{4.5}$$

A simple calculation gives

$$G_l(x;0) = \left(\frac{\theta_0}{2}\right) \frac{\cosh\left[(l - |x|)\theta_0\right]}{\sinh\left(l\theta_0\right)}\,, \qquad \theta_0 \equiv D^{-1/2}\,. \tag{4.6}$$

Therefore, we obtain that

$$u_0(x) = 1 - \frac{6}{\mathcal{A}^2 U} G(x;0)\,. \tag{4.7}$$

To determine the matching condition for the inner and outer solutions for u, we expand u_0 in (4.7) in right and left-sided limits as $x \to 0^{\pm}$. In this way, we obtain that U satisfies

$$U = 1 - \frac{6 G_l(0;0)}{\mathcal{A}^2 U}\,, \qquad G_l(0;0) = \left(\frac{\theta_0}{2}\right) \coth\left(\theta_0 l\right)\,, \tag{4.8}$$

and that the solution u_{i1} to (4.2b) must have the far field behavior

$$u_{i1} \sim -\frac{6 G_{lx}(0^{\pm};0)}{\mathcal{A}^2 U} y\,, \qquad \text{as} \quad y \to \pm\infty\,. \tag{4.9}$$

Next, we introduce new inner variables $\hat{u}_{i1}$ and $\hat{\nu}_{i1}$ defined by

$$u_{i1} = \frac{1}{D\mathcal{A}^2 U}\hat{u}_{i1}\,, \qquad \nu_{i1} = \frac{1}{D\mathcal{A}^3 U^3}\hat{\nu}_{i1}\,. \tag{4.10}$$

Substituting (4.10) and (4.9) into (4.2), and noting that $G_{lx}(0^{\pm};0) = \mp\theta_0^2/2$, we obtain an explicit two-term inner expansion. The outer expansion is obtained by substituting (4.8) into (4.7). Then, identifying $l = 1/k$, we obtain the following result for a k-spike pattern:

Proposition 4.1: *For $\varepsilon \to 0$, consider a k-spike equilibrium solution to (1.6). Then, when $\mathcal{A} > \mathcal{A}_{ke}$ there are two such solutions; the small solution u_-, ν_-, and the large solution u_+, ν_+. The two-term inner expansions in the core of each spike are*

$$\nu_{i\pm}(y) \sim \frac{1}{\mathcal{A}U_\pm}\left[w(y) + \frac{\varepsilon}{\mathcal{A}^2 U_\pm^2 D}\hat{\nu}_{i1}(y) + \cdots\right], \qquad u_{i\pm}(y) \sim U_\pm\left[1 + \frac{\varepsilon}{\mathcal{A}^2 U_\pm^2 D}\hat{u}_{i1}(y) + \cdots\right], \tag{4.11a}$$

where $U_\pm$ and $\mathcal{A}_{ke}$ are given by

$$U_\pm = \frac{1}{2}\left[1 \pm \sqrt{1 - \frac{\mathcal{A}_{ke}^2}{\mathcal{A}^2}}\right], \qquad \mathcal{A}_{ke} \equiv \sqrt{\frac{12\theta_0}{\tanh(\theta_0/k)}}, \qquad \theta_0 \equiv D^{-1/2}. \tag{4.11b}$$

Here $\hat{\nu}_{i1}(y)$ and $\hat{u}_{i1}(y)$ satisfy the parameter-independent inner problems on $-\infty < y < \infty$

$$\hat{\nu}_{i1}'' - \hat{\nu}_{i1} + 2w\hat{\nu}_{i1} = -\hat{u}_{i1}w^2, \qquad \hat{\nu}_{i1} \to 0, \quad \text{as} \quad |y| \to \infty, \tag{4.12a}$$

$$\hat{u}_{i1}'' = w^2, \qquad \hat{u}_{i1} \sim 3|y|, \quad \text{as} \quad |y| \to \infty. \tag{4.12b}$$

The corresponding outer solution for u on $-1 < x < 1$ is

$$u_0(x) = 1 - \frac{(1-U_\pm)}{a_g}\sum_{j=1}^{k} G(x;x_j), \qquad a_g = \left[2\sqrt{D}\tanh(\theta_0/k)\right]^{-1}, \tag{4.13}$$

where $G(x;x_j)$ is the (global) Green's function on $-1 < x < 1$ defined by (2.6).

From (4.11a), we obtain that the leading-order decoupling property for the inner solutions is valid provided that

$$\frac{\varepsilon}{\mathcal{A}^2 U_\pm^2 D} \ll 1. \tag{4.14}$$

Since $U_+ \to 1$ as $\mathcal{A} \to \infty$, the condition (4.14) holds uniformly as $\mathcal{A} \to \infty$ for the large solution u_+, ν_+. Alternatively, as $\mathcal{A} \to \infty$, we have for the small solution that

$$U_- \sim \frac{\mathcal{A}_{ke}^2}{4\mathcal{A}^2} + O\left(\frac{\mathcal{A}_{ke}^4}{\mathcal{A}^4}\right), \qquad \text{for} \quad \mathcal{A} \gg 1. \tag{4.15}$$

Substituting (4.15) into (4.14), and using (4.11b) for $\mathcal{A}_{ke}$, we obtain that the decoupling condition (4.14) holds when

$$O(1) \leq \mathcal{A} \ll 3\varepsilon^{-1/2}\coth[\theta_0/k], \qquad \theta_0 = D^{-1/2}. \tag{4.16}$$

Therefore, for $D = O(1)$, we must have that $\mathcal{A} \ll O(\varepsilon^{-1/2})$. Although, (4.16) appears to hold uniformly in D as $D \to 0$, this is misleading since the existence threshold $\mathcal{A}_{ke}$ in (2.12) yields $\mathcal{A}_{ke} = O(D^{-1/4})$ as $D \to 0$. Hence, for $D \ll 1$, the intermediate regime holds provided that $O(D^{-1/4}) \ll \mathcal{A} \ll \varepsilon^{-1/2}$. This requires that $D \gg O(\varepsilon^2)$. Hence, there is no intermediate range for $\mathcal{A}$ for the parameter regime studied in [26] where $D = O(\varepsilon^2)$.

We now derive certain scaling laws in the intermediate regime where $O(1) \ll \mathcal{A} \ll O(\varepsilon^{-1/2})$ and $D = O(1)$. In particular, in this regime we use (4.15) to obtain that the two-term inner expansion for the small solution in the core of each spike satisfies

$$\nu_{i-}(y) \sim \frac{\mathcal{A}\sqrt{D}}{3\coth(\theta_0/k)}\left[w(y) + \left(\frac{\varepsilon\mathcal{A}^2}{9\coth^2(\theta_0/k)}\right)\hat{\nu}_{i1}(y) + \cdots\right], \tag{4.17a}$$

$$u_{i-}(y) \sim \frac{3\coth(\theta_0/k)}{\mathcal{A}^2\sqrt{D}}\left[1 + \left(\frac{\varepsilon\mathcal{A}^2}{9\coth^2(\theta_0/k)}\right)\hat{u}_{i1}(y) + \cdots\right]. \tag{4.17b}$$

Moreover, for $\mathcal{A} \gg 1$, the outer solution (4.13) has the form

$$u_0(x) \sim 1 - \frac{1}{a_g}\left(1 - \frac{\mathcal{A}_{ke}^2}{4\mathcal{A}^2}\right)\sum_{j=1}^{k} G(x;x_j). \tag{4.18}$$

In the intermediate regime, we can use (4.17a) to calculate the norm $|\nu|_2$ in (2.13) as

$$|\nu|_2 \equiv \left(\varepsilon^{-1}\int_{-1}^{1}\nu^2\,dx\right)^{1/2} \sim \frac{\sqrt{6Dk}\mathcal{A}}{3\coth(\theta_0/k)}. \tag{4.19}$$

The expansions (4.17) and (4.18) do suggest the following scalings of the inner and outer solutions in the pulse-splitting regime $\mathcal{A} = O(\varepsilon^{-1/2})$ studied in the companion paper [13]:

$$\nu_i = O(\varepsilon^{-1/2}), \qquad u_i = O(\varepsilon), \qquad u_0 = u_{01} + O(\varepsilon). \tag{4.20}$$

The spectral results of §3 are valid when the decoupling condition (4.16) holds. We now derive scaling laws for the stability thresholds of the small solution u_-, ν_- in the intermediate regime. Using (4.15), we obtain for $\mathcal{A} \to \infty$ that

$$s \equiv \frac{1 - U_-}{U_-} = \frac{4\mathcal{A}^2}{\mathcal{A}_{ke}^2} - 2 + o(1), \qquad \text{as} \quad \mathcal{A} \to \infty. \tag{4.21}$$

To calculate the stability threshold D_{kL} for the large eigenvalues, we let $s \to \infty$ in (3.27a). Using (4.21), a simple calculation shows that for $\mathcal{A} \gg 1$ we have

$$D_{kL} = D_{kL}^{\infty} + O(1), \qquad D_{kL}^{\infty} = \frac{8\mathcal{A}^2}{k^2\gamma_k\mathcal{A}_{ke}^2} \qquad \text{where} \quad \gamma_k \equiv 1 + \cos\left(\frac{\pi}{k}\right). \tag{4.22}$$

Hence, $D_{kL}^{\infty} = O(\mathcal{A}^2)$ as $\mathcal{A} \to \infty$.

To determine the thresholds for a Hopf bifurcation in the intermediate regime, we let $s \to \infty$ in (3.19) to get that $C_j = (1/2) + O(s^{-1})$ when $\tau = O(1)$ and $D = O(1)$. In this range of τ and D, where $C_j \sim 1/2$, there are no eigenvalues in the right half-plane (cf. [36]). Therefore, for $D = O(1)$, an instability can only occur when $\tau \gg 1$. The correct scaling law is to introduce a new $O(1)$ parameter τ_0 by

$$\tau = \tau_0 \tanh^2(\theta_0/k)\, s^2. \tag{4.23}$$

Substituting (4.23) into (3.19), and assuming that $D = O(1)$, we let $s \to \infty$ to get

$$C_j(\lambda) \sim C_\infty(\lambda) + O(s^{-2}), \qquad C_\infty(\lambda) \equiv \frac{1}{2}\left[1 + \sqrt{\tau_0 \lambda}\right]. \tag{4.24}$$

Notice that $C_\infty(\lambda)$ is independent of j, D, and the number of spikes k. The nonlocal eigenvalue problem in the intermediate regime is then obtained by replacing χ in (3.12) with $\chi = 1/C_\infty$. Substituting (4.21) into (4.23), we obtain the key scaling law

$$\tau \sim \tau_\infty, \qquad \tau_\infty \equiv \frac{16\mathcal{A}^4}{\mathcal{A}_{ke}^4} \tanh^2(\theta_0/k)\, \tau_0 \left(1 - \frac{\mathcal{A}_{ke}^2}{2\mathcal{A}^2}\right)^2 + o(1). \tag{4.25}$$

Here $\mathcal{A}_{ke}$ are the existence thresholds of (2.12). This leads to the following result for the large eigenvalues in the intermediate regime $O(1) \ll \mathcal{A} \ll O(\varepsilon^{-1/2})$:

<u>Proposition 4.2</u>: *Assume that $0 < \varepsilon \ll 1$, $D = O(1)$, and $O(1) \ll \mathcal{A} \ll O(\varepsilon^{-1/2})$. Then, with $\Phi = \Phi(y)$, the $O(1)$ eigenvalues of (2.17) for a symmetric k-spike small solution u_-, ν_- satisfy, for each $j = 1, \ldots, k$, the following leading-order nonlocal eigenvalue problem:*

$$L_0\Phi - \chi_\infty w^2 \left(\frac{\int_{-\infty}^{\infty} w\Phi\, dy}{\int_{-\infty}^{\infty} w\, dy}\right) = \lambda\Phi, \qquad -\infty < y < \infty, \tag{4.26a}$$

with $\Phi \to 0$ as $|y| \to \infty$. Here the local operator L_0 is given in (3.12c), and the multiplier χ_∞ is

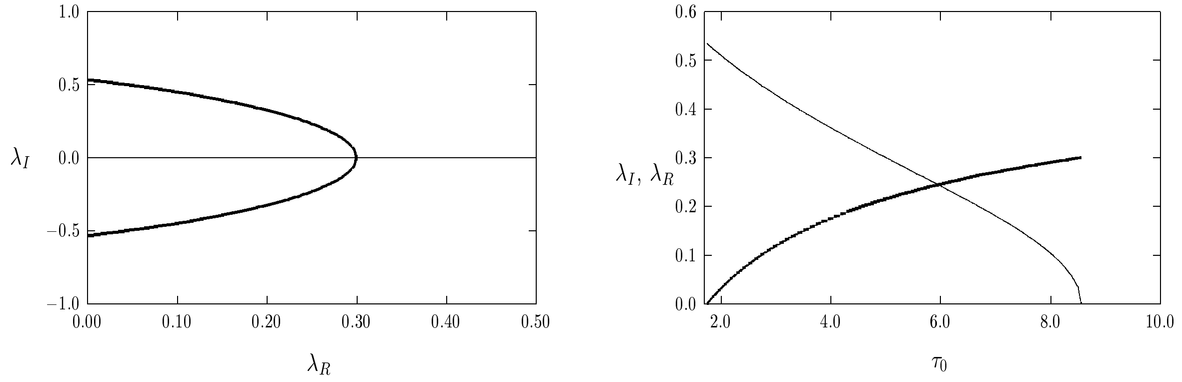
$$\chi_\infty \equiv \frac{2}{1 + \sqrt{\tau_0\lambda}}. \tag{4.26b}$$

The corresponding global eigenfunction $\phi(x)$, representing the perturbation in ν_-, is

$$\phi(x) \sim \sum_{i=1}^{k} c_i \Phi\left[\varepsilon^{-1}(x - x_i)\right]. \tag{4.27}$$

Here c_i for $i = 1, \ldots, k$ are the components of any one of the k independent eigenvectors in (3.11).

The eigenvalue problem (4.26) was derived and studied using a dynamical systems approach in [4] and [5]. In [4] and [5], hypergeometric functions were used to numerically compute a winding number criterion for (4.26) over a wedge-shaped region that includes part of the left half-plane. Since the continuous spectrum is on the negative real axis, with the approach in [5] there are many technical difficulties that were overcome to count the number of eigenvalues near the origin. In our approach of §3, by having derived properties of $f(\lambda)$ on the imaginary axis, we do not need to consider the left half-plane and the difficulties with the continuous spectrum intersecting the origin. The only effect of the continuous spectrum with our formulation is that $C_\infty'(0)$ is infinite. Even with this change in the property of C in Proposition 3.6, the theory of §3 applies directly to (4.26), and proves that there is a Hopf bifurcation value τ_{0h} (possibly non-unique) where (4.26) has complex conjugate eigenvalues on the imaginary axis at λ_{Ih}, and that there is another critical value τ_{0m} where complex conjugate eigenvalues in the right half-plane first merge onto the positive real at some λ_{Rm} as τ_0 is increased.



(a) path in complex plane

(b) $\lambda_R(\tau_0)$, $\lambda_I(\tau_0)$

Figure 20: Left figure: plot of the path of $\lambda = \lambda_R \pm i\lambda_I$ as τ_0 increases past $\tau_{0h} = 1.748$ until it merges onto the real axis at $\tau_{0m} = 8.567$. Right figure: λ_R (heavy solid curve) and λ_I (solid curve) versus τ_0.

In Fig. 20(a) we plot the numerically computed path of the eigenvalues for (4.26) as τ_0 is increased, up until these eigenvalues merge onto the real axis. The corresponding functions $\lambda_R = \lambda_R(\tau)$ and $\lambda_I = \lambda_I(\tau) > 0$ are shown in Fig. 20(b). These numerical computations suggest the stronger result that τ_{0h} is in fact unique, and that there are always two eigenvalues of (4.26) in the right half-plane when $\tau_0 > \tau_{0h}$. Our computations yield the critical values,

$$\tau_{0h} = 1.748\,, \quad \lambda_{Ih} = 0.534\,; \qquad \tau_{0m} = 8.567\,, \quad \lambda_{Rm} = 0.300\,. \tag{4.28}$$

As shown in §5, these values are consistent with those in [4]. To obtain our scaling laws, we substitute (2.12) for $\mathcal{A}_{ke}$ into (4.25). In this way, and in analogy with Proposition 3.15, we can then summarize our stability results for the small solution in the intermediate regime.

Proposition 4.3: *Let $\varepsilon \ll 1$, and consider the intermediate regime $O(1) \ll \mathcal{A} \ll O(\varepsilon^{-1/2})$ for a symmetric k-spike small solution u_-, ν_-. Then, when $D = O(1)$, the solution will be stable with respect to the large eigenvalues when $\tau < \tau_h^\infty$, where*

$$\tau_h^\infty \sim \frac{\mathcal{A}^4 D}{9} \tanh^4\left(\theta_0/k\right) \tau_{0h} \left(1 - \frac{6\theta_0}{\mathcal{A}^2 \tanh(\theta_0/k)}\right)^2 + o(1)\,. \tag{4.29}$$

Moreover, for $D = O(1)$, there are two eigenvalues on the positive real axis in the interval $0 < \lambda < 5/4$ when $\tau > \tau_m^\infty$, where

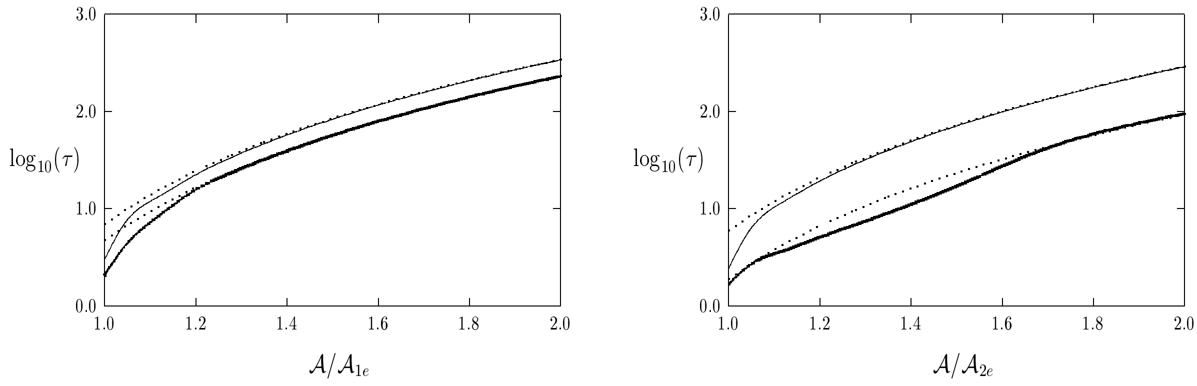
$$\tau_m^\infty \sim \frac{\mathcal{A}^4 D}{9} \tanh^4\left(\theta_0/k\right) \tau_{0m} \left(1 - \frac{6\theta_0}{\mathcal{A}^2 \tanh(\theta_0/k)}\right)^2 + o(1)\,. \tag{4.30}$$

Alternatively, suppose that $D \gg 1$, with $D > D_{kL}^\infty$, where D_{kL}^∞ is defined in (4.22). Then, the small solution u_-, ν_- is unstable for any $\tau > 0$.

As shown in §5, the leading term in the scaling law for τ_h^∞ and τ_m^∞ agrees with results in [4]. This result shows that in the intermediate regime the roots of each $g_j(\lambda) = 0$ for $j = 1, \ldots, k$, where g_j is defined in (3.18), are on the imaginary axis at asymptotically the same value of τ and λ. This suggests a certain complexity in the dynamics near this bifurcation point, since there are k possible eigenfunctions in (4.27) that occur when τ crosses past τ_h^∞. Since for a k spike solution each of the k unstable modes crosses into the right half-plane at asymptotically the same value of τ, there is no apriori guarantee that the spike oscillations will be synchronous in the intermediate regime.

The threshold for D can be written more explicitly by using (2.12) for $\mathcal{A}_{ke}$ in (4.22). In this way, we obtain that there are no eigenvalues on the positive real axis when τ is sufficiently small, provided that $D < D_{L\infty}$, where $D_{L\infty}$ is the unique root of

$$\frac{\sqrt{D}}{\tanh\left(\frac{1}{k\sqrt{D}}\right)} = \frac{2\mathcal{A}^2}{3k^2}\,. \tag{4.31}$$



(a) $k = 1$: $\log_{10}(\tau)$ versus $\mathcal{A}/\mathcal{A}_{1e}$

(b) $k = 2$: $\log_{10}(\tau)$ versus $\mathcal{A}/\mathcal{A}_{2e}$

Figure 21: Comparison of numerical Hopf bifurcation value $\log_{10}(\tau_{hL})$ (solid curve is $D = 0.1$, heavy solid curve is $D = 0.75$) with the asymptotic scaling law $\log_{10}(\tau_h^\infty)$, where τ_h^∞ given in (4.29) (dashed curves). Left figure: $k = 1$. Right figure: $k = 2$.

In Fig. 21 we compare the scaling law (4.29) for τ for the Hopf bifurcation in the intermediate regime with the corresponding numerical value computed from (3.12). In Fig. 21(a) and Fig. 21(b) we compare these critical values for $k = 1$ and $k = 2$, respectively. From these figures, we observe that the scaling law τ_h^∞ in (4.29) provides a good approximation to the numerically computed Hopf bifurcation value except near the existence threshold $\mathcal{A}_{ke}$.

As a remark, a similar scaling law for a Hopf bifurcation can be derived for the infinite-line problem studied in §3.4. Letting $s \to \infty$ in (3.53), we obtain that the Hopf bifurcation value τ_h^∞ is given by

$$\tau_h^\infty \sim \frac{\mathcal{A}^4 D}{9} \tau_{0h} \left(1 - \frac{6\theta_0}{\mathcal{A}^2}\right)^2 + o(1)\,, \tag{4.32}$$

where τ_{0h}, computed from (4.26), is given in (4.28). Setting $D = 1$, and writing $A = \varepsilon^{1/2}\mathcal{A}$, we can write the leading term in (4.32) as

$$\tau_h^\infty \sim 0.19422 A^4 \varepsilon^{-2}\,. \tag{4.33}$$

This result for the infinite-line problem is equivalent to equation (2.11) of [21].

5 Discussion and Comparisons

In this section we discuss more precisely the relationship between our results and those in the literature. We also summarize which of our results are rigorous, and we list some open technical problems together with a few interesting areas for further research.

In §3.2 and in §4 we compared our results with those of [21], which were based on the infinite-line problem in both the low feed-rate and intermediate regimes. Our results in those regimes are completely consistent with those of [21].

Next, we compare our results with those of [4] and [5]. In those works, the dimensionless formulation of the Gray-Scott model is

$$V_T = \delta^2 V_{XX} - \delta^\beta b V + UV^2\,, \qquad 0 < X < L\,, \quad T > 0\,, \tag{5.1a}$$

$$U_T = U_{XX} + \delta^2 a (1 - U) - UV^2\,, \qquad 0 < X < L\,, \quad T > 0\,, \tag{5.1b}$$

$$U_X = V_X = 0\,, \qquad X = 0, L\,. \tag{5.1c}$$

Here $0 < \beta < 1$, and $\delta \ll 1$. In [4], solutions of a spatial period $\mathcal{T}$ are constructed. For a k-spike solution, this determines L as $L = \mathcal{T}k$. In terms of our dimensionless groupings of (1.3), it follows that

$$D \equiv \frac{4}{\delta^2 a \mathcal{T}^2 k^2}\,, \qquad \varepsilon^2 \equiv \frac{4\delta^{2-\beta}}{b\mathcal{T}^2 k^2}\,, \qquad \tau \equiv \frac{\delta^{\beta-2} b}{a}\,, \qquad A \equiv \frac{\delta^{1-\beta}\sqrt{a}}{b}\,. \tag{5.2}$$

In equation (2.10) of [4], the mode m of the periodic pattern is related to the period $\mathcal{T}$ by

$$m = \frac{E^2 + 1}{E^2 - 1} = \frac{1}{\tanh(\log E)}\,, \qquad E \equiv e^{\mathcal{T}\delta\sqrt{a}/2}\,. \tag{5.3}$$

Using $L = \mathcal{T}k$ and (5.2), we can express m in terms of our notation as

$$m = \frac{1}{\tanh(\theta_0/k)}\,, \quad \theta_0 \equiv D^{-1/2}\,. \tag{5.4}$$

The only nonlocal eigenvalue problem studied in [4] and [5] is given in equation (4.14) of [4]. Using (5.2), we can readily show that equation (4.14) of [4] is exactly equivalent to our nonlocal eigenvalue problem (4.26) of Proposition 4.2, which holds in the intermediate regime. In [4], hypergeometric functions were used to study this nonlocal eigenvalue problem. Using this approach it was shown in equation (5.16) of [4] that there is a Hopf bifurcation at the critical value

$$b = b_h \equiv \frac{\sqrt{a}}{m}\delta^{1/2-\beta}(0.66)\,. \tag{5.5}$$

Furthermore, it was shown in [4] that two complex-conjugate eigenvalues merge onto the real positive axis of the complex λ plane when

$$b = b_c \equiv \frac{\sqrt{a}}{m}\delta^{1/2-\beta}(0.99)\,. \tag{5.6}$$

We now show that these results agree precisely with the leading-order terms for the scaling-laws τ_h^∞ and τ_m^∞ given in Proposition 4.3. To show this, we write the leading term in (4.29) as

$$\tau_h^\infty \sim \frac{\mathcal{A}^4 D}{9}\tanh^4\left(\theta_0/k\right)\tau_{0h} = \frac{\varepsilon^{-2}A^4 D}{9}\tanh^4\left(\theta_0/k\right)\tau_{0h}\,. \tag{5.7}$$

Using the change of variables (5.2), which relates our notation to that of [4], we obtain

$$\frac{\delta^{\beta-2}b}{a} = \frac{\delta^{-3\beta}a}{b^3}\tanh^4\left(\theta_0/k\right)\left(\frac{\tau_{0h}}{9}\right)\,. \tag{5.8}$$

Solving (5.8) for b, and recalling (5.4) for m, we obtain

$$b = \frac{\sqrt{a}}{m}\delta^{1/2-\beta}\left(\frac{\tau_{0h}}{9}\right)^{1/4}\,. \tag{5.9}$$

Recalling from (4.28) that $\tau_{0h} = 1.748$, we observe that (5.9) and (5.5) agree. Similarly, the leading-order term for τ_m^∞ in Proposition 4.2 agrees with (5.6).

Therefore, our results for the stability of k-spike patterns in the intermediate regime agree with those of the earlier formal asymptotic study of [4]. However, there are a few differences in the approach and in the results. Firstly, our rigorous analysis of the nonlocal eigenvalue problem in Proposition 4.2 provides a simple alternative proof to the study in [4] and [5], which uses dynamical systems techniques and computations involving hypergeometric functions to establish the existence of a Hopf bifurcation point. Secondly, the scaling laws in Proposition 4.3 incorporate an extra correction term not given in [4] (see Fig. 21). Without this term, only fair agreement is obtained with numerically computed Hopf bifurcation thresholds unless ε is very small.

The essential difference between our study and that of previous studies is our analysis of the low-feed rate regime in §2 and §3. The eigenvalue problem of Proposition 3.2 is new in the context of the Gray-Scott model. It leads to the existence of both competition and (typically) synchronous oscillatory instabilities. These instabilities have not been observed previously for the Gray-Scott model. As we have proved, competition instabilities cannot occur in the intermediate regime. In addition, Proposition 3.3 shows a clear equivalence between the Gray-Scott and GM models in the low feed-rate regime. There is no such equivalence principle in the intermediate parameter regime. In the intermediate regime, the stability analysis of k-spike patterns is essentially very simple, in that the scaling laws described above show that the stability threshold for a k-spike pattern is a multiple of the threshold for a one-spike pattern on the infinite line. This simplification does not occur for the Gray-Scott model in the low-feed rate regime, and consequently the stability analysis there is much more intricate than in the intermediate regime.

Next, we make some remarks concerning the rigor of our approach. Although we have presented only a formal derivation of the nonlocal eigenvalue problem (3.12), this derivation can be done rigorously, and with the same result, as was done for the GM model in [37]. All of the spectral results of §3.2 have been rigorously established.

One key open problem for both the low feed-rate and intermediate regimes is to prove a strict transversal crossing condition to guarantee that the complex conjugate eigenvalues on the imaginary axis when $\tau = \tau_{hL}$ remain in the right half-plane for any $\tau > \tau_{hL}$. Numerical evidence suggests that this is the case, but an analytical proof is not available.

There are many open problems concerning large-scale behavior away from bifurcation points. In particular, in the low feed-rate regime $\mathcal{A} = O(1)$, an open problem is to analyze the large-scale synchronous oscillatory instabilities and competition instabilities that occur after they are initiated. In particular, it is an open problem to prove that the oscillatory instability in the Gray-Scott model is subcritical. The oscillatory instabilities in the GM model studied in [35] often lead to stable large-scale time-periodic solutions. It would be interesting to calculate the normal form of these oscillations for both the Gray-Scott and GM models to show the differences in the behavior of the oscillations. Another open problem is to determine if there is a wide parameter regime where asynchronous oscillations in the spike amplitudes are possible. It would also be interesting to characterize the dynamics of quasi-equilibrium patterns in the low and intermediate feed-rate regimes. For the GM model (1.9), such an analysis of the dynamics of quasi-equilibrium patterns was done for a k-spike pattern with $\tau = 0$ in [10], and for a one-spike pattern with $\tau > 0$ in [32]. Two-spike evolutions were analyzed in [2] and [3] on the infinite line, and in [33] on the finite line. Finally, since our analysis does not rely on dynamical systems techniques, much of it is readily extended to treat the stability and dynamics of two-dimensional spot patterns for the Gray Scott model. Work in this direction is in progress.

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