The Stability of Spike Solutions to the One-Dimensional Gierer-Meinhardt Model

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Abstract

The stability properties of an N-spike equilibrium solution to a simplified form of the Gierer-Meinhardt activator-inhibitor model in a one-dimensional domain is studied asymptotically in the limit of small activator diffusivity ε. The equilibrium solution consists of a sequence of spikes of equal height. The two classes of eigenvalues that must be considered are the $O(1)$ eigenvalues and the $O(\varepsilon^2)$ eigenvalues, which are referred to as the large and small eigenvalues, respectively. The spike pattern is stable when the parameters in the Gierer-Meinhardt model are such that both sets of eigenvalues lie in the left-half plane. For a certain range of these parameters and for $N \geq 2$ and $\varepsilon \to 0$, it is shown the $O(1)$ eigenvalues are in the left half-plane only when $D < D_N$, where D_N is some explicit critical value of the inhibitor diffusivity D. For $N \geq 2$ and $\varepsilon \to 0$, it is also shown that the small eigenvalues are real and that they are negative only when $D < D_N^*$, where D_N^* is another critical value of D, which satisfies $D_N^* < D_N$. Thus, when $N \geq 2$ and $\varepsilon \ll 1$, the spike pattern is stable only when $D < D_N^*$. An explicit formula for D_N^* is given. For the special case $N = 1$, it is shown that a one-spike equilibrium solution is stable when $D < D_1(\varepsilon)$, where $D_1(\varepsilon)$ is exponentially large as $\varepsilon \to 0$, and is unstable when $D > D_1(\varepsilon)$. An asymptotic formula for $D_1(\varepsilon)$ is given. Finally, the dynamics of a one-spike solution is studied by deriving a differential equation for the trajectory of the center of the spike.

1 Introduction

In [17], Turing proposed that localized peaks in the concentration of a chemical substance, known as a morphogen, could be responsible for the process of morphogenesis, which describes the development of a complex organism from a single cell. Through the use of a linearized analysis, he showed how stable spatially complex patterns can develop from small perturbations of spatially homogeneous initial data for a coupled system of reaction-diffusion equations. Later, Gierer and Meinhardt [3] have demonstrated numerically the existence of stable spatially inhomogeneous equilibrium solutions in the fully nonlinear regime for the following dimensionless reaction-diffusion

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system of activator-inhibitor type:

$$A_t = \varepsilon^2 A_{xx} - A + \frac{A^p}{H^q}\,, \qquad -1 < x < 1\,, \quad t > 0\,, \tag{1.1a}$$

$$\tau H_t = D H_{xx} - \mu H + \frac{A^r}{H^s}\,, \qquad -1 < x < 1\,, \quad t > 0\,, \tag{1.1b}$$

$$A_x(\pm 1, t) = H_x(\pm 1, t) = 0\,. \tag{1.1c}$$

Here A, H, ε, $D > 0$, $\mu > 0$ and τ represent the scaled activator concentration, inhibitor concentration, activator diffusivity, inhibitor diffusivity, inhibitor decay rate and reaction time constant. The exponents (p, q, r, s) in the GM model (1.1) are assumed to satisfy

$$p > 1, \qquad q > 0, \qquad r > 0, \qquad s \geq 0, \qquad 0 < \frac{p-1}{q} < \frac{r}{s+1}\,. \tag{1.2}$$

For $\varepsilon \ll 1$, many numerical studies of the GM model (1.1) (i. e. [3], [5]) have shown that the solution to (1.1) can have one or many spikes in the activator concentration A. These spikes, which represent strong localized deviations from a constant background concentration, have a spatial extent of $O(\varepsilon)$ and are such that $A \to \infty$ as $\varepsilon \to 0$ in the core of each spike.

The main goal of this paper is to develop a formal asymptotic analysis to study the stability properties of an N-spike equilibrium solution to (1.1) when $\tau = 0$. This analysis is the first step towards understanding the more complicated dynamics and stability behavior that are likely to occur when $\tau > 0$. The equilibrium solution that we linearize around is the one for which the spikes have equal height. Other equilibrium spike-type solutions where the spikes do not have equal height also exist ([2]). Since the amplitude of a spike tends to infinity as $\varepsilon \to 0$, it is convenient to introduce new variables, as in [6], to ensure that the amplitude of the spike is $O(1)$ as $\varepsilon \to 0$ and that $h = O(1)$ as $\varepsilon \to 0$. The unique scaling that achieves this is to define a and h by

$$A = \varepsilon^{-\nu_a} a\,, \qquad H = \varepsilon^{-\nu_h} h\,, \tag{1.3a}$$

where ν_a and ν_h are given by

$$\nu_a = \frac{q}{(1-p)(1+s) + rq}\,, \qquad \nu_h = \frac{(p-1)}{(1-p)(1+s) + rq}\,. \tag{1.3b}$$

Setting $\tau = 0$, and writing (1.1) in terms of the new variables (1.3), we get

$$a_t = \varepsilon^2 a_{xx} - a + \frac{a^p}{h^q}\,, \qquad -1 < x < 1\,, \quad t > 0\,, \tag{1.4a}$$

$$0 = D h_{xx} - \mu h + \varepsilon^{-1} \frac{a^r}{h^s}\,, \qquad -1 < x < 1\,, \quad t > 0\,, \tag{1.4b}$$

$$a_x(\pm 1, t) = h_x(\pm 1, t) = 0\,. \tag{1.4c}$$

Most of the previous work on spike-type solutions of the GM model has been based on the study of the *shadow problem*, which results from taking the limit $D \to \infty$ in (1.4). In this limit, (1.4b) reduces to the nonlocal reaction-diffusion equation

$$a_t = \epsilon^2 a_{xx} - a + \frac{a^p}{h^q}\,, \qquad -1 < x < 1\,, \quad t > 0\,, \tag{1.5a}$$

$$h = \left(\frac{\varepsilon^{-1}}{2\mu} \int_{-1}^{1} a^r \, dx \right)^{\frac{1}{s+1}}\,, \qquad a_x(\pm 1, t) = 0\,. \tag{1.5b}$$

Before describing the contents of this paper, we summarize some previous work for (1.4) and (1.5). The equilibrium problem for (1.5) was studied using formal asymptotic methods in [9]. The delicate asymptotic problem of the determination of the equilibrium spike locations for the shadow problem (1.5) extended to a multi-dimensional spatial domain has been studied in many papers (see [4], [10], [18], [20] and the references therein). The stability of a boundary spike solution to (1.5) was studied in [21]. In [6] a formal asymptotic analysis was used to show that the dynamics of a one-spike solution to the shadow problem (1.5) is metastable for the usual parameter set $(p, q, r, s) = (2, 1, 2, 0)$. This metastability persists when (1.5) is extended to a multi-dimensional spatial domain [6]. The existence of this metastable behavior has been proved in [19] by examining the spectrum associated with the linearization and in [1] using inertial manifold techniques. There are only a few rigorous results for (1.4) in the large amplitude regime. In [16] the existence of an N-spike equilibrium solution to (1.4) in the limit $\varepsilon \ll 1$ with spikes of equal height was established. A survey of results for spike-type solutions in reaction-diffusion systems is given in [13].

The key feature that motivates our study is that an N-spike equilibrium solution to (1.5) (with $N \geq 1$), and its multi-dimensional counterpart, is unstable whenever the spikes are located strictly inside the domain rather than on the boundary. This instability is very weak for the case of one spike, since in this case the growth rate of infinitesimal perturbations is asymptotically exponentially small as $\varepsilon \to 0$ (see [6]). The conjecture of [6] and [7], based on numerical computations, is that the system (1.4) will stabilize an N-spike equilibrium solution whenever the inhibitor diffusivity D is sufficiently small. The goal of this paper is to investigate this conjecture analytically in the simple case of a one-dimensional spatial domain for equilibrium solutions with spikes of equal height. It is important to mention that our stability analysis is very different from the classical Turing-type stability analysis that is based on linearizing a reaction-diffusion system around a spatially homogeneous steady-state equilibrium solution. Our analysis is based on the study of the linearization of (1.4) around an N-spike equilibrium solution, which has a very high degree of spatial inhomogeneity. A similar analysis for the Fitzhugh-Nagumo model has been carried out in [15]. Some stability results for the case of one spike with $\tau \neq 0$ is given in [14].

We now give an outline of the paper and summarize some of the key results obtained. In §2 we use the method of matched asymptotic expansions to construct equilibrium solutions to (1.4) in the limit $\varepsilon \to 0$ that have $N \geq 1$ spikes of equal amplitude in the activator concentration. In §3 and §4 we study the spectrum of the eigenvalue problem associated with linearizing (1.4) around the equilibrium solution constructed in §2. In §3 we study the large eigenvalues of order $\lambda = O(1)$ in the spectrum, while in §4 we study the small eigenvalues of order $\lambda = O(\varepsilon^2)$. The N-spike solution is stable when both sets of eigenvalues lie in the left half-plane. For $N \geq 2$ and $\varepsilon \to 0$, in §3 we obtain an explicit critical value D_N such that the large $O(1)$ eigenvalues are in the left half-plane only when $D < D_N$. When this condition on D is satisfied, we say that the equilibrium solution is stable with respect to the $O(1)$ eigenvalues. In §4, for $N \geq 2$ and $\varepsilon \to 0$, we show that the small eigenvalues are always real and that they are negative only when $D < D_N^*$. An explicit formula for D_N^* is derived and it is found that $D_N^* < D_N$. Thus, for $N \geq 2$ and $\varepsilon \to 0$, an N-spike symmetric equilibrium spike pattern is stable when $D < D_N^*$ and is unstable otherwise. The results for D_N and D_N^* are given below in Propositions 7 and 11, respectively. The main stability results, summarized in propositions 5, 7, 8, 10, and 11, are obtained from a careful but formal asymptotic analysis. It would be of interest to establish these results rigorously.

Finally, in §5 we study the stability and dynamics of a solution to (1.4) with exactly one spike. For a certain range of exponents (p, q, r, s), we show that a one-spike equilibrium solution to (1.4) will be stable when $D < D_1(\varepsilon)$, where $D_1(\varepsilon)$ is exponentially large as $\varepsilon \to 0$. It is unstable when $D > D_1(\varepsilon)$. An asymptotic formula for $D_1(\varepsilon)$ is given in Proposition 13 of §5.2. This result is consistent with the result of [6] for the shadow problem (1.5) for which $D = \infty$, where it was shown that a one-spike equilibrium solution is unstable, but with an asymptotically exponentially small positive eigenvalue. In §5.1, we study the dynamics of a one-spike solution to (1.4) for finite D by deriving an asymptotic differential equation for the trajectory of the center of the spike using the method of matched asymptotic expansions. The asymptotic differential equation is given in Proposition 12 of §5.1 and is favorably compared in §5.1 with results from full numerical computations.

2 An Asymptotic Analysis of the Equilibrium Solution

For $\varepsilon \to 0$, we construct an N-spike equilibrium solution to (1.4) with equal amplitude using the method of matched asymptotic expansions. A solution with three spikes is shown in Fig. 1. The locations x_j, for $j = 0, \ldots, N-1$, of the spikes for an N-spike solution, which follows from symmetry considerations under the Neumann conditions (1.4c), satisfy

$$x_j = -1 + \frac{1+2j}{N}, \qquad j = 0, 1, \ldots, N-1. \tag{2.1}$$

At these points the equilibrium solution satisfies $a'(x_j) = 0$ and $h(x_j) = H$, where H is independent of j. For an N-peak equilibrium solution to (1.4), the activator concentration is localized in the inner regions defined near each x_j, and is exponentially small in the outer regions defined away from the spike locations.

In the inner region near the j^{th} spike we introduce new variables by

$$y_j = \varepsilon^{-1}(x - x_j), \qquad \tilde{h}(y_j) = h(x_j + \varepsilon y), \qquad \tilde{a}(y_j) = a(x_j + \varepsilon y), \tag{2.2a}$$

and we expand

$$\tilde{h}(y_j) = \tilde{h}_0(y_j) + \varepsilon \tilde{h}_1(y_j) + \cdots, \qquad \tilde{a}(y_j) = \tilde{a}_0(y_j) + O(\varepsilon). \tag{2.2b}$$

Substituting (2.2) into the equilibrium problem for (1.4), and collecting powers of ε, we get

$$\tilde{a}_0'' - \tilde{a}_0 + \tilde{a}_0^p / \tilde{h}_0^q = 0, \qquad -\infty < y_j < \infty, \tag{2.3a}$$

$$\tilde{h}_0'' = 0, \tag{2.3b}$$

$$D\tilde{h}_1'' = -\tilde{a}_0^r / \tilde{h}_0^s. \tag{2.3c}$$

The conditions at $y_j = 0$ are that $\tilde{a}_0'(0) = 0$, $\tilde{h}_0(0) = H$, and $\tilde{h}_1(0) = 0$. The conditions needed to match to the outer solution are that $\tilde{h}_0$ is bounded as $|y_j| \to \infty$ and $\tilde{a}_0 \to 0$ as $|y_j| \to \infty$. Thus, the solution to (2.3b) is $\tilde{h}_0 = H$. Next, we introduce u_c by

$$\tilde{a}_0 = H^\gamma u_c, \qquad \text{where} \qquad \gamma \equiv q/(p-1). \tag{2.4}$$

Then, (2.3a) and (2.3c) become

$$u_c'' - u_c + u_c^p = 0, \qquad -\infty < y_j < \infty, \tag{2.5a}$$

$$u_c \to 0 \quad \text{as} \quad |y_j| \to \infty; \qquad u_c'(0) = 0, \tag{2.5b}$$

$$D\tilde{h}_1'' = -u_c^r H^{\gamma r - s}. \tag{2.5c}$$

From phase-plane considerations, there is a unique positive solution to (2.5). In particular, when $p = 2$ we have

$$u_c(y) = \frac{3}{2}\operatorname{sech}^2\left(\frac{y}{2}\right). \tag{2.6}$$

Upon integrating (2.5c) from $y_j = -\infty$ to $y_j = \infty$ we obtain

$$\lim_{y_j \to +\infty} \tilde{h}_1^{'} - \lim_{y_j \to -\infty} \tilde{h}_1^{'} = -\frac{1}{D} H^{\gamma r - s} b_r \,, \qquad \text{where} \qquad b_r \equiv \int_{-\infty}^{\infty} [u_c(y)]^r \, dy \,. \tag{2.7}$$

This equation yields a jump condition for the outer solution.

In the outer region, defined away from $O(\varepsilon)$ regions near each x_j, a is exponentially small and h is expanded as

$$h(x) = h_0(x) + o(\varepsilon) \,. \tag{2.8}$$

Here h_0 satisfies $Dh_0^{''} - \mu h_0 = 0$ on the interval $[-1, 1]$ with suitable jump conditions imposed across the x_j. Upon matching to the inner solution constructed above, we obtain that h_0 is continuous across each x_j and that the jump in $h_0^{'}$ is given by the right-hand side of (2.7). Therefore, h_0 satisfies

$$Dh_0^{''} - \mu h_0 = -H^{\gamma r - s} b_r \sum_{k=0}^{N-1} \delta(x - x_k) \,, \qquad -1 < x < 1 \,, \tag{2.9a}$$

$$h_0^{'}(\pm 1) = 0 \,, \tag{2.9b}$$

where $\delta(y)$ is the Dirac delta function. To solve (2.9) we introduce the Green's function $G(x; x_k)$ satisfying

$$DG_{xx} - \mu G = -\delta(x - x_k) \,, \qquad -1 < x < 1 \,, \tag{2.10a}$$

$$G_x(\pm 1; x_k) = 0 \,. \tag{2.10b}$$

A simple calculation gives,

$$G(x; x_k) = \begin{cases} A_k \cosh\left[\theta(1 + x)\right] / \cosh\left[\theta(1 + x_k)\right] \,, & -1 < x < x_k \,, \\ A_k \cosh\left[\theta(1 - x)\right] / \cosh\left[\theta(1 - x_k)\right] \,, & x_k < x < 1 \,. \end{cases} \tag{2.11a}$$

Here

$$A_k = \frac{1}{\sqrt{\mu D}} \left(\tanh\left[\theta(1 - x_k)\right] + \tanh\left[\theta(1 + x_k)\right] \right)^{-1} \,, \qquad \theta \equiv (\mu / D)^{1/2} \,. \tag{2.11b}$$

In terms of $G(x; x_k)$, the solution to (2.9) is

$$h_0(x) = H^{\gamma r - s} b_r \sum_{k=0}^{N-1} G(x; x_k) \,. \tag{2.12}$$

Finally, to determine H we set $h_0(x_j) = H$ and use the fact that $\sum_{k=0}^{N-1} G(x_j; x_k)$ is independent of j when the locations satisfy (2.1). This can be shown directly either by using (2.11) and summing certain geometric series, or by using the matrix analysis given following Proposition 4 below. In either way, we get

$$H^{\gamma r-(s+1)} = \frac{1}{b_r a_g}\,, \qquad \text{where} \qquad a_g \equiv \sum_{k=0}^{N-1} G(x_j; x_k)\,. \tag{2.13}$$

This leads to the following equilibrium result:

Proposition 1: *For $\varepsilon \to 0$, an N-spike equilibrium solution to (1.4), which we label by $a_e(x)$ and $h_e(x)$, is given asymptotically by*

$$a_e(x) \sim H^{\gamma} \sum_{k=0}^{N-1} u_c\left[\varepsilon^{-1}(x - x_k)\right]\,, \tag{2.14a}$$

$$h_e(x) \sim \frac{H}{a_g} \sum_{k=0}^{N-1} G(x; x_k)\,. \tag{2.14b}$$

Here $u_c(y)$ is the positive solution to (2.5), H and a_g are defined in (2.13), G is given in (2.11), and x_k satisfies (2.1). The three-spike equilibrium solution plotted in Fig. 1 is obtained from (2.14).

To determine the stability properties of the equilibrium solution we introduce the perturbation

$$a(x,t) = a_e(x) + e^{\lambda t}\phi(x)\,, \qquad h(x,t) = h_e(x) + e^{\lambda t}\eta(x)\,, \tag{2.15}$$

where $\eta \ll 1$ and $\phi \ll 1$. Substituting (2.15) into (1.4) and linearizing, we obtain the eigenvalue problem

$$\varepsilon^2\phi_{xx} - \phi + \frac{p a_e^{p-1}}{h_e^q}\phi - \frac{q a_e^p}{h_e^{q+1}}\eta = \lambda\phi\,, \qquad -1 < x < 1\,, \tag{2.16a}$$

$$D\eta_{xx} - \mu\eta = -\varepsilon^{-1} r \frac{a_e^{r-1}}{h_e^s}\phi + \varepsilon^{-1} s \frac{a_e^r}{h_e^{s+1}}\eta\,, \qquad -1 < x < 1\,, \tag{2.16b}$$

$$\phi_x(\pm 1) = \eta_x(\pm 1) = 0\,. \tag{2.16c}$$

In §3 we analyze the spectrum of (2.16) corresponding to those eigenvalues that are bounded away from zero as $\varepsilon \to 0$. These eigenvalues are referred to as the large eigenvalues. In §4 we analyze the spectrum of (2.16) corresponding to those eigenvalues that approach zero as $\varepsilon \to 0$. These eigenvalues, referred to as the small eigenvalues, are shown to be $O(\varepsilon^2)$ as $\varepsilon \to 0$. The goal is to

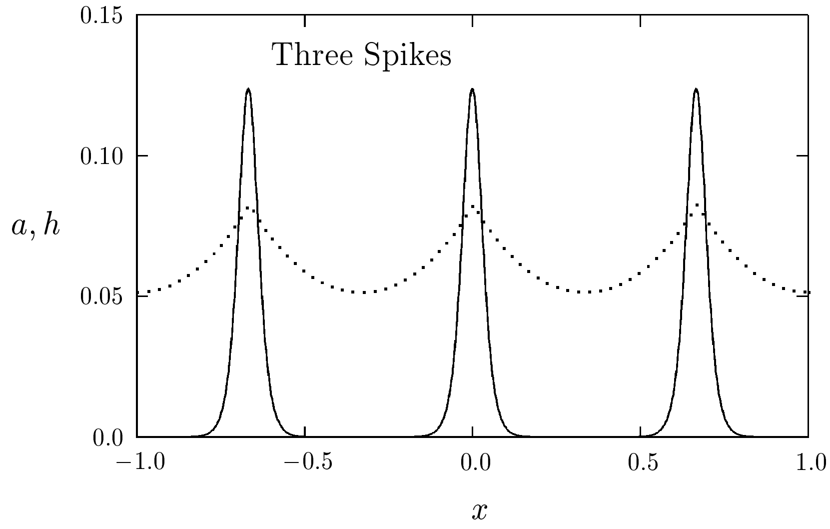


Figure 1: Plot of the activator and inhibitor concentration for a three-spike asymptotic symmetric equilibrium solution with $\epsilon = .02$, $D = .10$, $\mu = 1$, and $(p, q, m, s) = (2, 1, 2, 0)$. The solid curve is the activator concentration and the dotted curve is the inhibitor concentration.

determine the range of D as a function of N for which the large and the small eigenvalues both have negative real parts.

Qualitatively, the small eigenvalues arise from the near translation invariance property of the system. When $D = \infty$, then h_e and η are constants in (2.16a). In this special case, the resulting eigenvalue problem has N exponentially small eigenvalues. These exponentially small eigenvalues arise as a consequence of the near translation invariance property and an exponentially weak interaction between adjacent spikes (mediated by their tail behavior) and between the spikes and the boundary. The corresponding eigenfunction is, to within exponentially small terms, a linear combination of the first spatial derivative of $u_c\left[\varepsilon^{-1}(x - x_j)\right]$. However, when D is finite so that η is a slowly varying function of x near each spike, then these exponentially small eigenvalues are dominated by an algebraically small spike interaction mediated by the function $\eta(x)$. The leading term in the eigenfunction is still a linear combination of the first spatial derivative of u_c, but the expansion of the eigenfunction proceeds in powers of ϵ. When $D = \infty$ and $\eta = 0$, the operator in (2.16) has exactly one positive eigenvalue in the vicinity of each spike, and this eigenfunction is of one sign. Hence, when $D = \infty$ and $\eta = 0$, an N-spike solution is unstable on an $O(1)$ time scale. However, when D is decreased from infinity, the $O(1)$ positive eigenvalue near each spike can be pushed into the left-half plane owing to the dependence of η on D. This is the origin of the large $O(1)$ eigenvalues.

3 Analysis of the Large Eigenvalues

In this section we analyze the eigenvalues of (2.16) that do not approach zero as $\varepsilon \to 0$. In §3.1 we consider the case where $s = 0$ and in §3.2 we extend the analysis to treat $s > 0$. For ease of notation, the subscripts such as η_x shall indicate derivatives with respect to x, whereas the primes will generally refer to differentiation with respect to the stretched variable y.

3.1 Analysis for $s = 0$

To study the eigenvalue problem (2.16) it is convenient to introduce scaled variables defined by

$$a_e = H^\gamma u \,, \qquad h_e = Hv \,, \qquad \phi = H^\gamma \bar{\phi} \,, \qquad \eta = H\bar{\eta} \,, \tag{3.1}$$

where $\gamma \equiv q/(p-1)$. From (2.14a), we conclude that $u \sim \sum_{k=0}^{N-1} u_c \left[\varepsilon^{-1}(x - x_k)\right]$. Substituting (3.1) into (2.16) with $s = 0$, using (2.13) for $H^{\gamma r - 1}$, and dropping the overbar notation, we get

$$\varepsilon^2 \phi_{xx} - \phi + \frac{pu^{p-1}}{v^q}\phi - \frac{qu^p}{v^{q+1}}\eta = \lambda\phi \,, \qquad -1 < x < 1 \,, \tag{3.2a}$$

$$D\eta_{xx} - \mu\eta = -\frac{ru^{r-1}}{\varepsilon b_r a_g}\phi \,, \qquad -1 < x < 1 \,, \tag{3.2b}$$

$$\phi_x(\pm 1) = \eta_x(\pm 1) = 0 \,. \tag{3.2c}$$

Using the symmetry of the equilibrium solution and the localization of the coefficients in (3.2), we look for an eigenfunction for (3.2) in the form

$$\phi(x) \sim \sum_{k=0}^{N-1} c_k \Phi\left[\varepsilon^{-1}(x - x_k)\right] \,, \tag{3.3}$$

for some c_k, where $\Phi(y) \to 0$ as $|y| \to \infty$. The right-hand side of (3.2b) behaves like a sum of delta functions as $\varepsilon \to 0$. Thus, for $\varepsilon \to 0$, we calculate that η satisfies

$$D\eta_{xx} - \mu\eta = -\frac{r}{b_r a_g}\int_{-\infty}^{\infty} \left[u_c(y)\right]^{r-1} \Phi(y)\, dy \sum_{k=0}^{N-1} c_k \delta(x - x_k) \,, \qquad -1 < x < 1 \,, \tag{3.4a}$$

$$\eta_x(\pm 1) = 0 \,. \tag{3.4b}$$

The solution to (3.4) is written in terms of the Green's function $G(x;x_k)$ defined in (2.11) as

$$\eta(x) = \frac{r}{a_g b_r} \int_{-\infty}^{\infty} [u_c(y)]^{r-1} \Phi(y)\, dy \sum_{k=0}^{N-1} G(x;x_k) c_k \, . \tag{3.5}$$

Then, we substitute (3.3) and (3.5) into (3.2a), and use the fact that $v = 1 + O(\varepsilon)$ when $|x - x_j| = O(\varepsilon)$. The resulting eigenvalue problem, when written in terms of the stretched variable $y = \varepsilon^{-1}(x - x_j)$, becomes for $j = 0, \ldots, N-1$,

$$c_j \left(\Phi'' - \Phi + p u_c^{p-1} \Phi \right) - \frac{q r u_c^p}{a_g b_r} \int_{-\infty}^{\infty} [u_c(y)]^{r-1} \Phi(y)\, dy \sum_{k=0}^{N-1} G(x_j;x_k) c_k = c_j \lambda \Phi \, , \qquad -\infty < y < \infty \, , \tag{3.6}$$

with $\Phi \to 0$ as $|y| \to \infty$. This eigenvalue problem is the same for each j when $c_0, \ldots, c_{N-1}$ are the components of the eigenvector for the matrix problem

$$\mathcal{G} \boldsymbol{c} = \alpha \boldsymbol{c} \, , \qquad \boldsymbol{c} \equiv \begin{pmatrix} c_0 \\ \vdots \\ c_{N-1} \end{pmatrix} . \tag{3.7}$$

Here $\mathcal{G}$ is the $N \times N$ symmetric matrix whose entries are the coefficients $G(x_j;x_k)$. The eigenvalues of $\mathcal{G}$ are real. Then, using (2.7) for b_r, we get that (3.6) becomes the nonlocal eigenvalue problem

$$\Phi'' - \Phi + p u_c^{p-1} \Phi - \frac{\alpha q r u_c^p}{a_g} \left(\frac{\int_{-\infty}^{\infty} [u_c(y)]^{r-1} \Phi(y)\, dy}{\int_{-\infty}^{\infty} [u_c(y)]^{r}\, dy} \right) = \lambda \Phi \, , \qquad -\infty < y < \infty \, , \tag{3.8a}$$

$$\Phi \to 0 \quad \text{as} \quad |y| \to \infty \, . \tag{3.8b}$$

The goal is to determine conditions on D, μ and N for which the eigenvalue $\lambda_0 \neq 0$ of (3.8) with the largest real part satisfies $\text{Re}(\lambda_0) > 0$ for any eigenvalue α of the matrix problem (3.7).

The outline of the rest of the analysis is as follows. First, we obtain explicit formulae for the eigenvalues α_j and the eigenvectors $\boldsymbol{c}_j$ of $\mathcal{G}$. These eigenpairs depend on the values of D, μ and N. The next step is to use a key result of [19], which we restate below, that proves that the principal eigenvalue of (3.8), in the restricted subset for which $\lambda \neq 0$, has a positive real part when $\alpha < \alpha_c$ and a negative real part when $\alpha > \alpha_c$. Here $\alpha_c > 0$ is some specific threshold value. Hence, we conclude that there is no eigenvalue of (3.8) with a positive real part when the minimum eigenvalue α_1 of the matrix problem (3.7) satisfies $\alpha_1 > \alpha_c$. We show explicitly the range of parameter values D, μ and N for which this relation holds. We now carry out the details of this analysis.

We first calculate the eigenvalues of the full symmetric matrix $\mathcal{G}$. This is readily done since $\mathcal{G}^{-1}$ is a symmetric tridiagonal matrix. To see this, in Appendix A we solve (3.2b) on each subinterval $[x_{j-1}, x_j]$ and impose the following jump conditions across each $x = x_j$ that are associated with (3.2b):

$$[\eta]_j = 0\,, \qquad [D\eta_x]_j = -\omega_j\,, \qquad \omega_j = \frac{rc_j}{a_g b_r}\int_{-\infty}^{\infty} [u_c(y)]^{r-1}\Phi(y)\,dy\,. \tag{3.9}$$

Here $[a]_j \equiv a(x_{j+}) - a(x_{j-})$. This procedure then leads to a linear system for $\eta(x_j), j = 0, \ldots, N-1$ of the form

$$\mathcal{B}\boldsymbol{\eta} = (\mu D)^{-1/2}\,\boldsymbol{\omega}\,, \tag{3.10a}$$

where the $N \times N$ tridiagonal matrix $\mathcal{B}$ and the N-vectors $\boldsymbol{\eta}$ and $\boldsymbol{\omega}$ are defined by

$$\mathcal{B} \equiv \begin{pmatrix} d & f & 0 & \cdots & 0 & 0 & 0 \\ f & e & f & \cdots & 0 & 0 & 0 \\ 0 & f & e & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & e & f & 0 \\ 0 & 0 & 0 & \cdots & f & e & f \\ 0 & 0 & 0 & \cdots & 0 & f & d \end{pmatrix}, \qquad \boldsymbol{\eta} \equiv \begin{pmatrix} \eta(x_0) \\ \vdots \\ \eta(x_{N-1}) \end{pmatrix}, \qquad \boldsymbol{\omega} \equiv \begin{pmatrix} \omega_0 \\ \vdots \\ \omega_{N-1} \end{pmatrix}. \tag{3.10b}$$

Here d, e and f are defined by

$$d \equiv \coth(2\theta/N) + \tanh(\theta/N)\,, \qquad e \equiv 2\coth(2\theta/N)\,, \qquad f \equiv -\mathrm{csch}(2\theta/N)\,, \tag{3.10c}$$

where $\theta = (\mu/D)^{1/2}$. Note that $d = e + f$. Thus, $\boldsymbol{\eta}$ is given by $\boldsymbol{\eta} = \mathcal{B}^{-1}\boldsymbol{\omega}\,(\mu D)^{-1/2}$. Another way to determine $\boldsymbol{\eta}$ is to evaluate (3.5) at $x = x_j$, for $j = 0, \ldots, N-1$. The equivalence of these two representations of $\boldsymbol{\eta}$ yields

$$\mathcal{G} = \frac{\mathcal{B}^{-1}}{\sqrt{\mu D}}\,. \tag{3.11}$$

In Appendix B we show the explicit calculation that yields the following result for the eigenvalues κ_j and the eigenvectors $\boldsymbol{q}_j$ of $\mathcal{B}$:

Proposition 2: *The eigenvalues κ_j, ordered as $0 < \kappa_1 < \ldots < \kappa_N$, and the normalized eigenvectors $\boldsymbol{q}_j$ of $\mathcal{B}$ are*

$$\kappa_1 = e + 2f\,; \qquad \kappa_j = e + 2f\cos\left(\frac{\pi(j-1)}{N}\right), \quad j = 2, \ldots, N\,, \tag{3.12a}$$

$$\boldsymbol{q}_1^t = \frac{1}{\sqrt{N}}(1, \ldots, 1)\,; \quad q_{l,j} = \sqrt{\frac{2}{N}}\cos\left(\frac{\pi(j-1)}{N}(l-1/2)\right), \quad j = 2, \ldots, N\,. \tag{3.12b}$$

Here $\boldsymbol{q}^t$ denotes transpose and $\boldsymbol{q}_j^t = (q_{1,j}, \dots, q_{N,j})$.

Therefore, from (3.11) and since $e > 0$ and $f < 0$, the smallest eigenvalue of $\mathcal{G}$ is proportional to κ_N^{-1} and the corresponding eigenvector is proportional to $\boldsymbol{q}_N$. Relabeling this eigenpair we obtain:

Proposition 3: *The smallest eigenvalue α_1 of $\mathcal{G}$ and the corresponding (unnormalized) eigenvector $\boldsymbol{q}_1$ are*

$$\alpha_1 = \frac{(\mu D)^{-1/2}}{e - 2f\cos(\pi/N)}, \tag{3.13a}$$

$$q_{1,l} = \sin\left(\frac{\pi(N-1)}{2N}\right)\cos\left(\frac{\pi(N-1)(l-1/2)}{N}\right). \tag{3.13b}$$

Here e and f are defined in (3.10c) and $\boldsymbol{q}_1^t = (q_{1,1}, \dots, q_{1,N})$.

The following rigorous result, which is a special case of a Theorem in [19], pertains to the principal eigenvalue of (3.8) and is critical to our stability analysis.

Theorem(**Wei [19]**): *Consider the eigenvalue problem for $\gamma_0 \geq 0$*

$$\Phi'' - \Phi + pu_c^{p-1}\Phi - \gamma_0(p-1)u_c^p\left(\frac{\int_{-\infty}^{\infty}[u_c(y)]^{r-1}\Phi(y)\,dy}{\int_{-\infty}^{\infty}[u_c(y)]^r\,dy}\right) = \lambda\Phi, \qquad -\infty < y < \infty, \tag{3.14a}$$

$$\Phi \to 0 \quad \text{as} \quad |y| \to \infty, \tag{3.14b}$$

corresponding to eigenpairs for which $\lambda \neq 0$. Here u_c satisfies (2.5). Let $\lambda_0 \neq 0$ be the eigenvalue of (3.14) with the largest real part. Then, if $\gamma_0 < 1$, we conclude that

$$\mathrm{Re}(\lambda_0) > 0. \tag{3.15}$$

Alternatively, if $\gamma_0 > 1$ and if either of the following two conditions hold

$$\text{(i)} \quad r = 2, \; 1 < p \leq 5, \qquad \text{or} \qquad \text{(ii)} \quad r = p+1, \; p > 1, \tag{3.16a}$$

then

$$\mathrm{Re}(\lambda_0) < 0. \tag{3.16b}$$

The simple proof of (3.15) is given in Appendix E. The proof of (3.16) is given in Lemma A and Theorem 1.4 of [19]. For completeness we give it in Appendix F.

The assumption (3.16a) holds for the two common sets of parameter sets for the GM model $(p,q,r,s) = (2,1,2,0)$ and $(p,q,r,s) = (4,2,2,0)$. Comparing (3.8) with (3.14), the theorem above yields the following key result on the spectrum associated with (3.8):

Proposition 4: *Let $\lambda_0 \neq 0$ be the eigenvalue of (3.8) with the largest real part and assume condition (3.16a) holds. Then, Re*$(\lambda_0) > 0$ *when*

$$\alpha_1 < \alpha_c \qquad \text{where} \qquad \alpha_c = \frac{(p-1)a_g}{qr} \, . \tag{3.17}$$

Also Re$(\lambda_0) < 0$ *when* $\alpha_1 > \alpha_c$. *Here* α_1 *is the minimum eigenvalue of* $\mathcal{G}$ *given in (3.13a) and* a_g *is the constant row sum of* $\mathcal{G}$ *defined in (2.13).*

To get an explicit stability criterion we must calculate a_g. Since $\boldsymbol{q}_1^t = (1, \dots, 1)$ is an eigenvector of $\mathcal{B}$ with eigenvalue $\kappa_1 = e + 2f$ we can multiply both sides of (3.11) by $\boldsymbol{q}_1$ to get

$$\mathcal{G}\boldsymbol{q}_1 = a_g\,(1, \dots, 1)^t = \frac{1}{\sqrt{\mu D}}\mathcal{B}^{-1}\mathbf{q}_1 = \frac{1}{\kappa_1\sqrt{\mu D}}\,(1, \dots, 1)^t \, . \tag{3.18}$$

Hence,

$$a_g = \frac{1}{\sqrt{\mu D}}\,(e+2f)^{-1} = \frac{1}{2\sqrt{\mu D}}\frac{1}{\coth(2\theta/N) - \operatorname{csch}(2\theta/N)} \, . \tag{3.19}$$

Substituting (3.13a) and (3.19) into (3.17) we obtain that $\mathrm{Re}(\lambda_0) = 0$ when

$$\frac{e+2f}{e - 2f\cos(\pi/N)} = \frac{p-1}{qr} \, . \tag{3.20}$$

Using the definition (3.10c) for e and f, we calculate $e/(2f) = -\cosh(2\theta/N)$. Substituting this expression into (3.20) and solving for the critical value of θ we get the following main result:

Proposition 5: *Let $\lambda_0 \neq 0$ be the eigenvalue of (3.8) with the largest real part and assume condition (3.16a) holds. Then, Re*$(\lambda_0) < 0$ *when*

$$D < D_N \equiv \frac{\mu}{\theta_N^2} \, , \quad N = 1, 2, \dots \, , \tag{3.21a}$$

$$\theta_N \equiv \frac{N}{2}\ln\left[a + \sqrt{a^2 - 1}\right] \, , \qquad a \equiv 1 + \left[1 + \cos\left(\frac{\pi}{N}\right)\right]\left(\frac{qr}{p-1} - 1\right)^{-1} \, . \tag{3.21b}$$

Alternatively, when $D > D_N$ *then Re*$(\lambda_0) > 0$.

This result gives the stability criterion for the large eigenvalues of (2.16) when $s = 0$. For example, from this result we can conclude that a three-spike equilibrium solution is stable with respect to the large $O(1)$ eigenvalues only when when $D < D_3$. To stabilize one additional spike we need to decrease D below D_4.

We now examine (3.21) for the GM parameter set $(p, q, r, s) = (2, 1, 2, 0)$ for which

$$\theta_N = \frac{N}{2}\ln\left[2 + \cos\left(\frac{\pi}{N}\right) + \sqrt{\left(2 + \cos\left(\frac{\pi}{N}\right)\right)^2 - 1}\right] \, . \tag{3.22}$$

We then calculate the following sequence of critical values of D_N:

$$D_1 = \mu/\theta_1^2 = \infty\,, \qquad \theta_1 = 0\,, \tag{3.23a}$$
$$D_2 = \mu/\theta_2^2 = 0.5766\mu\,, \qquad \theta_2 = \ln\left(2+\sqrt{3}\right)\,, \tag{3.23b}$$
$$D_3 = \mu/\theta_3^2 = 0.1810\mu\,, \qquad \theta_3 = \frac{3}{2}\ln\left(\frac{5}{2}+\frac{\sqrt{21}}{2}\right)\,, \tag{3.23c}$$
$$D_4 = \mu/\theta_4^2 = 0.0915\mu\,, \qquad \theta_4 = 2\ln\left(2+\frac{\sqrt{2}}{2}+\sqrt{\frac{7}{2}+2\sqrt{2}}\right)\,. \tag{3.23d}$$

In the limit $N \gg 1$, we get

$$D_N \sim 4\mu N^{-2}\left(\ln\left[3+\sqrt{8}\right]\right)^{-2} + O\left(N^{-4}\right)\,. \tag{3.24}$$

For the analysis leading to (3.21) to be valid we require that $D/\varepsilon^2 \gg 1$ in order to ensure that h is slowly varying in the inner regions near each spike. Hence it follows that we require $N \ll 1/\varepsilon$, which limits the range of validity of (3.24).

For the other common parameter set $(p,q,r,s) = (4,2,2,0)$ we get the critical values

$$D_1 = \mu/\theta_1^2 = \infty\,, \qquad \theta_1 = 0\,, \tag{3.25a}$$
$$D_2 = \mu/\theta_2^2 = 0.2349\mu\,, \qquad \theta_2 = \ln\left(4+\sqrt{15}\right)\,, \tag{3.25b}$$
$$D_3 = \mu/\theta_3^2 = 0.0778\mu\,, \qquad \theta_3 = \frac{3}{2}\ln\left(\frac{11}{2}+\frac{\sqrt{117}}{2}\right)\,, \tag{3.25c}$$
$$D_4 = \mu/\theta_4^2 = 0.0401\mu\,, \qquad \theta_4 = 2\ln\left(4+\frac{3\sqrt{2}}{2}+\sqrt{\frac{39}{2}+12\sqrt{2}}\right)\,. \tag{3.25d}$$

The results in (3.23a) and (3.25a) suggest that the principle eigenvalue of (3.8) for a one-spike equilibrium solution will always have a negative real part for any value of D. This conclusion is true when D is independent of ε, but needs to be modified if we allow D to depend on ε. More specifically, we show in §5 that a one-spike equilibrium solution is stable only when $D < D_1(\varepsilon)$, where $D_1(\varepsilon)$ is exponentially large as $\varepsilon \to 0$ and satisfies $D_1(\varepsilon) = O\left(\varepsilon^2 e^{2/\varepsilon}\right)$ for $\varepsilon \ll 1$.

3.2 Analysis for $s > 0$

For $s > 0$ we again introduce the new variables (3.1) into (2.16) and use $H^{\gamma r-(s+1)} = 1/(b_r a_g)$ from (2.13), with the result

$$\varepsilon^2 \phi_{xx} - \phi + \frac{pu^{p-1}}{v^q}\phi - \frac{qu^p}{v^{q+1}}\eta = \lambda\phi\,, \qquad -1 < x < 1\,, \tag{3.26a}$$

$$D\eta_{xx} - \mu\eta - \frac{su^r}{\varepsilon b_r a_g v^{s+1}}\eta = -\frac{ru^{r-1}}{\varepsilon b_r a_g v^s}\phi\,, \qquad -1 < x < 1\,, \tag{3.26b}$$

$$\phi_x(\pm 1) = \eta_x(\pm 1) = 0\,. \tag{3.26c}$$

Here $u \sim \sum_{k=0}^{N-1} u_c\left[\varepsilon^{-1}(x - x_k)\right]$. Substitute the form for ϕ given in (3.3) into (3.26b) and use the facts that u is localized and that $v = 1 + O(\varepsilon)$ near each x_k. Then, in place of (3.4), (3.26b) and (3.26c) become

$$fD\eta_{xx} - \left[\mu + \frac{s}{a_g}\sum_{k=0}^{N-1}\delta(x - x_k)\right]\eta = -\frac{r}{a_g b_r}\int_{-\infty}^{\infty}\left[u_c(y)\right]^{r-1}\Phi(y)dy\sum_{k=0}^{N-1}c_k\delta(x - x_k),\ \ |x| < 1, \tag{3.27a}$$

$$\eta_x(\pm 1) = 0\,. \tag{3.27b}$$

Thus the term proportional to s in (3.27a) acts as a psuedo-potential and hence will modify the jump condition for η_x across each x_j. Since u is localized near each x_j, and $\eta(x)$ is slowly varying with respect to ε near each x_j, we need only calculate $\eta(x_j)$ and substitute into (3.26a) to obtain the eigenvalue problem.

To calculate $\eta(x_j)$ we proceed as follows. We introduce $\boldsymbol{\eta}$ and $\boldsymbol{\omega}$ as defined in (3.9) and (3.10). We then solve (3.27a) analytically on each subinterval in terms of hyperbolic functions and then patch the subinterval solutions together using the appropriate jump conditions

$$[\eta]_j = 0\,, \qquad [D\eta_x]_j = -\omega_j + \frac{s}{a_g}\eta(x_j)\,, \tag{3.28}$$

where ω_j was defined in (3.9). This calculation, which we omit, shows that the solution for $\boldsymbol{\eta}$ can be written in the form

$$\mathcal{B}_s\boldsymbol{\eta} = (\mu D)^{-1/2}\,\boldsymbol{\omega}\,, \tag{3.29}$$

where the matrix $\mathcal{B}_s$ is given by

$$\mathcal{B}_s = \mathcal{B} + \frac{s}{a_g\sqrt{\mu D}}I\,. \tag{3.30}$$

Here I is the $N \times N$ identity matrix and $\mathcal{B}$ is the matrix defined in (3.10b) and (3.10c). Therefore, using (3.9) and (3.10b), we obtain

$$\boldsymbol{\eta} = \frac{r}{a_g b_r \sqrt{\mu D}} \int_{-\infty}^{\infty} [u_c(y)]^{r-1} \, \Phi(y) \, dy \; \mathcal{B}_s^{-1} \boldsymbol{c} \,, \tag{3.31}$$

where $\boldsymbol{c}$ is defined in (3.7). In place of (3.6) we get, for $j = 0, \ldots, N-1$, that

$$c_j \left(\Phi'' - \Phi + p u_c^{p-1} \Phi \right) - \frac{q r u_c^p}{a_g b_r \sqrt{\mu D}} \int_{-\infty}^{\infty} [u_c(y)]^{r-1} \, \Phi(y) \, dy \; \left(\mathcal{B}_s^{-1} \boldsymbol{c} \right)_{j+1} = c_j \lambda \Phi \,, \qquad -\infty < y < \infty \,, \tag{3.32}$$

with $\Phi \to 0$ as $|y| \to \infty$. Here $\left(\mathcal{B}_s^{-1} \boldsymbol{c} \right)_j$ denotes the j^{th} component of the vector $\mathcal{B}_s^{-1} \boldsymbol{c}$.

Now let $\boldsymbol{c}$ be an eigenvector of the matrix eigenvalue problem

$$\mathcal{B}_s \boldsymbol{q} = \kappa \boldsymbol{q} \,. \tag{3.33}$$

Then, using (2.7) for b_r, (3.32) becomes

$$\Phi'' - \Phi + p u_c^{p-1} \Phi - \frac{q r u_c^p}{a_g \kappa \sqrt{\mu D}} \left(\frac{\int_{-\infty}^{\infty} [u_c(y)]^{r-1} \, \Phi(y) \, dy}{\int_{-\infty}^{\infty} [u_c(y)]^r \, dy} \right) = \lambda \Phi \,, \qquad -\infty < y < \infty \,, \tag{3.34a}$$

$$\Phi \to 0 \quad \text{as} \quad |y| \to \infty \,. \tag{3.34b}$$

Let $\lambda_0 \neq 0$ be the eigenvalue of (3.34) with the largest real part. Then, from comparing (3.14) and (3.34), we conclude from the theorem of [19] stated above that $\text{Re}(\lambda_0) < 0$ only when

$$\frac{1}{\kappa a_g \sqrt{\mu D}} > \frac{(p-1)}{q r} \,. \tag{3.35}$$

To obtain a condition in terms of the minimum eigenvalue of $\mathcal{G}$, we use (3.11) to get that $\mathcal{G} \boldsymbol{q} = \alpha \boldsymbol{q}$, where $\kappa a_g \sqrt{\mu D} = s + a_g / \alpha$. Substituting this relation between κ and α into (3.35), we obtain the following result in terms of the smallest eigenvalue α_1 of $\mathcal{G}$:

Proposition 6: *Let $\lambda_0 \neq 0$ be the eigenvalue of (3.8) with the largest real part and assume condition (3.16a) holds. Then, $Re(\lambda_0) > 0$ when*

$$\frac{\alpha_1}{a_g} < \left(\frac{q r}{p-1} - s \right)^{-1} \,. \tag{3.36}$$

Also $Re(\lambda_0) < 0$ when the inequality in (3.36) is reversed.

The right-hand side of (3.36) is always positive by the assumption (1.2) on the exponents. Setting $\alpha_1/a_g = [qr/(p-1)-s]^{-1}$, and using (3.13) and (3.19) for α_1 and a_g, respectively, we get the following main result for the stability of the equilibrium solution with regards to the large $O(1)$ eigenvalues:

Proposition 7: *Let $\lambda_0 \neq 0$ be the eigenvalue of (3.34) with the largest real part and assume condition (3.16a) holds. Then, $Re(\lambda_0) < 0$ when*

$$D < D_N \equiv \frac{\mu}{\theta_N^2}\,, \quad N = 1, 2, \dots\,, \tag{3.37a}$$

$$\theta_N \equiv \frac{N}{2}\ln\left[a + \sqrt{a^2-1}\right]\,, \qquad a \equiv 1 + \left[1 + \cos\left(\frac{\pi}{N}\right)\right]\left(\frac{qr}{p-1} - (s+1)\right)^{-1}\,. \tag{3.37b}$$

Alternatively, when $D > D_N$ then $Re(\lambda_0) > 0$.

From (1.2) we get that $a > 1$ since $qr/(p-1) > (s+1)$. In addition, D_N decreases as s increases, and so for each fixed N it follows that D must be made smaller as s increases in order to stabilize an N-spike equilibrium solution.

4 Analysis of the Small Eigenvalues

The results in §3 establish conditions for which the equilibrium solution is stable on an $O(1)$ time scale. Now, we examine the more difficult problem of determining conditions guaranteeing that the small eigenvalues with $\lambda = O(\varepsilon^2)$ lie in the left half-plane. The first step, done in §4.1, is to reduce (2.16) to the study of a matrix eigenvalue problem. In §4.2 we analyze this matrix eigenvalue problem to determine the small eigenvalues and their signs explicitly.

4.1 Deriving the Matrix Eigenvalue Problem

We begin by writing (2.16) in the form

$$L_\varepsilon\phi - \frac{qa_e^p}{h_e^{q+1}}\eta = \lambda\phi\,, \qquad -1 < x < 1\,, \tag{4.1a}$$

$$D\eta_{xx} - \mu\eta = -\varepsilon^{-1} r\frac{a_e^{r-1}}{h_e^s}\phi + \varepsilon^{-1} s\frac{a_e^r}{h_e^{s+1}}\eta\,, \qquad -1 < x < 1\,, \tag{4.1b}$$

$$\phi_x(\pm 1) = \eta_x(\pm 1) = 0\,, \tag{4.1c}$$

where

$$L_\varepsilon\phi \equiv \varepsilon^2\phi_{xx} - \phi + \frac{pa_e^{p-1}}{h_e^q}\phi\,. \tag{4.1d}$$

Here a_e and h_e are given by

$$a_e \sim \sum_{k=0}^{N-1} H^{\gamma} u_k \,; \qquad h_e \sim \frac{H}{a_g} \sum_{k=0}^{N-1} G(x; x_k) \,; \qquad H^{\gamma r - (1+s)} = \frac{1}{b_r a_g} \,. \tag{4.2}$$

We have defined $u_k(y) \equiv u_c \left[\varepsilon^{-1}(x - x_k)\right]$, where $u_c(y)$ satisfies (2.5). The equilibrium positions for x_j are such that

$$\langle h_{ex} \rangle_j = 0 \,, \qquad j = 0, .., N-1 \,. \tag{4.3}$$

Here and below we have defined $\langle \zeta \rangle_j \equiv (\zeta(x_{j+}) + \zeta(x_{j-}))/2$ and $[\zeta]_j \equiv \zeta(x_{j+}) - \zeta(x_{j-})$, where $\zeta(x_{j\pm})$ are the one-sided limits of $\zeta(x)$ as $x \to x_{j\pm}$.

If the inhibitor diffusivity was infinite and there only one spike, then by translation invariance we would obtain $L_\varepsilon a_{ex} = 0$. Here we expect that $L_\varepsilon a_{ex}$ is still small. To show this, we differentiate the equilibrium problem for (1.4a) with respect to x to get

$$L_\varepsilon a_{ex} = \frac{q a_e^p}{h_e^{q+1}} h_{ex} \,. \tag{4.4}$$

Thus, for x near x_j we get

$$L_\varepsilon u_j' \sim \frac{\varepsilon q H^q u_j^p}{h_e^{q+1}} h_{ex} \,. \tag{4.5}$$

This fact suggests that we expand

$$\phi = \phi_0 + \varepsilon \phi_1 + \cdots \,, \qquad \eta(x) = \varepsilon \eta_0(x) + \cdots \,, \tag{4.6a}$$

where

$$\phi_0 \equiv \sum_{j=0}^{N-1} c_j u_j' \left[\varepsilon^{-1}(x - x_j)\right] \,, \qquad \phi_1 \equiv \sum_{j=0}^{N-1} c_j \phi_{1j} \left[\varepsilon^{-1}(x - x_j)\right] \,, \tag{4.6b}$$

and the c_j are arbitrary coefficients.

We substitute (4.6a) into (4.1a) and use (4.5) and $\lambda = O(\varepsilon^2)$. For x near x_j, we get that $\phi_{1j}(y)$ satisfies

$$c_j L_\varepsilon \phi_{1j} \sim -\frac{q u_j^p H^q}{h_e^{q+1}} \left[c_j h_{ex}(x_j + \varepsilon y) - H^{\gamma} \eta_0(x_j + \varepsilon y)\right] \,. \tag{4.7}$$

Before solving this equation for ϕ_{1j} we need to determine an important continuity property of the right-hand side of (4.7).

Substituting (4.6a) into (4.1b), we get that η_0 satisfies

$$D\eta_{0xx} - \mu\eta_0 = -\varepsilon^{-2} r \frac{a_e^{r-1}}{h_e^s} (\phi_0 + \varepsilon\phi_1) + \varepsilon^{-1} s \frac{a_e^r}{h_e^{s+1}} \eta_0 \,, \qquad -1 < x < 1 \,. \tag{4.8}$$

Since ϕ_0 is a linear combination of $u_j^{'}$, it follows that the term multiplied by ϕ_0 on the right-hand side in (4.8) behaves like a dipole. Hence, for $\varepsilon \ll 1$, this term is a linear combination of $\delta^{'}(x - x_j)$ for $j = 0, .., N-1$, where $\delta(x)$ is the delta function. Thus, η_0 will be discontinuous across $x = x_j$. However, if we define the function $f(x)$ by

$$f(x) \equiv H^{\gamma} \eta_0(x) - c_j h_{ex}(x) \,, \tag{4.9}$$

then f is continuous across $x = x_j$. To see this, we differentiate (1.4b) for h_e with respect to x and subtract appropriate multiples of the resulting equation and (4.8) to find that the dipole term cancels exactly. Thus, f is continuous across $x = x_j$, and we have $\langle f \rangle_j = f(x_j)$. However, $\langle h_{ex} \rangle_j = 0$ from (4.3). Hence, $f(x_j) = H^{\gamma} \langle \eta_0 \rangle_j$. Therefore, for $\varepsilon \ll 1$, we get from (4.7) that ϕ_{1j} satisfies

$$c_j L_{\varepsilon} \phi_{1j} \sim q u_j^p H^{\gamma-1} \langle \eta_0 \rangle_j \,. \tag{4.10}$$

Since $L_{\varepsilon} u_j = (p-1) u_j^p + O(\varepsilon)$, (4.10) is easily solved to get

$$c_j \phi_{1j}(y) = \frac{q}{p-1} u_j(y) H^{\gamma-1} \langle \eta_0 \rangle_j + O(\varepsilon) \,. \tag{4.11}$$

This condition shows that ϕ_{1j} is continuous across $x = x_j$ and has the form of a spike. This implies that the term in (4.8) proportional to ϕ_1 behaves like a linear combination of $\delta(x - x_j)$ when $\varepsilon \ll 1$ and, most importantly, is *of the same order* in ε as the dipole term proportional to ϕ_0. This shows the fact that we need to determine the approximate eigenfunction for ϕ to both the $O(1)$ and $O(\varepsilon)$ terms in order to calculate an eigenvalue of order $O(\varepsilon^2)$.

Next, let $\varepsilon \to 0$ and use (4.6b) and (2.13) for $H^{\gamma r - (s+1)}$ to calculate for x near x_j that

$$-\varepsilon^{-2} r \frac{a_e^{r-1}}{h_e^s} \phi_0 \sim -\frac{H^{1-\gamma}}{a_g} c_j \delta^{'}(x - x_j) \,, \tag{4.12a}$$

$$-\varepsilon^{-1} r \frac{a_e^{r-1}}{h_e^s} \phi_{1j} \sim -\frac{r H^{1-\gamma} c_j}{a_g b_r} \int_{-\infty}^{\infty} u_c^{r-1} \phi_{1j} \, dy \, \delta(x - x_j) \,. \tag{4.12b}$$

Substituting (4.12) into (4.8), and using the formula (4.11) for ϕ_{1j}, we get

$$D\eta_{0xx} - \left[\mu + \frac{s}{a_g}\sum_{j=0}^{N-1}\delta(x-x_j)\right]\eta_0 = -\frac{H^{1-\gamma}}{a_g}\sum_{j=0}^{N-1} c_j\delta^{'}(x-x_j) - \frac{qr}{(p-1)a_g}\sum_{j=0}^{N-1}\langle\eta_0\rangle_j\delta(x-x_j)\,. \tag{4.13}$$

This problem is equivalent to

$$D\eta_{0xx} - \mu\eta_0 = 0\,, \qquad -1 < x < 1\,; \qquad \eta_{0x}(\pm 1) = 0\,, \tag{4.14a}$$

$$[D\eta_0]_j = -\left(\frac{\varepsilon c_j}{a_g}\right)H^{\gamma-1}\,; \qquad [D\eta_{0x}]_j = \frac{1}{a_g}\left(s - \frac{qr}{(p-1)}\right)\langle\eta_0\rangle_j\,. \tag{4.14b}$$

For convenience we introduce $\tilde{\eta}_0$ defined by

$$\eta_0 = H^{1-\gamma}\tilde{\eta}_0\,. \tag{4.15}$$

Next, we estimate the small eigenvalue. Substitute (4.6) and (4.15) into (4.1a) and multiply both sides of (4.1a) by $u_j^{'}$. Integrating the resulting equation across the domain, we get

$$\sum_{i=0}^{N-1}\left(u_j^{'}, c_i L_\varepsilon u_i^{'}\right) + \varepsilon\sum_{i=0}^{N-1}\left(u_j^{'}, c_i L_\varepsilon\phi_{1i}\right) - \varepsilon q H^{1-\gamma}\left(u_j^{'}, \frac{a_e^p\tilde{\eta}_0}{h_e^{q+1}}\right) \sim \lambda\sum_{i=0}^{N-1}\left(c_i u_i^{'}, u_j^{'}\right)\,. \tag{4.16}$$

Here we have defined $(f,g) \equiv \int_{-1}^{1} f(x)g(x)\,dx$. To within negligible exponentially small terms, the dominant contribution in the sum comes from $i=j$ since $u_j^{'}$ is exponentially localized near $x=x_j$. Thus, (4.16) becomes

$$c_j\left(u_j^{'}, L_\varepsilon u_j^{'}\right) + \varepsilon c_j\left(u_j^{'}, L_\varepsilon\phi_{1j}\right) - \varepsilon q H^{1+q}\left(u_j^{'}, \frac{u_j^p\tilde{\eta}_0}{h_e^{q+1}}\right) \sim \lambda c_j\left(u_j^{'}, u_j^{'}\right)\,. \tag{4.17}$$

Since L_ε is self-adjoint, we integrate by parts on the second term on the left-hand side of (4.17) and use (4.5) for $L_\varepsilon u_j^{'}$. The integrands are localized near $x = x_j$. Thus, writing the resulting integrals in terms of the stretched variable $y = \varepsilon^{-1}(x-x_j)$, we get

$$\varepsilon^2 q c_j H^q\int_{-\infty}^{\infty}\frac{u_j^p u_j^{'}}{h_e^{q+1}}h_{ex}\,dy - \varepsilon^2 q H^{1+q}\int_{-\infty}^{\infty}\frac{u_j^p u_j^{'}}{h_e^{q+1}}\tilde{\eta}_0\,dy$$
$$+\,\varepsilon^3 q c_j H^q\int_{-\infty}^{\infty}\frac{u_j^p\phi_{1j}}{h_e^{q+1}}h_{ex}\,dy \sim \varepsilon\lambda c_j\int_{-\infty}^{\infty}\left(u_c^{'}\right)^2 dy\,. \tag{4.18}$$

In this expression $\tilde{\eta}_0 = \tilde{\eta}_0(x_j+\varepsilon y)$, $h_e = h_e(x_j+\varepsilon y)$, and $h_{ex} = h_{ex}(x_j+\varepsilon y)$.

We now estimate each of the terms in (4.18). Since $[\phi_{1j}]_j = 0$, $\langle h_{ex} \rangle_j = 0$, and $u_j^{'}$ is odd, it follows that

$$\int_{-\infty}^{\infty} \frac{u_j^p \phi_{1j}}{h_e^{q+1}} h_{ex}\, dy = o(1) \qquad \text{as} \quad \varepsilon \to 0\,. \tag{4.19}$$

Hence, the third integral on the left-hand side of (4.18) will be $o(\varepsilon^3)$ and can be neglected. Next, we combine the first two terms on the left-hand side of (4.18) to get

$$\varepsilon^2 q c_j H^q \int_{-\infty}^{\infty} \frac{u_j^p u_j^{'}}{h_e^{q+1}} h_{ex}\, dy - \varepsilon^2 q H^{1+q} \int_{-\infty}^{\infty} \frac{u_j^p u_j^{'}}{h_e^{q+1}} \tilde{\eta}_0\, dy = -\varepsilon^2 q H^q \int_{-\infty}^{\infty} \frac{u_j^{'} u_j^p}{h_e^{q+1}} f(x_j + \varepsilon y)\, dy\,. \tag{4.20}$$

Here $f(x)$, defined in (4.9), is given in terms of $\tilde{\eta}_0$ by $f(x) = H\tilde{\eta}_0(x) - c_j h_{ex}(x)$. The function f is continuous across $x = x_j$ but its derivative is not. For $\varepsilon \ll 1$, we calculate

$$-\varepsilon^2 q H^q \int_{-\infty}^{\infty} \frac{u_j^{'} u_j^p}{h_e^{q+1}} f(x_j + \varepsilon y)\, dy \sim \varepsilon^3 q \frac{c_j h_{exx}(x_j)}{H} \int_{-\infty}^{\infty} y u_j^{'} u_j^p\, dy - \varepsilon^3 q \langle \tilde{\eta}_{0x} \rangle_j \int_{-\infty}^{\infty} y u_j^{'} u_j^p\, dy\,. \tag{4.21}$$

Upon integrating by parts in (4.21), and using $h_{exx}(x_j) = \mu H/D$, we get

$$\varepsilon^2 q H^{1+q} \int_{-\infty}^{\infty} \frac{u_j^{'} u_j^p}{h_e^{q+1}} \left(\frac{c_j h_{ex}}{H} - \tilde{\eta}_0 \right) dy \sim \frac{\varepsilon^3 q}{p+1} \left(\langle \tilde{\eta}_{0x} \rangle_j - \frac{c_j \mu}{D} \right) \int_{-\infty}^{\infty} [u_c(y)]^{p+1}\, dy\,. \tag{4.22}$$

Substituting (4.19) and (4.22) into (4.18), we obtain a formula for λ. We summarize the result (redefining η_0 for convenience) as follows:

<u>**Proposition 8**</u>: *The eigenvalues of order $O(\varepsilon^2)$ for (2.16) satisfy*

$$\lambda c_j \int_{-\infty}^{\infty} \left[u_c^{'}(y) \right]^2 dy \sim \frac{\varepsilon^2 q}{p+1} \int_{-\infty}^{\infty} [u_c(y)]^{p+1}\, dy \left(\langle \eta_x \rangle_j - \frac{c_j \mu}{D} \right), \qquad j = 0, .., N-1\,. \tag{4.23}$$

Here $\langle \eta_x \rangle_j$ is obtained from the solution to the boundary value problem

$$D\eta_{xx} - \mu\eta = 0\,, \qquad -1 < x < 1\,; \qquad \eta_x(\pm 1) = 0\,, \tag{4.24a}$$

$$[D\eta]_j = -\frac{\varepsilon c_j}{a_g}\,; \qquad [D\eta_x]_j = \frac{1}{a_g} \tilde{s} \langle \eta \rangle_j\,, \qquad \tilde{s} \equiv s - \frac{qr}{(p-1)}\,. \tag{4.24b}$$

4.2 Analyzing the Matrix Eigenvalue Problem

The next step in the derivation is to calculate $\langle \eta_x \rangle_j$ from the solution to (4.24). The solution to (4.24) can be decomposed as

$$\eta(x) = \frac{1}{a_g} \left(\sum_{k=0}^{N-1} c_k g(x; x_k) + \sum_{k=0}^{N-1} m_k G(x; x_k) \right) , \tag{4.25}$$

for some coefficients m_k, for $k = 0, \ldots , N-1$. Here G satisfies (2.10), and $g(x; x_k)$ is the dipole Green's function satisfying

$$D g_{xx} - \mu g = -\delta^{'} (x - x_k) , \qquad -1 < x < 1 , \tag{4.26a}$$

$$g_x(\pm 1; x_k) = 0 . \tag{4.26b}$$

Satisfying the jump conditions in (4.24b) we get the following matrix problem for the coefficients m_k:

$$\left(\frac{\tilde{s}}{a_g} \mathcal{G} + I \right) \boldsymbol{m} = -\frac{s}{a_g} \mathcal{P}_g \mathbf{c} . \tag{4.27}$$

Here $\mathcal{G}$ is the Green's function matrix defined in (3.7) with entries $G(x_j; x_k)$, and

$$\mathcal{P}_g \equiv \begin{pmatrix} \langle g(x_0; x_0) \rangle_0 & \cdots & g(x_0; x_{N-1}) \\ \vdots & \ddots & \vdots \\ g(x_{N-1}; x_0) & \cdots & \langle g(x_{N-1}; x_{N-1}) \rangle_{N-1} \end{pmatrix} , \qquad \boldsymbol{m} \equiv \begin{pmatrix} m_0 \\ \vdots \\ m_{N-1} \end{pmatrix} , \qquad \boldsymbol{c} \equiv \begin{pmatrix} c_0 \\ \vdots \\ c_{N-1} \end{pmatrix} . \tag{4.28}$$

The problem (4.27) determines $\boldsymbol{m}$ in terms of $\boldsymbol{c}$. Then, using (4.25), we can calculate $\langle \eta_x \rangle_j$, for $j = 0, \ldots , N-1$, from the matrix problem

$$\langle \boldsymbol{\eta_x} \rangle = \frac{1}{a_g} \left(\mathcal{G}_g \boldsymbol{c} + \mathcal{P} \boldsymbol{m} \right) , \tag{4.29}$$

where $\mathcal{G}_g$ is the Green's dipole matrix defined by

$$\mathcal{G}_g \equiv \begin{pmatrix} g_x(x_0; x_0) & \cdots & g_x(x_0; x_{N-1}) \\ \vdots & \ddots & \vdots \\ g_x(x_{N-1}; x_0) & \cdots & g_x(x_{N-1}; x_{N-1}) \end{pmatrix} , \tag{4.30}$$

and

$$\mathcal{P} \equiv \begin{pmatrix} \langle G_x(x_0;x_0)\rangle_0 & \cdots & G_x(x_0;x_{N-1}) \\ \vdots & \ddots & \vdots \\ G_x(x_{N-1};x_0) & \cdots & \langle G_x(x_{N-1};x_{N-1})\rangle_{N-1} \end{pmatrix}, \qquad \langle \boldsymbol{\eta_x} \rangle \equiv \begin{pmatrix} \langle \eta_x \rangle_0 \\ \vdots \\ \langle \eta_x \rangle_{N-1} \end{pmatrix}. \tag{4.31}$$

Next, we define σ by

$$\lambda = \frac{\varepsilon^2 q \sigma}{(p+1)a_g} \left(\frac{\int_{-\infty}^{\infty} [u_c(y)]^{p+1}\, dy}{\int_{-\infty}^{\infty} [u_c'(y)]^2\, dy} \right). \tag{4.32}$$

Substituting (4.29) and (4.32) into (4.23), we get a matrix eigenvalue problem for σ and $\boldsymbol{c}$

$$\mathcal{G}_g \boldsymbol{c} + \mathcal{P}\boldsymbol{m} = \left(\sigma + \frac{\mu a_g}{D}\right) \boldsymbol{c}\,. \tag{4.33}$$

Here $\boldsymbol{m}$ is determined in terms of $\boldsymbol{c}$ by (4.27).

The next step in the analysis is to reduce (4.27) and (4.33) to an equivalent generalized eigenvalue problem. This analysis involves matrices associated with G and g. To avoid confusion we have indicated with a subscript g those matrices associated with the dipole Green's function g.

In the analysis below we must find the eigenvalues of $\mathcal{G}_g$ explicitly. This is done as in §3 by showing that $\mathcal{G}_g^{-1}$ is a symmetric tridiagonal matrix. More specifically, in Appendix C we show that

$$\mathcal{G}_g = \frac{\theta}{D} \mathcal{B}_g^{-1}\,, \tag{4.34}$$

where B_g is a tridiagonal matrix with exactly the same form as in (3.10b), except that here the definitions of d, e and f in (3.10c) are to replaced with

$$d \equiv \coth(2\theta/N) + \coth(\theta/N)\,, \qquad e \equiv 2\coth(2\theta/N)\,, \qquad f \equiv -\mathrm{csch}(2\theta/N)\,, \tag{4.35}$$

where $d = e - f$. In Appendix D we calculate the eigenvalues and eigenvectors of $\mathcal{B}_g$ analytically. The result is summarized as follows:

Proposition 9: *The eigenvalues ξ_j, ordered as $0 < \xi_1 < \ldots < \xi_N$, of $\mathcal{B}_g$ and the associated normalized eigenvectors $\boldsymbol{v}_j$ of $\mathcal{B}_g$ are*

$$\xi_j = 2\coth\left(\frac{2\theta}{N}\right) - 2\mathrm{csch}\left(\frac{2\theta}{N}\right)\cos\left(\frac{\pi j}{N}\right) \qquad j = 1, \ldots, N\,, \tag{4.36a}$$

$$\boldsymbol{v}_N^t = \frac{1}{\sqrt{N}}\left(1, -1, 1, \ldots, (-1)^{N+1}\right)\;; \quad \boldsymbol{v}_{l,j} = \sqrt{\frac{2}{N}} \sin\left(\frac{\pi j}{N}(l - 1/2)\right)), \quad j = 1, \ldots, N-1\,. \tag{4.36b}$$

Here $\boldsymbol{v}^t$ denotes transpose and $\boldsymbol{v}_j^t = (v_{1,j}, \dots, v_{N,j})$.

Other key relations that we need are derived in Appendices A and C, where we show that

$$\mathcal{P}_g = \frac{1}{2D} \operatorname{csch}\left(\frac{2\theta}{N}\right) \mathcal{C}\mathcal{B}_g^{-1}, \qquad \mathcal{P} = -\frac{1}{2D} \operatorname{csch}\left(\frac{2\theta}{N}\right) \mathcal{C}^t \mathcal{B}^{-1}. \tag{4.37a}$$

Here the matrix $\mathcal{C}$ is defined by

$$\mathcal{C} \equiv \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & -1 \end{pmatrix}. \tag{4.37b}$$

From (4.37a) we obtain the result that

$$\mathcal{P}\mathcal{B} = -\left(\mathcal{P}_g \mathcal{B}_g\right)^t. \tag{4.37c}$$

We begin by solving (4.27) for $\boldsymbol{m}$. The matrix in (4.27) is invertible if $\tilde{s}\,(\alpha_1/a_g) + 1 < 0$, where α_1 is the minimum eigenvalue of $\mathcal{G}$. From (3.36) and the definition of $\tilde{s}$ in (4.24b), we see that this condition is satisfied when the large $O(1)$ eigenvalues are in the left half-plane. We will assume that $D < D_N$ so that this condition holds. Let $\boldsymbol{q}_j$, κ_j be the normalized eigenpairs of $\mathcal{B}$ as given in Proposition 2 for $j = 1, \dots, N$. Then,

$$\mathcal{B} = Q\mathcal{K}Q^t, \tag{4.38}$$

where Q is the orthogonal matrix whose columns are the normalized $\boldsymbol{q}_j$ and $\mathcal{K}$ is the diagonal matrix of the eigenvalues of $\mathcal{B}$. Since $\mathcal{G} = \mathcal{B}^{-1}/\sqrt{\mu D}$ from (3.11) and $Q^tQ = I$, we get

$$\left(\frac{\tilde{s}}{a_g}\mathcal{G} + I\right)^{-1} = Q\left(\frac{\tilde{s}}{a_g\sqrt{\mu D}}\mathcal{K}^{-1} + I\right)^{-1} Q^t. \tag{4.39}$$

Using $\theta = (\mu/D)^{1/2}$, we can solve for $\boldsymbol{m}$ in (4.27) in the form

$$\boldsymbol{m} = -\frac{\tilde{s}}{a_g} Q\left(\frac{\tilde{s}\theta}{a_g\mu}\mathcal{K}^{-1} + I\right)^{-1} Q^t \mathcal{P}_g \boldsymbol{c}. \tag{4.40}$$

We then substitute (4.40) and (4.34) into (4.33). This yields,

$$\mathcal{B}_g^{-1}\boldsymbol{c} - \tilde{s}D^2\left(\frac{\theta}{a_g\mu}\right)\mathcal{P}Q\left(\frac{\tilde{s}\theta}{a_g\mu}\mathcal{K}^{-1}+I\right)^{-1}Q^t\mathcal{P}_g\boldsymbol{c} = \left(\frac{D\sigma}{\theta}+\frac{\mu a_g}{\theta}\right)\boldsymbol{c}\,. \tag{4.41}$$

In (4.41) we use (4.37c) and (4.38) to replace $\mathcal{P}Q$ with

$$\mathcal{P}Q = \mathcal{P}\mathcal{B}\mathcal{B}^{-1}Q = -\left(\mathcal{P}_g\mathcal{B}_g\right)^t Q\mathcal{K}^{-1}\,. \tag{4.42}$$

We then introduce $\boldsymbol{u}$ and the diagonal matrix $\mathcal{D}$ defined by

$$\boldsymbol{u} \equiv \mathcal{B}_g^{-1}\boldsymbol{c}\,, \qquad \mathcal{D} \equiv \tilde{s}D^2\gamma\mathcal{K}^{-1}\left(\tilde{s}\gamma\mathcal{K}^{-1}+I\right)^{-1}\,. \tag{4.43}$$

Here we have defined γ by

$$\gamma \equiv \frac{\theta}{a_g\mu} = 2\left[\coth\left(\frac{2\theta}{N}\right) - \operatorname{csch}\left(\frac{2\theta}{N}\right)\right] = 2\tanh\left(\frac{\theta}{N}\right)\,. \tag{4.44}$$

Equation (4.44) is obtained from using the expression for a_g in (3.19). Using Proposition 2 for the eigenvalues κ_j of $\mathcal{K}$ we then calculate $\mathcal{D}$ as

$$\mathcal{D} \equiv \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_N \end{pmatrix}, \qquad \text{where} \qquad d_j = \frac{\tilde{s}D^2\gamma}{\kappa_j+\tilde{s}\gamma}\,, \qquad j=1,\ldots,N\,. \tag{4.45}$$

Substituting (4.42) and (4.43) into (4.41), we obtain the eigenvalue problem

$$\mathcal{B}_g\boldsymbol{u} = \omega\left(I+\mathcal{R}\right)\boldsymbol{u}\,. \tag{4.46a}$$

Here we have defined ω and $\mathcal{R}$ by

$$\omega \equiv \left(\frac{D\sigma}{\theta}+\frac{1}{\gamma}\right)^{-1}\,, \qquad \mathcal{R} \equiv \left(\mathcal{P}_g\mathcal{B}_g\right)^t Q\mathcal{D}Q^t\mathcal{P}_g\mathcal{B}_g\,. \tag{4.46b}$$

The assumption that the solution is stable with respect to the large $O(1)$ eigenvalues is equivalent to the condition that $\kappa_j + \tilde{s}\gamma < 0$ for $j = 1,..,N$. Under this condition, and since $\tilde{s} < -1$, we conclude that $\mathcal{D} > 0$. Hence, $I+\mathcal{R}$ is a positive-definite and symmetric matrix. This means that the eigenvalues ω_j, and consequently λ_j, for $j = 1,\ldots,N$ are real. The generalized eigenvalue problem (4.46) is equivalent to the combined problem (4.27) and (4.33).

The next step is to determine the spectrum of (4.46) analytically. To do so, we show that $\mathcal{R}$ has the same eigenvectors as $\mathcal{B}_g$. Hence, we claim that $\mathcal{R}$ can be written in terms of some positive diagonal matrix Σ as

$$\mathcal{R} = Q_g \Sigma Q_g^t \,. \tag{4.47}$$

Here Q_g is the eigenvector matrix associated with $\mathcal{B}_g$. The j^{th} column of Q_g is the eigenvector $\boldsymbol{v}_j$ given in Proposition 9. Using the formula for $\mathcal{P}_g\mathcal{B}_g$ in (4.37a), we can write $\mathcal{R}$ in (4.46b) as

$$\mathcal{R} = \frac{1}{4D^2}\text{csch}^2\left(\frac{2\theta}{N}\right)\mathcal{C}^t Q \mathcal{D} Q^t \mathcal{C} \,. \tag{4.48}$$

This is equivalent to

$$\mathcal{R} = \frac{1}{4D^2}\text{csch}^2\left(\frac{2\theta}{N}\right) Q_g \mathcal{M}\mathcal{D}\mathcal{M}^t Q_g^t \,, \tag{4.49a}$$

where the matrix $\mathcal{M}$ is defined by

$$\mathcal{M} = Q_g^t \mathcal{C}^t Q \,. \tag{4.49b}$$

Comparing (4.49a) and (4.47), we then define Σ by

$$\Sigma \equiv \frac{1}{4D^2}\text{csch}^2\left(\frac{2\theta}{N}\right)\mathcal{M}\mathcal{D}\mathcal{M}^t \,. \tag{4.50}$$

We now show that Σ defined in (4.50) is a diagonal matrix.

To show this, we first calculate the matrix $\mathcal{M}$ in (4.49b) using the explicit formulae for the eigenvectors of $\mathcal{B}$ and $\mathcal{B}_g$ given in Propositions 2 and 9. Let $m_{i,j}$ be the i,j^{th} entry of the matrix $\mathcal{M}$. Then, we calculate $m_{i,j}$ explicitly using (4.49b) and the definition of $\mathcal{C}$ in (4.37b), to get

$$m_{i,j} = \sum_{l=1}^{N} v_{l,i}\left(q_{l-1,j} - q_{l+1,j}\right) \,. \tag{4.51}$$

Here we have defined $q_{0,j} = q_{1,j}$ and $q_{N+1,j} = q_{N,j}$, where $q_{l,j}$ and $v_{l,j}$ are defined for $l = 1, \ldots, N$ and $j = 1, \ldots, N$ in (3.12b) and (4.36b), respectively. A tedious, but straightforward, calculation shows that $m_{i,j} = 0$ for $i \neq j-1$. However, the entry $m_{j-1,j}$ is non-zero. We calculate, for $j = 2, \ldots, N$, that

$$m_{j-1,j} = v_{1,j-1}\left(q_{1,j} - q_{2,j}\right) + v_{N,j-1}\left(q_{N-1,j} - q_{N,j}\right) + \sum_{l=2}^{N-1} v_{l,j-1}\left(q_{l-1,j} - q_{l+1,j}\right) \,. \tag{4.52}$$

Using (3.12b) and (4.36b), and standard trigonometric identities, we can reduce (4.52) to

$$m_{j-1,j} = \frac{4}{N}\sin\left(\frac{\pi(j-1)}{N}\right)\sum_{l=1}^{N}\sin^2\left(\frac{\pi(j-1)(l-1/2)}{N}\right), \qquad j = 2,\dots,N. \tag{4.53}$$

Therefore, we get the key result that

$$m_{j-1,j} = 2\sin\left(\frac{\pi(j-1)}{N}\right) \qquad j = 2,\dots N; \qquad m_{i,j} = 0 \quad \text{otherwise}. \tag{4.54}$$

Therefore, it is clear that the matrix product $\mathcal{M}\mathcal{D}\mathcal{M}^t$ in (4.50) is a diagonal matrix. This shows that $\mathcal{B}_g$ and $\mathcal{R}$ have the same eigenvectors. Then, by using (4.45) for the diagonal entries of $\mathcal{D}$, we calculate Σ in (4.50) explicitly as

$$\Sigma \equiv \begin{pmatrix} z_1 & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_N \end{pmatrix}, \tag{4.55a}$$

where

$$z_j = \frac{1}{4D^2}\text{csch}^2\left(\frac{2\theta}{N}\right)(m_{j,j+1})^2 d_{j+1}, \quad j = 1,\dots,N-1; \qquad z_N = 0. \tag{4.55b}$$

Finally, we use (4.54) for $m_{j,j+1}$, (4.45) for d_{j+1} and the result that $\kappa_{j+1} = \xi_j$ for $j = 1,\dots,N-1$, as obtained by comparing (3.12a) and (4.36a). In this way, we find that $z_j = z_j(\tilde{s})$, where

$$z_j = \frac{\tilde{s}\gamma}{\xi_j + \tilde{s}\gamma}\text{csch}^2\left(\frac{2\theta}{N}\right)\sin^2\left(\frac{\pi j}{N}\right), \qquad j = 1,\dots,N-1; \qquad z_N = 0. \tag{4.56}$$

Here γ was defined in (4.44). When $D < D_N$, so that the solution is stable with respect to the large $O(1)$ eigenvalues, then $\xi_j + \tilde{s}\gamma < 0$ for $j = 1,..,N-1$.

Since we have shown the crucial result that $\mathcal{B}_g$ and $\mathcal{R}$ have the same eigenvectors, it is easy to calculate the spectrum of (4.46). The eigenvalues ω_j of (4.46) are

$$\omega_j = \xi_j/(1+z_j), \qquad j = 1,\dots,N, \tag{4.57}$$

where ξ_j and z_j are given in (4.36a) and (4.56), respectively. Then, substituting (4.56) into the expression for σ in (4.46b), we get that $\sigma_j = \sigma_j(\tilde{s})$, where

$$\sigma_j = -\frac{\theta}{D\xi_j}\left(\frac{\xi_j}{\gamma} - 1 - z_j\right), \qquad j = 1,\dots,N. \tag{4.58}$$

Finally, we substitute (4.58) into (4.32) to obtain explicit formulae for the small eigenvalues $\lambda = O(\varepsilon^2)$. The main result is summarized as follows:

Proposition 10: *For $\varepsilon \ll 1$, consider the eigenvalues of (2.16) of order $\lambda = O(\varepsilon^2)$. The corresponding eigenfunction has the form (4.6) where $\boldsymbol{c}_j = \boldsymbol{v}_j$, with $\boldsymbol{v}_j$ defined in (4.36b). The explicit formula for the small eigenvalues is*

$$\lambda_j \sim -\frac{\varepsilon^2 q\mu}{D(p+1)} \left(\frac{\int_{-\infty}^{\infty} [u_c(y)]^{p+1}\, dy}{\int_{-\infty}^{\infty} [u_c'(y)]^2\, dy} \right) \left[\frac{1 - \cos(\pi j/N) - z_j\left(\cosh(2\theta/N) - 1\right)}{\cosh(2\theta/N) - \cos(\pi j/N)} \right], \tag{4.59}$$

for $j = 1, \ldots, N$. Here $z_j = z_j(\tilde{s})$ is defined in (4.56).

The final step in the analysis is to determine the sign of σ_j with respect to the parameter D. The condition $\sigma_j < 0$ for $j = 1, .., N$ holds when

$$\frac{\xi_j}{\gamma} - 1 - z_j > 0\,, \qquad j = 1, .., N\,. \tag{4.60}$$

Defining $w_j = \xi_j/\gamma$, we calculate from (4.36a) and (4.44), and from some standard trigonometric identities, that

$$w_j = 1 + \sin^2\left(\frac{\pi j}{2N}\right) \operatorname{csch}^2\left(\frac{\theta}{N}\right). \tag{4.61}$$

Since $z_N = 0$ and $w_N > 1$, (4.60) holds when $j = N$. Substituting (4.56) and (4.61) into (4.60), we see that $\sigma_j < 0$ when

$$\sin^2\left(\frac{\pi j}{2N}\right) \operatorname{csch}^2\left(\frac{\theta}{N}\right) \left(\frac{w_j}{\tilde{s}} + 1\right) > \sin^2\left(\frac{\pi j}{N}\right) \operatorname{csch}^2\left(\frac{2\theta}{N}\right), \qquad \text{for} \quad j = 1, .., N-1\,. \tag{4.62}$$

Using (4.61) and some standard identities, (4.62) reduces to

$$\left(1 + \tilde{s} + \operatorname{csch}^2\left(\frac{\theta}{N}\right)\right) \left(1 - \cos^2\left(\frac{\pi j}{2N}\right) \operatorname{sech}^2\left(\frac{\theta}{N}\right)\right) < 0\,. \tag{4.63}$$

The second bracketed term on the left-hand side of (4.63) is always positive for any $j = 1, .., N$. The first bracketed term is negative when D is very small since $\tilde{s} < -1$ and $\theta \gg 1$. However, this term will switch sign when D crosses through the critical value where

$$\operatorname{csch}^2\left(\frac{\theta}{N}\right) = -(1 + \tilde{s})\,. \tag{4.64}$$

Hence, $N-1$ of the small eigenvalues switch sign at the *same* value of D. Let D_N^* be the value of D satisfying (4.64). By solving (4.64) we obtain the following main result for the stability of the solution with respect to the small eigenvalues:

Proposition 11: *For $\varepsilon \ll 1$, consider the eigenvalues of (2.16) of order $\lambda = O(\varepsilon^2)$. These eigenvalues are negative only when $D < D_N^*$, where*

$$D_N^* \equiv \frac{\mu}{\left[N \ln(\sqrt{\beta} + \sqrt{\beta + 1})\right]^2}, \qquad \beta \equiv \left[\frac{qr}{p-1} - (1+s)\right]^{-1}. \tag{4.65}$$

There are $N-1$ small positive eigenvalues when $D > D_N^*$. When $D = D_N^*$, then $\lambda = 0$ is an eigenvalue of algebraic multiplicity $N-1$.

It is a simple exercise to show that, in general, these critical values are smaller than the critical values D_N given in Proposition 7 for the stability of the solution with respect to the large $O(1)$ eigenvalues. Thus, our final conclusion is that an N-spike equilibrium solution will be stable only when $D < D_N^*$. For the parameter sets $(p, q, r, s) = (2, 1, 2, 0)$ we get $\beta = 1$, and for $(p, q, r, s) = (4, 2, 2, 0)$ we get $\beta = 3$. From (4.65) we then calculate the critical values

$$N = 2 \quad \rightarrow \quad D_2 = 0.3218\mu \quad \beta = 1\,; \quad D_2 = 0.1441\mu \quad \beta = 3\,, \tag{4.66a}$$

$$N = 3 \quad \rightarrow \quad D_3 = 0.1430\mu \quad \beta = 1\,; \quad D_3 = 0.0641\mu \quad \beta = 3\,, \tag{4.66b}$$

$$N = 4 \quad \rightarrow \quad D_4 = 0.0805\mu \quad \beta = 1\,; \quad D_4 = 0.0361\mu \quad \beta = 3\,. \tag{4.66c}$$

The numerical computations of [7] of the time-dependent problem (1.4) with $(p, q, r, s) = (2, 1, 2, 0)$, starting with initial conditions close to an asymptotic equilibrium solution, suggested that $D_2 \approx 0.33$ and $D_3 \approx 0.14$. The detailed analysis presented above gives the theoretical basis for these numerical predictions.

5 The Dynamics of a One-Spike Solution

In this section we analyze the dynamics of a one-spike solution to (1.4). For finite inhibitor diffusivity D, in §5.1 we derive a differential equation determining the location $x_0(t)$ of the maximum of the activator concentration for a one-spike solution to (1.4). By linearizing this differential equation around the stable equilibrium location $x_0 = 0$, we show that the decay rate of infinitesimal perturbations coincides with the small eigenvalue result (4.59) when $N = 1$. Alternatively, when $D = \infty$, we know from [6] that the equilibrium solution $x_0 = 0$ for a one-spike solution is unstable. When $D = \infty$, the spike drifts exponentially slowly towards the closest endpoint of the domain (cf. [6]). To reconcile the finite D result with the infinite D analysis of [6], we show in §5.2 that the equilibrium location $x_0 = 0$ for a one-spike solution is stable when $D < D_1(\varepsilon)$, where D_1 is exponentially large as $\varepsilon \to 0$.

5.1 The Differential Equation for the Spike Location

In the inner region near the spike we introduce the new variables

$$y = \varepsilon^{-1}\left[x - x_0(\tau)\right] , \qquad \tilde{h}(y) = h(x_0 + \varepsilon y) , \qquad \tilde{a}(y) = a(x_0 + \varepsilon y) , \qquad \tau = \varepsilon^2 t , \tag{5.1a}$$

and we expand

$$\tilde{h}(y) = \tilde{h}_0(y) + \varepsilon \tilde{h}_1(y) + \cdots , \qquad \tilde{a}(y) = \tilde{a}_0(y) + \varepsilon \tilde{a}_1(y) + \cdots . \tag{5.1b}$$

Substituting (5.1) into (1.4), we find from the leading terms that a_0 and h_0 satisfy (2.3a) and (2.3b), respectively. Hence,

$$\tilde{a}_0(y) = H^{\gamma} u_c(y) , \qquad \tilde{h}_0(y) = H , \qquad \gamma = q/(p-1) , \tag{5.2}$$

where $u_c(y)$ satisfies (2.5). Here $H = H(\tau)$ is a function to be determined. Setting $\tilde{a}_1 = H^{\gamma} u_1$, we get to next order that

$$u_1'' - u_1 + p u_c^{p-1} u_1 = \frac{q u_c^p \tilde{h}_1}{H} - x_0' u_c' , \qquad -\infty < y < \infty , \tag{5.3a}$$

$$D \tilde{h}_1'' = -H^{\gamma r - s} u_c^r , \tag{5.3b}$$

with $u_1 \to 0$ exponentially as $|y| \to \infty$. The right-hand side of (5.3a) must be orthogonal to the homogeneous solution u_c' of (5.3a). From this solvability condition we obtain

$$x_0' = \frac{q}{H \int_{-\infty}^{\infty} \left[u_c'(y)\right]^2 \, dy} \int_{-\infty}^{\infty} u_c^p u_c' \tilde{h}_1 \, dy . \tag{5.4}$$

If we integrate (5.4) by parts twice, and use the facts that h_1'' and u_c are even functions, we get

$$x_0' = -\frac{q}{2H(p+1)} \left(\frac{\int_{-\infty}^{\infty} \left[u_c(y)\right]^{p+1} dy}{\int_{-\infty}^{\infty} \left[u_c'(y)\right]^2 \, dy} \right) \left[\lim_{y \to +\infty} \tilde{h}_1' + \lim_{y \to -\infty} \tilde{h}_1' \right] . \tag{5.5}$$

In the outer region away from the spike, a is exponentially small and, similar to the analysis in §2, we expand $h = h_0 + \cdots$, where h_0 satisfies

$$D h_0'' - \mu h_0 = -H^{\gamma r - s} b_r \delta(x - x_0) , \qquad -1 < x < 1 , \tag{5.6a}$$

$$h_0'(\pm 1) = 0 . \tag{5.6b}$$

Here b_r is defined in (2.7). To match with the inner solution we require that

$$h_0(x_0) = H\,, \qquad \lim_{y\to+\infty} \tilde{h}_1' + \lim_{y\to-\infty} \tilde{h}_1' = h_{0x}(x_{0+}) + h_{0x}(x_{0-})\,. \tag{5.7}$$

The solution to (5.6) is

$$h_0(x) = H^{\gamma r - s} b_r G(x; x_0)\,, \tag{5.8}$$

where the Green's function $G(x; x_0)$ satisfies (2.10). Substituting (5.8) into (5.7), and using (2.13) for $H^{\gamma r - s}$, we get

$$\lim_{y\to+\infty} \tilde{h}_1' + \lim_{y\to-\infty} \tilde{h}_1' = \frac{H}{G(x_0; x_0)} \left[G_x(x_{0+}; x_0) + G_x(x_{0-}; x_0)\right]\,, \tag{5.9a}$$

$$H = \left[\frac{1}{b_r G(x_0; x_0)}\right]^{1/\gamma r - (s+1)}\,. \tag{5.9b}$$

The solution $G(x; x_0)$ was given in (2.11). Using this solution we can calculate the right-hand side of (5.9). Then, substituting the result into (5.5) and (5.8), we obtain

Proposition 12: *For $\varepsilon \ll 1$, the dynamics of a one-spike solution to (1.4) is characterized by*

$$a(x,t) \sim H^{\gamma} u_c\left(\varepsilon^{-1}[x - x_0(t)]\right)\,, \tag{5.10a}$$

$$h(x,t) \sim HG\left[x; x_0(t)\right]/G\left[x_0(t); x_0(t)\right]\,, \tag{5.10b}$$

where $H = H(t)$ is given in (5.9b). The spike location $x_0(t)$ satisfies the differential equation

$$\frac{dx_0}{dt} \equiv F(x_0) \sim -\varepsilon^2 C\left[\tanh\left(\sqrt{\frac{\mu}{D}}(1 + x_0)\right) - \tanh\left(\sqrt{\frac{\mu}{D}}(1 - x_0)\right)\right]\,, \tag{5.10c}$$

where C is defined by

$$C \equiv \frac{q}{2(p+1)}\sqrt{\frac{\mu}{D}}\left(\frac{\int_{-\infty}^{\infty} [u_c(y)]^{p+1}\, dy}{\int_{-\infty}^{\infty} [u_c'(y)]^2\, dy}\right)\,. \tag{5.10d}$$

The equilibrium solution $x_0 = 0$ for this differential equation is stable for any D. The decay rate of infinitesimal perturbations around $x_0 = 0$ is

$$F'(0) \sim -\frac{\varepsilon^2 q\mu}{D(p+1)}\left(\frac{\int_{-\infty}^{\infty} [u_c(y)]^{p+1}\, dy}{\int_{-\infty}^{\infty} [u_c'(y)]^2\, dy}\right)\operatorname{sech}^2\left(\sqrt{\frac{\mu}{D}}\right)\,. \tag{5.11}$$

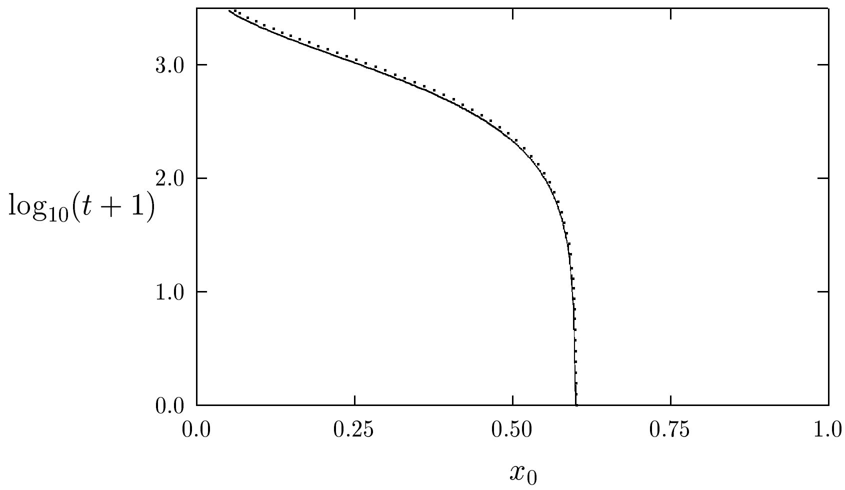


Figure 2: Plot of the trajectory $x_0(t)$ of the center of the spike for a one-spike solution with $\epsilon = .03$, $\mu = 1.0$, $D = 1.0$ and $(p, q, r, s) = (2, 1, 2, 0)$. The solid curve is the full numerical result and the dotted curve is the asymptotic result.

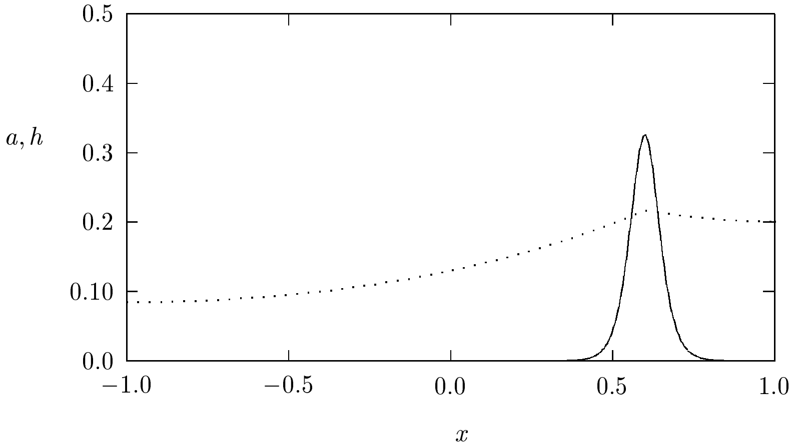


Figure 3: Plot of the initial condition for a one-spike solution corresponding to the parameter values shown in the caption of Fig. 2. The solid curve is the activator concentration and the dotted curve is the inhibitor concentration.

t	$\log_{10}(1+t)$	$x_0(t)$ (num.)	$x_0(t)$ (asy.)
12.0	1.114	0.5937	0.5942
96.0	1.987	0.5524	0.5552
204.0	2.312	0.5039	0.5091
486.0	2.688	0.3974	0.4073
864.0	2.937	0.2905	0.3035
1314.0	3.119	0.2008	0.2148
1884.0	3.275	0.1262	0.1392
2274.0	3.357	0.0919	0.1035

Table 1: A comparison of the asymptotic and numerical results for $x_0(t)$ corresponding to the parameter values shown in the caption of Fig. 2.

This result agrees precisely with the small eigenvalue result (4.59) when $N = j = 1$.

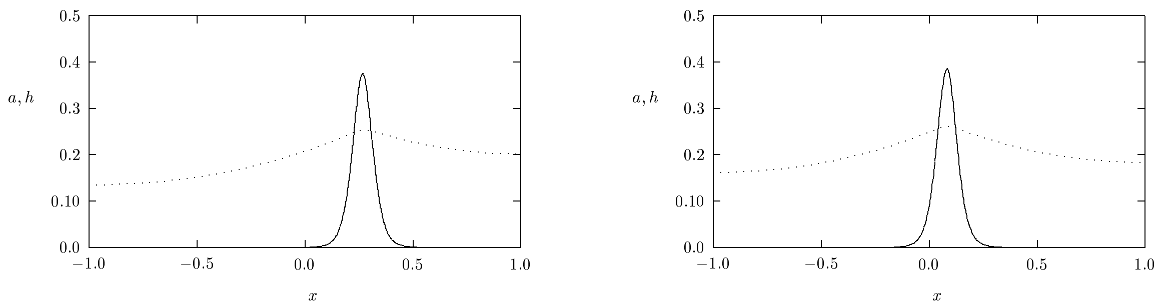
To verify (5.10c) for the parameter set $(p, q, r, s) = (2, 1, 2, 0)$ we compared the asymptotic result (5.10c) for $x_0(t)$ with the corresponding full numerical result computed from (1.4). The problem (1.4) was solved numerically using the routine D03PCF from the NAG library [12]. The initial condition was taken to be of the form (5.10a) and (5.10b) with $x_0(0) = 0.6$ and $\varepsilon = .03$, $\mu = 1.0$, and $D = 1.0$. An interpolation scheme was then used to locate the position of the maximum of a on the computational grid. In Fig. 2 and in Table 1 we compare this numerical result for x_0 with the corresponding asymptotic result obtained by solving the differential equation (5.10c) with the initial condition $x_0(0) = 0.6$. In solving the differential equation, the integrals in (5.10c) were evaluated using Romberg integration on a large but finite interval using the form for u_c given in (2.6). We find a close agreement between the asymptotic and numerical results for $x_0(t)$. In Fig. 3 we plot the initial condition used and then in Fig. 4(a) and Fig. 4(b) we plot the numerical solution to (1.4) at two different times showing the slow convergence to a one-spike equilibrium solution.

5.2 The Stability of a One-Spike Solution for $D \to \infty$

When $D = \infty$ it was shown in [6] that a one-spike solution is metastable and that the center $x_0(t)$ of the spike satisfies the asymptotic differential equation

$$\frac{dx_0}{dt} \equiv G(x_0) \sim \frac{2\alpha^2 \varepsilon}{\left(\int_{-\infty}^{\infty} \left[u_c'(y)\right]^2 \, dy\right)} \left(e^{-2(1-x_0)/\varepsilon} - e^{-2(1+x_0)/\varepsilon}\right) , \tag{5.12}$$

provided that x_0 is not within $O(\varepsilon)$ of the endpoints, i. e. (5.12) is valid when $1 - x_0 \gg O(\varepsilon)$ and $1 + x_0 \gg O(\varepsilon)$. Here α is defined by the limiting behavior $u_c(y) \sim \alpha e^{-|y|}$ as $|y| \to \infty$. This result



(a) $t = 960$

(b) $t = 2400$

Figure 4: Plot of a one-spike solution at two different times corresponding to the parameter values shown in the caption of Fig. 2. The solid curve is the activator concentration and the dotted curve is the inhibitor concentration.

shows that $x_0 = 0$ is unstable and that there is a metastable drift of the spike towards the closest endpoint of the domain. This result was obtained by analyzing the exponentially weak interaction between the tails of the spike and the boundaries $x = \pm 1$ of the domain. The analysis requires that (5.3a) be solved on the asymptotically large (but finite) domain $[-\varepsilon^{-1}(1+x_0), \varepsilon^{-1}(1+x_0)]$ with inhomogeneous boundary conditions for u_1 imposed at the endpoints. In this way, the result (5.12) was obtained in [6].

When D is asymptotically large we can superimpose the result (5.12) with (5.10c) to obtain

$$\frac{dx_0}{dt} \equiv G(x_0) + F(x_0) \,. \tag{5.13}$$

Here $F(x_0)$ and $G(x_0)$ are defined in (5.10c) and (5.12), respectively. This superposition is valid since the metastable interaction between the tails of the spike and the boundaries $x = \pm 1$ results in an additive term to the solvability condition that we impose on (5.3a). The stability property of the equilibrium solution for this differential equation is then given in the following result:

Proposition 13: *For $\varepsilon \ll 1$, a one-spike equilibrium solution to (1.3) is stable when $D < D_1(\varepsilon)$ and is unstable when $D > D_1(\varepsilon)$, where*

$$D_1(\varepsilon) \sim \frac{q\mu\varepsilon^2 e^{2/\varepsilon}}{8(p+1)\alpha^2} \int_{-\infty}^{\infty} [u_c(y)]^{p+1} \, dy \,. \tag{5.14}$$

Here α is defined by the limiting behavior $u_c \sim \alpha e^{-|y|}$ as $|y| \to \infty$, where $u_c(y)$ satisfies (2.5).

For the special case with $\mu = 1$ and $(p,q,r,s) = (2,1,2,0)$, where $u_c(y) = \frac{3}{2}\text{sech}^2(y/2)$ and $\alpha = 6$, we can calculate analytically that $D_1(\varepsilon) \sim \varepsilon^2 e^{2\varepsilon}/125.0$.

A Calculation of $\mathcal{B}$ and $\mathcal{P}$

Consider the boundary value problem

$$Dy'' - \mu y = 0\,, \qquad y'(\pm 1) = 0\,, \tag{A.1a}$$

$$[Dy]_j = 0\,, \qquad \left[Dy'\right]_j = -\omega_j\,, \tag{A.1b}$$

for $j = 0, \ldots, N-1$, where $[v]_j \equiv v(x_{j+}) - v(x_{j-})$ and x_j satisfies (2.1). The solution is

$$y(x) = \sum_{k=0}^{N-1} G(x; x_k)\omega_k\,, \tag{A.2}$$

where G satisfies (2.10). Define the N-vectors $\boldsymbol{y}$ and $\langle \boldsymbol{y}' \rangle$ by

$$\boldsymbol{y}^t = (y_0, \ldots, y_{N-1})\,, \qquad \langle \boldsymbol{y}' \rangle^t = \left(\langle y' \rangle_0, \ldots, \langle y' \rangle_{N-1} \right)\,, \tag{A.3}$$

where $y_j \equiv y(x_j)$ and $\langle y' \rangle_j \equiv \left(y'(x_{j+}) + y'(x_{j-}) \right)/2$. Then, we obtain from (A.2) that

$$\boldsymbol{y} = \mathcal{G}\boldsymbol{\omega}\,, \qquad \langle \boldsymbol{y}' \rangle = \mathcal{P}\boldsymbol{\omega}\,, \tag{A.4}$$

where $\boldsymbol{\omega}^t = (\omega_0, \ldots, \omega_{N-1})$. Here the matrices $\mathcal{G}$ and $\mathcal{P}$ are defined in (3.7) and (4.31), respectively. To determine these matrices explicitly we solve (A.1) analytically on each subinterval and impose the continuity of y to get

$$y(x) = \begin{cases} y_0 \frac{\cosh[\theta(1+x)]}{\cosh[\theta(1+x_0)]}\,, & -1 < x < x_0\,, \\ y_j \frac{\sinh[\theta(x_{j+1}-x)]}{\sinh[\theta(x_{j+1}-x_j)]} + y_{j+1} \frac{\sinh[\theta(x-x_j)]}{\sinh[\theta(x_{j+1}-x_j)]}\,, & x_j < x < x_{j+1}\,, \quad j = 0, \ldots, N-2\,, \\ y_{N-1} \frac{\cosh[\theta(1-x)]}{\cosh[\theta(1-x_{N-1})]}\,, & x_{N-1} < x < 1\,, \end{cases} \tag{A.5}$$

where $\theta = (\mu/D)^{1/2}$. To determine the relationship between $\boldsymbol{y}$ and $\boldsymbol{\omega}$, which yields $\mathcal{G}$, we use (A.5) and the jump condition $\left[Dy'\right]_j = -\omega_j$ in (A.1b) to get

$$\mathcal{B}\boldsymbol{y} = \frac{1}{D\theta}\boldsymbol{\omega}\,, \qquad \to \qquad \mathcal{G} = \frac{1}{D\theta}\mathcal{B}^{-1}\,, \tag{A.6}$$

where $\mathcal{B}$ is defined in (3.10b). Now using (A.5) we can calculate $\langle \boldsymbol{y}' \rangle$ in terms of $\boldsymbol{y}$ in the form

$$\langle \boldsymbol{y}' \rangle = -\frac{1}{2D} \operatorname{csch}\left(\frac{2\theta}{N}\right) \mathcal{C}^t \boldsymbol{y} \,, \tag{A.7}$$

where $\mathcal{C}$ is defined in (4.37b). Comparing (A.4) and (A.7), and using (A.6), we get the key relation

$$\mathcal{P} = -\frac{1}{2D} \operatorname{csch}\left(\frac{2\theta}{N}\right) \mathcal{C}^t \mathcal{B}^{-1} \,. \tag{A.8}$$

B Calculation of Matrix Eigenvalues of $\mathcal{B}$

In this appendix we calculate the eigenvalues κ_j and eigenvectors $\boldsymbol{q}_j$ of the matrix problem

$$\mathcal{B}\boldsymbol{q} = \kappa \boldsymbol{q} \,, \tag{B.1}$$

where the tridiagonal matrix $\mathcal{B}$ is defined in (3.10) and $\boldsymbol{q}^t = (q_1, \ldots, q_N)$. The calculation below is similar to that given in [8].

From (3.10c) it follows that $d = e + f$. Therefore, we get the following recursion relation for the coefficients q_l of the eigenvector $\boldsymbol{q}$:

$$f q_{l-1} + (e - \kappa) q_l + f q_{l+1} = 0 \,, \qquad l = 2, \ldots, N-1 \,, \tag{B.2a}$$

$$f q_1 + (e - \kappa) q_1 + f q_2 = 0 \,, \tag{B.2b}$$

$$f q_N + (e - \kappa) q_N + f q_{N-1} = 0 \,. \tag{B.2c}$$

Hence, to solve for the q_l we can use the relation (B.2a) for $l = 1$ and $l = N$ and then impose the end conditions

$$q_0 = q_1 \,, \qquad q_{N+1} = q_N \,. \tag{B.2d}$$

The solution to (B.2a) is

$$q_l = a\zeta_+^l + b\zeta_-^l \,, \qquad \zeta_\pm = \frac{1}{2f}\left(\kappa - e \pm \left[(\kappa - e)^2 - 4f^2\right]^{1/2}\right) \,. \tag{B.3}$$

The end conditions (B.2d) yield

$$a + b = a\zeta_+ + b\zeta_- \,, \tag{B.4a}$$

$$a\zeta_+^N + b\zeta_-^N = a\zeta_+^{N+1} + b\zeta_-^{N+1} \,. \tag{B.4b}$$

From (B.4) we get $\zeta_+ = \zeta_- = 1$ or $\zeta_+^N = \zeta_-^N$, which yields $\zeta_+ = \zeta_- \exp(2\pi i j/N)$, for $j = 1, \ldots, N-1$.

If $\zeta_+ = \zeta_- = 1$ we get $\kappa = e + 2f$ and $\boldsymbol{q}^t = (1, \ldots, 1)$. The other eigenvalues are calculated as in [8] to get $\kappa_j = e + 2f\cos(\pi(j-1)/N)$ for $j = 2, \ldots, N$ and $\zeta_\pm = \exp(\pm\pi i(j-1)/N)$. From (B.4b) we get $(1-\zeta_+)a + (1-\zeta_-)b = 0$. Substituting this relation into (B.3), and after rearranging the result, we obtain the unnormalized eigenvectors

$$q_{l,j} = \cos\left(\frac{\pi(j-1)}{N}(l-1/2)\right), \quad j = 2, \ldots, N. \tag{B.5}$$

Here $q_{l,j}$ is the l^{th} component of the eigenvector $\boldsymbol{q}_j$. These eigenvectors can be normalized and the result is summarized in Proposition 2 in §3.

C Calculation of $\mathcal{B}_g$ and $\mathcal{P}_g$

Consider the boundary value problem

$$Dy'' - \mu y = 0, \qquad y'(\pm 1) = 0, \tag{C.1a}$$

$$[Dy]_j = -\omega_j, \qquad \left[Dy'\right]_j = 0, \tag{C.1b}$$

for $j = 0, \ldots, N-1$, where $[v]_j \equiv v(x_{j+}) - v(x_{j-})$ and x_j satisfies (2.1). The solution is

$$y(x) = \sum_{k=0}^{N-1} g(x; x_k)\omega_k, \tag{C.2}$$

where g satisfies (4.26). In terms of the matrices $\mathcal{G}_g$ and $\mathcal{P}_g$, defined in (4.30) and (4.28), respectively, we have that

$$\boldsymbol{y}' = \mathcal{G}_g \boldsymbol{\omega}, \qquad \langle \boldsymbol{y} \rangle = \mathcal{P}_g \boldsymbol{\omega}, \tag{C.3}$$

where $\boldsymbol{\omega}^t = (\omega_0, \ldots, \omega_{N-1})$. Here $\boldsymbol{y}'$ and $\langle \boldsymbol{y} \rangle$ are defined by

$$\boldsymbol{y}'^t = \left(y_0', \ldots, y_{N-1}'\right), \qquad \langle \boldsymbol{y} \rangle^t = \left(\langle y \rangle_0, \ldots, \langle y \rangle_{N-1}\right), \tag{C.4}$$

where $y_j' \equiv y'(x_j)$ and $\langle y \rangle_j \equiv (y(x_{j+}) + y(x_{j-}))/2$. To determine $\mathcal{G}_g$ and $\mathcal{P}_g$ explicitly, we solve (C.1) analytically on each subinterval and impose the continuity of y' to get

$$y(x) = \begin{cases} \frac{y_0'}{\theta} \frac{\cosh[\theta(1+x)]}{\sinh[\theta(1+x_0)]}, & -1 < x < x_0, \\ \frac{y_{j+1}'}{\theta} \frac{\cosh[\theta(x-x_j)]}{\sinh[\theta(x_{j+1}-x_j)]} - \frac{y_j'}{\theta} \frac{\cosh[\theta(x_{j+1}-x)]}{\sinh[\theta(x_{j+1}-x_j)]}, & x_j < x < x_{j+1}, \quad j = 0, \ldots, N-2, \\ -\frac{y_{N-1}'}{\theta} \frac{\cosh[\theta(1-x)]}{\sinh[\theta(1-x_{N-1})]}, & x_{N-1} < x < 1, \end{cases} \tag{C.5}$$

where $\theta = (\mu/D)^{1/2}$. We then impose the jump condition $[Dy]_j = -\omega_j$ to obtain

$$\mathcal{B}_g \boldsymbol{y}' = \frac{\theta}{D}\boldsymbol{\omega}\,, \qquad \rightarrow \qquad \mathcal{G}_g = \frac{\theta}{D}\mathcal{B}_g^{-1}\,, \tag{C.6}$$

where $\mathcal{B}_g$ has the tridiagonal form given in (3.10b) with matrix entries defined in (4.35). Now we use (C.5) to calculate $\langle y \rangle_j$, and in this way we get

$$\langle \boldsymbol{y} \rangle = \frac{1}{2\theta}\text{csch}\left(\frac{2\theta}{N}\right)\mathcal{C}\boldsymbol{y}'\,, \tag{C.7}$$

where $\mathcal{C}$ is defined in (4.37b). Substituting (C.6) into (C.7), and comparing with (C.3), we obtain the key result

$$\mathcal{P}_g = \frac{1}{2D}\text{csch}\left(\frac{2\theta}{N}\right)\mathcal{C}\mathcal{B}_g^{-1}\,. \tag{C.8}$$

D Calculation of Matrix Eigenvalues of $\mathcal{B}_g$

In this appendix we calculate the eigenvalues ξ_j and eigenvectors $\boldsymbol{v}_j$ of the matrix problem

$$\mathcal{B}_g \boldsymbol{v} = \xi \boldsymbol{v}\,, \tag{D.1}$$

where the tridiagonal matrix $\mathcal{B}_g$ has the form given in (3.10b) with the coefficients d, e and f satisfying (4.35).

From (4.35) it follows that $d = e - f$. Therefore, we get the following recursion relation for the coefficients v_l of the eigenvector $\boldsymbol{v}$:

$$f v_{l-1} + (e - \xi)v_l + f v_{l+1} = 0\,, \qquad l = 1, \ldots, N\,, \tag{D.2a}$$

$$v_0 = -v_1\,, \qquad v_N = -v_{N+1}\,. \tag{D.2b}$$

The solution to (D.2a) is

$$v_l = a\zeta_+^l + b\zeta_-^l\,, \qquad \zeta_\pm = \frac{1}{2f}\left(\xi - e \pm \left[(\xi - e)^2 - 4f^2\right]^{1/2}\right)\,. \tag{D.3}$$

The end conditions (D.2b) yield

$$a + b = -a\zeta_+ - b\zeta_-\,, \tag{D.4a}$$

$$a\zeta_+^N + b\zeta_-^N = -a\zeta_+^{N+1} - b\zeta_-^{N+1}\,. \tag{D.4b}$$

From (D.4) we get $\zeta_+ = \zeta_- = -1$ or $\zeta_+ = \zeta_- \exp(2\pi i j/N)$, for $j = 1, \dots, N-1$. Substituting into (D.3) we get that the eigenvalues are

$$\xi_j = e + 2f \cos(\pi j/N) \ , \quad j = 1, \dots, N \, , \tag{D.5}$$

which are ordered as $0 < \xi_1 < \dots < \xi_N$ since $f < 0$. The corresponding unnormalized eigenvectors are found to be

$$v_{j,N} = \left(1, -1, 1, \dots, (-1)^{N+1}\right) \ ; \qquad v_{l,j} = \sin\left(\frac{\pi j}{N}(l - 1/2)\right) \ , \qquad j = 1, \dots, N-1 \, . \tag{D.6}$$

Here $v_{l,j}$ is the l^{th} component of the eigenvector $\boldsymbol{v}_j$. These eigenvectors can be normalized and the result is summarized in Proposition 9.

E Proof of (3.15)

We now prove (3.15). In fact, it is enough to find an eigenvalue λ_0 of (3.14) with $\lambda_0 > 0$ when $0 < \gamma_0 < 1$. Notice that this is equivalent to finding a positive zero of the function $\zeta(\lambda)$ defined by

$$\zeta(\lambda) \equiv \int_{-\infty}^{\infty} u_c^r \, dy - \gamma_0 (p-1) \int_{-\infty}^{\infty} u_c^{r-1} \left(L_0 - \lambda\right)^{-1} u_c^p \, dy \, , \tag{E.1}$$

where $L_0 \Phi \equiv \Phi'' - \Phi + p u_c^{p-1} \Phi$.

By Theorem 2.1 of [11], L_0 has a unique eigenvalue $\mu_1 > 0$ with an eigenfunction of constant sign. We now consider $\zeta(\lambda)$ in the interval $(0, \mu_1)$. Since $L_0^{-1} u_c^p = (p-1)^{-1} u_c$, we get from (E.1) that

$$\zeta(0) = (1 - \gamma_0) \int_{-\infty}^{\infty} u_c^p \, dy > 0 \, , \tag{E.2}$$

when $\gamma_0 < 1$. Next, as $\lambda \to \mu_1^-$, we have that

$$\int_{-\infty}^{\infty} u_c^{r-1} \left(L_0 - \lambda\right)^{-1} u_c^p \, dy \to +\infty \, . \tag{E.3}$$

Hence, we get from (E.1) and (E.3) that

$$\zeta(\lambda) \to -\infty \, , \qquad \text{as} \quad \lambda \to \mu_1^- \, , \tag{E.4}$$

when $0 < \gamma_0 < 1$. By (E.2) and (E.4) and the continuity of $\zeta(\lambda)$, we can find a $\lambda_0 \in (0, \mu_1)$ such that $\zeta(\lambda_0) = 0$ whenever $0 < \gamma_0 < 1$.

F Proof of (3.16)

In this appendix, we prove (3.16). Although this has been proved in [19], we include a proof here for the convenience of the readers. We present a proof which works in the general case of R^N. Let

$$r = 2, \quad 1 < p \leq 1 + \frac{4}{N}, \qquad \text{or} \qquad r = p + 1, \quad 1 < p < (\frac{N+2}{N-2})_+ \,,$$

where $(\frac{N+2}{N-2})_+ = \frac{N+2}{N-2}$ if $N \geq 3$ and $= +\infty$ if $N = 1, 2$. Define $w(|\boldsymbol{y}|)$, with $\boldsymbol{y} = (y_1, .., y_N)^t$, to be the unique positive solution to

$$w^{''} + \frac{N-1}{\rho} w^{'} - w + w^p = 0 \,, \qquad \rho > 0 \,,$$
$$w^{'}(0) > 0 \,, \qquad w(\rho) \sim \alpha \rho^{(1-N)/2} e^{-\rho} \,, \qquad \text{as} \quad \rho \to \infty \,.$$

When $N = 1$, then $w = u_c$, where u_c satisfies (2.5).

Suppose that (ϕ, λ_0), with $\lambda_0 \neq 0$, satisfies the following eigenvalue problem:

$$\Delta\phi - \phi + p w^{p-1}\phi - \gamma_0(p-1) \frac{\int_{R^N} w^{r-1}\phi}{\int_{R^N} w^r} w^p = \lambda_0 \phi \,, \qquad \phi \in H^2(R^1), \quad \gamma_0 > 1 \,. \tag{F.1}$$

When $N = 1$ this problem reduces to (3.14). Thus, the proof of (3.16) is complete once we show that

$$Re(\lambda_0) < 0 \,. \tag{F.2}$$

Let $\lambda_0 = \lambda_R + i\lambda_I, \phi = \phi_R + i\phi_I$.

We first introduce some notations and make some preparations. Set

$$L\phi := L_0\phi - \gamma_0(p-1) \frac{\int_{R^N} w^{r-1}\phi}{\int_{R^1} w^r} w^p \,, \qquad \phi \in H^2(R^1) \,,$$

where $\gamma_0 > 1$ and $L_0 := \Delta - 1 + p w^{p-1}$. Note that L is not self-adjoint if $r \neq p + 1$.

It is well-known that L_0 admits the following set of eigenvalues:

$$\mu_1 > 0, \qquad \mu_2 = 0 \,, ... \, \mu_{N+1} = 0 \,, \qquad \mu_{N+2} < 0 \,, \tag{F.3}$$

where the eigenfunction corresponding to μ_1 is of constant sign (see Theorem 2.1 of [11]).

Let

$$X_0 := kernel(L_0) = span\{\frac{\partial w}{\partial y_j}, j = 1, ..., N\}.$$

Then

$$L_0 w = (p-1)w^p\,, \qquad L_0(\frac{1}{p-1}w + \frac{1}{2}x\nabla w) = w\,, \tag{F.4}$$

and

$$\int_{R^N} (L_0^{-1}w)w = \int_{R^N} w(\frac{1}{p-1}w + \frac{1}{2}x\nabla w) = (\frac{1}{p-1} - \frac{N}{4})\int_{R^N} w^2, \tag{F.5}$$

$$\int_{R^N} (L_0^{-1}w)w^p = \int_{R^N} w^p(\frac{1}{p-1}w + \frac{1}{2}x\nabla w)$$

$$= \int_{R^N} (L_0^{-1}w)\frac{1}{p-1}L_0 w = \frac{1}{p-1}\int_{R^N} w^2. \tag{F.6}$$

We divide our proof into three cases:

Case 1: $r = 2, 1 < p < 1 + \frac{4}{N}$.

Since L is not self-adjoint, we introduce a new operator as follows:

$$L_1\phi := L_0\phi - (p-1)\frac{\int_{R^N} w\phi}{\int_{R^N} w^2}w^p - (p-1)\frac{\int_{R^N} w^p\phi}{\int_{R^N} w^2}w + (p-1)\frac{\int_{R^N} w^{p+1}\int_{R^N} w\phi}{(\int_{R^N} w^2)^2}w. \tag{F.7}$$

We have the following important lemma:

Lemma F.1 *(1) L_1 is self-adjoint and the kernel of L_1 (denoted by X_1) = span $\{w, \frac{\partial w}{\partial y_j}, j = 1, ..., N\}$. (2) There exists a positive constant $a_1 > 0$ such that*

$$L_1(\phi,\phi)$$

$$:= \int_{R^N} (|\nabla\phi|^2 + \phi^2 - pw^{p-1}\phi^2) + \frac{2(p-1)\int_{R^N} w\phi \int_{R^N} w^p\phi}{\int_{R^N} w^2} - (p-1)\frac{\int_{R^N} w^{p+1}}{(\int_{R^N} w^2)^2}(\int_{R^N} w\phi)^2$$

$$\geq a_1 d^2_{L^2(R^N)}(\phi, X_1)\,,$$

for all $\phi \in H^1(R^N)$, where $d_{L^2(R^N)}$ denotes distance in the L^2-norm.

Proof: By (F.7), L_1 is self-adjoint. Next we compute the kernel of L_1. It is easy to see that $w, \frac{\partial w}{\partial y_j}, j = 1, ..., N, \in kernel(L_1)$. On the other hand, if $\phi \in kernel(L_1)$, then by (F.4)

$$L_0\phi = c_1(\phi)w + c_2(\phi)w^p = c_1(\phi)L_0(\frac{1}{p-1}w + \frac{1}{2}x\nabla w) + c_2(\phi)L_0(\frac{w}{p-1})\,,$$

where

$$c_1(\phi) = (p-1)\frac{\int_{R^N} w^p \phi}{\int_{R^N} w^2} - (p-1)\frac{\int_{R^N} w^{p+1} \int_{R^N} w\phi}{(\int_{R^N} w^2)^2}, \qquad c_2(\phi) = (p-1)\frac{\int_{R^N} w\phi}{\int_{R^N} w^2}.$$

Hence

$$\phi - c_1(\phi)(\frac{1}{p-1}w + \frac{1}{2}x\nabla w) - c_2(\phi)\frac{1}{p-1}w \in kernel(L_0). \tag{F.8}$$

Note that

$$c_1(\phi) = (p-1)c_1(\phi)\frac{\int_{R^N} w^p(\frac{1}{p-1}w + \frac{1}{2}x\nabla w)}{\int_{R^N} w^2} - (p-1)c_1(\phi)\frac{\int_{R^N} w^{p+1}\int_{R^N} w(\frac{1}{p-1}w + \frac{1}{2}x\nabla w)}{(\int_{R^N} w^2)^2}$$

$$= c_1(\phi) - c_1(\phi)(\frac{1}{p-1} - \frac{N}{4})\frac{\int_{R^N} w^{p+1}}{\int_{R^N} w^2}$$

by (F.5) and (F.6). This implies that $c_1(\phi) = 0$. By (F.8), this proves (1).

It remains to prove (2). Suppose (2) is not true, then by (1) there exists (α, ϕ) such that (i) α is real and positive, (ii) $\phi \perp w$, $\phi \perp \frac{\partial w}{\partial y_j}$, $j = 1, \ldots, N$, and (iii) $L_1\phi = \alpha\phi$.

We show that this is impossible. From (ii) and (iii), we have

$$(L_0 - \alpha)\phi = (p-1)\frac{\int_{R^N} w^p\phi}{\int_{R^N} w^2}w. \tag{F.9}$$

We first claim that $\int_{R^N} w^p\phi \neq 0$. In fact if $\int_{R^N} w^p\phi = 0$, then $\alpha > 0$ is an eigenvalue of L_0. But by (F.3), $\alpha = \mu_1$ and ϕ has constant sign. This contradicts with the fact that $\phi \perp w$. Therefore $\alpha \neq \mu_1, 0$, and hence $L_0 - \alpha$ is invertible in $X_0^\perp$. So (F.9) implies

$$\phi = (p-1)\frac{\int_{R^N} w^p\phi}{\int_{R^N} w^2}(L_0 - \alpha)^{-1}w.$$

Thus

$$\int_{R^N} w^p\phi = (p-1)\frac{\int_{R^N} w^p\phi}{\int_{R^N} w^2}\int_{R^N}((L_0 - \alpha)^{-1}w)w^p,$$

$$\int_{R^N} w^2 = (p-1)\int_{R^N}((L_0 - \alpha)^{-1}w)w^p,$$

$$\int_{R^N} w^2 = \int_{R^N} ((L_0-\alpha)^{-1}w)((L_0-\alpha)w+\alpha w),$$

$$0 = \int_{R^N} ((L_0-\alpha)^{-1}w)w. \tag{F.10}$$

Let $h_1(\alpha) = \int_{R^N}((L_0-\alpha)^{-1}w)w$. Then, $h_1(0) = \int_{R^N}(L_0^{-1}w)w = \int_{R^N}(\frac{1}{p-1}w + \frac{1}{2}x\cdot\nabla w)w = (\frac{1}{p-1}-\frac{N}{4})\int_{R^N} w^2 > 0$ since $1 < p < 1+\frac{4}{N}$. Moreover $h_1^{'}(\alpha) = \int_{R^N}((L_0-\alpha)^{-2}w)w = \int_{R^N}((L_0-\alpha)^{-1}w)^2 > 0$. This implies $h_1(\alpha) > 0$ for all $\alpha \in (0,\mu_1)$. Clearly, also $h_1(\alpha) < 0$ for $\alpha \in (\mu_1,\infty)$ (since $\lim_{\alpha\to+\infty} h_1(\alpha) = 0$). A contradiction to (F.10)! This completes this part of the proof.

We now finish the proof of (3.16) in Case 1. Since $\lambda_0 \neq 0$, we can choose $\phi \perp kernel(L_0)$. Then we obtain two equations

$$L_0\phi_R - (p-1)\gamma_0\frac{\int_{R^N} w\phi_R}{\int_{R^N} w^2}w^p = \lambda_R\phi_R - \lambda_I\phi_I, \tag{F.11}$$

$$L_0\phi_I - (p-1)\gamma_0\frac{\int_{R^N} w\phi_I}{\int_{R^N} w^2}w^p = \lambda_R\phi_I + \lambda_I\phi_R. \tag{F.12}$$

Multiplying (F.11) by ϕ_R and (F.12) by ϕ_I, and adding the resulting expressions together, we obtain

$$-\lambda_R\int_{R^N}(\phi_R^2+\phi_I^2) = L_1(\phi_R,\phi_R) + L_1(\phi_I,\phi_I)$$

$$+(p-1)(\gamma_0-2)\frac{\int_{R^N} w\phi_R\int_{R^N} w^p\phi_R + \int_{R^N} w\phi_I\int_{R^N} w^p\phi_I}{\int_{R^N} w^2}$$

$$+(p-1)\frac{\int_{R^N} w^{p+1}}{(\int_{R^N} w^2)^2}[(\int_{R^N} w\phi_R)^2 + (\int_{R^N} w\phi_I)^2].$$

Multiplying (F.11) by w and (F.12) by w we obtain

$$(p-1)\int_{R^N} w^p\phi_R - \gamma_0(p-1)\frac{\int_{R^N} w\phi_R}{\int_{R^N} w^2}\int_{R^N} w^{p+1} = \lambda_R\int_{R^N} w\phi_R - \lambda_I\int_{R^N} w\phi_I, \tag{F.13}$$

$$(p-1)\int_{R^N} w^p\phi_I - \gamma_0(p-1)\frac{\int_{R^N} w\phi_I}{\int_{R^N} w^2}\int_{R^N} w^{p+1} = \lambda_R\int_{R^N} w\phi_I + \lambda_I\int_{R^N} w\phi_R. \tag{F.14}$$

Multiplying (F.13) by $\int_{R^N} w\phi_R$ and (F.14) by $\int_{R^N} w\phi_I$, and adding them together, we obtain

$$(p-1)\int_{R^N} w\phi_R \int_{R^N} w^p\phi_R + (p-1)\int_{R^N} w\phi_I \int_{R^N} w^p\phi_I$$

$$= (\lambda_R + \gamma_0(p-1)\frac{\int_{R^N} w^{p+1}}{\int_{R^N} w^2})((\int_{R^N} w\phi_R)^2 + (\int_{R^N} w\phi_I)^2).$$

Therefore we have

$$-\lambda_R \int_{R^N} (\phi_R^2 + \phi_I^2) = L_1(\phi_R,\phi_R) + L_1(\phi_I,\phi_I)$$

$$+(p-1)(\gamma_0-2)(\frac{1}{p-1}\lambda_R + \gamma_0\frac{\int_{R^N} w^{p+1}}{\int_{R^N} w^2})\frac{(\int_{R^N} w\phi_R)^2 + (\int_{R^N} w\phi_I)^2}{\int_{R^N} w^2}$$

$$+(p-1)\frac{\int_{R^N} w^{p+1}}{(\int_{R^N} w^2)^2}[(\int_{R^N} w\phi_R)^2 + (\int_{R^N} w\phi_I)^2].$$

Set

$$\phi_R = c_R w + \phi_R^\perp, \quad \phi_R^\perp \perp X_1, \qquad \phi_I = c_I w + \phi_I^\perp, \quad \phi_I^\perp \perp X_1.$$

Then

$$\int_{R^N} w\phi_R = c_R \int_{R^N} w^2, \qquad \int_{R^N} w\phi_I = c_I \int_{R^N} w^2,$$

$$d^2_{L^2(R^N)}(\phi_R, X_1) = \|\phi_R^\perp\|^2_{L^2}, \qquad d^2_{L^2(R^N)}(\phi_I, X_1) = \|\phi_I^\perp\|^2_{L^2}.$$

By some simple computations we have

$$L_1(\phi_R,\phi_R) + L_1(\phi_I,\phi_I)$$

$$+(\gamma_0-1)\lambda_R(c_R^2+c_I^2)\int_{R^N} w^2 + (p-1)(\gamma_0-1)^2(c_R^2+c_I^2)\int_{R^N} w^{p+1} + \lambda_R(\|\phi_R^\perp\|^2_{L^2} + \|\phi_I^\perp\|^2_{L^2}) = 0.$$

By Lemma F.1 (2)

$$(\gamma_0-1)\lambda_R(c_R^2+c_I^2)\int_{R^N} w^2$$

$$+(p-1)(\gamma_0-1)^2(c_R^2+c_I^2)\int_{R^N} w^{p+1}+(\lambda_R+a_1)(\|\phi_R^\perp\|_{L^2}^2+\|\phi_I^\perp\|_{L^2}^2)\le 0.$$

Since $\gamma_0>1$, we must have $\lambda_R<0$, which completes the proof of (3.16) in Case 1.

Case 2: $r=2, p=1+\frac{4}{N}$.

In this case we have

$$\int_{R^N}(L_0^{-1}w)w=\int_{R^N} w(\frac{1}{p-1}w+\frac{1}{2}x\nabla w)=0. \tag{F.15}$$

Set

$$w_0=\frac{1}{p-1}w+\frac{1}{2}x\nabla w. \tag{F.16}$$

We will follow the proof in Case 1. We just need to take care of w_0. We first have the following lemma which is similar to Lemma F.1: The proof is omitted.

Lemma F.2 *(1) The kernel of L_1 is given by $X_1=span\ \{w,w_0,\frac{\partial w}{\partial y_j}, j=1,...,N\}$. (2) There exists a positive constant $a_2>0$ such that*

$$L_1(\phi,\phi)=\int_{R^N}(|\nabla\phi|^2+\phi^2-pw^{p-1}\phi^2)$$

$$+\frac{2(p-1)\int_{R^N} w\phi\int_{R^N} w^p\phi}{\int_{R^N} w^2}-(p-1)\frac{\int_{R^N} w^{p+1}}{(\int_{R^N} w^2)^2}(\int_{R^N} w\phi)^2$$

$$\ge a_2 d^2_{L^2(R^N)}(\phi,X_1),\quad \forall\phi\in H^1(R^N).$$

Now we can finish the proof of (3.16) in Case 2. Similar to Case 1, we obtain two equations (F.11) and (F.12). We now decompose

$$\phi_R=c_Rw+b_Rw_0+\phi_R^\perp,\quad \phi_R^\perp\perp X_1\,,\qquad \phi_I=c_Iw+b_Iw_0+\phi_I^\perp,\quad \phi_I^\perp\perp X_1.$$

Similar to Case 1, we obtain

$$L_1(\phi_R,\phi_R)+L_1(\phi_I,\phi_I)$$

$$+(\gamma_0-1)\lambda_R(c_R^2+c_I^2)\int_{R^N} w^2+(p-1)(\gamma_0-1)^2(c_R^2+c_I^2)\int_{R^N} w^{p+1}$$

$$+\lambda_R(b_R^2(\int_{R^N} w_0^2)^2 + b_I^2(\int_{R^N} w_0^2)^2 + \|\phi_R^\perp\|_{L^2}^2 + \|\phi_I^\perp\|_{L^2}^2) \le 0\,.$$

By Lemma F.2 (2)

$$(\gamma_0 - 1)\lambda_R(c_R^2 + c_I^2)\int_{R^N} w^2 + (p-1)(\gamma_0-1)^2(c_R^2+c_I^2)\int_{R^N} w^{p+1}$$

$$+\lambda_R(b_R^2(\int_{R^N} w_0^2)^2 + b_I^2(\int_{R^N} w_0^2)^2) + (\lambda_R + a_2)(\|\phi_R^\perp\|_{L^2}^2 + \|\phi_I^\perp\|_{L^2}^2) \le 0\,.$$

If $\lambda_R \ge 0$, then necessarily we have

$$c_R = c_I = 0, \qquad \phi_R^\perp = 0, \qquad \phi_I^\perp = 0\,.$$

Hence $\phi_R = b_R w_0, \phi_I = b_I w_0$. This implies that

$$b_R L_0 w_0 = (b_R - b_I) w_0\,, \qquad b_I L_0 w_0 = (b_R + b_I) w_0,$$

which is impossible unless $b_R = b_I = 0$. A contradiction! This completes this part of the proof.

Case 3: $r = p+1, 1 < p < (\frac{N+2}{N-2})_+$.

Let $r = p+1$. L becomes

$$L = L_0 - \frac{qr}{s+1}\frac{\int_{R^N} w^p \cdot}{\int_{R^N} w^{p+1}} w^p.$$

We will follow the proof of Case 1. We need to define another operator.

$$L_3\phi := L_0\phi - (p-1)\frac{\int_{R^N} w^p\phi}{\int_{R^N} w^{p+1}} w^p. \tag{F.17}$$

We have the following lemma:

Lemma F.3 *(1) L_3 is self-adjoint and the kernel of L_3 (denoted by X_3) consists of $w, \frac{\partial w}{\partial y_j}, j = 1, ..., N$. (2) There exists a positive constant $a_3 > 0$ such that*

$$L_3(\phi,\phi) := \int_{R^N}(|\nabla\phi|^2 + \phi^2 - pw^{p-1}\phi^2) + \frac{(p-1)(\int_{R^N} w^p\phi)^2}{\int_{R^N} w^{p+1}}$$

$$\ge a_3 d_{L^2(R^N)}^2(\phi, X_3), \qquad \forall \phi \in H^1(R^N).$$

Proof: The proof of (1) is similar to that of Lemma F.1. We omit the details. It remains to prove (2). Suppose (2) is not true, then by (1) there exists (λ, ϕ) such that (i) λ is real and positive, (ii) $\phi \perp w, \phi \perp \frac{\partial w}{\partial y_j}$, $j = 1, \ldots, N$, and (iii) $L_3\phi = \lambda\phi$.

We show that this is impossible. From (ii) and (iii), we have

$$(L_0 - \lambda)\phi = \frac{(p-1)\int_{R^N} w^p \phi}{\int_{R^N} w^{p+1}} w^p. \tag{F.18}$$

Similar to the proof of Lemma F.1, we have that $\int_{R^N} w^p\phi \neq 0, \lambda \neq \mu_1, 0$, and hence $L_0 - \lambda$ is invertible in $X_0^\perp$. So (F.18) implies

$$\phi = \frac{(p-1)\int_{R^N} w^p \phi}{\int_{R^N} w^{p+1}} (L_0 - \lambda)^{-1} w^p.$$

Thus

$$\int_{R^N} w^p \phi = (p-1)\frac{\int_{R^N} w^p \phi}{\int_{R^N} w^{p+1}} \int_{R^N} ((L_0 - \lambda)^{-1} w^p) w^p,$$

$$\int_{R^N} w^{p+1} = (p-1)\int_{R^N} ((L_0 - \lambda)^{-1} w^p) w^p. \tag{F.19}$$

Let $h_3(\lambda) = (p-1)\int_{R^N}((L_0-\lambda)^{-1}w^p)w^p - \int_{R^N} w^{p+1}$, then $h_3(0) = (p-1)\int_{R^N}(L_0^{-1}w^p)w^p - \int_{R^N} w^{p+1} = 0$. Moreover $h_3'(\lambda) = (p-1)\int_{R^N}((L_0-\lambda)^{-2}w^p)w^p = (p-1)\int_{R^N}((L_0-\lambda)^{-1}w^p)^2 > 0$. This implies $h_3(\lambda) > 0$ for all $\lambda \in (0, \mu_1)$. Clearly, also $h_3(\lambda) < 0$ for $\lambda \in (\mu_1, \infty)$. A contradiction to (F.19)! This completes this part of the proof.

We now finish the proof of (3.16) in Case 3. Similar to case 1, we obtain two equations

$$L_0\phi_R - (p-1)\gamma_0 \frac{\int_{R^N} w^p \phi_R}{\int_{R^N} w^{p+1}} w^p = \lambda_R\phi_R - \lambda_I\phi_I, \tag{F.20}$$

$$L_0\phi_I - (p-1)\gamma_0 \frac{\int_{R^N} w^p \phi_I}{\int_{R^N} w^{p+1}} w^p = \lambda_R\phi_I + \lambda_I\phi_R. \tag{F.21}$$

Multiplying (F.20) by ϕ_R and (F.21) by ϕ_I and adding them together, we obtain

$$-\lambda_R \int_{R^N} (\phi_R^2 + \phi_I^2) = L_3(\phi_R, \phi_R) + L_3(\phi_I, \phi_I)$$

$$+(p-1)(\gamma_0-1)\frac{(\int_{R^N} w^p\phi_R)^2+(\int_{R^N} w^p\phi_I)^2}{\int_{R^N} w^{p+1}}.$$

By Lemma F.3 (2)

$$\lambda_R\int_{R^N}(\phi_R^2+\phi_I^2)+a_2 d_{L^2}^2(\phi,X_1)+(p-1)(\gamma_0-1)\frac{(\int_{R^N} w^p\phi_R)^2+(\int_{R^N} w^p\phi_I)^2}{\int_{R^N} w^{p+1}}\le 0\,,$$

which implies $\lambda_R<0$ since $\gamma_0>1$. Thus, (3.16) in Case 3 is proved.

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