ON POSITIVE SOLUTIONS CONCENTRATING ON SPHERES FOR THE GIERER-MEINHARDT SYSTEM

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ABSTRACT. We consider the stationary Gierer-Meinhardt system in a ball of $\mathbb{R}^N$:

$$\begin{cases} \varepsilon^2 \Delta u - u + \frac{u^p}{v^q} = 0 & \text{in } \ \Omega, \\ \Delta v - v + \frac{u^m}{v^s} = 0 & \text{in } \ \Omega, \\ u, v > 0, \text{ and } \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \ \partial\Omega \end{cases}$$

where $\Omega = B_R$ is a ball of $\mathbb{R}^N$ $(N \geq 2)$ with radius R, $\varepsilon > 0$ is a small parameter, and p, q, m, s satisfy the following condition:

$$p > 1, q > 0, m > 1, s \geq 0, \frac{qm}{(p-1)(1+s)} > 1.$$

We construct positive solutions which concentrate on a $(N-1)-$dimensional sphere for this system for all sufficiently small ε. More precisely, under some conditions on the exponents (p, q) and the radius R, it is proved the above problem has a radially symmetric positive solution $(u_\varepsilon, v_\varepsilon)$ with the property that $u_\varepsilon(r) \to 0$ in $\Omega \backslash \{r \neq r_0\}$ for some $r_0 \in (0, R)$. Existence of bound states in the whole $\mathbb{R}^N$ is also established.

1. INTRODUCTION

Of concern is the existence of positive solutions for the following generalized Gierer-Meinhardt system

$$\begin{cases} u_t = \varepsilon^2 \Delta u - u + \frac{u^p}{v^q} & \text{in } \ \Omega \times (0, +\infty), \\ \tau v_t = \Delta v - v + \frac{u^m}{v^s} & \text{in } \ \Omega \times (0, +\infty), \\ u, v > 0 \text{ in } \Omega \times (0, +\infty), \text{ and } \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \ \partial\Omega \times (0, +\infty) \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\varepsilon > 0$ is a small constant, $\tau \geq 0$ is a fixed nonnegative constant, $\Delta := \sum_{j=1}^N \frac{\partial^2}{\partial x_j \partial x_j}$ denotes the Laplace operator in $\mathbb{R}^N$, ν stands for the unit outer normal to $\partial\Omega$, and the exponents (p, q, m, s) satisfy the following condition:

$$p > 1, q > 0, m > 1, s \geq 0, \frac{qm}{(p-1)(1+s)} > 1. \tag{1.2}$$

1991 *Mathematics Subject Classification.* Primary 35B40, 35B45; Secondary 35J55, 92C15, 92C40.

Key words and phrases. Layered Solutions, Gierer-Meinhardt System, Pattern Formation, Singular Perturbations.

Gierer-Meinhardt system was used in [12] to model head formation of *hydra*, an animal of a few millimeters in length, made up of approximately 100,000 cells of about fifteen different types. It consists of a "head" region located at one end along its length. Typical experiments with *hydra* involve removing part of the "head" region and transplanting it to other parts of the body column. Then, a new "head" will form if the transplanted area is sufficiently far from the (old) head. These observations led to the assumption of the existence of two chemical substances—a *slowly* diffusing activator u and a *rapidly* diffusing inhibitor v. The ratio of their diffusion rates, denoted by ε, is assumed to be small.

The Gierer-Meinhardt system falls within the framework of a theory proposed by TURING [33] in 1952 as a mathematical model for the development of complex organisms from a single cell. He speculated that localized peaks in concentration of chemical substances, known as inducers or morphogens, could be responsible for a group of cells developing differently from the surrounding cells. Turing discovered through linear analysis that a large difference in relative size of diffusivities for activating and inhibiting substances carries instability of the homogeneous, constant steady state, thus leading to the presence of nontrivial, possibly stable stationary configurations. Activator-inhibitor systems have been used extensively in the mathematical theory of biological pattern formation, [20], [21]. Among them Gierer-Meinhardt system has been the object of extensive mathematical treatment in recent years. We refer the reader to the survey articles [22] for a description of progress made and references.

In particular, it has been a matter of high interest to study nonconstant positive steady states, namely, solutions of the following elliptic system

$$\begin{cases} \varepsilon^2\Delta u - u + \frac{u^p}{v^q} = 0 & \text{in } \Omega, \\ D\Delta v - v + \frac{u^m}{v^s} = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Problem (1.3) is quite difficult to solve, in general, since it does not have **variational structure**. One approach in studying (1.3) is to study its **shadow system**, an idea due to Keener [16], as a first step. Namely, we take $D = +\infty$ first. By suitable scaling, the study of steady-stae for the shadow system can be transformed to that of the scalar equation

$$\begin{cases} \varepsilon^2\Delta u - u + u^p = 0 & \text{in } \Omega, \\ u > 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

A number of interesting results concerning solutions with single or multiple peaks or spikes of this scalar problem have been derived in recent years. For the subcritical case, we refer the reader to the articles [6], [11], [14], [15], [26], [34], [36] and the references therein, starting with the pioneering works [17], [23], [24], [25], [31]. For the critical exponent case, we refer to the papers [3], [4], [13], [27], and the references therein. A review of the subject up to 1998 can be found in [22].

In the case of finite D and bounded domain case, Takagi [32] first constructed multiple symmetric peaks in the one-dimensional case. In higher dimensional case, Ni and Takagi [26] constructed multiple boundary spikes in the case of axially symmetric domains, assuming that D is large. Multiple interior spikes for finite D case in a bounded two dimensional domain are constructed in [37], [38], and [39]. The stability of multiple spikes as well as the dynamics of spikes are considered in [9], [28], [29], [30], [35], [38], [39] and the references therein.

In the case of $\Omega = \mathbb{R}^N$, problem (1.3) has been shown to exhibit single or multiple bump solutions in one or two dimensions. See [7], [8], [9], [10] and the references therein.

Solutions with peaks or spikes are called point-condensation solutions in the sense that they concentrate on a finite number of points.

It seems natural to ask whether solutions exist in higher dimensional case ($N \geq 3$), or if there are solutions other than point-condensations. Progress on the latter problem has been made recently for single equations: in [18], solutions concentrating on the whole $\partial\Omega$ are constructed for (1.4) along a subsequence $\varepsilon_n \to 0$. In [1] and [2], it is proved a radially symmetric solution concentrating on a sphere exists for (1.4) as well as for singularly perturbed nonlinear Schrödinger equation:

$$\varepsilon^2 \Delta u - V(|x|)u + u^p = 0 \text{ in } \Omega, \tag{1.5}$$

where $\Omega = B_1$ or $\Omega = \mathbb{R}^N$. In the case of (1.4), the effect of the boundary is analyzed. In the case of (1.5), the shape of the potential $V(|x|)$ determines the position of concentrating spheres. The existence of radially symmetric clustered layer solution is proved in [19]. All the above papers employ very heavily on the variational structure of single equations.

In this paper, we consider the case of of finite D and the domain being a ball in $\mathbb{R}^N$. (The case of the whole $\mathbb{R}^N$ is also included here.) By scaling, we may take $D = 1$ and obtain formally the following elliptic system

$$
\begin{cases}
\varepsilon^2\Delta u - u + \frac{u^p}{v^q} = 0 & \text{in } B_R, \\
\Delta v - v + \frac{u^m}{v^s} = 0 & \text{in } B_R, \\
u > 0, v > 0 & \text{in } B_R, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } B_R.
\end{cases}
\tag{1.6}
$$

Under some conditions on the parameters N, p, q, R , there are indeed positive solutions to system (1.6) concentrating on a $(N-1)$-dimensional sphere. In particular, this implies the existence of a positive solution in the case of $N \geq 3$, which is new and of independent interest.

2. Main Results

Before we state the main results of the paper, some notation have to be introduced.

We first define two functions, to be used later: let $J_1(r)$ be the radially symmetric solutions of the following problem

$$
J_1^{''} + \frac{N-1}{r} J_1^{'} - J_1 = 0, J^{'}(0) = 0, J_1(0) = 1, J_1 > 0. \tag{2.1}
$$

The second function, called $J_2(r)$, satisfies

$$
J_2^{''} + \frac{N-1}{r} J_2^{'} - J_2 + \delta_0 = 0, J_2 > 0, J_2(+\infty) = 0, \tag{2.2}
$$

where δ_0 is the Dirac measure at 0.

The functions $J_1(r)$ and $J_2(r)$ can be written in terms of modified Bessel's functions. In fact

$$
J_1(r) = c_1 r^{\frac{2-N}{2}} I_\nu(r), \quad J_2(r) = c_2 r^{\frac{2-N}{2}} K_\nu(r), \ \nu = \frac{N-2}{2} \tag{2.3}
$$

where c_1, c_2 are two positive constants and I_ν, K_ν are modified Bessel functions of order ν. See [page 378, [5]] for details. In the case of $N = 3$, J_1, J_2 can be computed explicitly:

$$
J_1 = \frac{\sinh r}{r}, \ J_2(r) = \frac{e^{-r}}{4\pi r}. \tag{2.4}
$$

Let $w(y)$ be the unique solution for the following ODE:

$$
w^{''} - w + w^p = 0 \text{ in } \mathbb{R}, \ w > 0, \ w(0) = \max_{y \in \mathbb{R}} w(y), w(y) \to 0 \text{ as } |y| \to \infty. \tag{2.5}
$$

Let $R > 0$ be a fixed constant. We define

$$
J_{2,R}(r) = J_2(r) - \frac{J_2^{'}(R)}{J_1^{'}(R)} J_1(r) \tag{2.6}
$$

and a Green's function $G_R(r;r')$

$$G_R'' + \frac{N-1}{r}G_R' - G_R + \delta_{r'} = 0, G_R'(0;r') = 0, G_R'(R;r') = 0. \tag{2.7}$$

Note that

$$J_{2,R}'(R) = 0, \quad \lim_{R\to+\infty} J_{2,R}(r) = J_2(r). \tag{2.8}$$

It is easy to see that

$$G_R(r;r') = \frac{1}{J_1'(r')J_{2,R}(r') - J_1(r')J_{2,R}'(r')} \begin{cases} J_{2,R}(r')J_1(r), & \text{for } r < r', \\ J_1(r')J_{2,R}(r), & \text{for } r > r'. \end{cases} \tag{2.9}$$

By computing the Wronskian of J_1, J_2, it is easy to see that

$$J_1'(r')J_{2,R}(r') - J_1(r')J_{2,R}'(r') = \frac{1}{c_0(r')^{N-1}} \tag{2.10}$$

for some constant $c_0 \neq 0$. Substituting (2.10) into (2.9), we obtain another formula for $G_R(r;r')$:

$$G_R(r;r') = c_0(r')^{N-1} \begin{cases} J_{2,R}(r')J_1(r), & \text{for } r < r', \\ J_1(r')J_{2,R}(r), & \text{for } r > r'. \end{cases} \tag{2.11}$$

For $t \in (0,R)$, set

$$M_R(t) := \frac{(N-1)(p-1)}{qt} + \frac{J_1'(t)}{J_1(t)} + \frac{J_{2,R}'(t)}{J_{2,R}(t)}. \tag{2.12}$$

When $R = +\infty$, $J_{2,+\infty}(r) = J_2(r)$. We denote $G_{+\infty}(r;r')$ as $G(r;r')$ and $M_{+\infty}(t)$ as $M(t)$.

We now state our main theorem in this paper:

Theorem 2.1. *Assume that there exist two points $0 < r_1 < r_2 < R$ such that*

$$M_R(r_1)M_R(r_2) < 0. \tag{2.13}$$

Then for ε sufficiently small, problem (1.6) has a solution $(u_{\varepsilon,R}, v_{\varepsilon,R})$ with the following properties:

(1) $u_{\varepsilon,R}, v_{\varepsilon,R}$ are radially symmetric,

(2) $u_{\varepsilon,R}(r) = \xi_{\varepsilon,R}^{\frac{q}{p-1}} w(\frac{r-t_\varepsilon}{\varepsilon})(1+o(1))$,

(3) $v_{\varepsilon,R}(r) = \xi_{\varepsilon,R}(G_R(t_\varepsilon;t_\varepsilon))^{-1}G_R(r;t_\varepsilon)(1+o(1))$, where $G_R(r;t_\varepsilon)$ satisfies (2.7), $\xi_{\varepsilon,R}$ is defined at the following

$$\xi_{\varepsilon,R} = \left(\varepsilon(\int_{\mathbb{R}} w^m)G_R(t_\varepsilon;t_\varepsilon)\right)^{\frac{(1+s)(p-1)-qm}{qm}} \tag{2.14}$$

and $t_\varepsilon \in (r_1, r_2)$ satisfying $\lim_{\varepsilon\to 0} M_R(t_\varepsilon) = 0$.

Remark: 1. Condition (2.13) guarantees the existence of a zero for the function $M_R(t)$ in the interval (r_1, r_2).

2. It seems interesting to see that the location of the concentration sphere depends only on (p, q, N, R) and does not depend on (m, s).

3. It remains to check condition (2.13), which can be verified numerically. Under some conditions on p, q, we can obtain the following corollary.

Corollary 2.2. *Assume that the following condition holds:*

$$\frac{(N-2)q}{N-1} + 1 < p < q + 1. \tag{2.15}$$

Then there exists an $R_0 > 0$ such that for $R > R_0$ and ε sufficiently small, problem (1.6) has **two** *radially symmetric solutions $(u^i_{\varepsilon,R}, v^i_{\varepsilon,R})$ concentrating on sphere $\{r = t_i\}$ with $M_R(t_i) = 0, i = 1, 2,$ and $0 < t_1 < t_2 < R, i = 1, 2$.*

We remark that Corollary 2.2 is the first rigorous result on the existence to (1.6) of positive solutions in dimension $N \geq 3$.

In the case of $N = 2, p = 2, q = 1$, numerical computations (by Maple) show that the critical number R_0 is 3.622.... More precisely, for $R < R_0$, $M_R(t) > 0$ for all $t \geq 0$; and for $R = R_0$, $M_R(t)$ has a unique zero; and for $R > R_0$, $M_R(t)$ has **two** zeroes $t_1 < t_2 < R$. A picture of $M_R(t)$ for $R = 3, 3.622, 5$ is depicted as follows.

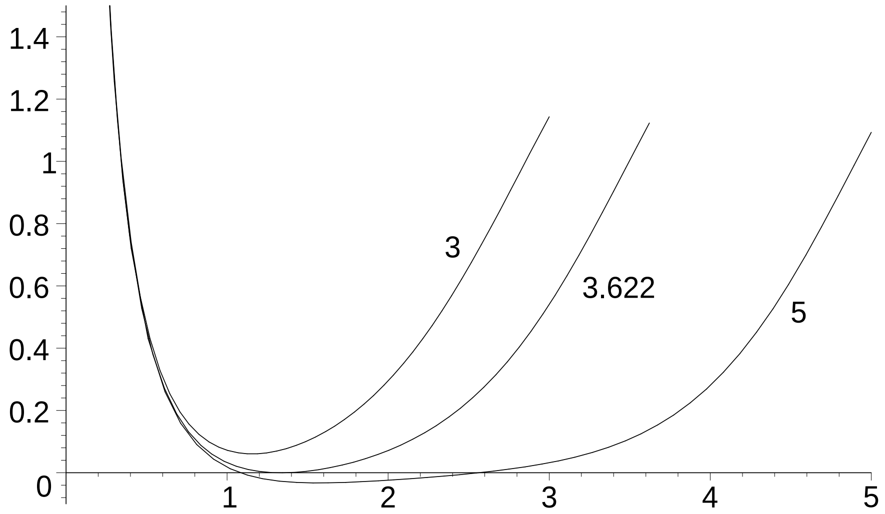

FIGURE 1. The graph of $M_R(t)$ for $N = 2, p = 2, q = 1, R = 3,\ 3.622,\ 5$.

Next we consider the existence of bound states. That is, we consider the following elliptic system in $\mathbb{R}^N$:

$$\begin{cases} \varepsilon^2 \Delta u - u + \frac{u^p}{v^q} = 0 & \text{in } \mathbb{R}^N, \\ \Delta v - v + \frac{u^m}{v^s} = 0 & \text{in } \mathbb{R}^N, \\ u, v > 0, \ u, v \to 0 & \text{as } |x| \to +\infty. \end{cases} \tag{2.16}$$

We have the following result.

Theorem 2.3. *Assume that there exist two points $0 < r_1 < r_2 < +\infty$ such that*

$$M(r_1) M(r_2) < 0. \tag{2.17}$$

Then for ε sufficiently small, problem (2.16) has a solution $(u_\varepsilon, v_\varepsilon)$ with the following properties:

(1) $u_\varepsilon, v_\varepsilon$ are radially symmetric,

(2) $u_\varepsilon(r) = \xi_\varepsilon^{\frac{q}{p-1}} w(\frac{r - r_\varepsilon}{\varepsilon})(1 + o(1))$,

(3) $v_\varepsilon(r) = \xi_\varepsilon (G(r_\varepsilon; r_\varepsilon))^{-1} G(r; r_\varepsilon)(1 + o(1))$, where ξ_ε is defined at the following

$$\xi_\varepsilon = \left(\varepsilon \left(\int_{\mathbb{R}} w^m \right) G(r_\varepsilon; r_\varepsilon) \right)^{\frac{(1+s)(p-1)-qm}{qm}} \tag{2.18}$$

and $r_\varepsilon \in (r_1, r_2)$ satisfying $\lim_{\varepsilon \to 0} M(r_\varepsilon) = 0$.

Similarly we have the following corollary.

Corollary 2.4. *Assume that $N \geq 2$ and that the condition (2.15) holds. Then for ε sufficiently small, problem (1.6) has a radially symmetric bound state solution $(u_\varepsilon, v_\varepsilon)$ which concentrates on a sphere $\{r = t_0\}$ where $M(t_0) = 0$.*

Numerical computations (by Maple) show that for $N \geq 3$, $M(t) > 0$ in the classical Gierer-Meinhardt case $(p, q, m, s) = (2, 1, 2, 0)$. A picture of $M(t)$ for $N = 2, 4$ is depicted in Figure 2.

By using the same method, it is not difficult to generalize the results of Theorem 2.1 to other symmetric domains, such as annulus or the exterior of a ball. We omit the details.

Several interesting questions are left open. First, can multiple layered solutions to (1.6) exist? Second, it would be an interesting question to study the stability of these "ring-like" solutions. Numerical computations in two dimension indicate that the "ring-like" solutions constructed in Theorem 2.1 are unstable and will break into several spots due to angular fluctuations.Third,

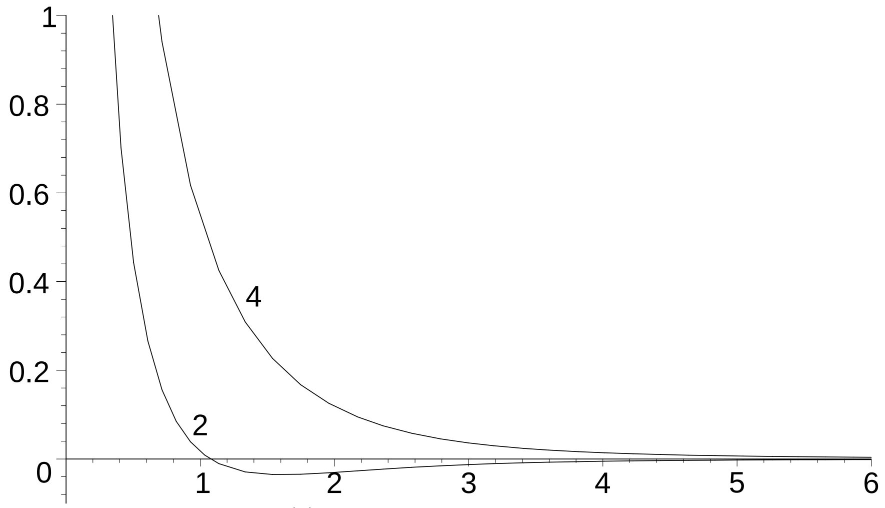

FIGURE 2. The graph of $M(t)$ for $N = 2, 4, p = 2, q = 1$. The left curve is for $N = 2$ and the right curve is for $N = 4$.

if we vary R from 0 to $+\infty$, what is the relation between the layered solution constructed in [2] [19] for the single equation (1.4) and the solutions in Theorem 2.1?

For simplicity, the case $s = 0$ in (1.6) will be treated first. The general case of $s > 0$ will be discussed in the last section.

Acknowledgments. Research supported in part by NSF grant DMS 0400452. The research of the second author is partly supported by an Earmarked Grant from RGC of Hong Kong. We thank Dr. Theodore Kolokolnikov for the computations on Maple.

3. Outline of the proof of Theorem 2.1

Our strategy of the proof of the main results is based on the idea of solving the second equation in (1.6) for v and then working with a nonlocal elliptic PDE rather than directly with the system. It is however convenient to do this by replacing first u by $\xi_\varepsilon^{\frac{q}{p-1}} u$ and v by $\xi_\varepsilon v$, which transforms (1.6) into the problem

$$\begin{cases} \varepsilon^2 \Delta u - u + \frac{u^p}{v^q} = 0 & \text{in } B_R, \\ \Delta v - v + c_{\varepsilon,t} u^m = 0 & \text{in } B_R, \\ u = u(|x|) > 0, v = v(|x|) > 0 \text{ in } B_R, \quad \text{and } \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } B_R. \end{cases} \tag{3.1}$$

where

$$c_{\varepsilon,t} = \xi_\varepsilon^{\frac{qm}{p-1} - 1} \tag{3.2}$$

and ξ_ε is to be chosen later.

We consider a point $t \in (0, R)$ which is a candidate for the location of concentration. Then we choose c_ε such that the following holds:

$$c_{\varepsilon,t}\varepsilon \int_{\mathbb{R}} w^m = \frac{1}{G_R(t;t)}. \tag{3.3}$$

In the sequel, by $\mathbb{T}[h]$ we denote the unique solution of the equation

$$\Delta v - v + c_{\varepsilon,t}h = 0 \text{ in } B_R,\ v(x) = v(|x|) \text{ and } \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial B_R \tag{3.4}$$

for $h \in L^2(B_R)$, namely $\mathbb{T} = c_{\varepsilon,t}(-\Delta + 1)^{-1}$.

Solving the second equation for v in (3.1) and rescaling $r = t + \varepsilon y$ we get $v = \mathbb{T}[u^m]$, which leads to the nonlocal PDE for u

$$u^{''} + \frac{\varepsilon(N-1)}{t+\varepsilon y}u^{'} - u + \frac{u^p}{(\mathbb{T}[u^m](t+\varepsilon y))^q} = 0,\ y \in (-\frac{t}{\varepsilon}, \frac{R-t}{\varepsilon}), \tag{3.5}$$

$$u^{'}(-\frac{t}{\varepsilon}) = u^{'}(\frac{R-t}{\varepsilon}) = 0.$$

Equation (3.4) can be solved by using the Green's function $G_R(r;r^{'})$ given by (2.11). In fact, the operator $\mathbb{T}$ can be written in the following way:

$$\mathbb{T}[h](r^{'}) = c_{\varepsilon,t}\int_0^R G_R(r;r^{'})(\frac{r}{r^{'}})^{N-1}h(r)dr. \tag{3.6}$$

We note that the function $G_R(r;r^{'})$ is not symmetric in r and $r^{'}$, but $r^{N-1}G_R(r;r^{'})$ is. So (3.6) can be written in another useful form

$$\mathbb{T}[h](r^{'}) = c_{\varepsilon,t}\int_0^R G_R(r^{'};r)h(r)dr. \tag{3.7}$$

We look for a solution to (3.5) in the form $u = w(\frac{r-t}{\varepsilon}) + \phi$, where ϕ is a lower order term. Then, formally, we have

$$\mathbb{T}[u^m] = \mathbb{T}[w^m] + m\mathbb{T}[w^{m-1}\phi] + l.o.t.$$

where $l.o.t.$ correspond to lower order terms.

Using (3.7) and the definition of $c_{\varepsilon,t}$ in (3.3), we find (formally) that near the t's,

$$\mathbb{T}[w^m](t+\varepsilon y) = 1 + l.o.t.$$

Arguing similarly we get

$$\mathbb{T}[w^{m-1}\phi] = \frac{1}{\int_{\mathbb{R}} w^m}\int_{\mathbb{R}} w^{m-1}\phi\, dx + l.o.t.$$

Then

$$\frac{u^p}{v^q}=\frac{w^p+pw^{p-1}\phi+l.o.t.}{(\mathbb{T}[w^m])^q+qm\mathbb{T}[w^{m-1}\phi]+l.o.t.}=w^p+pw^{p-1}\phi-qm\frac{\int_{\mathbb{R}}w^{m-1}\phi}{\int_{\mathbb{R}}w^m}w^p+l.o.t.$$

Substituting all this in (3.5) we obtain the equation for ϕ

$$\phi^{''}-\phi+pw^{p-1}\phi-\frac{qm\int_{\mathbb{R}}w^{m-1}\phi}{\int_{\mathbb{R}}w^m}w^p=\mathbb{E}_\varepsilon+\mathbb{M}_\varepsilon[\phi], \tag{3.8}$$

where $\mathbb{E}_\varepsilon=w^{''}+\frac{\varepsilon(N-1)}{t+\varepsilon y}w^{'}-w+\frac{w^p}{(\mathbb{T}[w^m])^q}$ and $\mathbb{M}_\varepsilon[\phi]$ represents higher order terms in ϕ.

Thus we have reduced the problem of finding solutions to (3.1) to the problem of solving (3.8) for ϕ.

Rather than solving directly problem (3.8), we consider first the following auxiliary problem: given point t, find a function ϕ such that for certain constants $\beta_\varepsilon(t)$ the following equation is satisfied:

$$\mathbb{L}\phi=\mathbb{E}_\varepsilon+\mathbb{M}_\varepsilon[\phi]+\beta_\varepsilon w^{p-1}w^{'},\ \int_{\mathbb{R}}\phi w^{p-1}w^{'}=0, \tag{3.9}$$

where

$$\mathbb{L}\phi:=\phi^{''}-\phi+pw^{p-1}\phi-\frac{qm\int_{\mathbb{R}}w^{m-1}\phi}{\int_{\mathbb{R}}w^m}w^p. \tag{3.10}$$

We will prove in §5 and §7 that this problem is uniquely solvable within a class of small functions ϕ. We will then get a solution of the original problem when the point t is adjusted in such a way that $\beta_\varepsilon(t)=0$. We show the existence of such a point in §8, where Theorem 2.1 is finally established. In §9, we consider the modifications of the whole $\mathbb{R}^N$ case and complete the proof of Theorem 2.3 and Corollary 2.4. The proof of Corollary 2.2 as well as the case of $s>0$ are discussed in the last section §10.

It is worth mentioning that there is one other important technical point we have to overcome: while working with $\mathbb{R}^N$, namely, we have to make sense the term $\frac{u^p}{v^q}$ in certain Sobolev spaces. To this end, we will use a weighted Sobolev space approach. Similar approach has been used in [8].

In the remainder of the paper we carry out the program outlined above. In particular, we will need to understand invertibility properties of the linear operator $\mathbb{L}$ first. We will do this in the next section.

4. A Linear operator

In this section, we consider some properties of the function w and the linear operator $\mathbb{L}$, defined by (3.10).

We first state the following facts of $w(y)$.

Lemma 4.1. *(1) The following identities hold*

$$\int_{\mathbb{R}} (w^{'})^2 = \frac{p-1}{2(p+1)} \int_{\mathbb{R}} w^{p+1}, \tag{4.1}$$

$$\int_{\mathbb{R}} w^{p+1}(y)(\int_{-\infty}^{y} w^m(z)dz)dy = \int_{\mathbb{R}} w^{p+1}(y)(\int_{y}^{\infty} w^m(z)dz)dy = \frac{1}{2}\int_{\mathbb{R}} w^{p+1} \int_{\mathbb{R}} w^m. \tag{4.2}$$

(2) The solution to the following problem

$$\mathbb{L}_0\phi := \phi^{''} - \phi + pw^{p-1}\phi = 0, |\phi| \leq C \tag{4.3}$$

for some $C > 0$ is given by $\phi = cw^{'}$ for some constant c.

Proof: Since w satisfies (2.5), we have $(w^{'})^2 = w^2 - \frac{2}{p+1}w^{p+1}$. Combining this with the following identity

$$\int_{\mathbb{R}} ((w^{'})^2 + w^2) = \int_{\mathbb{R}} w^{p+1}$$

we deduce (4.1). To prove (4.2), we observe that

$$\int_{\mathbb{R}} w^{p+1}(y)(\int_{-\infty}^{y} w^m(z)dz - \int_{y}^{\infty} w^m(z)dz)dy = 0$$

$$\int_{\mathbb{R}} w^{p+1}(y)(\int_{-\infty}^{y} w^m(z)dz + \int_{y}^{\infty} w^m(z)dz)dy = \int_{\mathbb{R}} w^{p+1} \int_{\mathbb{R}} w^m.$$

(4.2) then follows.

(2) follows from standard ODE theory. □

The following lemma characterizes the kernel of $\mathbb{L}$.

Lemma 4.2. *If $\frac{qm}{p-1} \neq 1$ and*

$$\mathbb{L}\phi = 0, \phi \in H^2(\mathbb{R})$$

then $\phi = cw^{'}$ for some constant c.

Proof: In fact, let $\mathbb{L}\phi = 0$. Then we have

$$\mathbb{L}_0(\phi - \frac{qm}{p-1}\frac{\int_{\mathbb{R}} w^{m-1}\phi}{\int_{\mathbb{R}} w^m}w) = 0.$$

Since $\phi \in H^2(\mathbb{R})$, by (2) of Lemma 4.1, we have

$$\phi - \frac{qm}{p-1}\frac{\int_{\mathbb{R}} w^{m-1}\phi}{\int_{\mathbb{R}} w^m}w = cw' \tag{4.4}$$

for some c. Multiplying (4.4) by w^{m-1} and integrating over $\mathbb{R}$, we obtain that

$$(1 - \frac{qm}{p-1})\int_{\mathbb{R}} w^{m-1}\phi = 0$$

and hence $\int_{\mathbb{R}} w^{m-1}\phi = 0$ which implies that $\phi = cw'$.

□

5. A Linear Problem

This section is devoted to a study of a linear problem, which forms the foundation of the finite dimension reduction.

Let $0 < r_1 < r_2$ be the two numbers given in Theorem 2.1 or Theorem 2.3. Let t be a point such that

$$r_1 < t < r_2. \tag{5.1}$$

We set

$$\mathbb{I}_{\varepsilon,t} := (-\frac{t}{\varepsilon}, \frac{R-t}{\varepsilon}). \tag{5.2}$$

Choose two fixed numbers $R_1 \in (0, r_1), R_2 \in (r_2, R)$. Let $\chi(s)$ be a function such that $\chi(s) = 1$ for $s \in [R_1, \frac{r_2+R_2}{2}]$ and $\chi(s) = 0$ for $s < \frac{R_1}{2}$ or $s > R_2$. Set

$$w_{\varepsilon,t}(y) = w(y)\chi(t + \varepsilon y). \tag{5.3}$$

For $u, v \in H^1(B_R)$, we equip it with the following scalar product:

$$(u, v)_\varepsilon = \int_{\mathbb{I}_{\varepsilon,t}} (u'v' + uv)(t + \varepsilon y)^{N-1} dy \tag{5.4}$$

(which is equivalent to the norm $\|u\|_{H^1(B_R)}$).

Then orthogonality to the function $w'_{\varepsilon,t}$ in that space is equivalent to the orthogonality to the function

$$Z_{\varepsilon,t} = w'''_{\varepsilon,t} + \frac{\varepsilon(N-1)}{t+\varepsilon y}w''_{\varepsilon,t} - w'_{\varepsilon,t} \tag{5.5}$$

in $L^2(\mathbb{I}_{\varepsilon,t})$, equipped with the following scalar product

$$< u, v >_\varepsilon = \int_{\mathbb{I}_{\varepsilon,t}} (uv)(t+\varepsilon y)^{N-1} dy \tag{5.6}$$

(which is equivalent to the norm $\|u\|_{B_R}$).

Then we consider the following problem: $h \in L^2(B_R) \cap L^\infty(\mathbb{I}_{\varepsilon,t})$ being given, find a function ϕ satisfying

$$\begin{cases} \mathbb{L}_{\varepsilon,t}[\phi] := \phi'' + \frac{\varepsilon(N-1)}{t+\varepsilon y}\phi' - \phi + pw_{\varepsilon,t}^{p-1}\phi - qm\frac{\int_{\mathbb{I}_{\varepsilon,t}} w_{\varepsilon,t}^{m-1}\phi}{\int_{\mathbb{I}_{\varepsilon,t}} w_{\varepsilon,t}^m} w_{\varepsilon,t}^p = h + cZ_{\varepsilon,t}, \\ \phi'(-\frac{t}{\varepsilon}) = \phi'(\frac{R-t}{\varepsilon}) = 0, < \phi, Z_{\varepsilon,t} >_\varepsilon = 0 \end{cases} \tag{5.7}$$

for some constant c.

Before we solve the system (5.7), we need the following result based on ODE estimates.

Lemma 5.1. *Let $\phi \in C^2(\bar{\mathbb{I}}_\varepsilon)$ satisfy*

$$\left|\phi''(y) + \frac{\varepsilon(N-1)}{t+\varepsilon y}\phi'(y) - \phi(y)\right| \le c_0 e^{-\mu|y|},$$

$$\phi'\Big(-\frac{t}{\varepsilon}\Big) = \phi'\Big(-\frac{R-t}{\varepsilon}\Big) = 0,$$

for some $c_0 > 0$ and $\mu \in (0,1)$. Then, provided that $\mu > 0$ is sufficiently small,

$$|\phi(y)| \le 2e^2(|\phi(0)| + c_0)e^{-\mu|y|} \quad \forall y \in \mathbb{I}_{\varepsilon,t}.$$

Proof. We use a comparison principle. Take $\eta(t)$ a smooth cut-off function such that

$$\eta(t) = 1 \text{ for } |t| \le 1, \quad \eta(t) = 0 \text{ for } \ |t| \ge 2, \quad 0 \le \eta \le 1.$$

Now consider the following auxiliary function:

$$\Phi(y) = A\Big[e^{\mu y} + (e^{\mu y_0} - e^{\mu y})\eta\Big(\mu\Big(y + \frac{t}{\varepsilon}\Big)\Big)\Big]$$

where

$$y_0 = -\frac{t}{\varepsilon} + \frac{1}{\mu}, \quad A = 2e(|\phi(0)| + c_0).$$

If $y \in (-\frac{t}{\varepsilon}, y_0)$, $\Phi(y) = Ae^{\mu y_0}$ and hence

$$\Phi'' + \frac{\varepsilon(N-1)}{t+\varepsilon y}\Phi' - \Phi = -Ae^{\mu y_0} \le -c_0 e^{\mu y}.$$

If $y \in (-\frac{t}{\varepsilon} + \frac{2}{\mu}, 0)$, $\Phi(y) = Ae^{\mu y}$ and $\frac{\varepsilon}{t+\varepsilon y} \le \frac{\mu}{2}$, hence

$$\Phi'' + \frac{\varepsilon(N-1)}{t+\varepsilon y}\Phi' - \Phi \le A[\frac{N+1}{e}\mu^2 - 1]e^{\mu y} \le -c_0 e^{\mu y}$$

provided that μ is sufficiently small. Finally it is easy to see that for $y \in (y_0, y_0 + \frac{1}{\mu})$

$$e^{\mu y_0} \le e^{\mu y} \le e e^{\mu y_0}, \quad \Phi(y) \ge A e^{\mu y_0}, \quad \frac{\mu}{2} \le \frac{\varepsilon}{t + \varepsilon y} \le \mu;$$

hence

$$\Phi'' + \frac{\varepsilon(N-1)}{t+\varepsilon y}\Phi' - \Phi \le AO(\mu^2)e^{\mu y} - Ae^{\mu y_0}$$
$$\le AO(\mu^2)e^{\mu y} - \frac{A}{e}e^{\mu y} \le -c_0 e^{\mu y}$$

provided that μ is sufficiently small.

In any case, we have that for $y \in (-\frac{t}{\varepsilon}, 0)$, $\Phi(y)$ satisfies

$$\Phi'' + \frac{\varepsilon(N-1)}{t+\varepsilon y}\Phi' - \Phi \le -c_0 e^{\mu y}, \quad \Phi'\Big(-\frac{t}{\varepsilon}\Big) = 0, \quad \Phi(0) \ge |\phi(0)|. \tag{5.8}$$

Combining (5.8) with the hypothesis we obtain

$$(\Phi - \phi)''(y) + \frac{\varepsilon(N-1)}{t+\varepsilon y}(\Phi-\phi)'(y) - (\Phi-\phi)(y) \le 0 \ \ \forall y \in \Big[-\frac{t}{\varepsilon}, 0\Big] \tag{5.9}$$

and

$$(\Phi - \phi)(0) > 0, \ \ (\Phi-\phi)'\Big(-\frac{t}{\varepsilon}\Big) = 0;$$

we claim that $(\Phi - \phi)(y) \ge 0$ for $y \in [-\frac{t}{\varepsilon}, 0)$. Otherwise, if we call $\bar{y}$ the minimum point of $\Phi - \phi$ in $[-\frac{t}{\varepsilon}, 0)$, then it would be $(\Phi-\phi)(\bar{y}) < 0$ and $(\Phi-\phi)'(\bar{y}) = 0$, $(\Phi-\phi)''(\bar{y}) \ge 0$, in contradiction with (5.9). Hence we have proved that $\phi \le \Phi$ in $[-\frac{t}{\varepsilon}, 0]$. On the other hand by (5.8) and the hypothesis we also get

$$(\Phi + \phi)''(y) + \frac{2\varepsilon}{t+\varepsilon y}(\Phi+\phi)'(y) - (\Phi+\phi)(y) \le 0 \ \ \forall y \in \Big[-\frac{t}{\varepsilon}, 0\Big]$$

and $(\Phi+\phi)(0) > 0$, $(\Phi+\phi)'(-\frac{t}{\varepsilon}) = 0$. Proceeding as before we conclude $\phi \ge -\Phi$ in $[-\frac{t}{\varepsilon}, 0]$.

For $y \in [0, \frac{R-t}{\varepsilon})$, we use $\hat{\Phi}(y) = A\frac{\cosh(\mu(\frac{R-t}{\varepsilon} - y))}{\cosh(\frac{\mu(R-t)}{\varepsilon})}$. Note that $\hat{\Phi}(0) = A, \hat{\Phi}'(\frac{R-t}{\varepsilon}) = 0$. It is easy to see that $\frac{A}{2}e^{-\mu y} < \hat{\Phi}(y) < 2Ae^{-\mu y}$ and hence

$$\hat{\Phi}'' + \frac{\varepsilon(N-1)}{t+\varepsilon y}\hat{\Phi}' - \hat{\Phi} \le -c_0 e^{-\mu y}$$

provided that μ is sufficiently small. By repeating the previous argument we obtain $|\phi| \le \hat{\Phi}$ in $[0, \frac{R-t}{\varepsilon})$ and the conclusion follows. □

Let $0 < \mu < \frac{1}{10}\min(p-1, m-1, 1)$ be a small number such that Lemma 5.1 holds. For every function $\phi : \mathbb{I}_{\varepsilon,t} \to \mathbb{R}$ define

$$\|\phi\|_* = \|e^{\mu <y>}\phi(y)\|_{L^\infty(\mathbb{I}_{\varepsilon,t})} \tag{5.10}$$

where $< y >= (1+y^2)^{\frac{1}{2}}$.

Since $\frac{\varepsilon(N-1)}{t+\varepsilon y} w''_{\varepsilon,t} = O(\varepsilon)e^{-|y|}$, we obtain

$$Z_{\varepsilon,t}(y) = w'''(y) - w'(y) + O(\varepsilon)e^{-\mu<y>} = -pw^{p-1}(y)w'(y) + O(\varepsilon)e^{-\mu<y>} \tag{5.11}$$

uniformly for $t \in [R_1, R_2]$.

The following proposition provides a priori estimates.

Proposition 5.2. *Let (ϕ, c) satisfy (5.7). Then for ε sufficiently small, $t \in [R_1, R_2]$, we have*

$$\|\phi\|_* \le C\|h\|_* \tag{5.12}$$

where C is a positive constant depending on R, N, p only.

Remark: A more precise inequality should be

$$\|\phi\|_* \le C\|h^\perp\|_*, \text{ where } h^\perp = h - \frac{< h, Z_{\varepsilon,t} >_\varepsilon}{< Z_{\varepsilon,t}, Z_{\varepsilon,t} >^2} Z_{\varepsilon,t}. \tag{5.13}$$

Proof. We prove it by contradiction. Suppose not. Then there exists sequences $\varepsilon_k \to 0, t_k \in [R_1, R_2]$ and a sequence of functions ϕ_{ε_k,t_k} satisfying (5.7) such that the following holds:

$$\|\phi_{\varepsilon_k,t_k}\|_* = 1, \|h_k\|_* = o(1), \int_{\mathbb{I}_{\varepsilon_k,t_k}} \phi_{\varepsilon_k,t_k} Z_{\varepsilon_k,t_k}(t_k + \varepsilon_k y)^{N-1} dy = 0.$$

For simplicity of notation, we drop the subindex k.

Multiplying the first equation of (5.7) by $w'_{\varepsilon,t}$ and integrating over $\mathbb{I}_{\varepsilon,t}$, we obtain that

$$c\int_{\mathbb{I}_{\varepsilon,t}} Z_{\varepsilon,t} w'_{\varepsilon,t} = -\int_{\mathbb{I}_{\varepsilon,t}} h w'_{\varepsilon,t} + \int_{\mathbb{I}_{\varepsilon,t}} (\mathbb{L}_\varepsilon[\phi_{\varepsilon,t}]) w'_{\varepsilon,t}. \tag{5.14}$$

The left hand side of (5.14) equals $c(-\int_{\mathbb{R}} pw^{p-1}(w')^2 + o(1))$ because of (5.11). The first term on the right hand side of (5.14) can be estimated by

$$\int_{\mathbb{I}_{\varepsilon,t}} h w'_{\varepsilon,t} = O(\|h\|_*).$$

The last term equals

$$\int_{\mathbb{I}_{\varepsilon,t}} (\mathbb{L}_\varepsilon[\phi_{\varepsilon,t}]) w'_{\varepsilon,t} = \int_{\mathbb{I}_{\varepsilon,t}} \left[\phi''_{\varepsilon,t} + \frac{\varepsilon(N-1)}{t+\varepsilon y}\phi'_{\varepsilon,t} - \phi_{\varepsilon,t} + pw^{p-1}_{\varepsilon,t}\phi_{\varepsilon,t} \right] w'_{\varepsilon,t} - qm \frac{\int_{\mathbb{I}_{\varepsilon,t}} w^{m-1}_{\varepsilon,t}\phi_{\varepsilon,t}}{\int_{\mathbb{I}_{\varepsilon,t}} w^m_{\varepsilon,t}} \int_{\mathbb{I}_{\varepsilon,t}} w^p_{\varepsilon,t} w'_{\varepsilon,t}$$

$$= o(\|\phi_{\varepsilon,t}\|_*).$$

Hence we obtain that

$$|c| = O(\|h\|_*) + o(\|\phi_{\varepsilon,t}\|_*), \quad \|h + cZ_{\varepsilon,t}\|_* = o(1). \tag{5.15}$$

Next we claim that $|\phi_{\varepsilon,t}(y)| \to 0$ in any compact interval of $\mathbb{R}$. In fact, we consider $\bar{\phi}_\varepsilon(y) = \phi_{\varepsilon,t}\chi(t+\varepsilon y)$ where χ is the cut-off function introduced at the beginning of this section. Then it is easy to see that $\|\bar{\phi}_\varepsilon\|_{H^2} \le C$ and hence $\bar{\phi}_\varepsilon \to \phi_0$ weakly in $H^2(\mathbb{R})$ and ϕ_0 satisfies

$$\mathbb{L}\phi_0 = 0, |\phi_0| \le Ce^{-\mu|y|}.$$

By Lemma 4.2, we must have $\phi_0 = cw'$. On the other hand, $\int_{\mathbb{I}_{\varepsilon,t}} \phi_{\varepsilon,t} Z_{\varepsilon,t}(t+\varepsilon y)^{N-1} dy = 0$ and hence $\int_{\mathbb{R}} \phi_0 w^{p-1} w' = 0$ which implies that $c = 0$ and hence $\phi_{\varepsilon,t} \to 0$ in any compact interval of $\mathbb{R}$. This shows that

$$\|w_{\varepsilon,t}^{p-1}\phi_{\varepsilon,t}\|_* = \sup_{y\in\mathbb{I}_{\varepsilon,t}} |e^{\mu<y>} w_{\varepsilon,t}^{p-1}(y) e^{-\mu<y>}\phi_{\varepsilon,t}(y)| = o(1). \tag{5.16}$$

On the other hand, by Lebesgue's Dominated Convergence Theorem, we have that

$$\int_{\mathbb{I}_{\varepsilon,t}} w_{\varepsilon,t}^{m-1}\phi_{\varepsilon,t}(t+\varepsilon y)^{N-1} \to 0$$

which implies

$$\|\frac{\int_{\mathbb{I}_{\varepsilon,t}} w_{\varepsilon,t}^{m-1}\phi_{\varepsilon,t}}{\int_{\mathbb{I}_{\varepsilon,t}} w_{\varepsilon,t}^{m}} w_{\varepsilon,t}^p\|_* = o(1). \tag{5.17}$$

Thus we have arrived at the following situation: $\phi_{\varepsilon,t}$ satisfies

$$\phi_{\varepsilon,t}^{''} + \frac{\varepsilon(N-1)}{t+\varepsilon y}\phi_{\varepsilon,t}^{'} - \phi_{\varepsilon,t} = o(e^{-\mu<y>}),\ \phi_{\varepsilon,t}^{'}(-\frac{t}{\varepsilon}) = \phi_{\varepsilon,t}^{'}(\frac{R-t}{\varepsilon}) = 0 \tag{5.18}$$

Since $\phi_{\varepsilon,t} \to 0$ in any compact interval, $\phi_{\varepsilon,t}(0) = o(1)$. Applying Lemma 5.1, we conclude that $\phi_{\varepsilon,t} = o(e^{-\mu|y|})$. A contradiction to the assumption that $\|\phi_{\varepsilon,t}\|_* = 1$.

This proves the proposition.

□

Finally we prove:

Proposition 5.3. *There exists an $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, $t \in [R_1, R_2]$, given any $h \in L^2(\mathbb{I}_{\varepsilon,t}) \cap L^\infty(\mathbb{I}_{\varepsilon,t})$, there exists a unique pair (ϕ, c) such that the following hold:*

$$\mathbb{L}_{\varepsilon,t}[\phi] = h + cZ_{\varepsilon,t}, \tag{5.19}$$

$$\phi^{'}(-\frac{t}{\varepsilon}) = \phi^{'}(\frac{R-t}{\varepsilon}) = 0,\ <\phi, Z_{\varepsilon,t}>_\varepsilon = 0. \tag{5.20}$$

Moreover, we have

$$\|\phi\|_* \le C\|h\|_*. \tag{5.21}$$

Proof: The existence follows from Fredholm alternatives. To this end, let us set

$$\mathbb{H} = \{u \in H^1(B_R) | (u, w_{\varepsilon,t}^{'})_\varepsilon = 0\}.$$

Observe that ϕ solves (5.19) and (5.20) if and only if $\phi \in H^1(B_R)$ satisfies

$$\int_{B_R} (\nabla\phi\nabla\psi + \phi\psi) - p < w_{\varepsilon,t}^{p-1}\phi, \psi >_\varepsilon$$

$$-qm\frac{\int_{\mathbb{I}_{\varepsilon,t}} w_{\varepsilon,t}^{m-1}\phi}{\int_{\mathbb{I}_{\varepsilon,t}} w_{\varepsilon,t}^{m}} < w_{\varepsilon,t}^{p}, \psi >_\varepsilon = < h, \psi >_\varepsilon, \ \forall \psi \in H^1(B_R).$$

This equation can be rewritten in the following form

$$\phi + \mathbb{S}(\phi) = \bar{h} \tag{5.22}$$

where $\mathbb{S}$ is a linear compact operator form $\mathbb{H}$ to $\mathbb{H}$, $\bar{h} = (\Delta - 1)^{-1}(h^\perp) \in \mathbb{H}$ and $\phi \in \mathbb{H}$.

Using Fredholm's alternatives, we will show equation (5.22) has a uniquely solvable solution for each $\bar{h}$ by proving that the equation has a unique solution for $\bar{h} = 0$, i.e., $h^\perp = 0$. To this end, we assume the contrary. That is, there exists (ϕ, c) such that

$$\mathbb{L}_\varepsilon[\phi] = cZ_{\varepsilon,t}, \tag{5.23}$$

$$\phi^{'}(-\frac{t}{\varepsilon}) = \phi^{'}(\frac{R-t}{\varepsilon}) = 0, \quad < \phi, Z_{\varepsilon,t} >_\varepsilon = 0. \tag{5.24}$$

From (5.23), it is easy to see that $\|\phi\|_* < +\infty$. So without loss of generality, we may assume that $\|\phi\|_* = 1$. But then this contradicts (5.13).

$\square$

6. A study of the operator $\mathbb{T}[h]$

In this section, we study the operator $\mathbb{T}[h]$, where we choose h to be

$$h = \left(w_{\varepsilon,t}(\frac{r-t}{\varepsilon}) + \phi(\frac{r-t}{\varepsilon}) \right)^m, \ \|\phi\|_* = O(\varepsilon^\sigma). \tag{6.1}$$

Let $c_{\varepsilon,t}$ be chosen so that

$$\varepsilon c_{\varepsilon,t} \int_{\mathbb{I}_{\varepsilon,t}} w_{\varepsilon,t}^m = \frac{1}{G_R(t;t)}. \tag{6.2}$$

According to (3.7), we have

$$\mathbb{T}[h](r^{'}) = c_{\varepsilon,t} \int_0^R G_R(r^{'}; r) h(r) dr.$$

Let $r^{'} = t + \varepsilon y$. We consider two regions of $r^{'}$. First we consider $|r^{'} - t| < \frac{\delta}{4}$ where $\delta = \min(R_1, R - R_2)$. Then we have

$$\mathbb{T}[(w_{\varepsilon,t} + \phi)^m](t + \varepsilon y) = c_{\varepsilon,t} \int_0^R G_R(t + \varepsilon y; r) \left(w_{\varepsilon,t}(\frac{r-t}{\varepsilon}) + \phi(\frac{r-t}{\varepsilon}) \right)^m dr$$

$$= \varepsilon c_{\varepsilon,t} \int_{\mathbb{I}_{\varepsilon,t}} G_R(t + \varepsilon y; t + \varepsilon z) \left(w_{\varepsilon,t}(z) + \phi(z) \right)^m dz$$

$$= E_1 + E_2$$

where

$$E_1 = \varepsilon c_{\varepsilon,t} \int_{\mathbb{I}_{\varepsilon,t}} G_R(t + \varepsilon y; t + \varepsilon z) w_{\varepsilon,t}^m dz,$$

$$E_2 = \varepsilon c_{\varepsilon,t} \int_{\mathbb{I}_{\varepsilon,t}} G_R(t + \varepsilon y; t + \varepsilon z) \left[(w_{\varepsilon,t}(z) + \phi(z))^m - (w_{\varepsilon,t}(z))^m \right] dz.$$

Observe that $G_R(r^{'}; r)$ can be expanded as follows: for $\bar{y} < \bar{z}$, we have

$$G_R(t + \bar{y}; t + \bar{z}) = G_R(t; t) \left(1 + (\frac{J_{2,R}^{'}(t)}{J_{2,R}(t)} + \frac{N-1}{t}) \bar{z} + \frac{J_1^{'}(t)}{J_1(t)} \bar{y} + O(|\bar{y}|^2 + |\bar{z}|^2) \right); \tag{6.3}$$

while for $\bar{z} < \bar{y}$, there is another expansion

$$G_R(t + \bar{y}; t + \bar{z}) = G_R(t; t) \left(1 + (\frac{J_1^{'}(t)}{J_1(t)} + \frac{N-1}{t}) \bar{z} + \frac{J_{2,R}^{'}(t)}{J_{2,R}(t)} \bar{y} + O(|\bar{y}|^2 + |\bar{z}|^2) \right). \tag{6.4}$$

Then we have, using (6.2) and (6.3)-(6.4),

$$E_1 = \varepsilon c_{\varepsilon,t} \int_{\mathbb{I}_{\varepsilon,t}} G_R(t + \varepsilon y; t + \varepsilon z) w^m(z) dz + O(\varepsilon^2)$$

$$= 1 + \varepsilon \rho(y) + O(\varepsilon^2 (1 + |y|^2)) \tag{6.5}$$

where $\rho(y)$ is defined by

(6.6)

$$\rho(y) = \frac{1}{\int_{\mathbb{R}} w^m} \left[\frac{J_{2,R}^{'}(t)}{J_{2,R}(t)} \left(y \int_{-\infty}^y w^m + \int_y^{\infty} z w^m(z) dz \right) + \frac{J_1^{'}(t)}{J_1(t)} \left(y \int_y^{\infty} w^m + \int_{-\infty}^y z w^m(z) dz \right) \right].$$

For E_2, we have

$$E_2 = \varepsilon c_{\varepsilon,t} \int_{\mathbb{I}_{\varepsilon,t}} G_R(t + \varepsilon y; t + \varepsilon z) [m w_{\varepsilon,t}^{m-1} \phi] dz + O(\|\phi\|_*^{1+\tau})$$

$$= \frac{m}{\int_{\mathbb{I}_{\varepsilon,t}} w_{\varepsilon,t}^m} \int_{\mathbb{I}_{\varepsilon,t}} w_{\varepsilon,t}^{m-1}\phi + O(\varepsilon\|\phi\|_* + \|\phi\|_*^{1+\tau})$$

where the constant τ is defined by

$$\tau = \min(1, p-1, m-1). \tag{6.7}$$

On the other hand, for $|r^{'} - t| \geq \frac{\delta}{4}$, we use

$$|\phi(\frac{r-t}{\varepsilon})| \leq C\|\phi\|_* e^{-\mu < \frac{r-t}{\varepsilon} >}. \tag{6.8}$$

This implies

$$\mathbb{T}[(w_{\varepsilon,t} + \phi)^m](r^{'})$$

$$\geq c_{\varepsilon,t}\int_0^R G_R(r^{'};r) w_{\varepsilon,t}^m dr - C\varepsilon^\sigma c_{\varepsilon,t}\int_0^R G_R(r^{'};r) e^{-\mu<\frac{r-t}{\varepsilon}>} dr \geq C. \tag{6.9}$$

Summarizing all the estimates, we have obtained the following lemma:

Lemma 6.1. *(1) For $|r^{'} - t| < \frac{\delta}{4}, r^{'} = t + \varepsilon y$, we have*

$$\mathbb{T}[(w_{\varepsilon,t} + \phi)^m](t+\varepsilon y) = 1 + \varepsilon\rho(y) + \frac{m}{\int_{\mathbb{I}_{\varepsilon,t}} w^m} \int_{\mathbb{I}_{\varepsilon,t}} w_{\varepsilon,t}^{m-1}\phi \tag{6.10}$$

$$+ O(\varepsilon^2(1+|y|^2) + \varepsilon\|\phi\|_* + \|\phi\|_*^{1+\tau}).$$

(2) For $|r^{'} - t| \geq \frac{\delta}{4}$, we then have

$$\mathbb{T}[(w_{\varepsilon,t} + \phi)^m](r^{'}) \geq C. \tag{6.11}$$

As a consequence of (6.11), for $|r^{'} - t| \geq \frac{\delta}{4}$, we have

$$\frac{(w_{\varepsilon,t} + \phi)^p}{(\mathbb{T}[(w_{\varepsilon,t} + \phi)^m])^q} \leq Ce^{-\mu\frac{|r^{'}-t|}{\varepsilon}}. \tag{6.12}$$

7. A Nonlinear Problem

In this section, we solve the following system of equations for (ϕ, β):

$$(w_{\varepsilon,t} + \phi)^{''} + \frac{\varepsilon(N-1)}{t+\varepsilon y}(w_{\varepsilon,t} + \phi)^{'} - (w_{\varepsilon,t} + \phi) + \frac{(w_{\varepsilon,t} + \phi)^p}{(\mathbb{T}[(w_{\varepsilon,t} + \phi)^m](t+\varepsilon y))^q} = \beta Z_{\varepsilon,t}, \tag{7.1}$$

$$\phi^{'}(-\frac{t}{\varepsilon}) = \phi^{'}(\frac{R-t}{\varepsilon}) = 0, \ \int_{\mathbb{I}_{\varepsilon,t}} \phi Z_{\varepsilon,t}(t+\varepsilon y)^{N-1} dy = 0. \tag{7.2}$$

The main result in this section is to show the following proposition:

Proposition 7.1. *For $t \in [R_1, R_2]$ and ε sufficiently small, there exists a unique pair $(\phi_{\varepsilon,t}, \beta_\varepsilon(t))$ satisfying (7.1)-(7.2). Furthermore, $(\phi_{\varepsilon,t}, \beta_\varepsilon(t))$ is continuous in t and we have the following estimate*

$$\|\phi_{\varepsilon,t}\|_* \leq \varepsilon^\sigma \tag{7.3}$$

where $\sigma \in (\frac{1}{1+\tau}, 1)$ is a constant.

Proof: We write (7.1) in the following form:

$$\mathbb{L}_{\varepsilon,t}[\phi] = \mathbb{E}_\varepsilon + \mathbb{M}_\varepsilon[\phi] + \beta Z_{\varepsilon,t} \tag{7.4}$$

and use contraction mapping theorem. Here

$$\mathbb{E}_\varepsilon = -w''_{\varepsilon,t} + w_{\varepsilon,t} - \frac{\varepsilon(N-1)}{t+\varepsilon y} w'_{\varepsilon,t} - \frac{w^p_{\varepsilon,t}}{(\mathbb{T}[w^m_{\varepsilon,t}])^q} \tag{7.5}$$

and $\mathbb{M}_\varepsilon[\phi]$ is given by

$$\mathbb{M}_\varepsilon[\phi] = \frac{(w_{\varepsilon,t}+\phi)^p}{(\mathbb{T}[(w_{\varepsilon,t}+\phi)^m])^q} - \frac{w^p_{\varepsilon,t}}{(\mathbb{T}[w^m_{\varepsilon,t}])^q} - p w^{p-1}_{\varepsilon,t}\phi + qm \frac{\int_{\mathbb{I}_{\varepsilon,t}} w^{m-1}_{\varepsilon,t}\phi}{\int_{\mathbb{I}_{\varepsilon,t}} w^m_{\varepsilon,t}} w^p_{\varepsilon,t}. \tag{7.6}$$

Since $w''_{\varepsilon,t} - w_{\varepsilon,t} = -w^p_{\varepsilon,t} + O(\varepsilon^2 e^{-\mu|y|})$, it is easy to see that

$$E_\varepsilon = -\frac{\varepsilon(N-1)}{t+\varepsilon y} w'_{\varepsilon,t} + w^p_{\varepsilon,t} \frac{(\mathbb{T}[w^m_{\varepsilon,t}])^q - 1}{(\mathbb{T}[w^m_{\varepsilon,t}])^q} + O(\varepsilon^2 e^{-\mu|y|}) \tag{7.7}$$

and hence by Lemma 6.1,

$$\|E_\varepsilon\|_* \leq C\varepsilon. \tag{7.8}$$

For $\mathbb{M}_\varepsilon$, we note that

$$\mathbb{M}_\varepsilon[\phi] = \frac{(w_{\varepsilon,t}+\phi)^p - w^p_{\varepsilon,t} - p w^{p-1}_{\varepsilon,t}\phi}{(\mathbb{T}[(w_{\varepsilon,t}+\phi)^m])^q}$$

$$-\left(\frac{w^p_{\varepsilon,t}}{(\mathbb{T}[w^m_{\varepsilon,t}])^q} - \frac{w^p_{\varepsilon,t}}{(\mathbb{T}[(w_{\varepsilon,t}+\phi)^m])^q} - qm \frac{\int_{\mathbb{I}_{\varepsilon,t}} w^{m-1}_{\varepsilon,t}\phi}{\int_{\mathbb{I}_{\varepsilon,t}} w^m_{\varepsilon,t}} w^p_{\varepsilon,t} \right)$$

$$-p w^{p-1}_{\varepsilon,t}\phi \left(1 - \frac{1}{(\mathbb{T}[(w_{\varepsilon,t}+\phi)^m])^q}\right).$$

Suppose that $\|\phi\|_* < \varepsilon^\sigma$. The first term in $\mathbb{M}_\varepsilon[\phi]$ can be estimated, using (6.12):

$$\left| \frac{(w_{\varepsilon,t}+\phi)^p - w^p_{\varepsilon,t} - p w^{p-1}_{\varepsilon,t}\phi}{(\mathbb{T}[(w_{\varepsilon,t}+\phi)^m])^q} \right| \leq C e^{-(p-1-\tau)<y>} |\phi|^{1+\tau}.$$

By Lemma 6.1, we have the second term and the last term in $\mathbb{M}_\varepsilon[\phi]$ can be bounded by $O(\varepsilon\|\phi\|_* + \|\phi\|_*^{1+\tau})$. Therefore, we have

$$\|\mathbb{M}_\varepsilon[\phi]\|_* \le C\Bigg(\varepsilon\|\phi\|_* + \|\phi\|_*^{1+\tau}\Bigg). \tag{7.9}$$

Set $\mathbb{B} = \{\phi \in \mathbb{H} \mid \|\phi\|_* < \varepsilon^\sigma\}$. Fix $\phi \in \mathbb{B}$ and let the map $\mathbb{A}_\varepsilon$ be the unique map $h \to \phi$ given by Proposition 5.3. Define

$$\mathbb{G}_\varepsilon = \mathbb{A}_\varepsilon(\mathbb{E}_\varepsilon + \mathbb{M}_\varepsilon[\phi]).$$

We now show that $\mathbb{G}_\varepsilon$ is a contraction map. In fact, by Proposition 5.3, we have

$$\|\mathbb{G}_\varepsilon[\phi]\|_* \le C\|\mathbb{E}_\varepsilon + \mathbb{M}_\varepsilon[\phi]\|_* \le C\varepsilon + \varepsilon^{\sigma(1+\tau)} \le C\varepsilon \tag{7.10}$$

since $\sigma(1+\tau) > 1$, and hence $\mathbb{G}_\varepsilon[\phi] \in \mathbb{B}$. Moreover, we also have that

$$\|\mathbb{G}_\varepsilon[\phi_1] - \mathbb{G}_\varepsilon[\phi_2]\|_* \le C\|\mathbb{M}_\varepsilon[\phi_1] - \mathbb{M}_\varepsilon[\phi_2]\|_* \le C(\varepsilon + \|\phi_1\|_*^\tau + \|\phi\|_*^\tau)\|\phi_1 - \phi_2\|_*. \tag{7.11}$$

(7.10) and (7.11) show that the map $\mathbb{G}_\varepsilon$ is a contraction map from $\mathbb{B}$ to $\mathbb{B}$. By the contraction mapping theorem, (7.4) has a unique solution $\phi \in \mathbb{B}$, called $\phi_{\varepsilon,t}$.

The continuity of $(\phi_{\varepsilon,t}, \beta_\varepsilon(t))$ follows from the uniqueness of $(\phi_{\varepsilon,t}, \beta_\varepsilon(t))$.

□

8. The reduced problem

In this section we solve the reduced problem and establish our main existence result given by Theorem 2.1. In particular, we prove that

Proposition 8.1. *For ε sufficiently small, $\beta_\varepsilon(t)$ is continuous in t and we have*

$$\beta_\varepsilon(t) = b_0\varepsilon(M_R(t)) + O(\varepsilon^{\sigma(1+\tau)}), \tag{8.1}$$

for some generic constant $b_0 \neq 0$.

From the Proposition 8.1, we can finish the proof of Theorem 2.1.

Proof of Theorem 2.1:

Since $\varepsilon^{-1}\beta_\varepsilon(t) = b_0 M_R(t) + O(\varepsilon^{\sigma(1+\tau)-1})$, for ε sufficiently small, we also have $\beta_\varepsilon(r_1)\beta_\varepsilon(r_2) < 0$, using assumption (2.13). By the continuity of $\beta_\varepsilon(t)$ and the mean value theorem, a zero of β_ε, denoted by t_ε, is thus guaranteed, which produces a solution $u_\varepsilon = w_{\varepsilon,t_\varepsilon} + \phi_{\varepsilon,t_\varepsilon}$ to (7.1)-(7.2). It is easy to verify that u_ε satisfies all the properties of Theorem 2.1.

□

We now prove Proposition 8.1. Observe that $\phi_{\varepsilon,t}$ satisfies (7.4).

Multiplying the equation (7.4) by $w'_{\varepsilon,t}$ and integrating over $\mathbb{I}_{\varepsilon,t}$, we obtain

$$\beta_\varepsilon(t)\int_{\mathbb{I}_{\varepsilon,t}} Z_{\varepsilon,t}w'_{\varepsilon,t} = \int_{\mathbb{I}_{\varepsilon,t}} \mathbb{L}_{\varepsilon,t}[\phi_{\varepsilon,t}]w'_{\varepsilon,t} + \int_{\mathbb{I}_{\varepsilon,t}}(-\mathbb{E}_\varepsilon w'_{\varepsilon,t}) + \int_{\mathbb{I}_{\varepsilon,t}}(-\mathbb{M}_\varepsilon[\phi_{\varepsilon,t}]w'_{\varepsilon,t}). \tag{8.2}$$

The left hand of of (8.2) can be computed by (5.11):

$$\beta_\varepsilon(t)\int_{\mathbb{I}_{\varepsilon,t}} Z_{\varepsilon,t}w'_{\varepsilon,t} = -\beta_\varepsilon(t)\int_{\mathbb{R}}((p-1)w^{p-1}(w')^2) + O(\varepsilon\beta_\varepsilon(t)). \tag{8.3}$$

We estimate each term on the right hand side of (8.2). For the first term, we use integration by parts:

$$\int_{\mathbb{I}_{\varepsilon,t}} \mathbb{L}_{\varepsilon,t}[\phi_{\varepsilon,t}]w'_{\varepsilon,t} = \int_{\mathbb{I}_{\varepsilon,t}}\left[\phi''_{\varepsilon,t} - \phi_{\varepsilon,t} + pw^{p-1}_{\varepsilon,t}\phi_{\varepsilon,t}\right]w'_{\varepsilon,t}$$

$$= \int_{\mathbb{I}_{\varepsilon,t}}\left[w'''_{\varepsilon,t} - w'_{\varepsilon,t} + pw^{p-1}_{\varepsilon,t}w'_{\varepsilon,t}\right]\phi_{\varepsilon,t} = O(\varepsilon^{1+\sigma}).$$

The last term in (8.2) gives, using (7.9),

$$\int_{\mathbb{I}_{\varepsilon,t}} \mathbb{M}_{\varepsilon,t}[\phi_{\varepsilon,t}]w'_{\varepsilon,t} = O(\varepsilon^{1+\sigma} + \varepsilon^{\sigma(1+\tau)}) = O(\varepsilon^{\sigma(1+\tau)}). \tag{8.4}$$

It remains to compute the second term at the right hand side of (8.2): by (7.7)

$$\int_{\mathbb{I}_{\varepsilon,t}}(-E_\varepsilon)w'_{\varepsilon,t} = \int_{\mathbb{I}_{\varepsilon,t}}\frac{\varepsilon(N-1)}{t+\varepsilon y}(w'_{\varepsilon,t})^2$$

$$+\int_{\mathbb{I}_{\varepsilon,t}} w^p_{\varepsilon,t}\frac{1-(\mathbb{T}[w^m_{\varepsilon,t}])^q}{(\mathbb{T}[w^m_{\varepsilon,t}])^q}w'_{\varepsilon,t} + O(\varepsilon^2).$$

It is easy to see that

$$\int_{\mathbb{I}_{\varepsilon,t}}\frac{\varepsilon(N-1)}{t+\varepsilon y}(w'_{\varepsilon,t})^2 = \varepsilon\frac{N-1}{t}\int_{\mathbb{R}}(w')^2 + O(\varepsilon^2). \tag{8.5}$$

Then we estimate, using Lemma 6.1,

$$\int_{\mathbb{I}_{\varepsilon,t}} w'_{\varepsilon,t}w^p_{\varepsilon,t}\left(\frac{1-(\mathbb{T}[w^m_{\varepsilon,t}])^q}{(\mathbb{T}[w^m_{\varepsilon,t}])^q}\right) = -q\varepsilon\left(\int_{\mathbb{R}} w'w^p\rho(y) + O(\varepsilon)\right). \tag{8.6}$$

From the definition of $\rho(y)$ at (6.6) we have

$$\int_{\mathbb{R}} w^m\int_{\mathbb{R}} w'w^p\rho(y) = \frac{J'_{2,R}(t)}{J_{2,R}(t)}\int_{\mathbb{R}} w'w^p\left(y\int_{-\infty}^y w^m + \int_{-\infty}^y zw^m(z)\right)$$

$$+\frac{J_1'(t)}{J_1(t)}\int_{\mathbb{R}} w^{'}w^p\left(y\int_y^\infty w^m+\int_y^\infty zw^m(z)\right)$$

$$=-\frac{1}{p+1}\frac{J_{2,R}'(t)}{J_{2,R}(t)}\int_{\mathbb{R}} w^{p+1}(y)\int_{-\infty}^y w^m-\frac{1}{p+1}\frac{J_1'(t)}{J_1(t)}\int_{\mathbb{R}} w^{p+1}\int_y^\infty w^m$$

$$=-\frac{1}{2(p+1)}\left(\frac{J_{2,R}'(t)}{J_{2,R}(t)}+\frac{J_1'(t)}{J_1(t)}\right)\int_{\mathbb{R}} w^{p+1}(y)\int_R w^m \tag{8.7}$$

where we have used (4.2) of Lemma 4.1.

Combining all together, we arrive at

$$-\beta_\varepsilon(t)\int_{\mathbb{R}}((p-1)w^{p-1}(w^{'})^2)=\varepsilon\frac{N-1}{t}\int_{\mathbb{R}}(w^{'})^2$$

$$+\varepsilon\frac{q}{2(p+1)}\int_{\mathbb{R}} w^{p+1}(\frac{J_{2,R}'(t)}{J_{2,R}(t)}+\frac{J_1'(t)}{J_1(t)})+O(\varepsilon^{\sigma(1+\tau)})$$

$$=\frac{q\varepsilon}{p-1}\int_{\mathbb{R}}(w^{'})^2\left[M_R(t)+O(\varepsilon^{\sigma(1+\tau)-1})\right]$$

using (4.1) of Lemma 4.1. Setting

$$b_0=-\frac{q\int_{\mathbb{R}}(w^{'})^2}{(p-1)^2\int_{\mathbb{R}} w^{p-1}(w^{'})^2},$$

we obtain (8.1).

□

9. The case of $\mathbb{R}^N$: Proofs of Theorem 2.3 and Corollary 2.4

In this section, we discuss the whole $\mathbb{R}^N$ case and prove Theorem 2.3 and Corollary 2.4.

Theorem 2.3 can be proved along the line of the proof of Theorem 2.1. First we replace $J_{2,R}$ by J_2, $G_R(r;r^{'})$ by $G(r;r^{'})$, and $M_R(t)$ by $M(t)$.

In this case, we still have the following representation formula:

$$\mathbb{T}[h](r^{'})=c_{\varepsilon,t}\int_0^\infty G(r^{'};r)h(r)dr, \tag{9.1}$$

where $G(r^{'};r)=\lim_{R\to+\infty}G_R(r^{'};r)$.

Second, we need to modify the statements of Lemma 6.1:

Lemma 9.1. *(1) For $|r^{'}-t|<\frac{t}{4}, r^{'}=t+\varepsilon y$, we have*

$$\mathbb{T}[(w_{\varepsilon,t}+\phi)^m](t+\varepsilon y)=1+\varepsilon\rho(y)+\frac{m}{\int_{\mathbb{I}_{\varepsilon,t}}w^m}\int_{\mathbb{I}_{\varepsilon,t}}w_{\varepsilon,t}^{m-1}\phi \tag{9.2}$$

$$+O(\varepsilon^2(1+|y|^2)+\varepsilon\|\phi\|_*+\|\phi\|_*^{1+\tau}).$$

(2) For $|r^{'}-t|\geq\frac{t}{4}$, we then have

$$\mathbb{T}[(w_{\varepsilon,t}+\phi)^m](r^{'})\geq Ce^{-2|r^{'}-t|}. \tag{9.3}$$

Proof: The proof of (9.2) is the same as before. We just need to show (9.3): In fact, from (2.11), we have that

$$\mathbb{T}[(w_{\varepsilon,t}+\phi)^m](r^{'})$$

$$=J_2(r^{'})c_{\varepsilon,t}\int_0^{r^{'}}c_0r^{N-1}J_1(r)(w_{\varepsilon,t}+\phi)^m dr+J_1(r^{'})c_{\varepsilon,t}\int_{r^{'}}^{\infty}c_0r^{N-1}J_2(r)(w_{\varepsilon,t}+\phi)^m dr$$

$$\geq C_1J_2(r^{'})-C_2J_1(r^{'})e^{-\frac{m\mu r^{'}}{2\varepsilon}}\geq CJ_2(r^{'})\geq Ce^{-2r^{'}}\geq Ce^{-2|r^{'}-t|}.$$

□

As a consequence of (9.3), for $|r^{'}-t|\geq\frac{t}{4}$, we have

$$\frac{(w_{\varepsilon,t}+\phi)^p}{(\mathbb{T}[(w_{\varepsilon,t}+\phi)^m])^q}\leq Ce^{-(p\mu-2\varepsilon)\frac{|r^{'}-t|}{\varepsilon}}\leq e^{-\mu\frac{|r^{'}-t|}{\varepsilon}}. \tag{9.4}$$

Once we have (9.4), the rest of the proof is exactly the same as those of Theorem 2.1. This proves Theorem 2.3.

Next we prove Corollary 2.4. To this end, we have to study the asymptotic behavior of J_1 and J_2 both at 0 and at ∞. We then have

Lemma 9.2. *For t large, we have*

$$J_1(t)=A_0t^{-\frac{N-1}{2}}e^t\left(1-\frac{(N-1)(N-3)}{8t}+O(\frac{1}{t^2})\right), \tag{9.5}$$

$$J_1^{'}(t)=A_0t^{-\frac{N-1}{2}}e^t\left(1-\frac{(N-1)(N+1)}{8t}+O(\frac{1}{t^2})\right), \tag{9.6}$$

$$J_2(t)=B_0t^{-\frac{N-1}{2}}e^{-t}\left(1+\frac{(N-1)(N-3)}{8t}+O(\frac{1}{t^2})\right), \tag{9.7}$$

$$J_2'(t) = B_0 t^{-\frac{N-1}{2}} e^t \left(1 - \frac{(N-1)(N+1)}{8t} + O(\frac{1}{t^2}) \right), \tag{9.8}$$

where A_0, B_0 are generic positive constants.

Proof: The proof follows from (2.3) and formulas 9.7.1-9.7.4 on page 378 of [5]. □

From (9.5), (9.6), (9.7) and (9.8), we see that for t large,

$$\frac{J_1'}{J_1} + \frac{J_2'}{J_2} = \frac{(J_1J_2)'}{J_1J_2} = -\frac{1}{t}\left[(N-1) + O(\frac{1}{t})\right].$$

and hence for t large

$$M(t) = \frac{(p-1)(N-1)}{qt} + \frac{J_1'}{J_1} + \frac{J_2'}{J_2}$$

$$= (\frac{p-1}{q} - 1)\frac{N-1}{t} + O(\frac{1}{t^2}). \tag{9.9}$$

So for $N \geq 2$ and $p < q+1$, we have

$$M(t) < 0 \text{ for } t \text{ large.} \tag{9.10}$$

On the other hand, for t near 0 and $N \geq 3$, it is easy to see that

$$J_1(t) = O(1), J_2(t) \sim t^{-(N-2)}. \tag{9.11}$$

and hence for t small

$$M(t) = \frac{(p-1)(N-1)}{qt} + \frac{J_1'}{J_1} + \frac{J_2'}{J_2} = \left[\frac{(p-1)}{q}(N-1) - (N-2)\right]\frac{1}{t} + O(1) > 0 \tag{9.12}$$

if $p > \frac{N-2}{N-1}q + 1$. For $N = 2$, t small, $J_2(t) \sim \log\frac{1}{t}$, $M(t) = \frac{(p-1)(N-1)}{qt} + O(1) + o(\frac{1}{t}) > 0$ if $p > 1$.

Now from (9.10) and (9.12), we apply Theorem 2.3 for $N \geq 2$ to conclude Corollary 2.4.

10. The proof of Corollary 2.2 and The case of $s > 0$

In this last section, we discuss the proof of Corollary 2.2 and the case of $s > 0$.

To prove Corollary 2.2, we first observe that

$$M_R(R) = \frac{(p-1)(N-1)}{qR} + \frac{J_1'(R)}{J_1(R)} > 0 \tag{10.1}$$

since $J_{2,R}'(R) = 0$ and $J_1(r)$ is strictly increasing in r.

On the other hand, for t fixed, we have

$$\lim_{R\to+\infty} M_R(t) = M(t). \tag{10.2}$$

From the proof of Corollary 2.4, we see that there exist $r_1 < r_2$ such that $M(r_1) > 0,\ M(r_2) < 0$. Thus for R large, $M_R(r_1) > 0,\ M_R(r_2) < 0$. Combining with (10.1), we see that there exist $r_1 < r_2 < r_3 < R$ such that $M_R(r_1)M_R(r_2) < 0,\ M_R(r_2)M_R(r_3) < 0$. Corollary 2.2 follows from Theorem 2.1.

Finally, we discuss how to modify the proof of Theorem 2.1 in the case of $s = 0$ to establish the case of $s > 0$.

First, by using upper and lower solution method, for any given $h = h(r) \in L^\infty(B_R) \cap L^2(B_R), h \geq 0$, there exists a unique solution, denoted by $\mathbb{T}_1[h]$, to the following nonlinear problem

$$\Delta v - v + \frac{h}{v^s} = 0 \text{ in } B_R,\ v(x) = v(|x|) \text{ and } \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial B_R. \tag{10.3}$$

In fact, let v_0 be the unique solution of (10.3) with $s = 0$. Then for ρ such that $\rho^{s+1} \max_{0 \leq r \leq R} v_0^s(r) < 1$, $v_1 = \rho v_0$ is a lower-solution to (10.3). On the other hand, $v_2 = (h_{max})^{\frac{1}{s+1}}$, where $h_{max} = \max_{0 \leq r \leq R} h(r)$, is an upper solution to (10.3). By choosing ρ small, we can have $v_1 < v_2$ and hence a solution to (10.3) between v_1 and v_2 is established. Note also that the solution to (10.3) is also unique, due to the fact that $s > 0$.

Now we let $c_{\varepsilon,t}$ be such that

$$c_{\varepsilon,t} = (\mathbb{T}_1[w_{\varepsilon,t}^m](t))^{-s-1} \tag{10.4}$$

and denote $\mathbb{T}[h]$ as the unique solution of

$$\Delta v - v + c_{\varepsilon,t}\frac{h}{v^s} = 0 \text{ in } B_R,\ v(x) = v(|x|) \text{ and } \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial B_R. \tag{10.5}$$

The rest of the proof goes exactly as in the case of $s = 0$ with only minor changes: for example, the linearized operator should become

$$\mathbb{L}_{\varepsilon,t}[\phi] := \phi'' + \frac{\varepsilon(N-1)}{t+\varepsilon y}\phi' - \phi + p w_{\varepsilon,t}^{p-1}\phi - \frac{qm}{s+1}\frac{\int_{\mathbb{I}_{\varepsilon,t}} w_{\varepsilon,t}^{m-1}\phi}{\int_{\mathbb{I}_{\varepsilon,t}} w_{\varepsilon,t}^m} w_{\varepsilon,t}^p,$$

and the estimate (6.10) should be modified as

(10.6)
$$\mathbb{T}[(w_{\varepsilon,t}+\phi)^m](t+\varepsilon y) = 1 + \varepsilon\rho(y) + \frac{m}{(s+1)\int_{\mathbb{I}_{\varepsilon,t}} w^m}\int_{\mathbb{I}_{\varepsilon,t}} w_{\varepsilon,t}^{m-1}\phi + O(\varepsilon^2(1+|y|^2) + \varepsilon\|\phi\|_* + \|\phi\|_*^{1+\tau}).$$

We omit the details.

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