GEOMETRY DRIVEN TYPE II HIGHER DIMENSIONAL BLOW-UP FOR THE CRITICAL HEAT EQUATION

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ABSTRACT. We consider the problem

$$
\begin{aligned}
&v_t = \Delta v + |v|^{p-1}v \quad \text{in } \ \Omega \times (0,T),\\
&v = 0 \quad \text{on } \partial\Omega \times (0,T),\\
&v > 0 \quad \text{in } \ \Omega \times (0,T).
\end{aligned}
$$

In a domain $\Omega \subset \mathbb{R}^d$, $d \geq 7$ enjoying special symmetries, we find the first example of a solution with type II blow-up for a power p less than the Joseph-Lundgren exponent

$$
p_{JL}(d) = \begin{cases} \infty, & \text{if } 3 \leq d \leq 10,\\ 1 + \frac{4}{d-4-2\sqrt{d-1}}, & \text{if } d \geq 11. \end{cases}
$$

No type II radial blow-up is present for $p < p_{JL}(d)$. We take $p = \frac{d+1}{d-3}$, the Sobolev critical exponent in one dimension less. The solution blows up on circle contained in a negatively curved part of the boundary in the form of a sharply scaled Aubin-Talenti bubble, approaching its energy density a Dirac measure for the curve. This is a completely new phenomenon for a diffusion setting.

1. Introduction

Perhaps the most studied model of singularity formation or blow-up in nonlinear parabolic problems is the semilinear heat equation

$$
\begin{aligned}
&v_t = \Delta v + |v|^{p-1}v \quad \text{in } \ \Omega \times (0,T), \qquad (1.1)\\
&v = 0 \quad \text{on } \partial\Omega \times (0,T),\\
&v > 0 \quad \text{in } \ \Omega \times (0,T).
\end{aligned}
$$

where $p > 1$. Here Ω be a smooth domain in $\mathbb{R}^d$ (or entire space) and $0 < T \leq +\infty$. A smooth solution $u(x,t)$ of Problem (1.1) is said to blow-up at time T if

$$
\lim_{t\to T} \|v(\cdot,t)\|_{L^\infty(\Omega)} = +\infty.
$$

The key issue in the study of blow-up phenomena is to understand how and where explosion can take place. The blow-up is said to be of type I if we have that

$$
\limsup_{t\to T} (T-t)^{\frac{1}{p-1}} \|v(\cdot,t)\|_{L^\infty(\Omega)} < +\infty \qquad (1.2)
$$

and of type II if

$$
\limsup_{t\to T} (T-t)^{\frac{1}{p-1}} \|v(\cdot,t)\|_{L^\infty(\Omega)} = +\infty. \qquad (1.3)
$$

Type I means that the blow-up takes place like that of the ODE $v_t = v^p$, so that in the explosion mechanism the nonlinearity plays the dominant role. In a related interpretation, the blow-up "respects" the natural scalings of the problem. The second alternative is rare and far less understood. The delicate interplay of diffusion, nonlinearity and geometry of the domain is responsible for that scenario.

The role of the Sobolev critical exponent

$$p_S(d) = \begin{cases} \infty, & \text{if } 1 \le d \le 2, \\ \frac{d+2}{d-2}, & \text{if } d \ge 3. \end{cases}$$

is well-known to be central in the possible types of blow-up for (1.1).

When $1 < p < p_S(d)$ solutions can only have type I blow-up, as it was first established by Giga and Kohn [22] for the case of Ω convex, and in [34] in for a general domain. This is also the case for $p = p_S(d)$ and radial solutions of (1.1) [14], or if Ω is star-shaped [2]. Type II blow-up radial sign-changing solutions exists for $p = p_S(4)$ [14, 37].

Refined asymptotics of Type I blow-up together with constructions and classification results have been obtained in many works, we refer the reader to [6, 23, 24, 30, 32, 35, 39] and references therein.

Type I is expected to be in any reasonable sense the "generic" way in which blow-up takes place for any $p > 1$, see [5, 14, 30, 29].

Type II blow-up solutions are much harder to be detected. The only examples known are for $d \ge 11$ and $p > p_{JL}(d)$ where $p_{JL}(d)$ is the *Joseph-Lundgren exponent* [28] defined as

$$p_{JL}(d) = \begin{cases} \infty, & \text{if } 3 \le d \le 10, \\ 1 + \frac{4}{d-4-2\sqrt{d-1}}, & \text{if } d \ge 11. \end{cases}$$

Herrero and Velázquez [26, 27] found a radial solution that blows-up with type II rate. The local profile locally resembles a time-dependent, asymptotically singular scaling of a positive radial solution of

$$\Delta w + w^p = 0 \quad \text{in } \mathbb{R}^d. \tag{1.4}$$

See also [33] for the case of a ball, and [3] for an arbitrary domain with the same profile profile when p is in addition an odd integer. A main ingredient in the constructions is the stability of radial solutions of (1.4) whenever $p > p_{JL}(d)$ [25]. No positive solution (radial or not) is stable for $p \le p_{JL}(d)$ [13].

In [29] Matano and Merle proved that in the radially symmetric case no Type II blow-up can take place if $p_S < p \le p_{JL}(d)$, a result that precisely complements that for the Herrero-Velazquez range. Recently in [5] an entire finite energy, axially symmetric type II blow-up solution with a singular set exactly being the symmetry axis was built for $d \ge 12$ and $p > p_{JL}(d-1) > p_{JL}(d)$.

A question that has remained conspicuously open for many years is whether or not type II blow-up solutions of (1.1) can exist in the Matano-Merle range $p_S(d) < p < p_{JL}(d)$. Such solutions must of course be non-radial. In this paper we prove that the answer is **yes** in dimension $d \ge 7$ and $p = \frac{d+1}{d-3} = p_S(d-1)$ in a class of domains with axial symmetry.

Let us identify $\mathbb{R}^d \simeq \mathbb{C} \times \mathbb{R}^{d-2}$ and consider the orthogonal transformations

$$Q_\alpha(z) = (e^{i\alpha}(z_1 + iz_2), z), \quad \pi_i(z, x') = (z_1 + iz_2, z_3, \dots, -z_i, \dots z_d). \tag{1.5}$$

We assume that Ω is a smooth, bounded domain with $0 \notin \Omega$, that is invariant under this transformations:

$$Q_\alpha(\Omega) = \Omega, \quad \pi_i(\Omega) = \Omega \quad \text{for all} \quad \alpha \in \mathbb{R}, \quad i = 3, \dots, d.$$

In other words Ω is a radial domain in the first two coordinates, even in the remaining ones which does not contain the origin. In Ω we consider the problem

$$\begin{aligned} &v_t = \Delta v + v^{\frac{d+1}{d-3}} \quad \text{in} \quad \Omega \times (0,T), \\ &v = 0 \quad \text{on } \partial\Omega \times (0,T), \\ &v > 0 \quad \text{in} \quad \Omega \times (0,T). \end{aligned} \tag{1.6}$$

Let $m = \inf\{|z_1 + iz_2| \ / \ (z_1 + iz_2, 0) \in \Omega\} > 0$ so that the curve

$$\Gamma := \{(z_1 + iz_2, 0) \ / \ |z_1 + iz_2| = m\}$$

is a circle contained in $\partial\Omega$.

Theorem 1. *Let $d \geq 7$ and Ω a domain as described above. For any sufficiently small $T > 0$, there exists a smooth solution $v(z,t)$ of problem* (1.6) *that remains uniformly bounded outside any neighborhood of the curve Γ while*

$$\lim_{t \to T} (T - t)^\gamma \|v(\cdot, t)\|_{L^\infty(\Omega)} > 0, \quad \gamma = \frac{(d-3)(d-4)}{2(d-5)}.$$

We notice that for $p = \frac{d+1}{d-3}$ we have $\frac{1}{p-1} = \frac{d-3}{4} < \gamma$ so that v exhibits type II blow-up.

The construction provides very accurate information on the solution. The principle is very simple We let $n = d - 1$ and consider the standard Aubin-Talenti function [38]

$$U(y) = \alpha_n \left(\frac{1}{1 + |y|^2} \right)^{\frac{n-2}{2}}, \quad \alpha_n = (n(n-2))^{\frac{1}{n-2}}, \tag{1.7}$$

which is a positive solution of $\Delta U + U^{\frac{n+2}{n-2}} = 0$. The solution has the form

$$v(z,t) \sim \frac{1}{\lambda(t)^{\frac{n-2}{2}}} U\left(\frac{x}{\lambda(t)} \right)$$

where x is the vector joining z and its closest point to a circle of the form $(1+d(t))\Gamma$ contained in the domain, with $\lambda(t) \to 0$ and $d(t) \to 0$ as $t \to T$. We have that the energy density $|\nabla u(z,t)|^2$ concentrates in the form of a Dirac mass for the curve Γ, a phenomenon usually called bubbling. Bubbling at points triggered by criticality is a feature known in several different contexts, including dispersive equations and geometric flows. There is a broad literature on that matter. We refer the reader for instance to [1, 7, 8, 9, 12, 23, 19, 20, 21, 31, 36] and their references.

The phenomenon of higher dimensional boundary bubbling here discovered is definitely triggered by geometry and is entirely new in the diffusion setting. It is worth mentioning that similar blow-up triggered by geometry of the boundary under axial symmetry has been numerically conjectured to hold for the three dimensional Euler equation in [18].

In an elliptic context, a result with resemblance to the current one was found in [11], methodologically connected with [10]. In fact we conjecture that a construction like the one here should be possible along a negatively curved closed geodesic of the boundary. We believe that geometry is essential and that in a convex domain the Matano-Merle range of exponents may still lead to non-existence of type II blow-up.

The 2-variable radial symmetry of the domain leads us to look for a solution with the same symmetry of a problem in a domain with one dimension less at the critical exponent where point bubbling is obtained. This problem is methodologically challenging. For instance the method in the construction in [37] of point type II blow up in the radial sign-changing context in dimension 4, based on the pioneering work by Merle, Raphael, Rodnianski [31], later applied to various blow-up problems, does not seem not apply here. See [3] for a difficult adaptation of that technique for a related problem in a non-radial setting, yet only valid for odd integer powers, which is never our case.

We close this introduction by mentioning that our proof applies equally well to an exterior domain of the same nature. Besides, within the symmetry class the phenomenon we obtain is codimension 1-stable (presumably highly unstable outside symmetry). We shall not elaborate in that issue, which is a rather direct consequence of our construction.

We devote the rest of this paper to the proof of Theorem 1.

2. Scheme of the proof

Let $d = n + 1$, and consider the change of variables

$$v(z_1, \dots, z_d, t) = u(\sqrt{z_1^2 + z_2^2}, z_3, \dots, z_d, t)$$

for some $u = u(x_1, \dots, x_n, t)$. In terms of u, solving (1.6) in the class of functions v that are invariant under the orthogonal transformations (1.5) translates into solving

$$\begin{aligned} &u_t = \Delta u + \frac{1}{x_1}\frac{\partial u}{\partial x_1} + u^p \text{ in } \quad \mathcal{D} \times (0, T), \quad p = \frac{n+2}{n-2} \\ &u > 0 \quad \text{in} \quad \mathcal{D} \times (0, T), \quad u = 0 \quad \text{on} \quad \partial\mathcal{D} \times (0, T), \end{aligned} \tag{2.1}$$

with $u(x_1, \dots, x_j, \dots, x_n) = u(x_1, \dots, -x_j, \dots, x_n)$ for any $j = 2, \dots, n$. Here $\mathcal{D}$ is the smooth bounded domain in $\mathbb{R}^n$ defined as

$$\Omega = \{(z_1, \dots, z_d) \in \mathbb{R}^d \, : \, (\sqrt{z_1^2 + z_2^2}, z_3, \dots, z_d) \in \mathcal{D}\},$$

with the properties

$$(x_1, \dots, x_j, \dots, x_n) \in \mathcal{D} \quad \Longleftrightarrow \quad (x_1, \dots, -x_j, \dots, x_n) \in \mathcal{D}, \quad j = 2, \dots, n,$$

and

$$\mathcal{D} \cup \{(x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} \, : \, \bar{x} = \bar{0}\} \neq \emptyset.$$

With no loss of generality, we assume that

$$\inf\{r > 0 \, : \, (r, \bar{0}) \in \mathcal{D}\} = 1.$$

The result contained in Theorem 1 is expressed in terms of Problem (2.1) in the following

Theorem 2. *For any T small enough, there exists a finite time blow-up solution to Problem* (2.1) *of the form*

$$u(x,t)=\lambda^{-\frac{n-2}{2}}(t)\left[U\left(\frac{x-\xi(t)}{\lambda(t)}\right)-U\left(\frac{x-\hat\xi(t)}{\lambda(t)}\right)\right](1+o(1))+O(1)$$

where $O(1)$ is uniformly bounded and $o(1)\to 0$ uniformly as $t\to T$. Here U is defined in (1.7)*, and*

$$\xi(t)=(1+d(t))\,\mathbf{e}_1,\quad \hat\xi=(1-d(t))\,\mathbf{e}_1,\quad \textit{where}\quad \mathbf{e}_1=(1,\bar 0) \tag{2.2}$$

with

$$d(t)=(T-t)\,(1+o(1))\,,\ \lambda=(T-t)^{\frac{n-3}{n-4}}\,(1+o(1))\,,\quad \textit{as}\quad t\to T$$

We find the solution to Problem (2.1) as predicted by Theorem 2 by constructing a sufficiently accurate approximation, and then an actual solution to the Problem as a *small* perturbation which is subtle to use in particular by the structure instability of the problem. Our solution has the form

$$u(x,t)=W_2(x,t)+\mathbf{w}(x,t) \tag{2.3}$$

where W_2 is an explicit approximation whose expression encodes the predicted asymptotic behavior as $t\to T$. Here $\mathbf{w}$ is a small correction in some appropriate topology.

In the rest of this section we describe $W_2(x,t)$ and the method of construction of an actual solution near W_2 which we call *the inner-outer gluing method.*

Construction of the approximation $W_2(x,t)$. We introduce two scalar functions $d,\ \lambda:(0,T)\to\mathbb{R}$, expressed respectively as

$$d(t)=d_0(t)+d_1(t),\quad \lambda(t)=\lambda_0(t)+\lambda_1(t), \tag{2.4}$$

where d_0 and λ_0 are explicitly given by

$$d_0(t)=(T-t),\quad \lambda_0(t)=\ell(T-t)^{1+\frac{1}{n-4}} \tag{2.5}$$

with ℓ a positive constant that we will define later. The functions λ_1 and d_1 are thought as parameter functions to be determined. For the moment, we assume that λ_1 and d_1 are controlled by λ_0 and d_0 in the whole interval $(0,T)$, in the following sense. For any scalar function $h(t)$, $t\in(0,T)$, and any real number δ, $\|h\|_\delta$ stands for the weighted L^∞-norm defined as

$$\|h\|_\delta=\|(T-t)^{-\delta}h(t)\|_{L^\infty(0,T)}. \tag{2.6}$$

We assume that

$$\begin{aligned}&\lambda_1(t):=\int_t^T\dot\lambda_1(s)\,ds,\quad d_1(t):=\int_t^T\dot d_1(s)\,ds,\quad \text{with}\\ &\qquad\qquad \|\dot d_1\|_{\frac{1+\sigma}{n-4}}+\|\dot\lambda_1\|_{\frac{1+\sigma}{n-4}}\lesssim 1,\quad \text{for some}\quad \sigma\in(\frac12,1).\end{aligned} \tag{2.7}$$

In fact, we can think at σ as close to 1. The final time $T>0$ will be chosen to be small enough so that $d(t)>0$, and $\lambda(t)>0$, in the whole interval $t\in(0,T)$.

As in the statement of Theorem 2, we proceed with the first step in the construction of W_2 in (2.3) and we introduce

$$W_1[\lambda_1,d_1](x,t)=W_0(x,t)-\bar W_0(x,t) \tag{2.8}$$

where

$$W_0(x,t) = \lambda^{-\frac{n-2}{2}}(t) U\left(\frac{x-\xi(t)}{\lambda(t)}\right), \quad \bar W_0(x,t) = \lambda^{-\frac{n-2}{2}}(t) U\left(\frac{x-\hat\xi(t)}{\lambda(t)}\right),$$

with U given by (1.7), and the points ξ, $\hat\xi$ described in (2.2). We recall that U solves

$$\Delta u + u^{\frac{n+2}{n-2}} = 0, \quad \text{in} \quad \mathbb{R}^n. \tag{2.9}$$

Since $d(t) > 0$ for all $t \in (0,T)$, we see that $\xi(t) \in \mathcal{D}$, $\hat\xi(t) \notin \mathcal{D}$, for any $t \in (0,T)$. In fact, since $(1,\bar 0) \in \partial\mathcal{D}$, the point $\hat\xi$ is the reflection of ξ through the boundary. In other words, $\frac{\xi(t)+\hat\xi(t)}{2} = (1,\bar 0)$. The radial symmetry of U implies that $W[\lambda_1, d_1](1,\bar 0) = 0$, for any possible election of λ_1 and d_1.

The way to establish whether W_1 is a good approximation is to measure the size of the error $S[W_1](x,t)$, where

$$S[u](x,t) = -u_t + \Delta u + \frac{1}{x_1}\frac{\partial u}{\partial x_1} + u^p.$$

Formally, one sees that, locally around a small neighborhood of ξ, the error $S[W_1]$ looks like, in the expanded variable $y = \frac{x-\xi}{\lambda}$,

$$\begin{aligned} S[W_1](\xi+\lambda y, t) \sim \lambda^{-\frac{n}{2}} &\left[\dot d + \frac{1}{1+d+\lambda y_1}\right] Z_1(y) \\ &+ \lambda^{-\frac{n}{2}}\dot\lambda Z_0(y) - \frac{p\alpha_n}{2^{n-2}}\lambda^{-\frac{n+2}{2}}\left(\frac{\lambda}{d}\right)^{n-2} U^{p-1}(y). \end{aligned} \tag{2.10}$$

We refer to (1.7) for the definition of the constant α_n and to (3.1) for the precise expression of $S[W_1]$.

The functions Z_1 and Z_0 that appear in (7.8) are

$$Z_0(y) = \frac{n-2}{s} U(y) + \nabla U(y)\cdot y, \quad Z_1(y) = \frac{\partial U}{\partial y_1}(y), \tag{2.11}$$

and they are the only bounded solutions to the linearized equation of (2.9) around U

$$L_0(\phi) := \Delta\phi + pU^{p-1}\phi = 0, \quad \text{in} \quad \mathbb{R}^n \tag{2.12}$$

in the class of functions that are even in the variable y_j, for any $j = 2,\dots,n$.

The definition of $d_0 = d_0(t)$ in (2.4)-(2.5) makes the biggest part of the function inside brackets in the first term in (2.10) at the point $y=0$ equals to zero, since

$$\dot d_0(t) + 1 = 0, \quad t \in (0,T), \quad d_0(T) = 0.$$

With this choice of d_0, the definition of λ_0 in (2.4)-(2.5) makes the integration of the second and third terms in (2.10) against Z_0 in $\mathbb{R}^n$ equals to zero. Indeed, λ_0 is the solution to the ordinary differential equation

$$\lambda_0\dot\lambda_0\left(\int_{\mathbb{R}^n} Z_0^2(y)\,dy\right) - \frac{p\,\alpha_n}{2^{n-2}}\left(\frac{\lambda_0}{d_0}\right)^{n-2}\left(\int_{\mathbb{R}^n} U^{p-1}(y)\,Z_0(y)\,dy\right) = 0, \tag{2.13}$$

with $\lambda_0(T) = 0$, provided the number ℓ in (2.5) is given by

$$\ell = \left[\frac{n-3}{n-4}\frac{2^{n-1}}{\alpha_n(n-2)}\frac{\int_{\mathbb{R}^n} Z_0^2(y)\,dy}{\int_{\mathbb{R}^n} U^p(y)\,dy}\right]^{\frac{1}{n-4}}, \tag{2.14}$$

since $-p\int_{\mathbb{R}^n} U^{p-1}(y)Z_0(y)\,dy = \frac{n-2}{2}\int_{\mathbb{R}^n} U^p(y)\,dy$.

A rigorous description of the error $S[W_1]$ in a region close to ξ is contained in Lemma 3.1. The main part of $S[W_1]$ turns out to be an explicit function of (x,t), independent of d_1 and λ_1. It is thus easy to correct W_1 to cancel the biggest part of the error, so we end up with a final approximation we called $W_2 = W_2[\lambda_1, d_1]$. The description of the second error $S[W_2]$ in a region close to the point ξ is contained in Lemma 3.2, while the description of part of the error $S[W_2]$ far from ξ is estimated in Lemma 3.3. In Lemma 3.3, we also provide a description of W_2 on the boundary $\partial\Omega$, which unfortunately is not identically zero. The correction of the boundary term and the construction of an actual solution to the equation is done in the second step of our argument, through the *inner-outer gluing method*.

Inner-outer gluing method. This method is a procedure to find the function $\mathbf{w}$ in (2.3). We expect that the function $\mathbf{w}$ corrects the approximation W_2 in a region far from the point ξ, adjusting of course the boundary conditions, and at the same time in a region close to ξ.

To organize this double role for $\mathbf{w}$, we introduce a smooth cut-off function η with $\eta(s) = 1$ for $s < 1$ and $= 0$ for $s > 2$, and we define

$$\eta_R(x,t) = \eta_R\left(\frac{x-\xi}{R\lambda_0}\right). \tag{2.15}$$

The radius R is independent of t and T, and we fix it arbitrarily large. We write

$$\mathbf{w}(x,t) = \psi(x,t) + \eta_R(x,t)\Phi(x,t). \tag{2.16}$$

In this decomposition, the term ψ is mainly influenced from the region far from ξ, while Φ reflects what is going on close to ξ.

In order that $u = u(x,t)$ defined in (2.3) is an actual solution to problem (2.1), the function $\mathbf{w}$ has to satisfy

$$\begin{aligned} &\mathbf{w}_t = \Delta\mathbf{w} + pW_2^{p-1}\mathbf{w} + \frac{1}{x_1}\frac{\partial\mathbf{w}}{\partial x_1} + S[W_2](x,t) + N(\mathbf{w}), \quad (x,t) \in \mathcal{D}\times(0,T)\\ &\mathbf{w} = -W_2, \quad (x,t) \in \partial\mathcal{D}\times(0,T). \end{aligned} \tag{2.17}$$

where

$$N(\mathbf{w}) = (W_2+\mathbf{w})^p - W_2^p - pW_2^{p-1}\mathbf{w}. \tag{2.18}$$

Thanks to (2.16), we proceed to decompose problem (2.17) into an *outer* and a *inner* problem.

Let $R' > R$ so that $\eta_{R'}\eta_R = \eta_R$. The equation in (2.17) is written explicitly in terms of ψ and Φ as follows

$$
\begin{aligned}
\psi_t + \underline{\eta_R \Phi_t} + (\eta_R)_t \Phi &= \Delta\psi + \frac{1}{x_1}\frac{\partial\psi}{\partial x_1} + \underline{\eta_R(\Delta\Phi + \frac{1}{x_1}\frac{\partial\Phi}{\partial x_1})} \\
&+ 2\nabla\eta_R\nabla\Phi + \Delta\eta_R\Phi + \frac{1}{x_1}\frac{\partial\eta_R}{\partial x_1}\Phi \\
&+ \underline{p\left[\lambda_0^{-\frac{n-2}{2}}U(\frac{x-\xi}{\lambda_0})\right]^{p-1}\eta_{R'}\eta_R(\psi+\Phi)} \\
&+ p\left[\lambda_0^{-\frac{n-2}{2}}U(\frac{x-\xi}{\lambda_0})\right]^{p-1}\eta_{R'}(1-\eta_R)\psi \\
&+ p\left[W_2^{p-1} - [\lambda_0^{-\frac{n-2}{2}}U(\frac{x-\xi}{\lambda_0})]^{p-1}\right]\eta_{R'}(\psi+\eta_R\Phi) \\
&+ pW_2^{p-1}(1-\eta_{R'})(\psi+\eta_R\Phi) + N[\mathbf{w}] + \underbrace{\bar{E}_2 + \underline{E_2\eta_R}}_{:=S[W_2]}.
\end{aligned}
$$

Here we have decomposed the error $S[W_2]$ into its principal part E_2 multiplied by the cut off η_R,

$$
S[W_2] = E_2\eta_R + \bar{E}_2 \tag{2.19}
$$

leaving all the rest into a term named $\bar{E}_2$. Observe that the terms which are underlined all go with the cut off function η_R in front. Define

$$
\begin{aligned}
V(x,t) &= p\left[\lambda_0^{-\frac{n-2}{2}}U(\frac{x-\xi}{\lambda_0})\right]^{p-1}\eta_{R'}(1-\eta_R) \\
&+ p\left[W_2^{p-1} - [\lambda_0^{-\frac{n-2}{2}}U(\frac{x-\xi}{\lambda_0})]^{p-1}\right]\eta_{R'} \\
&+ pW_2^{p-1}(1-\eta_{R'}).
\end{aligned} \tag{2.20}
$$

Then we observe that $\mathbf{w}$ defined in (2.16) solves (2.17) if the pair (ψ,Φ) solve the following system of coupled equations

$$
\begin{aligned}
\psi_t &= \Delta\psi + \frac{1}{x_1}\frac{\partial\psi}{\partial x_1} + V\psi + \left(\Delta - \frac{\partial}{\partial t}\right)\eta_R\,\Phi + 2\nabla\Phi\nabla\eta_R + \frac{1}{x_1}\frac{\partial\eta_R}{\partial x_1}\Phi \\
&+ p\left[W_2^{p-1} - [\lambda_0^{-\frac{n-2}{2}}U(\frac{x-\xi}{\lambda_0})]^{p-1}\right]\eta_{R'}\eta_R\Phi \\
&+ N[\mathbf{w}] + \bar{E}_2 \quad \text{in} \quad \mathcal{D}\times(0,T) \\
\psi &= -W_2, \quad \text{on} \quad \partial\mathcal{D}\times(0,T),
\end{aligned} \tag{2.21}
$$

and

$$
\begin{aligned}
\Phi_t &= \Delta\Phi + p\left[\lambda_0^{-\frac{n-2}{2}}U(\frac{x-\xi}{\lambda_0})\right]^{p-1}(\Phi+\psi) + \frac{1}{x_1}\frac{\partial\Phi}{\partial x_1} \\
&+ E_2 \quad \text{in} \quad B(\xi, 2R\lambda_0)\times(0,T).
\end{aligned} \tag{2.22}
$$

Problem (2.21) is referred to as the *outer problem*: ψ adjusts the boundary conditions, and takes care of the part of the error far from the concentration point ξ.

Problem (2.22) is referred to as the *inner problem*: Φ adjusts the error close to ξ.

To solve the outer and inner problems (2.21) and (2.22), we proceed as follows. For given parameters λ, d and functions Φ fixed in a suitable range, we solve for ψ Problem (2.21), for any small and smooth initial condition $\psi_0(x)$, in the form of a (nonlocal) operator $\psi = \Psi(\lambda, d, \Phi)$, provided the radius R in (2.15) is large enough and the final time T is small enough. We solve it developing a linear theory for an operator which resembles the characteristics of the heat equation. This is done in full details in Section 4.

We then replace the ψ we found into the inner problem (2.22). In order to get a cleaner expression for problem (2.22), it is convenient to perform two changes of variable for the function Φ. First, we perform a change of variable in the *space* variable, by setting

$$\Phi(x,t) = \lambda_0^{-\frac{n-2}{2}} \phi\left(\frac{x-\xi}{\lambda_0}, t\right). \tag{2.23}$$

In terms of ϕ, equation (2.22) gets the form

$$\begin{aligned} \lambda_0^2 \phi_t = L_0(\phi) + p\lambda_0^{\frac{n-2}{2}} U^{p-1} \psi(\lambda_0 y + \xi, t) + \lambda_0^{\frac{n+2}{2}} E_2(\lambda_0 y + \xi, t) \\ + B[\phi] \quad \text{in} \quad B(0,2R) \times (0,T), \end{aligned} \tag{2.24}$$

where L_0 is the linearized equation associated to the bubble U, introduced in (2.12), that we recall $L_0(\phi) = \Delta\phi + pU^{p-1}\phi$, and

$$B[\phi] = \lambda_0 \dot{\lambda}_0 \left[\frac{n-2}{2}\phi(y,t) + \nabla\phi(y,t)\cdot y\right] + \left[\lambda_0 \dot{d} + \frac{\lambda_0}{\lambda_0 y_1 + \xi}\right] \frac{\partial \phi}{\partial y_1}(y,t) \tag{2.25}$$

A second change of variable, in the *time* variable, is to define

$$\frac{dt}{d\tau} = \lambda_0^2(t), \quad \tau(t) = \frac{n-4}{(n-2)\ell}(T-t)^{-1-\frac{2}{n-4}} \tag{2.26}$$

where ℓ is the constant defined in (2.14). With this change in the time variable, equation (2.24) becomes

$$\phi_\tau = \Delta\phi + pU^{p-1}\phi + H[\lambda, d, \phi, \psi](y,\tau) \quad \text{in} \quad B(0,2R) \times (\tau_0, \infty), \tag{2.27}$$

for $\tau_0 = \tau(0)$ and

$$\begin{aligned} H[\lambda, d, \phi, \psi](y,\tau) = p\lambda_0^{\frac{n-2}{2}} U^{p-1} \psi(\lambda_0 y + \xi, t(\tau)) + \lambda_0^{\frac{n+2}{2}} E_2(\lambda_0 y + \xi, t(\tau)) \\ + B[\phi] \end{aligned} \tag{2.28}$$

Let us discuss how we treat Problem (2.27). The linear operator $L_1(\phi) := -\phi_\tau + L_0(\phi)$ is certainly not invertible, being all τ-independent elements of the kernel of L_0 also elements of the kernel of L_1. Thus, for solvability, one expects some orthogonality conditions to hold. Not only this. The solution ϕ we look for cannot grow exponentially in time. Recall that L_0 has a positive radially symmetric bounded eigenfunction Z associated to the only negative eigenvalue μ_0 to the problem

$$L_0(\phi) + \mu\phi = 0, \quad \phi \in L^\infty(\mathbb{R}^n). \tag{2.29}$$

It is known that μ_0 is a simple eigenvalue and that Z decays like

$$Z(y) \sim |y|^{-\frac{n-1}{2}} e^{-\sqrt{|\mu_0|}\,|y|} \quad \text{as} \quad |y| \to \infty.$$

To avoid exponential grow in time due to this instability, we construct a solution to (2.27) in the class of functions that are parallel to Z in the initial time τ_0.

To be more precise, we can construct a solution to the initial value problem

$$\begin{aligned} \phi_\tau &= \Delta\phi + pU^{p-1}\phi + H[\lambda, d, \phi, \psi](y,\tau) \quad \text{in} \quad B(0,2R)\times(\tau_0,\infty), \\ \phi(y,\tau_0) &= e_0 Z(y) \quad \text{in } B(0,2R), \end{aligned} \tag{2.30}$$

for some constant e_0. While no boundary conditions are specified, we shall request suitable time-space decay rates and, as already mentioned, some orthogonality conditions on the right-hand side $H[\lambda, d, \phi, \psi]$. In other words, one has solvability for (2.30) provided that the following orthogonality conditions

$$\int H[\lambda, d, \phi, \psi] Z_i(y)\, dy = 0, \quad i = 0,1 \quad \forall t \tag{2.31}$$

are fulfilled. It is at this point that we choose the parameters λ and d (as functions of the given ϕ) in such a way that these orthogonality conditions are satisfied. This is done in Section 5, for any R (see (2.15)) large enough, and any final time T small enough.

In Sections 6 we solve the inner problem (2.30): it is at this point that we find that there exists R sufficiently large for that, for any final time T small enough (or equivalently τ_0 large enough), the inner problem is solvable. We remark that the (small) initial condition required for ϕ should lie on a certain manifold locally described as a translation of the hyperplane orthogonal to $Z(y)$. This constraint defines a *codimension* 1 *manifold* of initial conditions which describes those for which the expected asymptotic bubbling behavior is possible.

In summary, the inner-outer gluing procedure allows us to show that: for any small and smooth initial condition ψ_0 for Problem (2.21), we find a solution ψ to (2.21), λ, d solutions to (2.31), and ϕ solution to (2.30), with initial condition belonging to a 1-codimensional space, so that $W_2(x,t) + \mathbf{w}(x,t)$ defined in (2.3)-(2.16) is a solution to (2.1) with the expected asymptotic bubbling behavior.

The rest of the paper is devoted to prove rigorously what we have described so far.

Notation. We use the symbol $'' \lesssim ''$ to indicate $'' \leq C''$, for a positive constant C, whose value may change from line to line, and also inside the same line, and which is independent of t and T.

3. Construction of a first approximation

We start with the description of the error function associated to the first approximation W_1, introduced in (2.8). We recall the definition of the error function

$$S[u](x,t) = -u_t + \Delta u + \frac{1}{x_1}\frac{\partial u}{\partial x_1} + u^p.$$

A direct computation gives

$$
\begin{aligned}
S[W_1](x,t) = & \underbrace{\lambda^{-\frac{n}{2}}\left[\dot d + \frac{1}{x_1}\right] Z_1\left(\frac{x-\xi}{\lambda}\right)}_{:=e_1(x,t)} \\
& + \underbrace{\lambda^{-\frac{n}{2}}\dot\lambda Z_0\left(\frac{x-\xi}{\lambda}\right) - pW_0^{p-1}\bar W_0}_{:=e_2(x,t)} \\
& + \underbrace{\lambda^{-\frac{n}{2}}\left[\dot d - \frac{1}{x_1}\right] Z_1\left(\frac{x-\hat\xi}{\lambda}\right) - \lambda^{-\frac{n}{2}}\dot\lambda Z_0\left(\frac{x-\hat\xi}{\lambda}\right)}_{:=e_3(x,t)} \\
& + \underbrace{\bar W_0^p + (W_0 - \bar W_0)^p - W_0^p + pW_0^{p-1}\bar W_0}_{:=e_4(x,t)} .
\end{aligned} \tag{3.1}
$$

We shall see that the main parts of the error function $S[W_1](x,t)$ are contained in the terms e_1 and e_2. Observe also that the term e_4 depends only on λ_1 and d_1, but it does not depend on $\dot\lambda_1$, nor on $\dot d_1$, while the term e_3 depends on all parameter functions λ_1, d_1, $\dot\lambda_1$ and $\dot d_1$.

Next Lemma contains a description of the error function $S[W_1](x,t)$ in a region close to ξ.

Lemma 3.1. *Assume the functions λ_1 and d_1 satisfy (2.7), and that T is small. Let $\delta > 0$ be a small fixed number and $y = \frac{x-\xi}{\lambda}$. In the region $|x-\xi| < \delta d$, the error of approximation $S[W_1](x,t)$ can be described as follows*

$$
\begin{aligned}
\lambda^{\frac{n+2}{2}} S[W_1](x,t) = & E_0(y,t) + E_\lambda[\lambda_1,\dot\lambda_1,d_1](y,t) \\
& + E_d[d_1,\dot d_1,\lambda_1](y,t) + E[\lambda_1,\dot\lambda_1,d_1,\dot d_1](y,t)
\end{aligned} \tag{3.2}
$$

where

$$
\begin{aligned}
E_0(y,t) &= \lambda_0\dot\lambda_0 Z_0(y) - \frac{p\,\alpha_n}{2^{n-2}}\left(\frac{\lambda_0}{d_0}\right)^{n-2} U^{p-1}(y), \\
E_\lambda[\lambda_1,\dot\lambda_1,d_1](y,t) &= (\lambda\dot\lambda_1 + \dot\lambda_0\lambda_1)\, Z_0(y) \\
&\quad - \frac{p\,(n-2)\,\alpha_n}{2^{n-2}}\left(\frac{\lambda_0}{d_0}\right)^{n-2}\left[\frac{\lambda_1}{\lambda_0} - \frac{d_1}{d_0}\right]\left[1 + q_1(\frac{\lambda_1}{\lambda_0},\frac{d_1}{d_0})\right] U^{p-1}(y) \\
E_d[d_1,\dot d_1,\lambda_1](y,t) &= \lambda\left[\dot d_1 - \frac{d_0 + d_1 + \lambda y_1}{1 + d + \lambda y_1}\right] Z_1(y) \\
&\quad + \frac{p\,(n-2)\,\alpha_n}{2^{n-1}}\left(\frac{\lambda}{d}\right)^{n-1} U^{p-1}(y)\, y_1 \\
E[\lambda_1,\dot\lambda_1,d_1,\dot d_1](y,t) &= \lambda\dot d_1\left(\frac{\lambda_0}{d_0}\right)^{n-1} f(y,\frac{\lambda_1}{\lambda_0},\frac{d_1}{d_0}) - \lambda\dot\lambda_1\left(\frac{\lambda_0}{d_0}\right)^{n-2} f(y,\frac{\lambda_1}{\lambda_0},\frac{d_1}{d_0}) \\
&\quad + \left(\frac{\lambda_0}{d_0}\right)^{n+2} f(y,\frac{\lambda_1}{\lambda_0},\frac{d_1}{d_0}).
\end{aligned}
$$

Here $f = f(y, \frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0})$ denotes a generic function, which is smooth and bounded for y in the considered region, and for λ_1 and d_1 satisfying (2.7), whose expression changes from line to line. With q_1 we denote a generic smooth real function, with the property that $q_1(0,0) = 0$, and $\nabla q_1(0,0) \neq 0$.

Remark 3.1. *A close look at the proof of Lemma 3.1 shows that the three functions E_0, E_λ and E_d originate from the terms e_1 and e_2 in (3.1), which, as already mentioned, are the main terms of $S[W_1]$.*

Proof. Let $\delta > 0$ be a small fixed number. To analyze $S[W_1](x,t)$ in the region $|x - \xi| < \delta d$, we introduce the variable $y = \frac{x-\xi}{\lambda}$ and we define

$$E_1(y,t) = \lambda^{\frac{n+2}{2}} S[W_1](\xi + \lambda y, t).$$

With abuse of notation, we will write $e_j(y,t) = e_j(\xi + \lambda y, t)$. The definition of d_0 in (2.5) gives that $\dot d_0 + 1 = 0$ in $(0,T)$, which simplifies the first term e_1 as follows

$$\lambda^{\frac{n+2}{2}} e_1(y,t) = \lambda \left[\dot d_1 - \frac{d_0 + d_1 + \lambda y_1}{1 + d + \lambda y_1} \right] Z_1(y). \tag{3.3}$$

We refer to (2.11) for the definition of $Z_1(y)$. Let us now describe e_2. In the region we are considering, $|y| < \delta \frac{d}{\lambda}$, we observe that

$$\lambda^{\frac{n-2}{2}} \bar W_0(\xi + \lambda y) = \frac{\alpha_n}{2^{n-2}} \left(\frac{\lambda}{d}\right)^{n-2} \left[1 - \frac{n-2}{2} y_1 \frac{\lambda}{d} + O(1 + |y|^2) q_2 \left(\frac{\lambda}{d}\right) \right]$$

where q_2 denotes a smooth function with the properties that $q_2(0) = q_2'(0) = 0$, $q_2''(0) \neq 0$. With this in mind, we get

$$\begin{aligned} \lambda^{\frac{n+2}{2}} e_2(y,t) &= \lambda \dot\lambda Z_0(y) - \frac{p\,\alpha_n}{2^{n-2}} \left(\frac{\lambda}{d}\right)^{n-2} U^{p-1}(y) \\ &\quad + \frac{p\,(n-2)\,\alpha_n}{2^{n-1}} \left(\frac{\lambda}{d}\right)^{n-1} U^{p-1}(y)\, y_1 + R[\lambda, d](y,t) \left(\frac{\lambda}{d}\right)^{n} \end{aligned} \tag{3.4}$$

where $R[\lambda, d](y,t)$ depends smoothly on λ and d, it does not depend on $\dot\lambda$, nor on $\dot d$, and satisfies the uniform estimates

$$|R[\lambda, d](y,t)| \leq \frac{C}{1 + |y|^2}, \tag{3.5}$$

for some constant C, independent of t and T. Replacing (2.13) in (3.4), we can write

$$\begin{aligned} \lambda^{\frac{n+2}{2}} e_2(y,t) &= \lambda_0 \dot\lambda_0 Z_0(y) - \frac{p\,\alpha_n}{2^{n-2}} \left(\frac{\lambda_0}{d_0}\right)^{n-2} U^{p-1}(y) \\ &\quad + (\lambda \dot\lambda_1 + \dot\lambda_0 \lambda_1)\, Z_0(y) \\ &\quad - \frac{p\,(n-2)\,\alpha_n}{2^{n-2}} \left(\frac{\lambda_0}{d_0}\right)^{n-2} \left[\frac{\lambda_1}{\lambda_0} - \frac{d_1}{d_0} \right] \left[1 + q_1(\frac{\lambda_1}{\lambda_0}) + q_1(\frac{d_1}{d_0}) \right] U^{p-1}(y) \\ &\quad + \frac{p\,(n-2)\,\alpha_n}{2^{n-1}} \left(\frac{\lambda}{d}\right)^{n-1} U^{p-1}(y)\, y_1 + \left(\frac{\lambda}{d}\right)^{n} R[\lambda, d](y,t), \end{aligned} \tag{3.6}$$

where R depends smoothly on λ and d, it does not depend on $\dot\lambda$, nor on $\dot d$, and satisfies the uniform estimate (3.5). In order to describe $\lambda^{\frac{n+2}{2}} e_3(y,t)$, we observe that in the region we are considering, we have

$$Z_1(y + 2\frac{d}{\lambda}\mathbf{e}_1) = \left(\frac{\lambda_0}{d_0}\right)^{n-1} f(y, \frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0})$$

and

$$Z_0(y + 2\frac{d}{\lambda}\mathbf{e}_1) = \left(\frac{\lambda_0}{d_0}\right)^{n-2} f(y, \frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0})$$

where $f = f(y, \frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0})$ denotes a generic function, which is smooth and bounded, whose expression changes from line to line. So, we get

$$\begin{aligned} \lambda^{\frac{n+2}{2}} e_3(y,t) &= \lambda \left[\dot d_1 - 2 - \frac{d+\lambda y_1}{1+d+\lambda y_1}\right] \left(\frac{\lambda_0}{d_0}\right)^{n-1} f(y, \frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}) \\ &- \lambda[\dot\lambda_0 + \dot\lambda_1] \left(\frac{\lambda_0}{d_0}\right)^{n-2} f(y, \frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}) \\ &= \lambda \dot d_1 \left(\frac{\lambda_0}{d_0}\right)^{n-1} f(y, \frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}) - \lambda\dot\lambda_1 \left(\frac{\lambda_0}{d_0}\right)^{n-2} f(y, \frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}) \\ &+ \left(\frac{\lambda_0}{d_0}\right)^{2(n-2)} f(y, \frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}). \end{aligned} \tag{3.7}$$

We finally observe that, for $|x-\xi| < \delta d$, we have

$$\lambda^{\frac{n+2}{2}} \bar W_0^p(\xi + \lambda y, t) = \left(\frac{\lambda_0}{d_0}\right)^{n+2} f(y, \frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0})$$

and, for $n = 6$

$$\lambda^{\frac{n+2}{2}} \left[(W_0 - \bar W_0)^p - W_0^p + pW_0^{p-1}\bar W_0\right](\xi + \lambda y, t) = \left(\frac{\lambda_0}{d_0}\right)^{2(n-2)} f(y, \frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0})$$

while for $n \geq 7$

$$\lambda^{\frac{n+2}{2}} \left[(W_0 - \bar W_0)^p - W_0^p + pW_0^{p-1}\bar W_0\right](\xi + \lambda y, t) = \left(\frac{\lambda_0}{d_0}\right)^{n+2} f(y, \frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}).$$

We thus conclude that

$$\lambda^{\frac{n+2}{2}} e_4(y,t) = \left(\frac{\lambda_0}{d_0}\right)^{n+2} f(y, \frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}) \tag{3.8}$$

where $f = f(y, \frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0})$ is a smooth bounded function. Putting together (3.3)-(3.4)-(3.6)-(3.7)-(3.8), and using the fact that $n \geq 6$, we obtain (3.2). □

Observe that the function E_0 in (3.2) is an explicit function of x and t, and it does not depend on the parameter functions λ_1, and d_1. It is convenient to slightly modify the approximate solution W_1, adding a correction that will eliminate the term E_0 in the error. To this purpose, we write

$$E_0(y,t) = \left(\frac{\lambda_0}{d_0}\right)^{n-2} \pi(y), \quad \pi(y) = \frac{p\,\alpha_n}{2^{n-2}} \left[\frac{\int_{\mathbb{R}^n} U^{p-1}(y) Z_0(y)\,dy}{\int_{\mathbb{R}^n} Z_0^2(y)\,dy} Z_0(y) - U^{p-1}(y)\right].$$

Let $h = h(y)$ be the radially symmetric, fast decaying solution to

$$\Delta h + pU^{p-1} h = \pi, \quad \text{in} \quad \mathbb{R}^n,$$

defined by the variation of parameters formula as follows. We denote by $\tilde{Z}$ a radial solution to $\Delta \tilde{Z} + pU^{p-1}\tilde{Z} = 0$ which is linearly independent to Z_0. One has that $\tilde{Z}(r) \sim r^{2-n}$ as $r \to 0$, while $\tilde{Z}(r) \sim 1$ as $r \to \infty$. Then h is given by

$$h(r) = cZ_0(r) \int_0^r \tilde{Z}(s)\pi(s)s^{n-1}\, ds - c\tilde{Z}(r) \int_0^r Z_0(s)\pi(s)s^{n-1}\, ds, \quad r = |y|,$$

for some constant c. One sees that

$$h(|y|) = O(|y|^{-2}), \quad \text{as} \quad |y| \to \infty. \tag{3.9}$$

Define

$$w(x,t) = \lambda^{-\frac{n-2}{2}} h\left(\frac{x-\xi}{\lambda}\right), \quad \text{and} \quad \bar{w}(x,t) = \lambda^{-\frac{n-2}{2}} h\left(\frac{x-\hat{\xi}}{\lambda}\right).$$

Observe that

$$\Delta w + p\left(\lambda^{-\frac{n-2}{2}} U\left(\frac{x-\xi}{\lambda}\right)\right)^{p-1} w = \lambda^{-\frac{n+2}{2}} \pi\left(\frac{x-\xi}{\lambda}\right). \tag{3.10}$$

A new approximate solution is defined to be

$$W_2[\lambda_1, d_1](x,t) = W_1[\lambda_1, d_1](x,t) - \left(\frac{\lambda_0}{d_0}\right)^{n-2} \underbrace{[w(x,t) - \bar{w}(x,t)]}_{:=W(x,t)}., \tag{3.11}$$

where W_1 is the function defined in (2.8). A direct computation gives that the new error function $S[W_2](x,t)$ is given by

$$\begin{aligned} S[W_2](x,t) = S[W_1](x,t) - \left(\frac{\lambda_0}{d_0}\right)^{n-2} \left[\Delta w + pW_0^{p-1} w\right] \\ + e_5(x,t) + e_6(x,t) \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} e_5(x,t) &= \left(\frac{\lambda_0}{d_0}\right)^{n-2} W_t, \\ e_6(x,t) &= \left(\frac{\lambda_0}{d_0}\right)^{n-2} \left[\Delta \bar{w} + pW_0^{p-1}\bar{w}\right] \\ &\quad + \frac{d}{dt}\left[\left(\frac{\lambda_0}{d_0}\right)^{n-2}\right] W - \left(\frac{\lambda_0}{d_0}\right)^{n-2} \frac{1}{x_1}\frac{\partial W}{\partial x_1} \\ &\quad + \left(W_1 - \left(\frac{\lambda_0}{d_0}\right)^{n-2} W\right)^p - W_1^p + p\left(\frac{\lambda_0}{d_0}\right)^{n-2} W_0^{p-1} W. \end{aligned}$$

Observe that the function e_6 depends only on λ_1 and d_1, but it does not depend on $\dot{\lambda}_1$, nor on $\dot{d}_1$. On the other hand, e_5 depends on all λ_1, d_1, $\dot{\lambda}_1$ and $\dot{d}_1$.

Next Lemma contains a description of the error function $S[W_2](x,t)$ in a region close to ξ. An immediate comparison between the expression of $S[W_1]$ in (3.2) and the one of $S[W_2]$ in (3.13) shows that the new approximate solution W_2 corrects the error E_0.

Lemma 3.2. *Assume the functions λ_1 and d_1 satisfy (2.7), and that T is small. Let $\delta > 0$ be a small fixed number and $y = \frac{x-\xi}{\lambda}$. In the region $|x-\xi| < \delta d$, the error of approximation $S[W_2](x,t)$ can be estimated as follows*

$$\begin{aligned}\lambda^{\frac{n+2}{2}} S[W_2](x,t) &= E_{2,\lambda}[\lambda_1, \dot\lambda_1, d_1](y,t) \\ &\quad + E_{2,d}[d_1, \dot d_1, \lambda_1](y,t) + E[\lambda_1, \dot\lambda_1, d_1, \dot d_1](y,t)\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}E_{2,\lambda}[\lambda_1, \dot\lambda_1, d_1](y,t) &= (\lambda\dot\lambda_1 + \dot\lambda_0\lambda_1)\, Z_0(y) \\ &\quad - (\lambda\dot\lambda_1 + \dot\lambda_0\lambda_1) \left(\frac{\lambda_0}{d_0}\right)^{n-2} \left(\frac{n-2}{2} h(y) + \nabla h(y)\cdot y\right) \\ &\quad - \frac{p\,(n-2)\,\alpha_n}{2^{n-2}} \left(\frac{\lambda_0}{d_0}\right)^{n-2} \left[\frac{\lambda_1}{\lambda_0} - \frac{d_1}{d_0}\right] \left[1 + q_1(\frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0})\right] U^{p-1}(y) \\ E_{2,d}[d_1, \dot d_1, \lambda_1](y,t) &= \lambda \left[\dot d_1 - \frac{d_0 + d_1 + \lambda y_1}{1 + d + \lambda y_1}\right] Z_1(y) \\ &\quad - \lambda \dot d_1 \left(\frac{\lambda_0}{d_0}\right)^{n-2} \frac{\partial h}{\partial y_1}(y) \\ &\quad + \frac{p\,(n-2)\,\alpha_n}{2^{n-1}} \left(\frac{\lambda}{d}\right)^{n-1} U^{p-1}(y)\, y_1 \\ E[\lambda_1, \dot\lambda_1, d_1, \dot d_1](y,t) &= \lambda \dot d_1 \left(\frac{\lambda_0}{d_0}\right)^{n-1} f(y, \frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}) - \lambda\dot\lambda_1 \left(\frac{\lambda_0}{d_0}\right)^{n-2} f(y, \frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}) \\ &\quad + \left(\frac{\lambda_0}{d_0}\right)^{n+2} f(y, \frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}).\end{aligned}$$

Here $f = f(y, \frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0})$ denotes a generic function, which is smooth and bounded for y in the considered region, and for λ_1 and d_1 satisfying (2.7), whose expression changes from line to line. With q_1 we denote a generic smooth real function, with the property that $q_1(0,0) = 0$, and $\nabla q_1(0,0) \neq 0$.

Proof. Let $y = \frac{x-\xi}{\lambda}$ and consider the region $|y| < \delta\frac{d}{\lambda}$, for some fixed number δ. The function $e_5(x,t)$ defined in (3.12), is explicitly given by

$$\begin{aligned}e_5(x,t) &= -\left(\frac{\lambda_0}{d_0}\right)^{n-2} \lambda^{-\frac{n}{2}} \left[\dot\lambda(\frac{n-2}{2} h(y) + \nabla h(y)\cdot y) + \dot d \frac{\partial h}{\partial y_1}(y)\right] \\ &\quad + \left(\frac{\lambda_0}{d_0}\right)^{n-2} \lambda^{-\frac{n}{2}} \left[\dot\lambda(\frac{n-2}{2} h(y + 2\frac{d}{\lambda}\mathbf{e}_1) - \nabla h(+2\frac{d}{\lambda}\mathbf{e}_1)\cdot(y + 2\frac{d}{\lambda}\mathbf{e}_1))\right] \\ &\quad + \left(\frac{\lambda_0}{d_0}\right)^{n-2} \lambda^{-\frac{n}{2}}\, \dot d\, \frac{\partial h}{\partial y_1}(y + 2\frac{d}{\lambda}\mathbf{e}_1).\end{aligned}$$

Taking advantage of the estimate (3.9), we can write

$$\begin{aligned}\lambda^{\frac{n+2}{2}} e_5(x,t) = &-\lambda\,\dot\lambda \left(\frac{\lambda_0}{d_0}\right)^{n-2} \left(\frac{n-2}{2} h(y) + \nabla h(y)\cdot y\right)\\ &-\lambda\,\dot d \left(\frac{\lambda_0}{d_0}\right)^{n-2} \frac{\partial h}{\partial y_1}(y)\\ &+\lambda\,\dot\lambda \left(\frac{\lambda_0}{d_0}\right)^{n} f(y,\frac{\lambda_1}{\lambda_0},\frac{d_1}{d_0}) + \lambda\,\dot d \left(\frac{\lambda_0}{d_0}\right)^{n+1} f(y,\frac{\lambda_1}{\lambda_0},\frac{d_1}{d_0})\end{aligned} \tag{3.14}$$

where $f = f(y, \frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0})$ denotes a generic function, which is smooth and bounded for y in the considered region, and for λ_1 and d_1 satisfying (2.7).

Next, we claim that

$$\lambda^{\frac{n+2}{2}} e_6(x,t) = \left(\frac{\lambda_0}{d_0}\right)^{n+2} f(y,\frac{\lambda_1}{\lambda_0},\frac{d_1}{d_0}), \tag{3.15}$$

for some f as before. To check the validity of (3.15), we start with the observation that

$$\begin{aligned}\left(\frac{\lambda_0}{d_0}\right)^{n-2}\left[\Delta\bar w + pW_0^{p-1}\bar w\right] &= \left(\frac{\lambda_0}{d_0}\right)^{n-2} p\left[W_0^{p-1} - \bar W_0^{p-1}\right]\bar w\\ &+ \left(\frac{\lambda_0}{d_0}\right)^{n-2}\lambda^{-\frac{n+2}{2}}\pi\left(\frac{x-\hat\xi}{\lambda}\right).\end{aligned}$$

Using again estimate (3.9), and a Taylor expansion, we get that

$$\lambda^{\frac{n+2}{2}}\left(\frac{\lambda_0}{d_0}\right)^{n-2}\left[\Delta\bar w + pW_0^{p-1}\bar w\right] = \left(\frac{\lambda_0}{d_0}\right)^{n+2} f(y,\frac{\lambda_1}{\lambda_0},\frac{d_1}{d_0}).$$

Observe now that

$$\lambda^{\frac{n+2}{2}}\frac{d}{dt}\left[\left(\frac{\lambda_0}{d_0}\right)^{n-2}\right] W = \lambda^2 (T-t)^{\frac{2}{n-4}} f(y,\frac{\lambda_1}{\lambda_0},\frac{d_1}{d_0}),$$

while

$$\lambda^{\frac{n+2}{2}}\left(\frac{\lambda_0}{d_0}\right)^{n-2}\frac{1}{x_1}\frac{\partial W}{\partial x_1} = \lambda\left(\frac{\lambda_0}{d_0}\right)^{n-2} f(y,\frac{\lambda_1}{\lambda_0},\frac{d_1}{d_0}).$$

A direct computation thus gives that both terms

$$\lambda^{\frac{n+2}{2}}\frac{d}{dt}\left[\left(\frac{\lambda_0}{d_0}\right)^{n-2}\right] W \quad\text{and}\quad \lambda^{\frac{n+2}{2}}\left(\frac{\lambda_0}{d_0}\right)^{n-2}\frac{1}{x_1}\frac{\partial W}{\partial x_1}$$

can be described as $\left(\frac{\lambda_0}{d_0}\right)^{n+2} f(y,\frac{\lambda_1}{\lambda_0},\frac{d_1}{d_0})$, for $f = f(y,\frac{\lambda_1}{\lambda_0},\frac{d_1}{d_0})$ smooth and bounded for y in the considered region, and for λ_1 and d_1 satisfying (2.7). Taylor expanding in $\left(W_1 - \left(\frac{\lambda_0}{d_0}\right)^{n-2} W\right)^p - W_1^p + p\left(\frac{\lambda_0}{d_0}\right)^{n-2} W_0^{p-1}W$, and using again (3.9), one gets a similar expression also for this last term. Putting all the above information together, (3.15) is proven.

The proof of expansion (3.12) thus directly follows from (3.14), (3.15) combined with expansion (3.1) in Lemma 3.1 and the definition of $W(x,t)$ in (3.11). $\square$

In the remaining part of the Section we describe the error function $S[W_2](x,t)$ far from the concentration point ξ, and also the approximation $W_2(x,t)$ itself when evaluated in the boundary of $\mathcal{D}$.

We write the error function $S[W_2]$ as in (2.19)

$$S[W_2](x,t) = \left[\underbrace{e_1 + e_2 - \left(\frac{\lambda_0}{d_0}\right)^{n-2} \left[\Delta w + pW_0^{p-1} w\right]}_{:=E_2} \right] \eta_R(x,t) + \bar{E}_2(x,t), \quad (3.16)$$

where η_R is the cut off function introduced in (2.15). In this way, $\bar{E}_2$ encodes the information of the error $S[W_2]$ regarding the lower order terms and the part of the main terms far away from the concentrating point ξ. For later purpose, we need to estimate this part of the error, $\bar{E}_2$, in certain weighted L^∞ norm.

Let $\alpha \in (0, \frac{1}{2})$ be a positive number, that we can think as very close to 0. For any smooth function $f = f(x,t)$, $x \in \mathcal{D}$ and $t \in (0,T)$, we define the norm

$$\begin{aligned} \|f\|_{**,\alpha} := \inf\{M > 0 \,:\, \lambda^{\frac{n-2}{2}} |f(\lambda y + \xi, t)| \\ \leq M \left(\frac{\lambda_0}{d_0}\right)^{n-2+\sigma} \frac{\lambda^{-2}}{1+|y|^{2+\alpha}}.\} \end{aligned} \quad (3.17)$$

For any smooth function $g = g(x,t)$ defined in $\partial\mathcal{D} \times (0,T)$, we introduce the norm on the boundary

$$\|g\|_{\partial\mathcal{D}} := \|\lambda^{\frac{n-2}{2}}(s) \left(\frac{\lambda_0}{d_0}(s)\right)^{-n+2-\sigma-\alpha} g(x,s)\|_{L^\infty(\partial\mathcal{D}\times(0,T))}. \quad (3.18)$$

In the next Lemma, we describe the part of the error we called $\bar{E}_2$ in the whole $\mathcal{D} \times (0,T)$, and its Lipschitz dependence on λ_1 and d_1. When needed, to emphasize the dependence of $\bar{E}_2$ on the parameter functions λ_1 and d_1 we use the notation

$$\bar{E}_2(x,t) = \bar{E}_2[\lambda_1, d_1](x,t).$$

Lemma 3.3. *Assume the functions λ_1 and d_1 satisfy* (2.7). *Let R be large and fixed and let T be small. Then,*

$$\|\bar{E}_2\|_{**,\alpha} \lesssim \max\{T^{\frac{1-\sigma}{(n-4)}}, R^{-2}\}, \quad \|W_2\|_{\partial\mathcal{D}} \lesssim T^{\frac{1-\sigma}{(n-4)}}. \quad (3.19)$$

Moreover, there exist positive numbers $\varepsilon_1, \varepsilon_2 > 0$ such that, for any parameter functions d_1, and λ_1^1, λ_1^2 satisfying (2.7), *one has*

$$\|\bar{E}_2[\lambda_1^1, d_1] - \bar{E}_2[\lambda_1^2, d_1]\|_{**,\alpha} \lesssim \max\{T^{\varepsilon_1}, R^{-\varepsilon_2}\} \|\dot{\lambda}_1^1 - \dot{\lambda}_1^2\|_{\frac{1+\sigma}{n-4}}, \quad (3.20)$$

and

$$\|W_2[\lambda_1^1, d_1] - W_2[\lambda_1^2, d_1]\|_{\partial\mathcal{D}} \lesssim T^{\varepsilon_1} \|\dot{\lambda}_1^1 - \dot{\lambda}_1^2\|_{\frac{1+\sigma}{n-4}}, \quad (3.21)$$

Also: for any parameter functions λ_1, and d_1^1, d_1^2 satisfying (2.7), *one has*

$$\|\bar{E}_2[\lambda_1, d_1^1] - \bar{E}_2[\lambda_1, d_1^2]\|_{**,\alpha} \lesssim \max\{T^{\varepsilon_1}, R^{-\varepsilon_2}\} \|\dot{d}_1^1 - \dot{d}_1^2\|_{\frac{1+\sigma}{n-4}} \quad (3.22)$$

and

$$\|W_2[\lambda_1, d_1^1] - W_2[\lambda_1, d_1^2]\|_{\partial\mathcal{D}} \lesssim T^{\varepsilon_1} \|\dot{d}_1^1 - \dot{d}_1^2\|_{\frac{1+\sigma}{n-4}} \quad (3.23)$$

Proof. We start the analysis of the second estimate in (3.19). If $x \in \partial\mathcal{D}$, then $|x-\hat\xi| \geq |x-\xi| > rd$, for some $r>0$, so that a Taylor expansion gives

$$W_2(x,t) = \lambda^{\frac{n-2}{2}}\left(1+O(\lambda^2)\right), \quad g(x) := \frac{1}{|x-\xi|^{n-2}} - \frac{1}{|x-\hat\xi|^{n-2}},$$

uniformly for $x \in \partial\mathcal{D}$. We claim that

$$g(x) = O(\frac{1}{d^{n-1}}), \quad \text{uniformly on} \quad \partial\mathcal{D}.$$

This is certainly true if x is a point of the boundary, far from $p := (1,\bar 0)$, say if $d(x,p) > r_0\sqrt{d}$, for some constant r_0. Observe now that if $x \in \partial\mathcal{D}$ is such that $d(x,(1,\bar 0)) \leq \sqrt{d}$, then we can assume that $x = (\phi(\bar x), \bar x)$, with ϕ a smooth function so that $\phi(\bar 0) = 1$, $\nabla\phi(\bar 0) = 0$, and $D^2\phi(\bar 0) \neq 0$. Thus, for x in this region, a simple Taylor expansion gives the existence of a constant c so that $|g(x)| \leq c\frac{d^2}{|x-\xi|^n}$, for $x \neq p$, $g(p) = 0$. We can conclude that, for any $x \in \partial\mathcal{D}$, one has

$$|W_2(x,t)| \leq c\frac{1}{\lambda^{\frac{n-2}{2}}}\left(\frac{\lambda}{d}\right)^{n-2} d.$$

The second estimate in (3.19) follows right away.

Let us check (3.21). Let d_1, and λ_1^1, λ_1^2 satisfy (2.7). For any $x \in \partial\mathcal{D}$, a Taylor expansion gives

$$\begin{aligned}\left|W_2[\lambda_1^1,d_1](x,t) - W_2[\lambda_1^2,d_1](x,t)\right| &\leq \lambda^{-\frac{n}{2}}\left|Z_0(\frac{x-\xi}{\lambda}) - Z_0(\frac{x-\hat\xi}{\lambda})\right| |\lambda_1^1-\lambda_1^2| \\ &+ \lambda^{-\frac{n}{2}}\left|\pi_0(\frac{x-\xi}{\lambda}) - \pi_0(\frac{x-\hat\xi}{\lambda})\right| |\lambda_1^1-\lambda_1^2|\end{aligned} \tag{3.24}$$

for some $\lambda = \lambda_0 + \bar\lambda$, with $\bar\lambda$ satisfying (2.7), where $\pi_0(y,t) = \frac{n-2}{2}h(y) + \nabla h(y)\cdot y$. We refer to (3.11) for the definition of h. Let us analyze the first term in the right hand side of the above formula. Arguing as before, and using (2.7), we get

$$\begin{aligned}\lambda^{-\frac{n}{2}}\left|Z_0(\frac{x-\xi}{\lambda}) - Z_0(\frac{x-\hat\xi}{\lambda})\right| |\lambda_1^1-\lambda_1^2| &\lesssim \lambda^{-\frac{n}{2}}\left(\frac{\lambda_0}{d_0}\right)^{n-2} d(T-t)^{1+\frac{1+\sigma}{n-4}}\|\dot\lambda_1^1-\dot\lambda_1^2\|_{\frac{1+\sigma}{n-4}} \\ &\lesssim \lambda^{-\frac{n-2}{2}}\left(\frac{\lambda_0}{d_0}\right)^{n-2+\sigma} T\,\|\dot\lambda_1^1-\dot\lambda_1^2\|_{\frac{1+\sigma}{n-4}}.\end{aligned}$$

Similarly, one can treat the second term in (3.24). This concludes the proof of (3.21). In a similar way, one can show the validity of (3.23).

Let us show the validity of the first estimate in (3.19). We write $\bar E_2$ explicitly

$$\begin{aligned}\bar E_2 = &\underbrace{\left[e_1 + e_2 - \left(\frac{\lambda_0}{d_0}\right)^{n-2}\left[\Delta w + pW_0^{p-1}w\right]\right](1-\eta_R(x,t))}_{:=e_{out}} \\ &+ \sum_{j=3}^{6} e_j(x,t).\end{aligned} \tag{3.25}$$

We refer to formulas (3.1) and (3.12) for the definition of e_j, $j = 1,\dots,6$.

We start analyzing e_{out}. Observe that this function is not zero only for $|x-\xi| > R\lambda_0$. We decompose

$$\mathcal{D} \cap \{x \,:\, |x-\xi| > R\lambda_0\} = \mathcal{D}_1 \cup \mathcal{D}_2, \quad \mathcal{D}_1 = \mathcal{D} \cap \{x \,:\, R\lambda_0 < |x-\xi| < \delta d\}.$$

To describe e_{out} in $\mathcal{D}_1$ we make use of the result of Lemma 3.2. In fact, using expansion (3.12), we see that, for $x \in \mathcal{D}_1$, we can estimate e_{out} with

$$\begin{aligned} |e_{out}(x,t)| \le \frac{C}{\lambda^{\frac{n+2}{2}}} \Bigg[\left(\frac{\lambda_0}{d_0}\right)^{n-2} \frac{\lambda_1}{\lambda_0} U^{p-1}(y) + \left(\frac{\lambda_0}{d_0}\right)^{n-1} U^{p-1}(y)|y| \\ + \left(\frac{\lambda_0}{d_0}\right)^{n+2} \Bigg] (1-\eta_R(\lambda y+\xi,t)) = t_1+t_2+t_3, \end{aligned} \tag{3.26}$$

for some constant C, independent of t, and of R. Here we use the variable $y = \frac{x-\xi}{\lambda}$. Observe now that, when $x \in \mathcal{D}_1$,

$$|t_1| \le C \frac{R^{-2-\alpha}}{\lambda^{\frac{n-2}{2}}} \left(\frac{\lambda_0}{d_0}\right)^{n-2} \frac{\lambda_1}{\lambda_0} \frac{\lambda^{-2}}{(1+|y|)^{2+\alpha}}, \quad |t_2| \le C \frac{R^{-1-\alpha}}{\lambda^{\frac{n-2}{2}}} \left(\frac{\lambda_0}{d_0}\right)^{n-1} \frac{\lambda^{-2}}{(1+|y|)^{2+\alpha}},$$

and

$$|t_3| \le C \frac{1}{\lambda^{\frac{n-2}{2}}} \left(\frac{\lambda_0}{d_0}\right)^{n-\alpha} \frac{\lambda^{-2}}{(1+|y|)^{2+\alpha}}.$$

Thanks to condition (2.7) on the parameter functions λ_1 and d_1, we get that

$$\left(\frac{\lambda_0}{d_0}\right)^{n-2} \frac{\lambda_1}{\lambda_0} \sim \left(\frac{\lambda_0}{d_0}\right)^{n-2+\sigma},$$

and hence

$$t_1+t_2+t_3 \le \frac{C}{\lambda^{\frac{n-2}{2}}} \left(\frac{\lambda_0}{d_0}\right)^{n-2-\sigma} \max\{T^{\frac{1-\sigma}{n-4}}, R^{-2}\} \frac{\lambda^{-2}}{(1+|y|)^{2+\alpha}} \tag{3.27}$$

for some constant C, independent of t, of T and of R.

We next discuss the size of e_{out} in $\mathcal{D}_2$. In this region, we think that $|y| = |\frac{x-\xi}{\lambda}| \ge \delta \frac{d}{\lambda}$, and we refer to the explicit expression of e_1, e_2 as in (3.1), and to $\left(\frac{\lambda_0}{d_0}\right)^{n-2} \left[\Delta w + pW_0^{p-1} w\right]$ as in (3.10). We analyze e_1, leaving to the interested reader the estimates of the other two terms, which can be done in a similar way. From (3.1), we obtain that

$$|e_1(x,t)| \le C \frac{\dot{d}_1}{\lambda^{\frac{n}{2}}} \frac{1}{(1+|y|)^{n-1}} \le \frac{C}{\lambda^{\frac{n-2}{2}}} \left(\frac{\lambda}{d}\right)^{n-2+\sigma} \lambda (\frac{\lambda}{d})^{\alpha} \frac{\lambda^{-2}}{(1+|y|)^{2+\alpha}}.$$

We readily get

$$|e_1(x,t)| \le \frac{C}{\lambda^{\frac{n-2}{2}}} \left(\frac{\lambda_0}{d_0}\right)^{n-2-\sigma} T^{\frac{1-\sigma}{n-4}} \frac{\lambda^{-2}}{(1+|y|)^{2+\alpha}}. \tag{3.28}$$

Collecting (3.26), (3.27) and (3.28), we conclude that

$$\|e_{out}\|_{**,\alpha} \le C \max\{T^{\frac{1-\sigma}{n-4}}, R^{-2}\}.$$

In order to estimate the remaining terms $e_3, \dots, e_6$, for each one of them we decompose the domain $\mathcal{D}$ into the region where $|x-\xi| \le \delta d$, and its complement.

To analyze these terms in the first region, we use the result of Lemma 3.2. For instance, we can see that, for $|x-\xi|\leq \delta d$, we have

$$|e_3(x,t)| \lesssim \frac{\lambda|\dot\lambda_1|}{\lambda^{\frac{n+2}{2}}}\left(\frac{\lambda_0}{d_0}\right)^{n-2} \lesssim \frac{C}{\lambda^{\frac{n-2}{2}}}\left(\frac{\lambda_0}{d_0}\right)^{n-2-\sigma} T^{\frac{1-\sigma}{n-4}}\frac{\lambda^{-2}}{1+|y|^{2+\alpha}}.$$

Similar estimates follows for the other terms $e_4,\ldots,e_6$ in this region. Consider now the complementary region, where $|x-\xi|>\delta d$. In this case, it is convenient to look at the explicit definition of the terms $e_3,\ldots,e_6$. For instance, consider e_3 as defined in (3.1). In this region, far from ξ, we estimate

$$|e_3(x,t)| \lesssim \frac{1}{\lambda^{\frac{n}{2}}}\frac{1}{(1+|y|)^{n-1}} \lesssim \frac{C}{\lambda^{\frac{n-2}{2}}}\left(\frac{\lambda_0}{d_0}\right)^{n-2-\sigma} T^{\frac{1-\sigma}{n-4}}\frac{\lambda^{-2}}{1+|y|^{2+\alpha}}.$$

In a very similar way, one can treat the other terms. We leave the details to the reader.

The Lipschitz dependence of $\bar E_2$ with respect to the topology of the set to which λ_1 and d_1 belong, as stated in (3.20) and (3.22), follows from the analysis of each one of the terms of $\bar E_2$ in (3.25). One has to study them both in a region relative close to ξ, where one takes advantage of the results contained in Lemma 3.2, and in a region far from ξ, where the explicit expressions collected in (3.1) and (3.12) are of use. □

4. Solving the outer problem

This section is devoted to solve in $\psi=\psi(x,t)$ the *outer problem* (2.21) in the form of a non linear non local operator

$$\psi(x,t)=\Psi[\lambda_1,d_1\phi]\,(x,t)$$

of the parameter functions λ_1 and d_1 satisfying the bounds (2.7), and of the function ϕ defined in (2.16)-(2.23) and chosen in the following range.

For $a>0$ and for functions $f=f(x,t)$ defined in $\mathcal{D}\times(0,T)$, define

$$\begin{aligned}\|f\|_a := \inf\{M>0\,:\,&\lambda^{\frac{n-2}{2}}|f(\lambda y+\xi,t)| \\ &\leq M\left(\frac{\lambda_0}{d_0}\right)^{n-2+\sigma}\frac{1}{1+|y|^a}\}\end{aligned}\tag{4.1}$$

and

$$\begin{aligned}\|f\|_{*,a} := \inf\{M>0\,:\,&\lambda^{\frac{n-2}{2}}|f(\lambda y+\xi,t)| \\ &\leq M\left(\frac{\lambda_0}{d_0}\right)^{n-2+\sigma}\frac{\lambda^{-1}}{1+|y|^{1+a}}\}.\end{aligned}\tag{4.2}$$

Let $\alpha\in(0,\frac12)$ be the positive number fixed in the definition of the norm (3.17). We recall that α may be thought as close to 0. We take a positive, small and $a>\alpha$. We assume that ϕ in (2.16)-(2.23) satisfies the following bound

$$\|\phi\|_{in} := \|\lambda^{-\frac{n-2}{2}}\phi\|_a + \|\lambda^{-\frac{n-2}{2}}\nabla\phi\|_{*,a} \lesssim \max\{T^{\frac{1-\sigma}{(n-4)}},R^{-2}\}\tag{4.3}$$

where we refer to (4.9) for the definition of β, and to (2.7) for the definition of σ.

For parameter functions λ_1, d_1 satisfying (2.7), and for functions ϕ satisfying (4.3), we find a solution ψ to the initial value Problem

$$\begin{aligned} \psi_t &= \Delta\psi + \frac{1}{x_1}\frac{\partial\psi}{\partial x_1} + V\psi + \left(\Delta - \frac{\partial}{\partial t}\right)\eta_R\,\Phi + 2\nabla\Phi\nabla\eta_R + \frac{1}{x_1}\frac{\partial\eta_R}{\partial x_1}\Phi \\ &+ p\left[W_2^{p-1} - [\lambda_0^{-\frac{n-2}{2}}U(\frac{x-\xi}{\lambda_0})]^{p-1}\right]\eta_{R'}\eta_R\Phi \\ &+ N[\mathbf{w}] + \bar E_2 \quad \text{in} \quad \mathcal{D}\times(0,T) \\ \psi &= -W_2, \quad \text{on} \quad \partial\mathcal{D}\times(0,T), \quad \psi = \psi_0, \quad \text{in} \quad \mathcal{D}\times\{t=0\}. \end{aligned} \tag{4.4}$$

This solution will have $\|\cdot\|_{*,\beta,\alpha}$-norm bounded, for any small and smooth initial condition ψ_0.

We have the validity of

Proposition 4.1. *Assume that the parameters λ_1 and d_1 satisfy* (2.7)*, and the function ϕ satisfies the constraint* (4.3)*. Assume furthermore that $\psi_0 \in C_0^2(\bar{\mathcal{D}})$ and*

$$\|\psi_0\|_{L^\infty(\bar{\mathcal{D}})} + \|\nabla\psi_0\|_{L^\infty(\bar{\mathcal{D}})} \lesssim \max\{T^{\frac{1-\sigma}{(n-4)}}, R^{-2}\}. \tag{4.5}$$

Assume that the radius R is large and fixed, and that T is small. Then Problem (4.4) *has a unique solution $\psi = \Psi(\lambda_1, d_1, \phi)$, so that, for $y = \frac{x-\xi}{\lambda}$,*

$$\lambda^{\frac{n-2}{2}}\,|\psi(x,t)| \;\lesssim\; \left(\frac{\lambda_0}{d_0}\right)^{n-2+\sigma}\frac{1}{|y|^\alpha+1}\,\max\{T^{\frac{1-\sigma}{(n-4)}}, R^{-2}\}, \tag{4.6}$$

and, for $|y|<R$,

$$\lambda^{\frac{n-2}{2}}\,|\nabla_x\psi(x,t)| \;\lesssim\; \left(\frac{\lambda_0}{d_0}\right)^{n-2+\sigma}\frac{\lambda^{-1}}{|y|^{\alpha+1}+1}\,\max\{T^{\frac{1-\sigma}{(n-4)}}, R^{-2}\}. \tag{4.7}$$

To prove this result, we shall estimate, for given functions $f(x,t)$, $g(x,t)$, $h(x)$ the unique solution of the linear problem

$$\begin{aligned} \partial_t\psi &= \Delta\psi + \frac{1}{x_1}\frac{\partial\psi}{\partial x_1} + V\psi \; + f(x,t) \quad \text{in } \mathcal{D}\times(0,T), \\ \psi &= g \quad \text{on } \partial\mathcal{D}\times(0,T), \quad \psi(\cdot,0) = h, \end{aligned} \tag{4.8}$$

where the function V is defined in (2.20). To this end, we define $\beta>0$ to be

$$\frac{1}{\lambda^{\frac{n-2}{2}}}\left(\frac{\lambda_0}{d_0}\right)^{n-2+\sigma} \sim (T-t)^{-\beta}, \quad \text{as} \quad t\to T, \tag{4.9}$$

where $\sigma\in(0,1)$ is the number fixed in (2.7). Thanks to (2.4), (2.5) and condition (2.7), we have that

$$\beta = \frac{n-2}{2} - \frac{n-2+2\sigma}{2(n-4)} > 0, \quad \text{for any} \quad n\geq 6.$$

We also assume that $\beta - \frac{\alpha}{n-4} > 0$.

Lemma 4.1. *Let c be a positive constant, independent of t and T, such that $\|f\|_{**,\alpha} < c$, $\|h\|_{L^\infty(\mathcal{D})} < c$ and $\|g\|_{\partial\mathcal{D}} < c$. Let $\psi = \psi[f,g,h]$ be the unique solution of Problem* (4.8)*. If α is chosen sufficiently small, then, for all (x,t),*

$$|\psi(x,t)| \;\lesssim\; \left(\|f\|_{**,\alpha} + \|h\|_{L^\infty(\mathcal{D})} + \|g\|_{\partial\mathcal{D}}\right)\frac{(T-t)^{-\beta}}{1+|y|^\alpha}, \tag{4.10}$$

where $y = \frac{x-\xi}{\lambda}$ *and the norm* $\|\cdot\|_b$ *is defined in* (3.18). *Moreover, we have the following local estimate on the gradient*

$$|\nabla_x \psi(x,t)| \ \lesssim \ \left(\|f\|_{**,\alpha} + \|h\|_{L^\infty(\mathcal{D})} + \|g\|_{\partial\mathcal{D}}\right) \frac{\lambda^{-1}(T-t)^{-\beta}}{1+|y|^{\alpha+1}}, \tag{4.11}$$

for $|y| \le R$.

Proof. To prove the result, we construct a super solution for (4.8). To this end, let $q(|z|) = \frac{1}{1+|z|^{2+\alpha}}$ and let $Q(|z|)$ be the radial positive solution of

$$\Delta Q + 4q = 0 \quad \text{in } \mathbb{R}^n, \quad \text{given by} \quad Q(r) = 4\int_r^\infty \frac{d\rho}{\rho^{n-1}} \int_0^\rho q(s)s^{n-1}ds.$$

Observe that $Q(z) \sim \frac{1}{1+|z|^\alpha}$ in $\mathbb{R}^n$. One has

$$\Delta Q + \frac{\delta}{1+|z|^2} Q + 2q \le 0 \quad \text{in } \mathbb{R}^n$$

provided δ is small enough. Define $\bar Q(x) := Q\left(\frac{x-\xi}{\lambda}\right)$ and $\bar q(x) := \frac{1}{\lambda^2} q\left(\frac{x-\xi}{\lambda}\right)$. For a possibly smaller δ, one has

$$\Delta_x \bar Q + \lambda^{-2} \frac{\delta}{1+\left|\frac{x-\xi}{\lambda}\right|^2} \bar Q + \frac{3}{2}\bar q \le 0 \quad \text{in } \mathbb{R}^n.$$

Observe now that

$$|V(x,t)| \le A \frac{\lambda^{-2}R^{-2}}{1+|y|^2},$$

for some constant A independent of t and T, as a direct consequence of the definition of V given in (2.20), and the bounds (2.7) on the parameter functions λ_1 and d_1. Moreover,

$$|\frac{1}{x_1}\frac{\partial \bar Q}{\partial x_1}(x,t)| \lesssim \frac{\lambda^{-1}}{|(\xi+\lambda y)_1|} \frac{1}{1+|y|^{1+\alpha}}.$$

From the above estimates, we obtain that

$$\Delta \bar Q + \frac{1}{x_1}\frac{\partial \bar Q}{\partial x_1} + V\bar Q + \frac{3}{2}\bar q \le 0$$

thanks to the fact that R is large.

Define $\psi_0(x,t) = (T-t)^{-\beta}\bar Q(x)$. We have

$$(\psi_0)_t = \beta(T-t)^{-\beta-1}\bar Q - (T-t)^{-\beta}\nabla \bar Q(\frac{x-\xi}{\lambda}) \cdot [\frac{x-\xi}{\lambda}\frac{\dot\lambda}{\lambda} + \frac{\dot d}{\lambda}],$$

and

$$\Delta\psi_0 + \frac{1}{x_1}\frac{\partial \psi_0}{\partial x_1} + V\psi_0 + f(x,t) \le -\tilde a \frac{\lambda^{-2}}{(1+|y|)^{2+\alpha}}(T-t)^{-\beta}$$

for some positive, possibly small, $\tilde a$. Thus we get, for some positive c_1

$$\begin{aligned} -(\psi_0)_t + \Delta\psi_0 + \frac{1}{x_1}\frac{\partial\psi_0}{\partial x_1} + V\psi_0 + f(x,t) &\le -\beta(T-t)^{-\beta-1}\bar Q \\ &+ c_1(T-t)^{-\beta-1}\nabla\bar Q(\frac{x-\xi}{\lambda}) \cdot \frac{x-\xi}{\lambda} + (T-t)^{-\beta}\nabla\bar Q(\frac{x-\xi}{\lambda}) \cdot \frac{\dot d}{\lambda} \\ &- \tilde a \frac{\lambda^{-2}}{(1+|y|)^{2+\alpha}}(T-t)^{-\beta}. \end{aligned}$$

Observe now that, for some constant $A > 0$,

$$\left|\nabla\bar{Q}(\frac{x-\xi}{\lambda})\cdot\frac{x-\xi}{\lambda}\right| \le \frac{A\alpha}{(1+|y|^{\alpha})}$$

and

$$\left|\nabla\bar{Q}(\frac{x-\xi}{\lambda})\cdot\frac{\dot{d}}{\lambda}\right| \le A\frac{\alpha\lambda^{-2}}{(1+|y|^{2+\alpha})}|\dot{d}|.$$

Choosing α small, we get

$$-(\psi_0)_t + \Delta\psi_0 + V\psi_0 + f(x,t) \le 0.$$

Moreover, one has $|g(x,t)| \le M\psi_0(x,t)$, for $(x,t) \in \partial\mathcal{D}\times(0,T)$, and $|h(x)| \le M\psi_0(x,0)$, for $x\in\mathcal{D}$, provided the constant $M>0$ is properly chosen, and α is close to 0. Thus $M\psi_0$ is a positive super solution for (4.8). Estimate (4.10) thus follows from parabolic comparison.

To get the gradient estimate in (4.11) we scale around ξ letting

$$\psi(x,t) := \tilde{\psi}\left(\frac{x-\xi}{\lambda}, \tau(t)\right)$$

where $\dot{\tau}(t) = \lambda(t)^{-2}$. We choose T small so that $\tau \ge 2$. Then $\tilde{\psi}$ satisfies for $|z| \le \delta\lambda^{-1}$, with sufficiently small δ,

$$\partial_\tau\tilde{\psi} = \Delta_z\tilde{\psi} + a(z,t)\cdot\nabla_z\tilde{\psi} + b(z,t)\tilde{\psi} + \tilde{f}(z,\tau)$$

where $\tilde{f}(z,\tau) = \lambda^2 f(\xi+\lambda z, t(\tau))$, and the uniformly small coefficients $a(z,t)$ and $b(z,t)$ are given by

$$a(z,t) := [\lambda\dot{\lambda}z + \dot{\xi}\lambda], \quad b(z,t) = V(\xi+\lambda z) = O(R^{-4})(1+|z|)^{-4}.$$

Our assumption in f implies that in this region

$$|\tilde{f}(z,\tau)| \;\lesssim\; (T-t(\tau))^{-\beta}\frac{\|f\|_{**,\alpha}}{1+|z|^{2+\alpha}}$$

while we have already established that

$$|\tilde{\psi}(z,\tau)| \lesssim (T-t(\tau))^{-\beta}\frac{\|f\|_{**,\alpha}}{1+|z|^{\alpha}}.$$

Let us now fix $0<\eta<1$. By standard parabolic estimates we get that for $\tau_1 \ge \tau(t_0)+2$,

$$\begin{aligned}
[\nabla_z\tilde{\psi}(\tau_1,\cdot)]_{\eta,B_{10}(0)} &+ \|\nabla_z\tilde{\psi}(\tau_1,\cdot)\|_{L^\infty(B_{10}(0))}\\
&\lesssim\; \|\tilde{\psi}\|_{L^\infty(B_{20}(0))\times(\tau_1-1,\tau_1)} + \|\tilde{f}\|_{L^\infty(B_{20}(0))\times(\tau_1-1,\tau_1)}\\
&\lesssim (T-t(\tau_1-1))^{-\beta}\|f\|_{*,\beta,2+\alpha} \lesssim (T-t(\tau_1))^{-\beta}\|f\|_{**,\alpha}.
\end{aligned}$$

provided that $\tau_1 \ge 2$. Translating this estimate to the original variables (x,t) we find that for any $t \ge c_n t_0$, for a suitable constant c_n,

$$(R\lambda)^{1+\eta}[\nabla_x\psi(t,\cdot)]_{\eta,B_{10R\lambda}(\xi)} + R\lambda\|\nabla_x\psi(t,\cdot)\|_{L^\infty(B_{10R\lambda}(\xi))} \;\lesssim\; (T-t)^{-\beta}\|f\|_{**,\alpha}.$$

Using similar parabolic estimate up to the initial condition ψ_0 at 0 for ψ yields the validity of the above estimate, and hence of (4.11), for any $t\ge 0$. The proof is complete. □

We now give the

Proof of Proposition 4.1. Lemma 4.1 defines a linear bounded operator $S(f,g,h) = \psi$, which is the solution to (4.8) and the existence of a constant $c>0$ such that

$$\|S(f,g,h)\|_\alpha \leq c\left(\|f\|_{**,\alpha} + \|h\|_{L^\infty(\mathcal{D})} + \|g\|_{\partial\mathcal{D}}\right). \tag{4.12}$$

We establish the existence of a solution ψ to (4.4), satisfying (4.6), as a fixed point for the Problem

$$\psi = \mathbf{S}(\psi), \quad \mathbf{S}(\psi) := S(f,g,h), \tag{4.13}$$

where

$$\begin{aligned} f = & \left(\Delta - \frac{\partial}{\partial t}\right)\eta_R\,\Phi + 2\nabla\Phi\nabla\eta_R + \frac{1}{x_1}\frac{\partial\eta_R}{\partial x_1}\Phi \\ & + p\left[W_2^{p-1} - [\lambda_0^{-\frac{n-2}{2}}U(\frac{x-\xi}{\lambda_0})]^{p-1}\right]\eta_{R'}\eta_R\Phi \\ & + N[\mathbf{w}] + \bar{E}_2, \quad g = -W_2, \quad h = \psi_0. \end{aligned} \tag{4.14}$$

We claim that there exists a fixed point ψ for (5.8) in the set

$$B_M = \{\psi : \|\psi\|_\alpha \leq M\max\{T^{\frac{1-\sigma}{(n-4)}}, R^{-2}\}\}, \quad \text{for some} \quad M>0,$$

as a consequence of the Contraction Mapping Theorem. Indeed, by Lemma 4.1, there exists a constant c such that, for any $\psi \in B_M$

$$\|\mathbf{S}(\psi)\|_\alpha \leq c\left(\|f\|_{**,\alpha} + \|h\|_{L^\infty(\mathcal{D})} + \|g\|_{\partial\mathcal{D}}\right)$$

From the second estimate in (3.19), we get that $\|g\|_{\partial\mathcal{D}} \lesssim T^{\frac{1-\sigma}{(n-4)}}$. Thus the map $\mathbf{S}$ sends the set B_M into B_M provided that $\|f\|_{**,\alpha} \lesssim T^{\frac{1-\sigma}{(n-4)}}$. This last inequality follows from the fact that

$$\begin{aligned} & \|\left(\Delta - \frac{\partial}{\partial t}\right)\eta_R\,\Phi + 2\nabla\Phi\nabla\eta_R + \frac{1}{x_1}\frac{\partial\eta_R}{\partial x_1}\Phi\|_{**,\alpha} \lesssim \max\{T^{\frac{1-\sigma}{(n-4)}}, R^{-2}\} \\ & \|p\left[W_2^{p-1} - [\lambda_0^{-\frac{n-2}{2}}U(\frac{x-\xi}{\lambda_0})]^{p-1}\right]\eta_{R'}\eta_R\Phi + N[\mathbf{w}]\|_{**,\alpha} \lesssim \max\{T^{\frac{1-\sigma}{(n-4)}}, R^{-2}\} \end{aligned} \tag{4.15}$$

combined with the first estimate in (3.19), that was already proven in Lemma 3.3. We postpone the proof of (5.10).

We claim that

$$\|\mathbf{S}(\psi_1) - \mathbf{S}(\psi_2)\|_\alpha \lesssim \mathbf{c}\|\psi_1 - \psi_2\|_\alpha, \tag{4.16}$$

with $0<\mathbf{c}<1$ if R is large and T small. Thus, the map $\mathbf{S}$ is a contraction, provided $\mathbf{S}$ is chosen small. This concludes the proof of the existence of ψ solution to (4.4), satisfying estimate (4.6). Estimate (4.7) follows from Lemma 4.1 and estimate (4.11).

The rest of this proof is devoted to establish the validity of (5.10) and (4.16)

We now prove (5.10). Recall that

$$\Phi(x,t) = \lambda_0^{-\frac{n-2}{2}}\phi\left(\frac{x-\xi}{\lambda_0}, t\right)$$

We star with the first estimate. Since we are assuming the bound (4.3) in the *inner* function ϕ, we observe that

$$\lambda^{\frac{n-2}{2}} \left|(\Delta - \frac{\partial}{\partial t})\eta_R \Phi\right| \lesssim \left(|\frac{\eta''}{R^2\lambda^2}| + |\frac{\eta'}{R\lambda}\frac{\dot\lambda}{\lambda}| + |\eta' \frac{\dot d}{\lambda}| \right) \|\phi\|_{in} \left(\frac{\lambda_0}{d_0}\right)^{n-2+\sigma} \frac{1}{1+|y|^a},$$

so that, in the region where it is not zero (that is $R \leq |y| \leq 2R$), we get

$$\lambda^{\frac{n-2}{2}} \left|(\Delta - \frac{\partial}{\partial t})\eta_R \Phi\right| \lesssim R^{-a+\alpha} \|\phi\|_{in} \left(\frac{\lambda_0}{d_0}\right)^{n-2+\sigma} \frac{\lambda^{-2}}{1+|y|^{2+\alpha}}.$$

Analogous estimate holds for the term $\frac{1}{x_1}\frac{\partial \eta_R}{\partial x_1}\Phi$. Similarly, one has

$$\lambda^{\frac{n-2}{2}} |2\nabla\Phi\nabla\eta_R| \lesssim |\frac{\eta'}{\lambda R}| \|\phi\|_{in} \left(\frac{\lambda_0}{d_0}\right)^{n-2+\sigma} \frac{\lambda^{-1}}{1+|y|^{1+\alpha}},$$

so that, in the region where it is not zero, we get

$$\lambda^{\frac{n-2}{2}} |2\nabla\Phi\nabla\eta_R| \lesssim R^{-a+\alpha} \|\phi\|_{in} \left(\frac{\lambda_0}{d_0}\right)^{n-2+\sigma} \frac{\lambda^{-2}}{1+|y|^{2+\alpha}}.$$

We conclude that

$$\| \left(\Delta - \frac{\partial}{\partial t}\right) \eta_R \, \Phi + 2\nabla\Phi\nabla\eta_R + \frac{1}{x_1}\frac{\partial \eta_R}{\partial x_1}\Phi\|_{**,\alpha} \lesssim R^{-a+\alpha}\|\phi\|_{in}. \tag{4.17}$$

This gives right away the validity of the first estimate in (5.10).

Next we consider the second estimate in (5.10). We have

$$\begin{aligned}
&\lambda^{\frac{n-2}{2}} \left| p\left[W_2^{p-1} - [\lambda_0^{-\frac{n-2}{2}} U(\frac{x-\xi}{\lambda_0})]^{p-1}\right] \eta_{R'}\eta_R\Phi \right| \\
&\lesssim \lambda^{-2} U^{p-1}(y) \, \|\phi\|_{in} \left(\frac{\lambda_0}{d_0}\right)^{n-2+\sigma} \frac{1}{1+|y|^a} |\eta_R| \\
&\lesssim R^{-2} \|\phi\|_{in} \left(\frac{\lambda_0}{d_0}\right)^{n-2+\sigma} \frac{\lambda^{-2}}{1+|y|^{2+\alpha}}.
\end{aligned}$$

In order to estimate $N(\mathbf{w})$, we write

$$W_2(x,t) = \frac{1}{\lambda^{\frac{n-2}{2}}}[U(y) + \rho(y)]. \tag{4.18}$$

From (2.18), we get

$$\begin{aligned}
\lambda^{\frac{n-2}{2}} |N(\mathbf{w})|(\lambda y + \xi) &\lesssim \frac{1}{\lambda^2} \Bigg[(U + \rho + \lambda^{\frac{n-2}{2}}[\psi + \eta_R\Phi])^p - (U+\rho)^p \\
&- p(U+\rho)^{p-1}\lambda^{\frac{n-2}{2}}[\psi + \eta_R\Phi] \Bigg] \lesssim \frac{1}{\lambda^2}\left(|\lambda^{\frac{n-2}{2}}\psi|^p + |\lambda^{\frac{n-2}{2}}\eta_R|\phi|^p \right) \\
&\lesssim \left(\frac{\lambda_0}{d_0}\right)^{(n-2+\sigma)p} \left(\frac{\|\psi\|_\alpha^p}{1+|y|^{\alpha p}} + \frac{\|\phi\|_{in}^p}{1+|y|^{ap}} \right) \\
&\lesssim \left(\frac{\lambda_0}{d_0}\right)^{(n-2+\sigma)(p-1)} \left(\|\psi\|_\alpha^{p-1} + \|\phi\|_{in}^{p-1} \right) \left(\frac{\lambda_0}{d_0}\right)^{n-2+\sigma} \frac{\lambda^{-2}}{1+|y|^{2+\alpha}}.
\end{aligned}$$

This concludes the proof of the second estimate in (5.10).

We now prove (4.16). Observe that, for any pair of functions ψ_1, $\psi_2 \in B_M$, we have

$$\|\mathbf{S}(\psi_1) - \mathbf{S}(\psi_2)\|_\alpha \leq c\|N(\mathbf{w}_1) - N(\mathbf{w}_2)\|_{**,\alpha}$$

since g and h, as defined in (5.7), do not depend on ψ. Here we denote

$$\mathbf{w}_j = \psi_j(x,t) + \eta_R(x,t)\Phi(x,t), \quad j = 1, 2.$$

We refer to (2.18) for the definition of N. Using again (4.18), we can write

$$\lambda^{\frac{n-2}{2}} |N(\mathbf{w}_1) - N(\mathbf{w}_2)| (\lambda y + \xi, t) \lesssim \lambda^{-2} \Big[(U + \rho + \lambda^{\frac{n-2}{2}} (\psi_1 + \eta_R \Phi))^p$$

$$- (U + \rho + \lambda^{\frac{n-2}{2}} (\psi_2 + \eta_R \Phi))^p - p(U+\rho)^{p-1} \lambda^{\frac{n-2}{2}} |\psi_1 - \psi_2| \Big]$$

$$\lesssim \left(\frac{\lambda_0}{d_0}\right)^{(n-2+\sigma)p} \frac{\lambda^{-2}}{1 + |y|^{2+\alpha}} \|\psi_1 - \psi_2\|_\alpha^p.$$

From here, we get the validity of (4.16), thanks to the fact that R is large and T is small. This concludes the proof of the Proposition.

□

We further observe that the solution $\psi = \psi[\lambda_1, d_1, \phi]$ to Problem (4.4) clearly depends on the parameter functions λ_1, d_1 and ϕ. Next Proposition clarifies that ψ is Lipschitz with respect to λ_1, d_1 and ϕ and their respective topologies.

Lemma 4.2. *Assume the validity of the hypothesis in Proposition 4.1. Taking R large and T small, there exists $\mathbf{c} \in (0,1)$ small so that, for any λ_1^1, λ_1^2 satisfying* (2.7)*, we have*

$$\|\psi[\lambda_1^1, d_1, \phi] - \psi[\lambda_1^2, d_1, \phi]\|_\alpha \leq \mathbf{c}\|\dot\lambda_1^1 - \dot\lambda_1^2\|_{\frac{1+\sigma}{n-4}}, \tag{4.19}$$

for any d_1^1, d_1^2 satisfying (2.7)*,*

$$\|\psi[\lambda_1, d_1^1, \phi] - \psi[\lambda_1, d_1^2, \phi]\|_\alpha \leq \mathbf{c}\|\dot d_1^1 - \dot d_1^2\|_{\frac{1+\sigma}{n-4}}, \tag{4.20}$$

and, for any ϕ_1, ϕ_2 satisfying (4.3)

$$\|\psi[\lambda_1, d_1, \phi_1] - \psi[\lambda_1, d_1, \phi_2]\|_\alpha \leq \mathbf{c}\|\phi_1 - \phi_2\|_{in}, \tag{4.21}$$

Proof. Estimates (4.19) and (4.20) follows from the Lipshitz bound on the error function $\bar E_2$ contained in (3.20) and (3.22), and from the Lipschitz bound on the value of W_2 on the boundary $\partial\mathcal{D}$ as described in (3.21) and (3.23). We leave the details to the reader.

We shall prove (4.21). As in the argument to show the first estimate in (5.10), we need to chose the number a in the definition of the norm (4.1) and the number α in the definition of the norm (3.17) so that $a > \alpha$.

Let λ_1 and d_1 be fixed, and let ϕ_1 and ϕ_2 satisfying (4.3). Let $\psi_i = \psi[\phi_i]$ the solution corresponding to ϕ_i, with the same λ_1 and d_1. Define $\psi = \psi_1 - \psi_2$. It solves

$$\begin{aligned}\psi_t &= \Delta\psi + \frac{1}{x_1}\frac{\partial\psi}{\partial x_1} + V\psi + \left(\Delta - \frac{\partial}{\partial t}\right)\eta_R\,\Phi + 2\nabla\Phi\nabla\eta_R + \frac{1}{x_1}\frac{\partial\eta_R}{\partial x_1}\Phi\\ &+ p\left[W_2^{p-1} - [\lambda_0^{-\frac{n-2}{2}}U(\frac{x-\xi}{\lambda_0})]^{p-1}\right]\eta_{R'}\eta_R\Phi\\ &+ N[\mathbf{w}_1] - N[\mathbf{w}_2] \quad \text{in} \quad \mathcal{D}\times(0,T)\\ \psi &= 0, \quad \text{on} \quad \partial\mathcal{D}\times(0,T), \quad \psi = 0, \quad \text{in} \quad \mathcal{D}\times\{t=0\}.\end{aligned} \tag{4.22}$$

where

$$\mathbf{w}_1(x,t) = \psi_1(x,t) + \eta_R(x,t)\Phi_1(x,t), \quad \Phi_i = \lambda_0^{-\frac{n-2}{2}}\phi_i\left(\frac{x-\xi}{\lambda_0}, t\right), \quad \Phi = \Phi_1 - \Phi_2.$$

To get (4.21), it is convenient to decompose ψ into a first part that it is linear in Φ, and the rest. We write

$$\psi = \bar\psi + \hat\psi$$

with $\hat\psi$ solution to

$$\begin{aligned}\psi_t &= \Delta\psi + \frac{1}{x_1}\frac{\partial\psi}{\partial x_1} + V\psi + \left(\Delta - \frac{\partial}{\partial t}\right)\eta_R\,\Phi + 2\nabla\Phi\nabla\eta_R + \frac{1}{x_1}\frac{\partial\eta_R}{\partial x_1}\Phi\\ &+ p\left[W_2^{p-1} - [\lambda_0^{-\frac{n-2}{2}}U(\frac{x-\xi}{\lambda_0})]^{p-1}\right]\eta_{R'}\eta_R\Phi \quad \text{in} \quad \mathcal{D}\times(0,T)\\ \psi &= 0, \quad \text{on} \quad \partial\mathcal{D}\times(0,T), \quad \psi = 0, \quad \text{in} \quad \mathcal{D}\times\{t=0\}.\end{aligned}$$

Arguing as in the proof of (4.17), we get

$$\| \left(\Delta - \frac{\partial}{\partial t}\right)\eta_R\,\Phi + 2\nabla\Phi\nabla\eta_R + \frac{1}{x_1}\frac{\partial\eta_R}{\partial x_1}\Phi\|_{**,\alpha} \lesssim R^{-a+\alpha}\|\phi_1 - \phi_2\|_{in}.$$

Arguing as in the proof of the second estimate in (5.10), we get

$$\|p\left[W_2^{p-1} - [\lambda_0^{-\frac{n-2}{2}}U(\frac{x-\xi}{\lambda_0})]^{p-1}\right]\eta_{R'}\eta_R\Phi\|_{**,\alpha} \lesssim T^{\varepsilon}\|\phi_1 - \phi_2\|_{in},$$

for some $\varepsilon > 0$. Applying Lemma 4.1, we obtain

$$\|\hat\psi\|_\alpha \le \mathbf{c}\|\phi_1 - \phi_2\|_{in} \tag{4.23}$$

with the constant $\mathbf{c} \in (0,1)$ if we choose R large and T small. In order to estimate $\bar\psi$, we observe that

$$\begin{aligned}N[\mathbf{w}_1] - N[\mathbf{w}_2] &= (W_2 + \psi_1 + \eta_R\Phi_1)^p - (W_2 + \psi_2 + \eta_R\Phi_2)^p\\ &\quad - pW_2^{p-1}\left(\psi_1 + \eta_R\Phi_1 - \psi_2 - \eta_R\Phi_2\right).\end{aligned}$$

Using again the notation introduced in (4.18), we get

$$\begin{aligned}\lambda^{\frac{n-2}{2}}|N[\mathbf{w}_1]-N[\mathbf{w}_2]|(\lambda y+\xi) &= \lambda^{-2}\Big|\Big(U+\rho+\lambda^{\frac{n-2}{2}}(\psi_1+\eta_R\Phi_1)\Big)^p \\ &- \Big(U+\rho+\lambda^{\frac{n-2}{2}}(\psi_2+\eta_R\Phi_2)\Big)^p \\ &- p(U+\rho)^{p-1}\lambda^{\frac{n-2}{2}}\left(\psi_1+\eta_R\Phi_1-\psi_2-\eta_R\Phi_2\right)\Big| \\ &\lesssim \lambda^{-2}\left[|\lambda^{\frac{n-2}{2}}(\psi_1-\psi_2)|^p+|\lambda^{\frac{n-2}{2}}\eta_R(\Phi_1-\Phi_2)|^p\right] \\ &\lesssim \left(\frac{\lambda_0}{d_0}\right)^{(n-2+\sigma)p}\left[\frac{\lambda^{-2}}{1+|y|^{2+\alpha}}\|\psi\|_\alpha^p+\frac{\lambda^{-2}}{1+|y|^{2+\alpha}}\|\Phi_1-\Phi_2\|_{in}^p\right].\end{aligned}$$

Thanks to the above estimate, the non linear equation satisfied by $\bar\psi$ can be solved applying Lemma 4.1 and a fixed point argument of contraction type for functions $\bar\psi$ satisfying

$$\|\bar\psi\|_\alpha \le \mathbf{c}\|\phi_1-\phi_2\|_{in}$$

for some $\mathbf{c}\in(0,1)$, provided T is chosen small and R large. This fact, together with (4.23), give the validity of (4.21) □

A last remark in in order.

Remark 4.1. *Proposition 4.1 defines the solution to Problem* (4.4) *as a function of the initial condition* ψ_0*, in the form of an operator* $\psi=\bar\Psi[\psi_0]$*, from a neighborhood of* 0 *in the Banach space* $C_0(\mathcal{D})$ *equipped with the* C^1 *norm* $\|\psi_0\|_{L^\infty(\mathcal{D})}+\|\nabla\psi_0\|_{L^\infty(\mathcal{D})}$ *into the Banach space of functions* $\psi\in L^\infty(\mathcal{D})$ *equipped with the norm* $\|\psi\|_{**,\beta,\alpha}$ *, defined in* (4.1)*.*

A closer look to the proof of Proposition 4.1, and the Implicit Function Theorem give that $\psi_0\to\bar\Psi[\psi_0]$ *is a diffeomorphism, and that*

$$\|\bar\Psi[\psi_0^1]-\bar\Psi[\psi_0^2]\|_\alpha\le c\left[\|\psi_0^1-\psi_0^2\|_{L^\infty(\mathcal{D})}+\|\nabla\psi_0^1-\nabla\psi_0^2\|_{L^\infty(\mathcal{D})}\right],$$

for some positive constant c*.*

5. Finding the parameter functions

As mentioned in Section 2, we can solve the *inner* Problem (2.27), provided that certain orthogonality condition of the "right-hand side" as in (3.24) are satisfied. In this Section we first derive the system of ordinary differential equations in λ_1 and d_1 that is equivalent to get the orthogonality conditions satisfied. Then we find parameter functions λ_1 and d_1 which solve these ODEs. This is done, for any ϕ fixed, and satisfying (4.3), while ψ is already fixed as the solution of the *outer* problem (2.21), as stated in Proposition 4.1. We conclude the Section showing that the solution λ_1 and d_1 Lipstitz depends on ϕ.

We start with

Lemma 5.1. *Assume that the parameters* λ_1 *and* d_1 *satisfy* (2.7)*, that* R *is large and* T *is small, and that the function* ϕ *satisfies the constraint* (4.3)*. Let* ψ *be the solution to Problem* (2.21)*, whose existence and properties are stated in Proposition*

4.1 and Lemma 4.2. Let $H = H[\lambda, d, \phi, \psi]$ be the function defined in (2.28)*. Then, for any T small, we have the validity of the following expansions*

$$
\begin{aligned}
\int_{B(0,2R)} H(y,t) Z_1(y)\, dy &= \lambda \dot d_1 \left(\int_{\mathbb{R}^n} Z_1^2 \right) a_{0,R} \left(1 + p(t) + \bar q_1 (\frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}, \phi) \right) \\
&+ \left(\frac{\lambda_0}{d_0} \right)^{n-2+\sigma} a_{0,R} \left(1 + p(t) + q_1 (\frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}, \phi) \right) \\
&+ \lambda \dot\lambda_1 \left(\frac{\lambda_0}{d_0} \right)^{n-2} R\, a_{0,R} \left(1 + p(t) + q_1 (\frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}, \phi) \right)
\end{aligned}
\tag{5.1}
$$

and

$$
\begin{aligned}
\int_{B(0,2R)} H(y,t) Z_0(y)\, dy &= \lambda \dot\lambda_1 a_{0,R} A \left(1 + p(t) + \bar q_1 (\frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}, \phi) \right) \\
&- \left(\frac{\lambda_0}{d_0} \right)^{n-2} B \left[\frac{\lambda_1}{\lambda_0} - \frac{d_1}{d_0} \right] a_{0,R} \left(1 + p(t) + q_1 (\frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}, \phi) \right) \\
&+ \left(\frac{\lambda_0}{d_0} \right)^{n-2+\sigma} a_{0,R} \left(1 + p(t) + q_1 (\frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}, \phi) \right) \\
&+ \lambda \dot d_1 \left(\frac{\lambda_0}{d_0} \right)^{n-1} R^2\, a_{0,R}\, q_1 (\frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}, \phi),
\end{aligned}
\tag{5.2}
$$

where A and B are the constants given by

$$
A = \int_{\mathbb{R}^n} Z_0^2(y)\, dy, \quad B = \frac{p(n-3)\alpha_n}{2^{n-2}} \left(\int_{\mathbb{R}^n} U^{p-1} Z_0 \right). \tag{5.3}
$$

Here $\sigma \in (0,1)$ is the number fixed in (2.7)*, which can be thought as close to* 1*. With $a_{0,R}$ we denote generic constants (i.e. independent of t) with $a_{0,R} = 1 + o(R^{-1})$, as $R \to \infty$. Here $p = p(t)$ denotes a generic function, which is smooth for $t \in (0,T)$ so that, for some $\sigma > 0$, $\|p\|_\sigma$ is uniformly bounded, as $T \to 0$. We refer to* (2.6) *for the definition of the $\| \cdot \|_\sigma$-norm. The explicit expression of $p = p(t)$ changes from line to line. Moreover, $q_1 = q_1(\eta_1, \eta_2, \phi)$ denotes another generic function, which is smooth in its variable, uniformly bounded, as $t \to T$, for η_1 , $\eta_2 \in L^\infty(0,T)$, and ϕ satisfying* (4.3)*, with $q_1(0,0,0) = 0$, and for any $t \in (0,T)$*

$$
\left| q_1[\eta_1^1, \eta_2, \phi](t) - q_1[\eta_1^2, \eta_2, \phi](t) \right| \lesssim \|\eta_1^1 - \eta_1^2\|_{L^\infty(0,T)} \tag{5.4}
$$

$$
\left| q_1[\eta_1, \eta_2^1, \phi](t) - q_1[\eta_1, \eta_2^2, \phi](t) \right| \lesssim \|\eta_2^1 - \eta_2^2\|_{L^\infty(0,T)} \tag{5.5}
$$

$$
|q_1[\eta_1, \eta_2, \phi_1](t) - q_1[\eta_1, \eta_2, \phi_2](t)| \lesssim T^a \|\phi_1 - \phi_2\|_{in}, \tag{5.6}
$$

for some $a > 0$ small. The explicit expression of q_1 also changes from line to line. The function $\bar q_1$ share the same properties as q_1, and moreover $\bar q_1(\eta_1, \eta_2, \bar\phi + \hat\phi) = \bar q_1(\eta_1, \eta_2, \bar\phi) + \bar q_1(\eta_1, \eta_2, \hat\phi)$.

Proof. We write

$$
H = \sum_{j=1}^3 H_j, \quad H_1 = p \lambda_0^{\frac{n-2}{2}} U^{p-1} \psi(\lambda_0 y + \xi, t(\tau)), \quad H_3 = B[\phi]. \tag{5.7}
$$

The proof of (5.1) is consequence of the following three expansions, as $T \to 0$,

$$\int_{B(0,2R)} H_2 Z_1\, dy = \lambda \dot d_1 \left(\int_{\mathbb{R}^n} Z_1^2 \right) a_{0,R} \left(1 + q_1(\frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}, 0) \right)$$
$$- \left(\frac{\lambda_0}{d_0} \right)^{n-1} \frac{p(n-2)\alpha_n}{2^{n-1}} \left(\int_{\mathbb{R}^n} U^{p-1} y_1 Z_1 \right) a_{0,R} \left(1 + p(t) + q_1(\frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}, 0) \right) \quad (5.8)$$
$$+ \lambda \dot\lambda_1 \left(\frac{\lambda_0}{d_0} \right)^{n-2} R\, a_{0,R} \left(1 + p(t) + q_1(\frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}, 0) \right),$$

$$\int_{B(0,2R)} H_1 Z_1\, dy = \left(\frac{\lambda_0}{d_0} \right)^{n-2+\sigma} a_{0,R} \left(1 + p(t) + q_1(\frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}, \phi) \right), \quad (5.9)$$

and

$$\int_{B(0,2R)} H_3 Z_1\, dy = \left(\frac{\lambda_0}{d_0} \right)^{2n-2+\sigma} a_{0,R}\, O(R^{-2})\, \bar q_1(0,0,\phi)$$
$$+ \lambda_0 \left(\frac{\lambda_0}{d_0} \right)^{n-2+\sigma} \dot d_1\, O(R^{-1}) \bar q_1(0,0,\phi) \quad (5.10)$$

where we are using the same notations as in the statement of the Lemma.

Proof of (5.8). In the region $y \in B(0,2R)$, the function

$$H_2 = \lambda_0^{\frac{n+2}{2}} \left[e_1 + e_2 - \left(\frac{\lambda_0}{d_0} \right)^{n-2} \left[\Delta w + p W_0^{p-1} w \right] \right] (\lambda_0 y + \xi, t(\tau))$$

has been described in Lemma 3.2. Referring to (3.13), we see immediately that $\int_{B(0,2R)} E_{2,\lambda}(y,t) Z_1(y)\, dy = 0$, for all t, because of symmetry. We get

$$\int_{B(0,2R)} H_2 Z_1\, dy = \lambda\, [\dot d_1 - \frac{d_0}{1+d}] \int_{B(0,2R)} Z_1^2(y)\, dy$$
$$+ \lambda \dot d_1 \left(\frac{\lambda_0}{d_0} \right)^{n-2} \int_{B(0,2R)} \frac{\partial h}{\partial y_1} Z_1(y)\, dy$$
$$- \left(\frac{\lambda_0}{d_0} \right)^{n-1} \frac{p(n-2)\alpha_n}{2^{n-1}} \left(\int_{B(0,2R)} U^{p-1} y_1 Z_1 \right)$$
$$+ \lambda \dot d_1 \left(\frac{\lambda_0}{d_0} \right)^{n-1} O\left(\int_{B(0,2R)} |Z_1(y)|\, dy \right)$$
$$+ \lambda \lambda_1 \left(\frac{\lambda_0}{d_0} \right)^{n-2} O\left(\int_{B(0,2R)} |Z_1(y)|\, dy \right)$$
$$+ \left(\frac{\lambda_0}{d_0} \right)^{n+2} O\left(\int_{B(0,2R)} |Z_1(y)|\, dy \right).$$

Expansion (5.8) follows after we observe that

$$\int_{B(0,2R)} Z_1^2(y)\, dy = (\int_{\mathbb{R}^n} Z_1^2(y)\, dy)(1+O(R^{2-n})), \quad \int_{B(0,2R)} \frac{\partial h}{\partial y_1} Z_1(y)\, dy = O(R^{-2}),$$

$$\int_{B(0,2R)} U^{p-1} y_1 Z_1 = \left(\int_{\mathbb{R}^n} U^{p-1} y_1 Z_1 \right) (1 + O(R^{-2})), \quad \int_{B(0,2R)} |Z_1| = O(R),$$

for R large.

Proof of (5.9). From the result of Proposition 4.1, and more specifically estimate (4.7), we expand

$$\psi(\xi+\lambda_0 y,t)=\psi(\xi)+\nabla\psi(\bar\xi)\lambda_0 y$$

for some $\bar\xi$. By symmetry, the integral of the first term is zero, so that we get

$$\int_{B(0,2R)} H_1 Z_1 = (\frac{\lambda_0}{d_0})^{n-2+\sigma}\left(1+p(t)+q_1(\frac{\lambda_1}{\lambda_0},\frac{d_1}{d_0},0)\right)(\int_{B(0,2R)} U^{p-1} y_1 Z_1).$$

Thus we get the validity of (5.9).

Proof of (5.10). From the definition of the function $B[\phi]$, we immediately observe that this is a linear function of ϕ, and it does not depend on λ_1. A direct computation and the use of the estimate on ϕ given in (4.3) gives

$$\int_{B(0,2R)} H_3 Z_1 = \lambda_0\dot\lambda_0\int_{B(0,2R)}\left[\frac{n-2}{2}\phi(y,t)+\nabla\phi(y,t)\cdot y\right]Z_1+\lambda_0\dot d\int_{B(0,2R)}\frac{\partial\phi}{\partial y_1}(y,t)Z_1$$
$$=\left(\frac{\lambda_0}{d_0}\right)^{2n-2+\sigma} a_{0,R}\,O(R^{-1+\alpha})\,q_1(0,0,\phi)+\lambda_0\left(\frac{\lambda_0}{d_0}\right)^{n-2+\sigma}\dot d_1\,\bar q_1(0,0,\phi).$$

The proof of (5.2) is consequence of the following three expansions, as $T\to 0$,

$$\begin{aligned}\int_{B(0,2R)} H_2 Z_0\,dy &= \lambda\dot\lambda_1 a_{0,R}\left(1+p(t)+q_1(\frac{\lambda_1}{\lambda_0},\frac{d_1}{d_0},0)\right)\int_{\mathbb{R}^n} Z_0^2(y)\,dy\\ &-\left(\frac{\lambda_0}{d_0}\right)^{n-2}\frac{p(n-3)\alpha_n}{2^{n-2}}\left[\frac{\lambda_1}{\lambda_0}-\frac{d_1}{d_0}\right]\left(\int_{\mathbb{R}^n} U^{p-1}Z_0\right)\times\\ &\times a_{0,R}\,(1+p(t)+q_1(\frac{\lambda_1}{\lambda_0},\frac{d_1}{d_0},0))\\ &+\lambda\dot d_1\left(\frac{\lambda_0}{d_0}\right)^{n-1}O(R^2)\,q_1(\frac{\lambda_1}{\lambda_0},\frac{d_1}{d_0},0)+\left(\frac{\lambda_0}{d_0}\right)^{n+2}O(R^2)\,q_1(\frac{\lambda_1}{\lambda_0},\frac{d_1}{d_0},0),\end{aligned}\tag{5.11}$$

$$\int_{B(0,2R)} H_1 Z_0\,dy=\left(\frac{\lambda_0}{d_0}\right)^{n-2+\sigma} a_{0,R}\,\left(1+p(t)+q_1(\frac{\lambda_1}{\lambda_0},\frac{d_1}{d_0},\phi)\right),\tag{5.12}$$

and

$$\begin{aligned}\int_{B(0,2R)} H_3 Z_0\,dy&=\left(\frac{\lambda_0}{d_0}\right)^{2n-2+\sigma} a_{0,R}\,O(R^{-1+\alpha})\,q_1(0,0,\phi)\\ &+\lambda_0\left(\frac{\lambda_0}{d_0}\right)^{n-2+\sigma}\dot d_1\,\bar q_1(0,0,\phi),\end{aligned}\tag{5.13}$$

where we are using the same notations as in the statement of the Lemma. The proofs of (5.12) and (5.13) are similar to the ones of (5.9) and (5.10) respectively, so we leave them to the reader.

Proof of (5.11). Referring again to (3.13) for the expression of H_2 in the region we are considering, we see immediately that $\int_{B(0,2R)} E_{2,d}(y,t)Z_0(y)\,dy=0$, for all

t, because of symmetry. We get

$$
\begin{aligned}
\int_{B(0,2R)} H_2 Z_0\, dy &= [\lambda\dot\lambda_1 + \dot\lambda_0\lambda_1] \int_{B(0,2R)} Z_0^2(y)\, dy \\
&+ [\lambda\dot\lambda_1 + \dot\lambda_0\lambda_1] \left(\frac{\lambda_0}{d_0}\right)^{n-2} \int_{B(0,2R)} (\frac{n-2}{2} h + \nabla h\cdot y) Z_0(y)\, dy \\
&- \left(\frac{\lambda_0}{d_0}\right)^{n-2} \frac{p(n-2)\alpha_n}{2^{n-2}} \left[\frac{\lambda_1}{\lambda_0} - \frac{d_1}{d_0}\right] \left(\int_{B(0,2R)} U^{p-1} Z_0\right) \times \\
&\times (1 + p(t) + q_1(\frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}, 0)) \\
&+ \left[\lambda \dot d_1 \left(\frac{\lambda_0}{d_0}\right)^{n-1} + \lambda\dot\lambda_1 \left(\frac{\lambda_0}{d_0}\right)^{n-2}\right] O(R^2)\, q_1(\frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}, 0) \\
&+ \left(\frac{\lambda_0}{d_0}\right)^{n+2} O(R^2)\, q_1(\frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}, 0).
\end{aligned}
$$

Using (2.13), that is $\dot\lambda_0 \int_{\mathbb{R}^n} Z_0^2 = \frac{p\alpha_n}{2^{n-2}} (\frac{\lambda_0}{d_0})^{n-2} \int_{\mathbb{R}^n} U^{p-1} Z_0$, we get

$$
\begin{aligned}
\int_{B(0,2R)} H_2 Z_0\, dy &= \lambda\dot\lambda_1 a_{0,R} \left(1 + p(t) + q_1(\frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}, 0)\right) \int_{\mathbb{R}^n} Z_0^2(y)\, dy \\
&- \left(\frac{\lambda_0}{d_0}\right)^{n-2} \frac{p(n-3)\alpha_n}{2^{n-2}} \left[\frac{\lambda_1}{\lambda_0} - \frac{d_1}{d_0}\right] \left(\int_{\mathbb{R}^n} U^{p-1} Z_0\right) \times \\
&\times a_{0,R}\, (1 + p(t) + q_1(\frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}, 0)) \\
&+ \lambda \dot d_1 \left(\frac{\lambda_0}{d_0}\right)^{n-1} O(R^2)\, q_1(\frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}, 0) + \left(\frac{\lambda_0}{d_0}\right)^{n+2} O(R^2)\, q_1(\frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}, 0).
\end{aligned}
$$

Thus (5.11) follows. □

Next we get the existence and Lipschtz properties of λ_1 and d_1 that make the required orthogonality conditions.

Proposition 5.1. *Let ψ be the solution to Problem* (2.21)*, whose existence and properties are stated in Proposition 4.1 and Lemma 4.2. Let $H = H[\lambda, d, \phi, \psi]$ be the function defined in* (2.28)*. For any function ϕ satisfying the constraint* (4.3)*, there exist functions $\lambda_1 = \lambda_1[\phi]$ and $d_1 = d_1[\phi]$, which satisfy the bound* (2.7)*, for which*

$$
\begin{aligned}
&\int_{B(0,2R)} H(y,t)\, Z_0(y)\, dy = 0 \quad \textit{for all} \quad t \in (0,T) \\
&\int_{B(0,2R)} H(y,t)\, Z_1(y)\, dy = 0, \quad \textit{for all} \quad t \in (0,T).
\end{aligned}
\tag{5.14}
$$

Moreover, if ϕ_1 and ϕ_2 satisfy (4.3)*, one has*

$$
\|\dot\lambda_1[\phi_1] - \dot\lambda_1[\phi_2]\|_{\frac{1+\sigma}{n-4}} + \|\dot d_1[\phi_1] - \dot d_1[\phi_2]\|_{\frac{1+\sigma}{n-4}} \le \mathbf{c} \|\phi_1 - \phi_2\|_{in} \tag{5.15}
$$

for some $\mathbf{c} \in (0,1)$, provided T is small and R is large.

Proof. The result of Lemma 5.1 is telling us that solving equation (5.14) is equivalent to solving a certain non linear non local system of ordinary differential equation of first order in λ_1 and d_1. Indeed, from (5.1) and using the fact that $(\frac{\lambda_0}{d_0})^{n-2+\sigma}\frac{1}{\lambda_0} = c(T-t)^{\frac{1+\sigma}{n-4}}$ (see (2.5)), for some constant c, we get that the second equation in (5.14) is equivalent to

$$\begin{aligned} \dot d_1 + A_R(T-t)^{\frac{1+\sigma}{n-4}} &= (T-t)^{\frac{1+\sigma}{n-4}}\left(p(t) + q_1\left(\frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}, \phi\right)\right) \\ &\quad + (T-t)^{1+\frac{1}{n-4}}\dot\lambda_1\left(1 + p(t) + q_1\left(\frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}, \phi\right)\right) \end{aligned} \tag{5.16}$$

where A_R is a constant (independent of t). The functions p and q_1 have the same properties as stated in Lemma 5.1.

We next look at (5.2) to get the differential equation corresponding to the first equation in (5.14). Using (2.5)-(2.14)-(2.13), we get

$$\begin{aligned} \dot\lambda_1 - \frac{(n-3)}{(T-t)}\lambda_1 &= (T-t)^{\frac{1+\sigma}{n-4}} f(t) + \frac{\lambda_1}{(T-t)} q_1\left(\frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}, \phi\right) \\ &\quad + (T-t)^{1+\frac{1}{n-4}}\dot d_1\left(1 + p(t) + q_1\left(\frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}, \phi\right)\right) \end{aligned}$$

Here $f = f(t)$ stands for a uniformly bounded in $(0,T)$, as $T \to 0$. It is convenient to multiply the above equation against $(T-t)^{n-3}$, and re-write it as

$$\begin{aligned} \frac{d}{dt}\left((T-t)^{n-3}\lambda_1\right) &= (T-t)^{n-3+\frac{1+\sigma}{n-4}} f(t) + (T-t)^{n-4}\,\lambda_1\, q_1\left(\frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}, \phi\right) \\ &\quad + (T-t)^{n-2+\frac{1}{n-4}}\dot d_1\left(1 + p(t) + q_1\left(\frac{\lambda_1}{\lambda_0}, \frac{d_1}{d_0}, \phi\right)\right) \end{aligned} \tag{5.17}$$

Let $\mathbf{d} = \mathbf{d}(t)$ and $\Lambda = \Lambda(t)$ be the solution to

$$\begin{aligned} \dot{\mathbf{d}} + A_R(T-t)^{\frac{1+\sigma}{n-4}} &= (T-t)^{\frac{1+\sigma}{n-4}} p(t) \\ \frac{d}{dt}\left((T-t)^{n-3}\Lambda\right) &= (T-t)^{n-3+\frac{1+\sigma}{n-4}} f(t), \end{aligned} \tag{5.18}$$

given by

$$\begin{aligned} \mathbf{d}(t) &= \int_t^T \left[-A_R(T-s)^{\frac{2}{n-4}} + (T-s)^{\frac{2}{n-4}} p(s)\right] ds \\ \Lambda(t) &= \frac{1}{(T-t)^{n-3}}\int_t^T (T-s)^{n-3+\frac{2}{n-4}} f(s)\, ds. \end{aligned} \tag{5.19}$$

These functions satisfy the bound (2.7). Then $d_1 = \mathbf{d} + d$, $\lambda_1 = \Lambda + \lambda$ solves the system (5.16)-(5.17) if

$$
\begin{aligned}
d(t) &= \int_t^T (T-s)^{\frac{1+\sigma}{n-4}} q_1 \left(\frac{\Lambda+\lambda}{\lambda_0}, \frac{\mathbf{d}+d}{d_0}, \phi \right)(s)\, ds \\
&\quad + \int_t^T (T-s)^{1+\frac{1}{n-4}} (\dot\Lambda + \dot\lambda)(s) \left(1 + p(s) + q_1 \left(\frac{\Lambda+\lambda}{\lambda_0}, \frac{\mathbf{d}+d}{d_0}, \phi \right) \right) ds \\
\lambda(t) &= \frac{1}{(T-t)^{n-3}} \int_t^T (T-s)^{n-4} \lambda_1(s)\, q_1 \left(\frac{\Lambda+\lambda}{\lambda_0}, \frac{\mathbf{d}+d}{d_0}, \phi \right) ds \\
&\quad + \frac{1}{(T-t)^{n-3}} \int_t^T (T-s)^{n-2+\frac{1}{n-4}} \dot d_1(s) \left(1 + p(s) + q_1 \left(\frac{\Lambda+\lambda}{\lambda_0}, \frac{\mathbf{d}+d}{d_0}, \phi \right) \right).
\end{aligned} \tag{5.20}
$$

Using again the result of Lemma 5, and in particular (5.4)-(5.5), one can solve (5.20) with a fixed point argument based on the Contraction Mapping Theorem. Estimate (5.15) follows from (5.20) and (5.6).

□

6. Solving the inner problem

The last step in the proof of our result is to solve the *inner* Problem (2.27), after we already defined the *outer* solution ψ, whose existence and properties are contained in Proposition 4.1 and Lemma 4.2 in Section 5, and the parameter functions d_1, λ_1, as in Proposition 5.1 in Section 5.

The key ingredient to solve (2.27) for functions ϕ satisfying (4.3) is the resolution of following linear problem: Given a sufficiently large number $R > 0$, construct a solution (ϕ, e_0) to the initial value problem

$$
\phi_\tau = \Delta\phi + pU(y)^{p-1}\phi + h(y,\tau) \quad \text{in } B_{2R} \times (\tau_0, \infty) \tag{6.1}
$$

$$
\phi(y, \tau_0) = e_0 Z(y) \quad \text{in } B_{2R}
$$

provided that h satisfies certain time-space decay rate and certain orthogonality conditions. Here Z is the positive radially symmetric bounded eigenfunction associated to the only negative eigenvalue to the linear problem (2.29). We recall that $\tau = \tau(t)$ is given in (2.26), as

$$
\tau(t) = \frac{n-4}{(n-2)\ell} (T-t)^{-1-\frac{2}{n-4}}, \quad \text{and} \quad \tau_0 = \tau(0).
$$

In the τ-variable, the bound (4.3) on ϕ reads as

$$
\begin{aligned}
\|\phi\|_{in} = & \sup_{\tau>\tau_0,\, y\in B(0,2R)} \tau^\nu (1+|y|^a) |\phi(y,\tau)| \\
& + \sup_{\tau>\tau_0,\, y\in B(0,2R)} \tau^\nu \lambda (1+|y|^{1+a}) |\nabla\phi(y,\tau)| \lesssim \max\{T^{\frac{1-\sigma}{(n-4)}}, R^{-2}\}
\end{aligned} \tag{6.2}
$$

where $\nu = \frac{n-2+\sigma}{n-2}$ so to have

$$
\tau^{-\nu} \sim \lambda^{\frac{n-2}{2}} (T-t)^{-\beta} \sim \left(\frac{\lambda_0}{d_0}\right)^{n-2+\sigma} \sim (T-t)^{\frac{n-2+\sigma}{n-4}}.
$$

Here $\sigma \in (\frac{1}{2}, 1)$ is the constant (which can be thought close to 1) introduced in (2.7). The solution for Problem (6.1) we build has R-dependent uniform bounds for right hand-side h with L^∞-weighted norms of the type

$$\|h\|_{\nu,2+a} := \sup_{\tau>\tau_0} \sup_{y\in B_{2R}} \tau^\nu \, \lambda^2 \, (1+|y|^{2+a}) \, |h(y,\tau)|. \tag{6.3}$$

Also, for a function $p = p(\tau)$ we denote

$$\|p\|_\nu := \sup_{\tau>\tau_0} \tau^\nu |p(\tau)|.$$

We have the validity of

Proposition 6.1. *Let $R > 0$ be large enough. For any τ_0 sufficiently large (depending on R), for any $h = h(y,\tau)$ with $\|h\|_{\nu,2+a} < +\infty$ that satisfies for all $j = 0, 1$*

$$\int_{B(0,2R)} h(y,\tau)\, Z_j(y)\, dy \;=\; 0 \quad \text{for all} \quad \tau \in (\tau_0, \infty), \tag{6.4}$$

there exist $\phi = \phi[h]$ and $e_0 = e_0[h]$ which solve Problem (6.1). *They define linear operators of h that satisfy the estimates*

$$(1+|y|)|\nabla\phi(y,\tau)| + |\phi(y,\tau)| \;\lesssim\; \tau^{-\nu} \frac{R^{n+1-a}}{1+|y|^a} \|h\|_{\nu,2+a}, \tag{6.5}$$

and

$$|e_0[h]| \;\lesssim\; \|h\|_{\nu,2+a}. \tag{6.6}$$

The proof of this Proposition is an adaptation to our symmetric setting of the result contained in Proposition 7.1 in [7]. For completeness, we will give a resumed proof of this Proposition in Section 7.

Proposition 6.1 states the existence of a linear operator $\mathbf{S}$ which to any function $h(y,\tau)$, with $\|h\|_{\nu,2+a}$-bounded and satisfying (7.2), associates the solution (ϕ, e_0) to (6.1). Furthermore, it states that $\mathbf{S}$ is continuous between L^∞ spaces equipped with the topologies described by (6.5)-(6.6).

We want to use Proposition 6.1 to solve the *inner* problem (2.27). Up to this moment in our argument, the radius R was chosen large and the final time T was chosen small, one independently from the other. Thus, let R be fixed arbitrarily large. We claim that, for any T small enough (or equivalently for any $\tau_0 = \frac{n-4}{(n-2)\ell} T^{-1-\frac{2}{n-4}}$ large enough) in terms of R, Problem (2.27) has a solution. Indeed, we observe first that the parameter functions λ_1 and d_1 as defined in Section 5 are such that the right-hand side $H(y,t)$ satisfies the orthogonality condition (7.2), for any $t \in (0,T)$ (or equivalently for any $\tau > \tau_0$). Thus, the existence and properties of ϕ and e_0 solution to (2.27) are reduced to find a fixed point for

$$\phi = \mathcal{A}(\phi), \quad \text{where} \quad \mathcal{A}(\phi) := \mathbf{S}\left(H(\lambda_1[\phi], d_1[\phi], \phi, \psi[\phi])\right)$$

in a proper set of functions. We recall the definition of H given in (2.28)

$$\begin{aligned} H[\lambda,d,\phi,\psi](y,\tau) &= p\lambda_0^{\frac{n-2}{2}} U^{p-1}\psi(\lambda_0 y + \xi, t(\tau)) \\ &\quad + \lambda_0^{\frac{n+2}{2}} E_2(\lambda_0 y + \xi, t(\tau)) + B[\phi], \end{aligned}$$

where we recall that $E_2 = e_1 + e_2 - \left(\frac{\lambda_0}{d_0}\right)^{n-2}\left[\Delta w + pW_0^{p-1}w\right]$. From Lemma 3.2, we get that

$$\|\lambda_0^{\frac{n+2}{2}} E_2(\lambda_0 y + \xi, t(\tau))\|_{\nu,2+a} \lesssim \max\{T^{\frac{1-\sigma}{(n-4)}}, R^{-2}\},$$

while from Proposition 4.1 we obtain

$$\|p\lambda_0^{\frac{n-2}{2}} U^{p-1}\psi(\lambda_0 y + \xi, t(\tau))\|_{\nu,2+a} \lesssim \max\{T^{\frac{1-\sigma}{(n-4)}}, R^{-2}\}.$$

This estimates suggest to search for a fixed point for the map $\mathcal{A}$ in the set of functions ϕ so that

$$\mathcal{C} := \{\phi \,:\, \|\phi\|_{in} \leq \, r\, R^{n+1-\alpha} \, \max\{T^{\frac{1-\sigma}{(n-4)}}, R^{-2}\}\},$$

for some r large, independent of T and R. From (2.25), we easily get

$$\|B[\phi]\|_{\nu,2+a} \lesssim T^{1+\frac{1}{n-4}}\|\phi\|_{in}. \tag{6.7}$$

This implies that, provided the constant r is chosen large, one has $\mathcal{A}(C) \subset C$. We next prove that $\mathcal{A}$ is a contraction mapping, provided R is (possibly) larger (and thus T smaller). We shall emphasize the fact that ψ depends from ϕ in a non linear and non local way, recalling that

$$\psi = \psi[\phi] = \psi[\lambda_1(\phi), d_1(\phi), \phi].$$

Combining (4.19), (4.20), (4.21) and (5.15), one gets

$$\|\psi[\phi_1] - \psi[\phi_2]\|_\alpha \leq \mathbf{c}\|\phi_1 - \phi_2\|_{in} \tag{6.8}$$

for some $\mathbf{c} \in (0,1)$, which can be done arbitrarily small, provided R is chosen large (and consequently T small). We claim that there exists $\mathbf{c} \in (0,1)$ so that

$$\|\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)\|_{in} \leq \mathbf{c}\|\phi_1 - \phi_2\|_{in} \tag{6.9}$$

for any ϕ_1, $\phi_2 \in \mathcal{C}$. From Proposition 6.1 we get that

$$\|\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)\|_{in} \leq c\, R^{n+1-\alpha}\|H[\lambda_1, d_1, \phi_1, \psi_1] - H[\lambda_2, d_2, \phi_2, \psi_2]\|_{\nu,2+a}$$

where $\lambda_i = \lambda[\phi_i]$, $d_i = d[\phi_i]$ and $\psi_i = \psi[\phi_i]$. Consider first

$$H_1[\phi] := p\lambda_0^{\frac{n-2}{2}} U^{p-1}\psi[\phi](\lambda_0 y + \xi[\phi], t(\tau)).$$

We write

$$\begin{aligned}|H_1[\phi_1] - H_1[\phi_2]| &\lesssim p\lambda_0^{\frac{n-2}{2}} U^{p-1}\, |\psi[\phi_1](\lambda_0 y + \xi[\phi_1], t(\tau)) - \psi[\phi_2](\lambda_0 y + \xi[\phi_1], t(\tau))| \\ &\quad + p\lambda_0^{\frac{n-2}{2}} U^{p-1}\, |\psi[\phi_2](\lambda_0 y + \xi[\phi_1], t(\tau)) - \psi[\phi_2](\lambda_0 y + \xi[\phi_1], t(\tau))| \\ &= h_1 + h_2.\end{aligned}$$

Observe that, thanks to (6.8) and using that $\lambda_0(t) = (T-t)^{1+\frac{1}{n-4}}$,

$$|h_1(y,\tau)| \leq c\,\mathbf{c}\, T^{\frac{n-2}{2}(1+\frac{1}{n-4})}\frac{\tau^{-\nu}}{1+|y|^4}\|\phi_1 - \phi_2\|_{in},$$

for some constants $c > 0$, and $\mathbf{c} \in (0,1)$. Also, thanks to (5.15), we have

$$\begin{aligned}|h_2(y,\tau)| &\leq c\, T^{\frac{n-2}{2}(1+\frac{1}{n-4})}\frac{\tau^{-\nu}}{1+|y|^4}\|\dot{d}[\phi_1] - \dot{d}[\phi_2]\|_{\frac{1+\sigma}{n-4}} \\ &\leq c\,\mathbf{c}\, T^{\frac{n-2}{2}(1+\frac{1}{n-4})}\frac{\tau^{-\nu}}{1+|y|^4}\|\phi_1 - \phi_2\|_{in}.\end{aligned}$$

Choosing, if necessary, R even larger (and automatically T smaller), we get

$$cR^{n+1-\alpha}\|H_1[\phi_1]-H_1[\phi_2]\|_{\nu,2+a}\leq \mathbf{c}\|\phi_1-\phi_2\|_{in} \tag{6.10}$$

for some $\mathbf{c}\in(0,1)$. Next, we consider $E_2[\phi](\lambda_0 y+\xi[\phi],t(\tau))$. Since the part of E_2 given by $-\left(\frac{\lambda_0}{d_0}\right)^{n-2}\left[\Delta w+pW_0^{p-1}w\right]$ does not depend on ϕ, we have

$$\begin{aligned}
\lambda_0^{\frac{n+2}{2}}E_2[\phi](\lambda_0 y+\xi[\phi],t(\tau)) &= \lambda_0^{\frac{n+2}{2}}(e_1+e_2)[\phi](\lambda_0 y+\xi[\phi])\\
&=\left(\frac{\lambda_0}{\lambda[\phi]}\right)^{\frac{n}{2}}\lambda_0\left[\dot d_1[\phi]+\frac{\lambda_0 y+d_0+d_1[\phi]}{\lambda_0 y+1+d_0+d_1[\phi]}\right]Z_1\left(\frac{\lambda_0}{\lambda[\phi]}y\right)\\
&+\left(\frac{\lambda_0}{\lambda[\phi]}\right)^{\frac{n}{2}}\lambda_0\left[\dot\lambda_0+\dot\lambda_1[\phi]\right]Z_0\left(\frac{\lambda_0}{\lambda[\phi]}y\right)\\
&+p\left(\frac{\lambda_0}{\lambda[\phi]}\right)^{\frac{n+2}{2}}U^{p-1}\left(\frac{\lambda_0}{\lambda[\phi]}\right)U\left(\frac{\lambda_0}{\lambda[\phi]}y+\frac{2d[\phi]}{\lambda[\phi]}\mathbf{e}_1\right).
\end{aligned}$$

Recall that we are in the region $|y|<2R$. We claim that

$$\|\lambda_0^{\frac{n+2}{2}}\left[E_2[\phi_1]-E_2[\phi_2]\right]\|_{\nu,2+a}\leq cT^{\frac{\sigma}{n-4}}\|\phi_1-\phi_2\|_{in}. \tag{6.11}$$

Assuming the validity of (6.11), and choosing, if necessary, R even larger (and automatically T smaller), we get

$$cR^{n+1-\alpha}\|\lambda_0^{\frac{n+2}{2}}\left[E_2[\phi_1]-E_2[\phi_2]\right]\|_{\nu,2+a}\leq \mathbf{c}\|\phi_1-\phi_2\|_{in} \tag{6.12}$$

for some $\mathbf{c}\in(0,1)$. To prove (6.11), we just consider the term

$$g[\phi]=\left(\frac{\lambda_0}{\lambda[\phi]}\right)^{\frac{n}{2}}\lambda_0\left[\dot\lambda_0+\dot\lambda_1[\phi]\right]Z_0\left(\frac{\lambda_0}{\lambda[\phi]}y\right)$$

in the expression of $\lambda_0^{\frac{n+2}{2}}E_2[\phi](\lambda_0 y+\xi[\phi],t(\tau))$. The estimates for the other two terms can be obtained in a similar way, and we leave them to the interested reader. We write

$$\begin{aligned}
g[\phi_1]-g[\phi_2]&=\left[\left(\frac{\lambda_0}{\lambda[\phi_1]}\right)^{\frac{n}{2}}-\left(\frac{\lambda_0}{\lambda[\phi_2]}\right)^{\frac{n}{2}}\right]\lambda_0\left[\dot\lambda_0+\dot\lambda_1[\phi_1]\right]Z_0\left(\frac{\lambda_0}{\lambda[\phi_1]}y\right)\\
&+\left(\frac{\lambda_0}{\lambda[\phi_2]}\right)^{\frac{n}{2}}\lambda_0\left[\dot\lambda_1[\phi_1]-\dot\lambda_1[\phi_2]\right]Z_0\left(\frac{\lambda_0}{\lambda[\phi]}y\right)\\
&+\left(\frac{\lambda_0}{\lambda[\phi_2]}\right)^{\frac{n}{2}}\lambda_0\left[\dot\lambda_0+\dot\lambda_1[\phi_2]\right]\left[Z_0\left(\frac{\lambda_0}{\lambda[\phi_1]}y\right)-Z_0\left(\frac{\lambda_0}{\lambda[\phi_2]}y\right)\right].
\end{aligned}$$

Observe that, for ϕ_1, $\phi_2\in\mathcal{C}$,

$$\left|\left[\left(\frac{\lambda_0}{\lambda[\phi_1]}\right)^{\frac{n}{2}}-\left(\frac{\lambda_0}{\lambda[\phi_2]}\right)^{\frac{n}{2}}\right]\right|\leq\left(\frac{\lambda_0}{\lambda[\phi]}\right)^{\frac{n}{2}}T^{\frac{\sigma}{n-4}}\|\dot\lambda_1[\phi_1]-\dot\lambda_2[\phi_2]\|_{\frac{1+\sigma}{n-4}},$$

$$\left|\dot\lambda_1[\phi_1]-\dot\lambda_1[\phi_2]\right|\leq\dot\lambda_0\,T^{\frac{\sigma}{n-4}}\|\dot\lambda_1[\phi_1]-\dot\lambda_2[\phi_2]\|_{\frac{1+\sigma}{n-4}},$$

and

$$\left|Z_0\left(\frac{\lambda_0}{\lambda[\phi_1]}y\right)-Z_0\left(\frac{\lambda_0}{\lambda[\phi_2]}y\right)\right|\leq|\nabla Z_0(\frac{\lambda_0}{\lambda}y)\cdot(\frac{\lambda_0}{\lambda}y)|\,T^{\frac{\sigma}{n-4}}\|\dot\lambda_1[\phi_1]-\dot\lambda_2[\phi_2]\|_{\frac{1+\sigma}{n-4}}$$

for some λ satisfying (2.7). We thus conclude that

$$\|g[\phi_1] - g[\phi_2]\|_{\nu,2+a} \le c\,T^{\frac{\sigma}{n-4}} \|\dot\lambda_1[\phi_1] - \dot\lambda_2[\phi_2]\|_{\frac{1+\sigma}{n-4}}$$

for some constant c. Estimate (6.11) for the term $g[\phi]$ follows directly from the Lipschitz dependent of λ on ϕ, as stated in (5.15).

Write now $B[\phi] = B_1[\phi] + B_2[\phi]$, where

$$B_1[\phi] = \lambda_0\dot\lambda_0 \left[\frac{n-2}{2}\phi(y,t) + \nabla\phi(y,t)\cdot y\right],$$

and

$$B_2[\phi] = \left[\lambda_0 \dot d[\phi] + \frac{\lambda_0}{\lambda_0 y_1 + \xi[\phi]}\right] \frac{\partial \phi}{\partial y_1}(y,t).$$

For both terms, we have

$$|B_i[\phi_1] - B_i[\phi_2]| \le cT^{1+\frac{1}{n-4}} R(T^{\frac{1}{n-4}}R + 1)\frac{\tau^{-\nu}}{1+|y|^{2+a}}\|\phi_1 - \phi_2\|_{in}, \quad i = 1,2.$$

Choosing, if necessary, T even smaller, we get

$$cR^{n+1-\alpha}\|B[\phi_1] - B[\phi_2]_{\nu,2+a} \le \mathbf{c}\|\phi_1 - \phi_2\|_{in} \tag{6.13}$$

for some $\mathbf{c} \in (0,1)$.

Estimates (6.10), (6.12) and (6.13) give the contraction property for $\mathcal{A}$ in the set $\mathcal{C}$. Thus we proved the existence of a solution to the *inner* problem (2.27). This fact concludes the proof of the existence of the solution predicted by Theorem 2, with the expected properties.

7. Proof of Proposition 6.1

We are interested in the construction of a solution to Problem (6.1) for any given right-hand side h with $\|h\|_{\nu,2+a} < +\infty$.

To describe our construction, we consider an orthonormal basis ϑ_m, $m = 0,1,\dots,$ in $L^2(S^2)$ of spherical harmonics, namely eigenfunctions of the problem

$$\Delta_{S^2}\vartheta_m + \lambda_m\vartheta_m = 0 \quad \text{in } S^2$$

so that $0 = \lambda_0 < \lambda_1 = \ldots = \lambda_n = 2 < \lambda_{n+1} \le \ldots$. For simplicity, we use the notation $B_{2R} = B(0,2R)$. Let $h \in L^2(B_{2R})$. We decompose it into the form

$$h(y,\tau) = \sum_{j=0}^{\infty} h_j(r,\tau)\vartheta_j(y/r), \quad r = |y|, \quad h_j(r,\tau) = \int_{S^2} h(r\theta,\tau)\vartheta_j(\theta)\,d\theta.$$

In addition, we write $h = h^0 + h^1 + h^\perp$ where

$$h^0 = h_0(r,\tau), \quad h^1 = \sum_{j=1}^{n} h_j(r,\tau)\vartheta_j, \quad h^\perp = \sum_{j=n+1}^{\infty} h_j(r,\tau)\vartheta_j.$$

Observe that $h^1 = h^\perp = 0$ if h is radially symmetric in the y variable. Consider also the analogous decomposition for ϕ into $\phi = \phi^0 + \phi^1 + \phi^\perp$. We build the solution ϕ of Problem (6.1) by doing so separately for the pairs (ϕ^0, h^0), (ϕ^1, h^1) and $(\phi^\perp, h^\perp)$.

We also need to recall that the operator $L_0(\phi) = \Delta\phi + pU^{p-1}\phi$ has an $n+1$ dimensional kernel generated by the bounded functions Z_0, Z_1 defined in (2.11) and also by

$$Z_i(y) = \frac{\partial U}{\partial y_i}, \quad i = 2, \dots, n. \tag{7.1}$$

Proposition 6.1 is a direct consequence of the following

Proposition 7.1. *Let ν, a be given positive numbers with $0 < a < 1$. Then, for all sufficiently large $R > 0$, there exists τ_0 so that, for any $h = h(y,\tau)$ with $\|h\|_{\nu,2+a} < +\infty$ that satisfies for all $j = 0, 1, \dots, n$*

$$\int_{B_{2R}} h(y,\tau)\, Z_j(y)\, dy \;=\; 0 \quad \textit{for all} \quad \tau \in (\tau_0, \infty) \tag{7.2}$$

there exist $\phi = \phi[h]$ and $e_0 = e_0[h]$ which solve Problem (6.1). They define linear operators of h that satisfy the estimates

$$\begin{aligned} |\phi(y,\tau)| \;\lesssim\; \tau^{-\nu}\Big[&\frac{R^n}{1+|y|^n}\, \|h^0\|_{\nu,2+a} + \frac{R^{n+1-a}}{1+|y|^{n+1}}\, \|h^1\|_{\nu,2+a} \\ &+ \frac{\|h\|_{\nu,2+a}}{1+|y|^a}\Big], \end{aligned} \tag{7.3}$$

$$\begin{aligned} |\nabla_y\phi(y,\tau)| \;\lesssim\; \tau^{-\nu}\Big[&\frac{R^n}{1+|y|^{n+1}}\, \|h^0\|_{\nu,2+a} + \frac{R^{n+1-a}}{1+|y|^{n+2}}\, \|h^1\|_{\nu,2+a} \\ &+ \frac{\|h\|_{\nu,2+a}}{1+|y|^{a+1}}\Big], \end{aligned} \tag{7.4}$$

and

$$|e_0[h]| \;\lesssim\; \|h\|_{\nu,2+a}. \tag{7.5}$$

We refer to (6.3) for the definition of the $\|\cdot\|_{\nu,2+a}$. Proposition 6.1 is a direct consequence of Proposition 7.1. Indeed, if h is even in the y_i variable, $i = 2, \dots, n$, (7.2) is automatically satisfied for $j = 2, \dots, n$..

The result contained in Proposition 7.1 follows from next Proposition, which refers to the following problem

$$\begin{aligned} &\phi_\tau = \Delta\phi + pU^{p-1}(y)\phi + h(y,\tau) - c(\tau)Z(y) \quad \text{in } B(0,2R)\times(\tau_0,\infty), \\ &\phi(y,\tau_0) = 0 \quad \text{in } B(0,2R). \end{aligned} \tag{7.6}$$

We have the validity of the following

Proposition 7.2. *Let ν, a be given positive numbers with $0 < a < 1$. Then, for all sufficiently large $R > 0$ and any h with $\|h\|_{\nu,2+a} < +\infty$ and satisfying the orthogonality conditions (7.2), there exist $\phi = \phi[h]$ and $c = c[h]$ which solve Problem (7.6), and define linear operators of h. The function $\phi[h]$ satisfies estimate (7.3), (7.4) and for some $\gamma > 0$*

$$\left| c(\tau) - \int_{B_{2R}} hZ \right| \lesssim \tau^{-\nu}\left[R^{2-a} \left\| h - Z\int_{B_{2R}} hZ \right\|_{\nu,2+a} + e^{-\gamma R}\|h\|_{\nu,2+\alpha}\right]. \tag{7.7}$$

Assuming the validity of Proposition 7.2, we easily get

Proof of Proposition 7.1. Let ϕ_1 be the solution of Problem (7.6) predicted by Proposition 7.2. Let us write

$$\phi(y,\tau) = \phi_1(y,\tau) + e(\tau)Z(y), \tag{7.8}$$

for some $e \in C^1([\tau_0,\infty))$. We find

$$\partial_\tau\phi = \Delta\phi + pU^{p-1}\phi + h(y,\tau) + [\, e'(\tau) - \lambda_0 e(\tau) - c(\tau)\,]\; Z(y).$$

We choose $e(\tau)$ to be the unique bounded solution of the equation

$$e'(\tau) \,-\, \lambda_0 e(\tau) \,=\, c(\tau), \quad \tau \in (\tau_0,\infty)$$

which is explicitly given by

$$e(\tau) = \int_\tau^\infty \exp(\sqrt{\lambda_0}(\tau - s))\, c(s)\, ds\,.$$

The function e depends linearly on h. Besides, we clearly have from (7.7), $|e(\tau)| \lesssim \tau^{-\nu}\|h\|_{\nu,2+a}$. and thus, from the fact that ϕ_1 satisfies estimates (7.3), (7.4), so does ϕ given by (7.8). Thus ϕ satisfies Problem (6.1) with initial condition $\phi(y,\tau_0) = e(\tau_0)Z(y)$. □

The rest of the Section is devoted to the

Proof of Proposition 7.2. The proof is divided in two steps.

Step 1. We claim that for all sufficiently large $R > 0$ and any h with $\|h\|_{\nu,2+a} < +\infty$ there exists ϕ and c which solve Problem

$$\phi_\tau = \Delta\phi + pU^{p-1}\phi + h(y,\tau) - c(\tau)Z(y) \quad \text{in } B_{2R}\times(\tau_0,\infty) \tag{7.9}$$
$$\phi = 0 \quad \text{on } \partial B_{2R}\times(\tau_0,\infty), \quad \phi(\cdot,\tau_0) = 0 \quad \text{in } B_{2R}.$$

Moreover,

$$|\phi(y,\tau)| \;\lesssim\; \tau^{-\nu}\Big[\frac{R^{n-2}\|h^0\|_{\nu,2+a}}{1+|y|^{n-2}} + \frac{R^n\|h^1\|_{\nu,2+a}}{1+|y|^{n-1}} + \frac{\|h\|_{\nu,2+a}}{1+|y|^{n-2}} + \frac{\|h\|_{\nu,2+a}}{1+|y|^a}\Big] \tag{7.10}$$

and for some $\gamma > 0$

$$\left|c(\tau) - \int_{B_{2R}} hZ\right| \lesssim \tau^{-\nu}\,\Big[\,\left\|h - Z\int_{B_{2R}} hZ\right\|_{\nu,2+a} + e^{-\gamma R}\|h\|_{\nu,2+a}\Big]. \tag{7.11}$$

We do the construction of the solution mode by mode.

Construction at mode 0. We solve Problem (7.9) in the radial case, with $h = h_0(r,\tau)$.

To this purpose, let $\chi(s)$ be a smooth cut-off function with $\chi(s) = 1$ for $s < 1$ and $\chi(s) = 0$ for $s > 2$, and consider $\chi_M(y) = \chi(|y| - M)$, for a large but fixed number M independently of R. By standard parabolic theory, there exists a unique solution $\phi_*[\bar h_0]$ to

$$\phi_\tau = \Delta\phi + pU^{p-1}(1-\chi_M)\phi + \bar h_0(y,\tau) \quad \text{in } B_{2R}\times(\tau_0,\infty) \tag{7.12}$$
$$\phi = 0 \quad \text{on} \quad \partial B_{2R}\times(\tau_0,\infty), \quad \phi(\cdot,\tau_0) = 0 \quad \text{in } B_{2R},$$

where

$$\bar h_0 = h_0 - c_0(\tau)Z, \quad c_0(\tau) = \int_{B_{2R}} hZ.$$

The function $\phi_*[\bar h_0]$ is radial and satisfies the bound

$$\left|\phi_*[\bar h_0]\right| \lesssim \tau^{-\nu}\frac{\|h\|_{\nu,2+a}}{1+|y|^a}.$$

This can be proved with the use of a special super solution, arguing as in Lemma 7.3 in [7]. Setting $\phi = \phi_*[\bar h_0] + \tilde\phi$ and $c(\tau) = c_0(\tau) + \tilde c(\tau)$, Problem (7.9) gets reduced to

$$\tilde\phi_\tau = \Delta\tilde\phi + pU^{p-1}\tilde\phi + \tilde h_0(r,\tau) - \tilde c(\tau)Z \quad \text{in } B_{2R}\times(\tau_0,\infty) \tag{7.13}$$
$$\tilde\phi = 0 \quad \text{on} \quad \partial B_{2R}\times(\tau_0,\infty), \quad \tilde\phi(\cdot,\tau_0) = 0 \quad \text{in } B_{2R},$$

where $\tilde h_0 = pU^{p-1}\chi_M\phi_*[\bar h_0]$. Observe that $\tilde h_0$ is radial, it is compactly supported and with size controlled by that of $\bar h_0$. In particular we have that for any $m>0$,

$$|\tilde h_0(r,\tau)| \lesssim \frac{\tau^{-\nu}}{1+r^m}\left[\sup_{\tau>\tau_0}\tau^\nu\|\phi_*[\bar h_0](\cdot,\tau)\|_{L^\infty}\right]. \tag{7.14}$$

We shall next solve Problem (7.13) under the additional orthogonality constraint

$$\int_{B_{2R}} \tilde\phi(\cdot,\tau)\, Z \;=\; 0 \quad \text{for all} \quad \tau\in(\tau_0,\infty). \tag{7.15}$$

Problem (7.13)-(7.15) is equivalent to solving just (7.13) for $\tilde c$ given by the explicit linear functional $\tilde c := \tilde c[\tilde\phi, \tilde h_0]$ determined by the relation

$$\tilde c(\tau)\int_{B_{2R}} Z^2 = \int_{B_{2R}} \tilde h_0(\cdot,\tau)Z + \int_{\partial B_{2R}} \partial_r\tilde\phi(\cdot,\tau)Z. \tag{7.16}$$

If the function $c = c(\tau)$ in Problem (7.13) were independent of ϕ, standard linear parabolic theory would give the existence of a unique solution. On the other hand, a close look to (7.16) shows that the dependence of $c = c(\tau)$ on ϕ is small for instance in an L^∞-$C^{1+\alpha,\frac{1+\alpha}{2}}$ setting, since $Z(R) = O(e^{-\gamma R})$ for some $\gamma>0$. A contraction argument applies to yield existence of a unique solution to (7.13)-(7.15) defined at all times. To get the estimates, we assume smoothness of the data so that integrations by parts and differentiations can be carried over, and then argue by approximations. Testing (7.13)-(7.15) against $\tilde\phi$ and integrating in space, we obtain the relation

$$\partial_\tau\int_{B_{2R}}\tilde\phi^2 + Q(\tilde\phi,\tilde\phi) = \int_{B_{2R}} g\tilde\phi, \quad g = \tilde h_0 - \tilde c(\tau)Z_0,$$

where Q is the quadratic form defined by

$$Q(\phi,\phi) := \int\left[|\nabla\phi|^2 - pU^{p-1}|\phi|^2\right]. \tag{7.17}$$

In [7], it is proven that there exists $\gamma>0$ such that, for any ϕ with $\int\phi Z = 0$, the following inequality holds

$$Q(\phi,\phi) \ge \frac{\gamma}{R^{n-2}}\int\phi^2.$$

Thus we have

$$\partial_\tau\int_{B_{2R}}\tilde\phi^2 + \frac{\gamma}{R^{n-2}}\int_{B_{2R}}\tilde\phi^2 \lesssim R^{n-2}\int_{B_{2R}} g^2. \tag{7.18}$$

We observe that from (7.16) and (7.14) for $m=0$ we get that

$$|\tilde c(\tau)|\le \tau^{-\nu}K, \quad K:=\left[\sup_{\tau>\tau_0}\tau^{\nu}\|\phi_*[\bar h_0](\cdot,\tau)\|_{L^\infty}\right]+e^{-\gamma R}\left[\sup_{\tau>\tau_0}\tau^{\nu}\|\nabla\phi_*[\bar h_0](\cdot,\tau)\|_{L^\infty}\right].$$

Besides, using again estimate (7.14) for a sufficiently large m, we get

$$\int_{B_{2R}} g^2 \ \lesssim\ \tau^{-2\nu}K^2.$$

Using that $\tilde\phi(\cdot,\tau_0)=0$ and Gronwall's inequality, we readily get from (7.18) the L^2-estimate

$$\|\tilde\phi(\cdot,\tau)\|_{L^2(B_{2R})}\ \lesssim\ \tau^{-\nu}R^{n-2}K, \tag{7.19}$$

for all $\tau>\tau_0$. Now, using standard parabolic estimates in the equation satisfied by $\tilde\phi$ we obtain then that on any large fixed radius $M>0$,

$$\|\tilde\phi(\cdot,\tau)\|_{L^\infty(B_M)}\ \lesssim\ \tau^{-\nu}R^{n-2}\,K \quad \text{for all} \quad \tau>\tau_0.$$

Since the data in the equation has arbitrarily fast space decay, we can dominate the solution outside B_M by a barrier of the order $\tau^{-\nu}|y|^{-(n-2)}$. As a conclusion, also using local parabolic estimates for the gradient, we find that

$$(1+|y|)\,|\nabla_y\tilde\phi(y,\tau)|+|\tilde\phi(y,\tau)|\ \lesssim\ \tau^{-\nu}\,R^{n-2}\,K\,|y|^{-(n-2)},$$

thus from the definition of K we finally get

$$(1+|y|)\,|\nabla_y\tilde\phi(y,\tau)|+|\tilde\phi(y,\tau)|\ \lesssim \tau^{-\nu}\frac{R^{n-2}}{1+|y|^{n-2}}\left[\sup_{\tau>\tau_0}\tau^{\nu}\|\phi_*[\bar h_0](\cdot,\tau)\|_{L^\infty}\right]. \tag{7.20}$$

It clearly follows from this estimate and inequality (7.14) that the function

$$\phi^0[h^0]:=\tilde\phi+\phi_*[\bar h_0] \tag{7.21}$$

solves Problem (7.9) for $h=h_0$ and satisfies

$$|\phi_0(y,\tau)|\ \lesssim\ \tau^{-\nu}\frac{R^{n-2}}{1+|y|^{n-2}}\|h^0\|_{\nu,2+a}.$$

Finally, from (7.16) we see that we have that

$$c(\tau)=\int_{B_{2R}}hZ+\int_{B_{2R}}pU^{p-1}\chi_M\phi_*[\bar h_0]\,Z+O(e^{-\gamma R})\|h\|_{*,\nu,2+a}.$$

From here we find the validity of estimate

$$\left|c(\tau)-\int_{B_{2R}}h_0Z\right|\lesssim \tau^{-\nu}\Big[\ \left\|h_0-Z\int_{B_{2R}}h_0Z\right\|_{\nu,2+a}+e^{-\gamma R}\|h_0\|_{\nu,2+a}\Big].$$

Hence estimates (7.10) and (7.11) hold. The construction of the solution at mode 0 is concluded.

Construction at modes 1 *to* n. Here we consider the case $h=h^1$ where $h^1(y,\tau)=\sum_{j=1}^n h_j(r,\tau)\vartheta_j$. The function

$$\phi^1[h^1]:=\sum_{j=1}^n\phi_j(r,\tau)\vartheta_j, \tag{7.22}$$

solves the initial-boundary value problem

$$\phi_\tau=\Delta\phi+pU^{p-1}\phi+h^1(y,\tau)\quad \text{in } B_{2R}\times(\tau_0,\infty) \tag{7.23}$$

$$\phi = 0 \quad \text{on } \partial B_{2R} \times (\tau_0, \infty), \quad \phi(\cdot, \tau_0) = 0 \quad \text{in } B_{2R},$$

if the functions $\phi_j(r,\tau)$ solves

$$\partial_\tau \phi_j = \mathcal{L}_1[\phi_j] + h_j(r,\tau) \quad \text{in } (0,2R) \times (\tau_0, \infty) \tag{7.24}$$

$$\partial_r \phi_j(0,\tau) = 0 = \phi_j(R,\tau) \quad \text{for all} \quad \tau \in (\tau_0,\infty), \quad \phi_j(r,\tau_0) = 0 \quad \text{for all} \quad r \in (0,R),$$

where

$$\mathcal{L}_1[\phi_j] := \partial_{rr}\phi_j + (n-1)\frac{\partial_r \phi_j}{r} - (n-1)\frac{\phi_j}{r^2} + pU^{p-1}\phi_j. \tag{7.25}$$

Let us assume that $\|h_j\|_{\nu,2+a} < +\infty$, so that $|h^1(r,\tau)| \le \tau^{-\nu}\|h^1\|_{\nu,2+a}(1+r)^{-2-a}$. Let us consider the solution of the stationary problem $\mathcal{L}_1[\phi] + (1+r)^{-(2+a)} = 0$ given by the variation of parameters formula

$$\bar\phi(r) = Z(r)\int_r^{2R} \frac{1}{\rho^{n-1}Z(\rho)^{n-1}} \int_0^\rho (1+s)^{-(2+a)} Z(s)^{n-2} s^{n-1}\, ds$$

where $Z(r) = w_r(r)$. Since $w_r(r) \sim r^{-n+1}$ for large r, we find the estimate $|\bar\phi(r)| \lesssim \frac{R^{n-a}}{1+r^{n-1}}$. Then $2\|h_j\|_{\nu,2+a}\tau^{-\nu}\bar\phi(r)$ is a positive super-solution of Problem (7.24) if τ_0 is large, and thus we find $|\phi_j(r,\tau)| \lesssim \tau^{-\nu}\frac{R^{n-a}}{1+r^{n-1}}\|h_j\|_{\nu,2+a}$. Hence $\phi^1[h^1]$ given by (7.22) satisfies

$$|\phi^1[h^1](y,\tau)| \lesssim \frac{R^{n-a}}{1+|y|^{n-1}}\|h^1\|_{\nu,2+a}.$$

Construction at higher modes. We consider now the case of higher modes,

$$\phi_\tau = \Delta\phi + pU^{p-1}\phi + h^\perp \quad \text{in } B_{2R} \times (\tau_0,\infty) \tag{7.26}$$

$$\phi = 0 \quad \text{on} \quad \partial B_{2R} \times (\tau_0,\infty), \quad \phi(\cdot,\tau_0) = 0 \quad \text{in } B_{2R},$$

where $h = h^\perp = \sum_{j=n+1}^\infty h_j(r)\Theta_j$ whose solution has the form $\phi^\perp = \sum_{j=n+1}^\infty \phi_j(r,\tau)\Theta_j$. We have that for $\phi^\perp \in H_0^1(B_{2R})$

$$\int_{B_{2R}} \frac{|\phi^\perp|^2}{r^2} \lesssim Q(\phi^\perp,\phi^\perp). \tag{7.27}$$

We refer to [7] for the proof of this fact. Let $\phi_*[h^\perp]$ be the solution to

$$\phi_\tau = \Delta\phi + pU^{p-1}(1-\chi_M)\phi + \bar h^\perp(y,\tau) \quad \text{in } B_{2R}\times(\tau_0,\infty)$$
$$\phi = 0 \quad \text{on} \quad \partial B_{2R}\times(\tau_0,\infty), \quad \phi(\cdot,0) = 0 \quad \text{in } B_{2R},$$

where $\bar h^\perp = h^\perp - c^\perp Z$, and $c^\perp = \int_{B_{2R}} h^\perp Z$. By writing $\phi = \phi_*[h^\perp] + \tilde\phi$, Problem (7.26) reduces to solving

$$\tilde\phi_\tau = \Delta\tilde\phi + pU^{p-1}(y)\tilde\phi + \tilde h \quad \text{in } B_{2R}\times(\tau_0,\infty)$$
$$\tilde\phi = 0 \quad \text{on} \quad \partial B_{2R}\times(\tau_0,\infty), \quad \tilde\phi(\cdot,\tau_0) = 0 \quad \text{in } B_{2R},$$

where $\tilde h = pU^{p-1}\chi_M\phi_*[h^\perp]$, for a sufficiently large M. The function

$$\phi^\perp[h^\perp] := \tilde\phi + \phi_*[h^\perp] \tag{7.28}$$

solves (7.26) and satisfies

$$|\phi^\perp[h^\perp](y,\tau)| \lesssim \tau^{-\nu}\left[(1+|y|)^{-n+2} + (1+|y|)^{-a}\right]\|h^\perp\|_{\nu,2+a} \quad \text{in } B_{2R}.$$

We simply let

$$\phi[h] := \phi^0[h^0] + \phi^1[h^1] + \phi^\perp[h^\perp]$$

for the functions defined in (7.21), (7.28). By construction, $\phi[h]$ solves Equation (7.9). It defines a linear operator of h and satisfies (7.10). The proof of *Step 1* is concluded.

Step 2. We complete the proof of Proposition 7.2. As before, we decompose h in modes, $h = h^0 + h^1 + h^\perp$, and define separately associated solutions of (7.6) in a decomposition $\phi = \phi^0 + \phi^1 + \phi^\perp$.

Construction at mode 0. For a bounded radial $h = h(|y|)$ defined in B_{2R} with $\int_{B_{2R}} hZ_0 = 0$ the equation

$$\Delta H + pU^{p-1}(y)H + \tilde h_0(|y|) = 0 \quad \text{in } \mathbb{R}^n, \quad H(y) \to 0 \quad \text{as} \quad |y| \to \infty$$

where $\tilde h$ designates the extension of h as zero outside B_{2R}, has a solution $H =: L_0^{-1}[h]$ represented by the variation of parameters formula

$$H(r) = \tilde Z(r) \int_r^\infty \tilde h(s)\, Z_0(s)\, s^{n-1}\, ds + Z_0(r) \int_r^\infty \tilde h(s)\, \tilde Z(s)\, s^{n-1}\, ds \tag{7.29}$$

where $\tilde Z(r)$ is a suitable second linearly independent radial solution of $L_0[\tilde Z] = 0$. If we consider a function $h_0 = h_0(|y|, \tau)$ defined in B_{2R} with $\|h_0\|_{\nu,2+a} < +\infty$ and $\int_{B_{2R}} h_0 Z_0 = 0$ for all τ, then $H_0 = L_0^{-1}[h_0(\cdot, \tau)]$ satisfies

$$\|H_0\|_{\nu,a} \lesssim \|h_0\|_{\nu,2+a}.$$

Let us consider the boundary value problem in B_{3R}

$$\Phi_\tau = \Delta\Phi + pU^{p-1}\Phi + H_0(|y|, \tau) - c_0(\tau) Z \quad \text{in } B_{3R} \times (\tau_0, \infty) \tag{7.30}$$

$$\Phi = 0 \quad \text{on} \quad \partial B_{3R} \times (\tau_0, \infty), \quad \Phi(\cdot, \tau_0) = 0 \quad \text{in } B_{3R}.$$

Thanks to the result in Step 1, we find a radial solution $\Phi_0[h_0]$ to this problem, which defines a linear operator of h_0 and satisfies the estimates

$$|\Phi_0(y, \tau)| \;\lesssim\; \frac{\tau^{-\nu} R^{n-2}}{1 + |y|^{n-2}} R^2 \|H_0\|_{\nu,2+a}, \tag{7.31}$$

where for some $\gamma > 0$

$$\left| c_0(\tau) - \int_{B_{2R}} H_0 Z \right| \lesssim \tau^{-\nu} \left[R^2 \left\| H_0 - Z \int_{B_{2R}} H_0 Z \right\|_{\nu,2+a} + e^{-\gamma R} \|h_0\|_{\nu,2+a} \right]. \tag{7.32}$$

At this point we observe that since $L_0[Z] = (-\mu_0) Z$ (one has $-\mu_0 > 0$, see (2.29)) then

$$(-\mu_0) \int_{B_{2R}} H_0 Z = \int_{B_{2R}} H_0 L_0[Z] = \int_{B_{2R}} L_0[H_0]\, Z + \int_{\partial B_{2R}} (Z \partial_\nu H_0 - H_0 \partial_\nu Z),$$

and hence

$$\int_{B_{2R}} H_0 Z = (-\mu_0)^{-1} \int_{B_{2R}} h_0\, Z + O(e^{-\gamma R}) \tau^{-\nu} \|h_0\|_{\nu,2+a}.$$

Also, from the definition of the operator L_0^{-1} we see that $Z = (-\mu_0) L_0^{-1}[Z]$. Thus

$$\left\| H_0 - Z \int_{B_{2R}} H_0 Z \right\|_{\nu,a} = \left\| L_0^{-1} \Big[h_0 - (-\mu_0) Z \int_{B_{2R}} H_0 Z \Big] \right\|_{\nu,a}$$

$$\lesssim \left\| h_0 - Z \int_{B_{2R}} h_0 Z \right\|_{\nu,2+a} + e^{-\gamma R} \|h_0\|_{\nu,2+a}.$$

Let us fix now a vector e with $|e| = 1$, a large number $\rho > 0$ with $\rho \le 2R$ and a number $\tau_1 \ge \tau_0$. Consider the change of variables

$$\Phi_\rho(z,t) := \Phi(\rho e+\rho z, \tau_1+\rho^2 t), \quad H_\rho(z,t) := \rho^2 [H_0(\rho e+\rho z, \tau_1+\rho^2 t) - c_0(\tau_1+\rho^2 t) Z_0(\rho e+\rho z)].$$

Then $\Phi_\rho(z,t)$ satisfies an equation of the form

$$\partial_t \Phi_\rho = \Delta_z \Phi_\rho + B_\rho(z,t)\Phi_\rho + H_\rho(z,t) \quad \text{in } B_1(0) \times (0,2).$$

where $B_\rho = O(\rho^{-2})$ uniformly in $B_2(0) \times (0,\infty)$. Standard parabolic estimates yield that for any $0 < \alpha < 1$

$$\|\nabla_z \Phi_\rho\|_{L^\infty(B_{\frac{1}{2}}(0)\times(1,2))} \lesssim \|\Phi_\rho\|_{L^\infty(B_1(0)\times(0,2))} + \|H_\rho\|_{L^\infty(B_1(0)\times(0,2))}.$$

Moreover

$$\|H_\rho\|_{L^\infty(B_1(0)\times(0,2))} \lesssim \rho^{2-a} \tau_1^{-\nu} \|H_0\|_{\nu,a}, \quad \|\Phi_\rho\|_{L^\infty(B_1(0)\times(0,2))} \lesssim \tau_1^{-1} K(\rho)$$

where

$$K(\rho) = \frac{R}{\rho} R^{2-a} \|h_0\|_{\nu,2+a} \tag{7.33}$$

This yields in particular that

$$\rho |\nabla_y \Phi(\rho e, \tau_1 + \rho^2)| = |\nabla \tilde{\phi}(0,1)| \lesssim \tau_1^{-\nu} K(\rho).$$

Hence if we choose $\tau_0 \ge R^2$, we get that for any $\tau > 2\tau_0$ and $|y| \le 3R$

$$(1+|y|)\, |\nabla_y \Phi(y,\tau)| \lesssim \tau^{-\nu} K(|y|) \tag{7.34}$$

We obtain that these bounds are as well valid for $\tau < 2\tau_0$ by the use of similar parabolic estimates up to the initial time (with condition 0).

Now, we observe that the function H_0 is of class C^1 in the variable y and $\|\nabla_y H_0\|_{1+a,\nu} \le \|h_0\|_{2+a,\nu}$. It follows that we have the estimate

$$(1+|y|^2)\, |D_y^2 \Phi(y,\tau)| \lesssim \tau^{-\nu} K(|y|)$$

for all $\tau > \tau_0$, $|y| \le 2R$. where K is the function in (7.33). The proof follows simply by differentiating the equation satisfied by Φ, rescaling in the same way we did to get the gradient estimate, and apply the bound already proven for $\nabla_y \Phi$.

$$(1+|y|^2)|D^2\Phi(y,\tau)| + (1+|y|)|\nabla\Phi(y,\tau)| + |\Phi(y,\tau)|$$
$$\lesssim \tau^{-\nu} \|h_0\|_{\nu,2+a} \frac{R^n}{1+|y|^{n-2}} \quad \text{in } B_{2R}.$$

This yields in particular

$$|L_0[\Phi](\cdot,\tau)| \lesssim \tau^{-\nu} \|h_0\|_{\nu,2+a} \frac{R^n}{1+|y|^n} \quad \text{in } B_{2R}$$

We define

$$\phi^0[h_0] := L_0[\Phi] \Big|_{B_{2R}}.$$

Then $\phi^0[h_0]$ solves Problem (7.6) with

$$c(\tau) := (-\mu_0) c_0(\tau). \tag{7.35}$$

$\phi^0[h_0]$ satisfies the estimate

$$|\phi^0[h_0](y,\tau)| \lesssim \tau^{-\nu}\|h_0\|_{\nu,2+a}\frac{R^n}{1+|y|^n} \quad \text{in } B_{2R}. \tag{7.36}$$

and from (7.32), estimate (7.7) holds too.

Construction for modes 1 *to* n. We consider now $h^1(y,\tau) = \sum_{j=1}^n h_j(r,\tau)\vartheta_j$ with $\|h^1\|_{\nu,2+a} < +\infty$ that satisfies for all $i = 1,\dots,n$ $\int_{B_{2R}} h^1 Z_i = 0$ for all $\tau \in (\tau_0,\infty)$. We will show that there is a solution

$$\phi^1[h^1] = \sum_{j=1}^n \phi_j(r,\tau)\vartheta_j(\frac{y}{r})$$

to Problem (7.6) for $h = h^1$, which define a linear operator of h^1 and satisfies the estimate

$$|\phi^1(y,\tau)| \lesssim \frac{R^{n+1}}{1+|y|^{n+1}}R^{-a}\|h\|_{\nu,2+a}. \tag{7.37}$$

Let us fix $1 \le j \le n$. For a function $h = h_j(r)\vartheta_j(\frac{y}{r})$ defined in B_{2R}, we let $H = L_0^{-1}[h] := H_j(r)\vartheta_j(\frac{y}{r})$ be the solution of the equation

$$\Delta H + pU^{p-1}H + \tilde h_j\vartheta_j = 0 \quad \text{in } \mathbb{R}^n, \quad H(y)\to 0 \quad \text{as} \quad |y|\to\infty$$

where $\tilde h_j$ designates the extension of h_j as zero outside B_{2R}, represented by the variation of parameters formula

$$H_j(r) = w_r(r)\int_r^{2R}\frac{1}{\rho^{n-1}w_r(\rho)^{n-1}}\int_\rho^\infty \tilde h_j(s)\,w_r(s)^{n-2}s^{n-1}\,ds$$

If we consider a function $h^j = h_j(r,\tau)\vartheta_j$ defined in B_{2R} with $\|h^j\|_{\nu,2+a} < +\infty$ and $\int_{B_{2R}} h^j Z_j = 0$ for all τ, then $H_j = L_0^{-1}[h^j(\cdot,\tau)]$ satisfies the estimate

$$\|H_j\|_{\nu,a} \lesssim \|h_j\|_{\nu,2+a}.$$

Let us consider the boundary value problem in B_{3R}

$$\Phi_\tau = \Delta\Phi + pU(y)^{p-1}\Phi + H_j(r)\vartheta_j(y) \quad \text{in } B_{3R}\times(\tau_0,\infty) \tag{7.38}$$

$$\Phi = 0 \quad \text{on } \partial B_{3R}\times(\tau_0,\infty), \quad \Phi(\cdot,\tau_0) = 0 \quad \text{in } B_{3R}.$$

As consequence of Step 1, we find a solution $\Phi_j[h]$ to this problem, which defines a linear operator of h_j and satisfies the estimates

$$|\Phi_j(y,\tau)| \lesssim \frac{\tau^{-\nu}R^n}{1+|y|^{n-1}}R^{1-a}\|h_j\|_{\nu,2+a}, \tag{7.39}$$

Arguing by scaling and parabolic estimates, we find as in the construction for mode 0,

$$|L[\Phi_j](\cdot,\tau)| \lesssim \tau^{-\nu}\|h\|_{\nu,2+a}\frac{R^{n+1-a}}{1+|y|^{n+1}} \quad \text{in } B_{2R}.$$

We define $\phi_j[h_j] := L[\Phi_j]\Big|_{B_{2R}}$. Then $\phi_j[h]$ solves the equation (7.6) and satisfies

$$|\phi_j[h_j](y,\tau)| \lesssim \tau^{-\nu}\|h_j\|_{\nu,2+a}\frac{R^{n+1}}{1+|y|^{n+1}}R^{-a} \quad \text{in } B_{2R}.$$

We then define $\phi^1[h^1] := \sum_{j=1}^3 \phi_j[h_j]\vartheta_j$. This function solves (7.6) for $h = h^1$ and satisfies

$$|\phi^1[h^1](y,\tau)| \lesssim \tau^{-\nu}\|h_j\|_{\nu,2+a}\frac{R^{n+1}}{1+|y|^{n+1}}R^{-a} \quad \text{in } B_{2R}. \tag{7.40}$$

Construction at higher modes. In order to deal with the higher modes, for $h = h^\perp = \sum_{j=n+1}^\infty h_j(r)\Theta_j$ we let $\phi^\perp[h^\perp]$ be just the unique solution of the problem

$$\phi_\tau = \Delta\phi + pU(y)^{p-1}\phi + h^\perp \quad \text{in } B_{2R}\times(\tau_0,\infty) \tag{7.41}$$

$$\phi = 0 \quad \text{on} \quad \partial B_{2R}\times(\tau_0,\infty), \quad \phi(\cdot,\tau_0) = 0 \quad \text{in } B_{2R},$$

which is estimated as

$$|\phi^\perp[h^\perp](y,\tau)| \lesssim \tau^{-\nu}\frac{\|h^\perp\|_{\nu,2+a}}{1+|y|^a} \quad \text{in } B_{2R}. \tag{7.42}$$

We just let

$$\phi[h] := \phi^0[h^0] + \phi^1[h^1] + \phi^\perp[h^\perp]$$

be the functions constructed above. According to estimates (7.36) and (7.42) we find that this function solves Problem (7.6) for $c(\tau)$ given by (7.16), with bounds (7.3), (7.4), (7.7) as required. The proof is concluded.

□

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