Euler's Type Equation

$$at^2 y'' + bty' + cy = 0, \quad a, b, c \text{ constants}$$

Corresponding Characteristic Equation

$$y = t^r$$

$$ar(r-1) + br + c = 0 \qquad \text{——} (1)$$

Case 1: (1) has two real distinct roots $r_1 \neq r_2$

$$y_1 = t^{r_1}, \quad y_2 = t^{r_2}$$

Case 2: (1) has two complex roots $r_1 = \lambda + i\mu$

$$y_1 = t^\lambda \cos(\mu \ln t), \quad y_2 = t^\lambda \sin(\mu \ln t)$$

Case 3: (1) has equal real roots, $r_1 = r_2 = r$

$$y_1 = t^r, \quad y_2 = t^r \ln t$$

Ex. 1 $t^2 y'' - 2ty' - 4y = 0 \Rightarrow r(r-1) - 2r - 4 = 0 \Rightarrow r^2 - 3r - 4 = 0$

$r_1 = 4$, $r_2 = -1$, $\quad y_1 = t^4$, $y_2 = t^{-1}$

Ex. 2. $t^2 y'' + 3ty' + y = 0 \Rightarrow r(r-1) + 3r + 1 = 0 \Rightarrow r^2 + 2r + 1 = 0$

$r_1 = r_2 = -1 \Rightarrow y_1 = t^{-1}$, $y_2 = t^{-1} \ln t$

Ex. 3 $t^2 y'' + 2ty' + y = 0 \Rightarrow r^2 + r + 1 = 0 \Rightarrow r_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$y_1 = t^{-\frac{1}{2}} \cos\left(\frac{\sqrt{3}}{2} \ln t\right)$, $y_2 = t^{-\frac{1}{2}} \sin\left(\frac{\sqrt{3}}{2} \ln t\right)$