QUALITATIVE ANALYSIS OF RUPTURE SOLUTIONS FOR AN MEMS PROBLEM

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Abstract. We prove a sharp Hölder continuity estimates of rupture sets for sequences of solutions of the following nonlinear problem with negative exponent

$$\Delta u = \frac{1}{u^p}, \ p > 1, \ \text{in } \Omega.$$

As a consequence, we prove the existence of rupture solutions with isolated ruptures in a bounded convex domain in $\mathbb{R}^2$.

1. The setting and main results

Of concern is the following MEMS problem in a bounded domain $\Omega \subset \mathbb{R}^n$

$$\Delta u = u^{-p} \text{ in } \ \Omega \tag{1.1}$$

where $p > 1$.

Problem (1.1) arises in modeling an electrostatic Micro-Electromechanical System (MEMS) device. We refer to the books by Pelesko-Bernstein [11] for physical derivations and Esposito-Ghoussoub-Guo [5] for mathematical analysis.

Of special interest are solutions that give rise to singularities in the equation, that is such that $u \approx 0$ in some region, which in the physical model represents a **rupture** in the device. The main result of this paper is to give a sharp estimate on the Hölder continuity of solutions near the ruptures and estimates on Hausdorff dimensions of such rupture sets under natural energy assumptions.

We now state our main results. Let u_i be a sequence of positive solutions to (1.1) in $B_2(0)$, satisfying

$$\sup_i \int_{B_2(0)} |\nabla u_i|^2 + u_i^{1-p} + u_i^2 = M < +\infty. \tag{1.2}$$

Here $B_2(0) \subset \mathbb{R}^n$ is the open ball of radius 2.

Theorem 1.1.

- *u_i are uniformly bounded in $C^{\frac{2}{p+1}}(\overline{B_1})$;*
- *Up to subsequence, u_i converges uniformly to u_∞ in B_1, strongly in $H^1(B_1)$, and u_i^{-p} converges to u_∞^{-p} in $L^1(B_1)$;*
- *Outside $\{u_\infty = 0\}$, u_i converges to u_∞ in any C^k norm;*
- *u_∞ is a* stationary *solution of* (1.1).

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By a solution we mean that $u \in H^1$, $u^{-p} \in L^1$ and satisfies (1.1) in the sense of distributions. We say a solution $u \in H^1 \cap L^{1-p}$ is stationary if for any smooth vector field Y with compact support,

$$\int \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{p-1}u^{1-p}\right) \mathrm{div}Y - DY(\nabla u, \nabla u) = 0. \tag{1.3}$$

Next we consider the partial regularity problem for stationary solutions.

Theorem 1.2. *Assume u is a $C^{\frac{2}{p+1}}$ continuous, stationary solution of* (1.1)*. Then $\{u = 0\}$ is a closed set with Hausdorff dimension no more than $n - 2$. Moreover, if $n = 2$, $\{u = 0\}$ is a discrete set.*

For related estimates on the zero set of solutions see [8, 7, 3, 2]. The dimension estimate in Theorem 1.2 is the best compared to these previous results, although with different hypotheses.

As an application of the preceding theorems, we consider the original MEMS problem in a bounded domain

$$-\Delta v = \frac{\lambda}{(1-v)^p} \text{ in } \Omega, v = 0 \text{ on } \partial\Omega \tag{1.4}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain. Here rupture means $v = 1$.

It is known that there exists a critical parameter $\lambda_* > 0$ such that for $\lambda < \lambda_*$, problem (1.4) has a minimal solution and for $\lambda > \lambda_*$ there are no positive solutions. In [4], Esposito-Ghoussoub-Guo showed that when $n \leq 7$, the extremal solution at λ_* is smooth and hence there is a secondary bifurcation near λ_*. When the domain is convex, it is known that the only solutions for λ small is the minimal solutions. Thus by Rabinowitz's bifurcation theorem [12], there exists a sequence of $\lambda_i \geq c_0 > 0$ and a sequence of solutions $\{u_i = 1 - v_i\}$ such that $\min u_i \to 0$. By convexity of Ω and the moving plane method, there is a neighborhood Ω_δ of $\partial\Omega$ such that u_i remains uniformly positive in Ω_δ (see Lemma 3.2 in [7]). As a consequence of Theorem 1.1, u_i are uniformly bounded in $C^{\frac{2}{p+1}}(\overline{\Omega})$ and hence converges uniformly to a Hölder continuous function u_∞ with nonempty rupture set $\{u_\infty = 0\}$. Applying Theorem 1.2 we obtain the following result.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^2$ be a convex set. Then there exists a $\lambda^* > 0$ such that the following problem*

$$\Delta u = \frac{\lambda^*}{u^p} \ \textit{in } \Omega, \ u = 1 \ \textit{on } \partial\Omega \tag{1.5}$$

admits a weak solution u such that u is Hölder continuous and the rupture set of u consists a finite number of points.

Theorem 1.3 was proved by Guo and the third author [7] under the condition that $p < 3$ and that the domain has two axes of symmetries.

The proof of the uniform Hölder estimate for positive solutions in Theorem 1.1 is inspired by the work of Noris, Tavares, Terracini and Verzini [10], where uniform Hölder estimates are established for a strongly competitive Schrödinger system. A contradiction argument leads after scaling to a globally Hölder stationary nontrivial solution of

$$u\Delta u = 0, \quad u \geq 0 \ \text{ in } \mathbb{R}^n. \tag{1.6}$$

But a Liouville theorem of [10] says that u is trivial. The argument is carried out in Section 2 and we give the Liouville theorem in the Appendix for completeness.

The proof of the remaining statements of Theorem 1.1 is given in Section 4, after some preliminaries in Section 3. The proof actually applies to a sequence of stationary solutions having a uniform Hölder bound. Section 5 contains the proof of Theorem 1.2.

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2. The uniform Hölder continuity

In this section we prove

Theorem 2.1. *Let u_i be a sequence of positive solutions to* (1.1) *in B_4 with*

$$\sup_i \int_{B_4} u_i < +\infty.$$

Then

$$\sup_i \|u_i\|_{C^{\frac{2}{p+1}}(\overline{B}_1)} < +\infty.$$

The remaining part of this section will be devoted to the proof of this theorem.

Note that because u_i is subharmonic and positive,

$$\sup_i \|u_i\|_{L^\infty(B_2(0))} < +\infty.$$

Take $\eta \in C^\infty(\mathbb{R}^n)$ such that $\eta \equiv 1$ in $B_1(0)$, $\{\eta > 0\} = B_2(0)$, $\eta = 0$ in $\mathbb{R}^n \setminus B_2(0)$. Denote

$$\hat{u}_i = u_i \eta.$$

We will actually prove that

$$\sup_i \|\hat{u}_i\|_{C^{\frac{2}{p+1}}(\bar{B}_2(0))} < +\infty.$$

Assume this is not true. Because $\hat{u}_i$ are smooth in B_2, there exist $x_i, y_i \in B_2(0)$ such that as $i \to +\infty$,

$$L_i = \frac{|\hat{u}_i(x_i) - \hat{u}_i(y_i)|}{|x_i - y_i|^{\frac{2}{p+1}}} = \max_{x,y \in B_2(0), x \neq y} \frac{|\hat{u}_i(x) - \hat{u}_i(y)|}{|x - y|^{\frac{2}{p+1}}} \to +\infty. \tag{2.1}$$

Note that because $\hat{u}_i$ are uniformly bounded, as $i \to +\infty$, $|x_i - y_i| \to 0$.

Denote $r_i = |x_i - y_i|$ and $z_i = (y_i - x_i)/r_i$. Since $|z_i| = 1$, we can assume that $z_i \to z_\infty \in \mathbb{S}^{n-1}$. Define

$$\widetilde{u}_i(x) := L_i^{-1} r_i^{-\frac{2}{p+1}} \hat{u}_i(x_i + r_i x) = L_i^{-1} r_i^{-\frac{2}{p+1}} u_i(x_i + r_i x)\eta(x_i + r_i x),$$

and

$$\bar{u}_i(x) := L_i^{-1} r_i^{-\frac{2}{p+1}} u_i(x_i + r_i x)\eta(x_i).$$

These functions are defined in $\Omega_i = \frac{1}{r_i}(B_2(0) - x_i)$. Note that Ω_i converges to Ω_∞, which may be the entire space or an half space.

We first present some facts about these rescaled functions, which will be used below. By definition we have

$$\widetilde{u}_i(x) = \frac{\eta(x_i + r_i x)}{\eta(x_i)} \bar{u}_i(x), \tag{2.2}$$

and

$$\begin{aligned} \nabla\widetilde{u}_i(x) &= \frac{r_i\nabla\eta(x_i+r_ix)}{\eta(x_i)}\bar{u}_i(x)+\frac{\eta(x_i+r_ix)}{\eta(x_i)}\nabla\bar{u}_i(x) \\ &= L_i^{-1}r_i^{\frac{p-1}{p+1}}u_i(x_i+r_ix)\nabla\eta(x_i+r_ix)+\frac{\eta(x_i+r_ix)}{\eta(x_i)}\nabla\bar{u}_i(x) \\ &= \frac{\eta(x_i+r_ix)}{\eta(x_i)}\nabla\bar{u}_i(x)+O(L_i^{-1}r_i^{\frac{p-1}{p+1}}). \end{aligned} \tag{2.3}$$

By (2.1) and noting that $|z_i|=1$, we have

$$1=|\widetilde{u}_i(0)-\widetilde{u}_i(z_i)|=\max_{x,y\in\Omega_i,x\neq y}\frac{|\widetilde{u}_i(x)-\widetilde{u}_i(y)|}{|x-y|^{\frac{2}{p+1}}}. \tag{2.4}$$

Next, because η is Lipschitz continuous in $\overline{B_2(0)}$, for $x\in\Omega_i$, we have a constant C which depends only on $\sup_{B_2(0)}u_i$ and the Lipschitz constant of η, such that

$$\begin{aligned} |\widetilde{u}_i(x)-\bar{u}_i(x)| &\le \frac{C}{L_ir_i^{\frac{2}{p+1}}}|\eta(x_i+r_ix)-\eta(x_i)| \\ &\le CL_i^{-1}r_i^{\frac{p-1}{p+1}}|x|. \end{aligned} \tag{2.5}$$

This converges to 0 uniformly on any compact set of Ω_∞ as $i\to+\infty$. By the Lipschitz continuity of η, we also have

$$\widetilde{u}_i(x)\le CL_i^{-1}r_i^{\frac{p-1}{p+1}}\operatorname{dist}(x,\partial\Omega_i). \tag{2.6}$$

Finally, we note that $\bar{u}_i$ satisfies

$$\Delta\bar{u}_i=\varepsilon_i\bar{u}_i^{-p}. \tag{2.7}$$

Here $\varepsilon_i=L_i^{-p-1}\eta(x_i)^{p+1}\to 0$ as $i\to+\infty$.

We divide the proof into two cases.

Case 1. $A_i:=\widetilde{u}_i(0)\to+\infty$.
By (2.6),

$$\operatorname{dist}(0,\partial\Omega_i)\ge cL_ir_i^{-\frac{p-1}{p+1}}A_i\to+\infty.$$

Hence Ω_i converges to $\mathbb{R}^n$. By (2.4), we can assume that (after passing to a subsequence of i) $\widetilde{u}_i-A_i$ converges to $\bar{u}_\infty$ uniformly on any compact set of $\mathbb{R}^n$. By (2.5), $\bar{u}_i-A_i$ converges to the same $\bar{u}_\infty$ uniformly on any compact set of $\mathbb{R}^n$.

For any $R>0$, if i large, (2.4) and (2.5) imply that

$$\inf_{B_R(0)}\bar{u}_i\ge\inf_{B_R(0)}\widetilde{u}_i-CL_i^{-1}r_i^{\frac{p-1}{p+1}}R\ge A_i-R^{\frac{2}{p+1}}-CL_i^{-1}r_i^{\frac{p-1}{p+1}}R\ge\frac{A_i}{2}.$$

So

$$0\le\Delta(\bar{u}_i-A_i)\le 2^p\varepsilon_iA_i^{-p}\to 0.$$

By standard $W^{2,q}$ estimates, for any $q\in(1,+\infty)$, $\bar{u}_i-A_i$ are uniformly bounded in $W^{2,q}_{loc}(\mathbb{R}^n)$. Then by the Sobolev embedding theorem, for any $\alpha\in(0,1)$, $\bar{u}_i-A_i$ are uniformly bounded in $C^{1,\alpha}_{loc}(\mathbb{R}^n)$. By letting $i\to+\infty$ in (2.7), we see $\bar{u}_\infty$ is a harmonic function on $\mathbb{R}^n$.

By the uniform convergence of $\bar{u}_i-A_i$, we can take the limit in (2.4) to get

$$1=|\bar{u}_\infty(0)-\bar{u}_\infty(z_\infty)|=\max_{x,y\in\Omega_i}\frac{|\bar{u}_\infty(x)-\bar{u}_\infty(y)|}{|x-y|^\alpha}.$$

The first equality implies that $\bar{u}_\infty$ is non-constant, while the second one implies that $\bar{u}_\infty$ is globally $2/(p+1)-$Hölder continuous, hence a constant by the Liouville theorem for harmonic functions. This is a contradiction.

Case 2. $A_i := \widetilde{u}_i(0) \to A_\infty \in [0, +\infty)$.
By the first equality in (2.4),

$$1 \leq \widetilde{u}_i(0) + \widetilde{u}_i(z_i). \tag{2.8}$$

Then by (2.6),

$$cL_i r_i^{-\frac{p-1}{p+1}} \leq \text{dist}(0, \partial\Omega_i) + \text{dist}(z_i, \partial\Omega_i) \leq 2\text{dist}(0, \partial\Omega_i) + 1.$$

So we still have $\text{dist}(0, \partial\Omega_i) \to +\infty$, and $\Omega_\infty = \mathbb{R}^n$.

By (2.4), we can assume that (by passing to a subsequence of i) $\widetilde{u}_i$ converges to $\bar{u}_\infty$ uniformly on any compact set of $\mathbb{R}^n$. By (2.5), $\bar{u}_i$ converges to the same $\bar{u}_\infty$ uniformly on any compact set of $\mathbb{R}^n$. By this uniform convergence, we can take the limit in (2.8) to get

$$1 \leq \bar{u}_\infty(0) + \bar{u}_\infty(z_\infty).$$

So the open set $D := \{\bar{u}_\infty > 0\}$ is non-empty.

In any compact set $D' \subset\subset D$, there exists a $\delta > 0$ such that $\inf_{D'} \bar{u}_\infty = 2\delta$. Then if i large,

$$\inf_{D'} \bar{u}_i \geq \delta.$$

By the same argument as in Case 1, we see

$$\Delta \bar{u}_\infty = 0 \quad \text{in } D.$$

Hence $\bar{u}_\infty$ is smooth in D. In particular, if $\{\bar{u}_\infty = 0\} = \emptyset$, we can use the same argument in Case 1 to get a contradiction.

In the following we assume $\{\bar{u}_\infty = 0\} \neq \emptyset$. Without loss of generality, assume that $\bar{u}_\infty(0) = 0$.

Lemma 2.2. *$\bar{u}_i$ converges strongly to $\bar{u}_\infty$ in $H^1_{loc}(\mathbb{R}^n)$. $\varepsilon_i \bar{u}_i^{1-p}$ converges to 0 in $L^1_{loc}(\mathbb{R}^n)$.*

Proof. Take a function $\eta \in C_0^\infty(\mathbb{R}^n)$. Testing the equation of $\bar{u}_i$ with $\bar{u}_i\eta^2$, we get

$$\int_{\mathbb{R}^n} |\nabla \bar{u}_i|^2 \eta^2 + \varepsilon_i \bar{u}_i^{1-p} \eta^2 + 2\bar{u}_i \eta \nabla \bar{u}_i \nabla \eta = 0. \tag{2.9}$$

First, by applying the Cauchy inequality to the last term, we have

$$\int_{\mathbb{R}^n} |\nabla \bar{u}_i|^2 \eta^2 + \varepsilon_i \bar{u}_i^{1-p} \eta^2 \leq 4 \int_{\mathbb{R}^n} \bar{u}_i^2 |\nabla \eta|^2.$$

Because $\bar{u}_i$ are uniformly bounded in any compact set of $\mathbb{R}^n$, $\bar{u}_i$ are uniformly bounded in $H^1_{loc}(\mathbb{R}^n)$. By the uniform convergence of $\bar{u}_i$, they must converges weakly to $\bar{u}_\infty$ in $H^1_{loc}(\mathbb{R}^n)$.

By taking limit in (2.9), we obtain

$$\lim_{i \to +\infty} \int_{\mathbb{R}^n} |\nabla \bar{u}_i|^2 \eta^2 - |\nabla \bar{u}_\infty|^2 \eta^2 + \varepsilon_i \bar{u}_i^{1-p} \eta^2 = -\int_{\mathbb{R}^n} |\nabla \bar{u}_\infty|^2 \eta^2 + 2\bar{u}_\infty \eta \nabla \bar{u}_\infty \nabla \eta.$$

On the other hand, take a $\sigma > 0$ small so that $\{\bar{u}_\infty = \sigma\}$ is a smooth hypersurface. Then because $\bar{u}_\infty$ is harmonic in $\{\bar{u}_\infty > \sigma\}$,

$$\int_{\{\bar{u}_\infty > \sigma\}} |\nabla \bar{u}_\infty|^2 \eta^2 + 2\bar{u}_\infty \eta \nabla \bar{u}_\infty \nabla \eta \quad = \quad \int_{\{\bar{u}_\infty = \sigma\}} \frac{\partial \bar{u}_\infty}{\partial \nu} \bar{u}_\infty \eta^2$$

$$\begin{aligned} &= \sigma \int_{\{\bar{u}_\infty=\sigma\}} \frac{\partial \bar{u}_\infty}{\partial \nu}\eta^2 \\ &= \sigma \int_{\{\bar{u}_\infty>\sigma\}} \nabla \bar{u}_\infty \nabla \eta^2 \\ &= O(\sigma). \end{aligned}$$

Here ν is the outward unit normal vector to $\partial\{\bar{u}_\infty > \sigma\}$. By letting $\sigma \to 0$, we see

$$\int_{\mathbb{R}^n} |\nabla \bar{u}_\infty|^2\eta^2 + 2\bar{u}_\infty \eta \nabla \bar{u}_\infty \nabla \eta = 0.$$

Hence

$$\lim_{i\to+\infty} \int_{\mathbb{R}^n} |\nabla \bar{u}_i|^2\eta^2 - |\nabla \bar{u}_\infty|^2\eta^2 + \varepsilon_i \bar{u}_i^{1-p}\eta^2 = 0. \qquad \square$$

Remark 2.3. *An essential point in this proof is the fact that*

$$\bar{u}_\infty \Delta \bar{u}_\infty = 0.$$

This is well defined, because $\Delta\bar{u}_\infty$ is a Radon measure and $\bar{u}_\infty$ is continuous. From this we also get, in the distributional sense

$$\Delta \bar{u}_\infty^2 = 2|\nabla \bar{u}_\infty|^2. \tag{2.10}$$

Because $\bar{u}_i > 0$ in Ω_i, it is smooth. Then by standard domain variation calculation, for any vector field $Y \in C_0^\infty(\Omega_i, \mathbb{R}^n)$,

$$\int_{\Omega_i} \left(\frac{1}{2}|\nabla \bar{u}_i|^2 - \frac{\varepsilon_i}{p-1}\bar{u}_i^{1-p}\right) \mathrm{div} Y - DY(\nabla \bar{u}_i, \nabla \bar{u}_i) = 0.$$

By the previous lemma, we can take the limit to get

$$\int_{\mathbb{R}^n} \frac{1}{2}|\nabla \bar{u}_\infty|^2 \mathrm{div} Y - DY(\nabla \bar{u}_\infty, \nabla \bar{u}_\infty) = 0,$$

for any vector field $Y \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$.

Now we can apply Theorem A.1 in the appendix, which says $\bar{u}_\infty$ is a constant. This is a contradiction because both $\{\bar{u}_\infty > 0\}$ and $\{\bar{u}_\infty = 0\}$ are nonempty.

In conclusion, the assumption (2.1) does not hold. So $\hat{u}_i$ are uniformly bounded in $C^{\frac{2}{p+1}}(\overline{B_2})$. Since $\hat{u}_i = u_i$ in B_1, this finishes the proof of Theorem 2.1.

3. Some tools

In this section we first present some consequences of the uniform Hölder continuity, which we will use to prove Theorems 1.1 and 1.2. Therefore, throughout this section we assume that u_i is a sequence of stationary solutions of (1.1) in $B_2(0)$ satisfying

$$\sup_i \|u_i\|_{C^{\frac{2}{p+1}}(\overline{B}_{3/2}(0))} < +\infty. \tag{3.1}$$

By Theorem 2.1, this includes the case that u_i are positive solutions of (1.1) in $B_2(0)$ satisfying (1.2).

Lemma 3.1. *There exists a constant C such that for any i, $x \in B_1$ and $r \in (0, 1/2)$,*

$$\int_{B_r(x)} u_i^{-p} \leq C r^{n-2\frac{p}{p+1}}.$$

Proof. Take a nonnegative function $\eta \in C_0^\infty(B_{2r}(x))$ such that $\eta \equiv 1$ in $B_r(x)$ and $|\Delta\eta| \le Cr^{-2}$. Then

$$\int u_i^{-p}\eta = \int (u_i - u_i(x))\,\Delta\eta \le Cr^{n-2+\frac{2}{p+1}}.$$

Here we have used the uniform $2/(p+1)-$Hölder continuity of u_i, which implies that

$$\sup_{B_r(x)} |u_i - u_i(x)| \le Cr^{\frac{2}{p+1}}. \tag{3.2}$$

□

Lemma 3.2. *There exists a constant C depending only on M, such that for any i, $x \in B_1$ and $r \in (0,1/2)$,*

$$\int_{B_r(x)} |\nabla u_i|^2 + u_i^{1-p} \le Cr^{n-2\frac{p-1}{p+1}}.$$

Proof. First by the previous lemma and Hölder inequality,

$$\int_{B_r(x)} u_i^{1-p} \le \left(\int_{B_r(x)} u_i^{-p}\right)^{\frac{p-1}{p}} |B_r(x)|^{\frac{1}{p}} \le Cr^{n-2\frac{p-1}{p+1}}.$$

Take an nonnegative function $\eta \in C_0^\infty(B_{2r}(x))$ such that $\eta \equiv 1$ in $B_r(x)$ and $|\nabla\eta| \le 2r^{-1}$. Testing the equation of u_i with $(u_i - u_i(x))\eta^2$, we get

$$\int |\nabla u_i|^2\eta^2 + u_i^{-p}(u_i - u_i(x))\eta^2 = -2\int \nabla u_i \nabla\eta(u_i - u_i(x))\eta.$$

The Cauchy inequality gives

$$\int |\nabla u_i|^2\eta^2 \le \int u_i^{-p}|u_i - u_i(x)|\eta^2 + 8\int |\nabla\eta|^2(u_i - u_i(x))^2.$$

Then using the previous lemma and (3.2) we have

$$\begin{aligned}\int |\nabla u_i|^2\eta^2 &\le \sup_{B_r(x)} |u_i - u_i(x)| \int u_i^{-p}\eta^2 + 8\sup_{B_r(x)} |u_i - u_i(x)|^2 \int |\nabla\eta|^2 \\ &\le Cr^{n-2\frac{p-1}{p+1}}.\end{aligned}$$

□

The following result holds for any $2/(p+1)-$Hölder continuous solutions.

Lemma 3.3. *If $x \in \{u > 0\}$,*

$$|\nabla u(x)| \le Cu(x)^{-\frac{p-1}{2}}.$$

Proof. Denote $h^{\frac{2}{p+1}} = u(x) > 0$. By the Hölder continuity, $u \ge \frac{h^{\frac{2}{p+1}}}{2}$ in $B_{\delta h}(x)$, where δ depends only the $C^{2/(p+1)}$ norm of u. Note that we also have $u \le 2h^{\frac{2}{p+1}}$ in $B_{\delta h}(x)$.

Define $\bar{u}(y) = h^{-\frac{2}{p+1}}u(x + hy)$. Then in $B_\delta(0)$, $1/2 \le \bar{u} \le 2$, and $\bar{u}$ satisfies the equation (1.1). By standard elliptic estimates, there exists a constant C depending only on δ and n so that

$$|\nabla\bar{u}(0)| \le C.$$

Rescaling back we get the required claim. □

This estimates implies that $|\nabla u^{\frac{p+1}{2}}| \le C$ in $\{u > 0\}$. Thus we get

Corollary 3.4. $u^{\frac{p+1}{2}}$ *is Lipschitz continuous.*

The next result is taken from [9], and it can be viewed as a non-degeneracy result.

Lemma 3.5. *There exists a constant c depending only on M, such that for any i, $x \subset B_1$ and $r \in (0, 1/2)$,*

$$\int_{B_r(x)} u_i \geq c r^{n+\frac{2}{p+1}}.$$

Proof. By the Hölder inequality,

$$\int_{B_r(x)} 1 = \int_{B_r(x)} u_i^{-\frac{p}{p+1}} u_i^{\frac{p}{p+1}} \leq \left(\int_{B_r(x)} u_i^{-p}\right)^{\frac{1}{p+1}} \left(\int_{B_r(x)} u_i\right)^{\frac{p}{p+1}}.$$

Substituting Lemma 3.1 into this we get the estimate. □

Finally let us recall the monotonicity formula for stationary solutions.

Theorem 3.6. *Let u be a stationary solution of* (1.1) *in B_1. Then for any $B_R(x) \subset B_1$ and $r \in (0, R)$,*

$$E(r; x, u) = r^{-n+2\frac{p-1}{p+1}} \int_{B_r(x)} \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{p-1} u^{1-p}\right) - \frac{r^{-n+2\frac{p-1}{p+1}-1}}{p+1} \int_{\partial B_r(x)} u^2$$

is nondecreasing in r. Moreover, if $E(r; x, u) \equiv const.$, then u is homogeneous with respect to x

$$u(x + \lambda y) = \lambda^{\frac{2}{p+1}} u(x + y), \quad y \in B_R(x), \ \lambda \in (0, 1).$$

Proof. By the proof in [6], we have

$$\frac{d}{dr} E(r; x, u) = c(n, p) r^{2\frac{p-1}{p+1}-n} \int_{\partial B_r(x)} \left(\frac{\partial u}{\partial r} - \frac{2}{p+1} r^{-1} u\right)^2 \geq 0. \tag{3.3}$$

This also characterizes the case of equality. □

By the equation we have

$$\int_{B_r(x)} |\nabla u|^2 + u^{1-p} - \int_{\partial B_r(x)} u u_r = 0.$$

Multiplying this with $\frac{2}{p-3} r^{2\frac{p-1}{p+1}-n}$, and adding it into $E(r; x, u)$, we get another form for $E(r; x, u)$

$$\begin{aligned} E(r; x, u) &= r^{-n+2\frac{p-1}{p+1}} \int_{B_r(x)} \left(\frac{1}{2} + \frac{2}{p-3}\right) |\nabla u|^2 + \left(\frac{2}{p-3} - \frac{1}{p-1}\right) u^{1-p} \\ &\quad - \frac{1}{p-3} \frac{d}{dr} \left[r^{-n+2\frac{p-1}{p+1}} \int_{\partial B_r(x)} u^2 \right]. \end{aligned}$$

4. The convergence

Let u_i be a sequence of stationary $C^{\frac{2}{p+1}}$ Hölder solutions of (1.1) in $B_2(0)$ satisfying the uniform estimate (3.1).

Let us list the results we obtained in the previous sections. There exists a constant C independent of i, such that:

(1) For any $x \in B_1$ and $r \in (0,1/2)$,
$$\int_{B_r(x)} |\nabla u_i|^2 + u_i^{1-p} \le C r^{n-2\frac{p-1}{p+1}}. \tag{4.1}$$

(2) For any $x \in B_1$ and $r \in (0,1/2)$,
$$\int_{B_r(x)} u_i^{-p} \le C r^{n-2\frac{p}{p+1}}. \tag{4.2}$$

(3) For any $x, y \in B_1$,
$$|u_i(x) - u_i(y)| \le C|x-y|^{\frac{2}{p+1}}. \tag{4.3}$$

(4) For any $x \in B_1$ and $r \in (0,1/2)$,
$$\int_{B_r(x)} u_i \ge \frac{1}{C} r^{\frac{2}{p+1}}. \tag{4.4}$$

By (4.3), we can assume that, up to a subsequence of i, u_i converges uniformly to a function u_∞ in B_1. Then with (4.1), u_i are also uniformly bounded in $H^1(B_1)$, and we can assume that it converges to u_∞ weakly in $H^1(B_1)$. By the uniform convergence, we see u_∞ also satisfies the estimate (4.3) and (4.4).

By standard elliptic estimates, for any domain $\Omega \subset\subset \{u_\infty > 0\} \cap B_1$ and k, u_i converges to u_∞ in $C^k(\Omega)$.

Lemma 4.1. $H^{n-2\frac{p}{p+1}}(\{u_\infty = 0\} \cap B_1) = 0$.

Proof. First by (4.4), for any $x \in \{u_\infty = 0\} \cap B_1$ and $r \in (0,1/2)$,
$$\sup_{B_r(x)} u_\infty \ge c r^{\frac{2}{p+1}}.$$
Then by the Hölder continuity (4.3) for u_∞, there exists a ball $B_{\delta r}(y) \subset B_r(x)$ (δ depends on the Hölder constant of u_∞) such that
$$u_\infty \ge c r^{\frac{2}{p+1}} \quad \text{in} B_{\delta r}(y).$$
In particular, $B_{\delta r}(y) \subset \{u_\infty > 0\}$. This means for any $x \in \{u_\infty = 0\} \cap B_1$ and $r \in (0,1/2)$,
$$\frac{|\{u_\infty = 0\} \cap B_r(x)|}{|B_r(x)|} \le 1 - c\delta.$$
By the Lebesgue differentiation theorem, $|\{u_\infty = 0\} \cap B_1| = 0$.

Then because u_i^{-p} converges to u^{-p} uniformly in any compact set of $\{u_\infty > 0\} \cap B_1$, u_i^{-p} converges to u^{-p} a.e. in B_1. By the Fatou lemma,
$$\int_{B_1} u_\infty^{-p} \le \liminf_{i \to +\infty} \int_{B_1} u_\infty^{-p} \le C. \tag{4.5}$$

For any $\varepsilon > 0$, take a maximal $\varepsilon-$separated set $\{x_i, 1 \le i \le N\}$ of $\{u_\infty = 0\} \cap B_1$. By definition, $B_{\varepsilon/2}(x_i)$ are disjoint, and
$$\{u_\infty = 0\} \cap B_1 \subset \cup_i^N B_\varepsilon(x_i).$$

Note that every $B_\varepsilon(x_i)$ belongs to the $\varepsilon-$neighborhood $\mathcal{N}_\varepsilon$ of $\{u_\infty = 0\}\cap B_1$. Hence

$$\sum_{i=1}^{N}\int_{B_{\varepsilon/2}(x_i)} u_\infty^{-p} \leq \int_{\mathcal{N}_\varepsilon} u_\infty^{-p}, \tag{4.6}$$

which goes to 0 as $\varepsilon \to 0$. Because $x_i \in \{u_\infty = 0\}$, by (4.3),

$$\sup_{B_{\varepsilon/2}(x_i)} u_\infty \leq C\varepsilon^{\frac{2}{p+1}}.$$

Thus

$$\int_{B_{\varepsilon/2}(x_i)} u_\infty^{-p} \geq C\varepsilon^{n-2\frac{p}{p+1}}.$$

Substituting this into (4.6), we see

$$\begin{aligned}\sum_{i=1}^{N}\left(\mathrm{diam}(B_\varepsilon(x_i))\right)^{n-2\frac{p}{p+1}} &\leq C\sum_{i=1}^{N}\int_{B_{\varepsilon/2}(x_i)} u_\infty^{-p} \\ &\leq C\int_{\mathcal{N}_\varepsilon} u_\infty^{-p}.\end{aligned}$$

By letting $\varepsilon \to 0$, we get $H^{n-2\frac{p}{p+1}}(\{u_\infty = 0\}\cap B_1) = 0$. □

Since u_i^{-1} converges to u_∞^{-1} a.e. in B_1, by passing limit in (4.1) and (4.2) and using the Fatou lemma, we see u_∞ also satisfies (4.1) and (4.2). (The estimate of $|\nabla u_\infty|$ is a direct consequence of weak convergence in $H^1(B_1)$.)

Lemma 4.2. *u_i^{-p} converges to u_∞^{-p} in $L^1(B_1)$.*

Proof. By the Fatou lemma, we always have

$$\int_{B_1} u_\infty^{-p} \leq \liminf_{i\to+\infty}\int_{B_1} u_i^{-p}.$$

Thus we only need to prove the reverse inequality

$$\int_{B_1} u_\infty^{-p} \geq \limsup_{i\to+\infty}\int_{B_1} u_i^{-p}.$$

By the previous lemma, for any $\varepsilon > 0$, there exists a covering of $\{u_\infty = 0\}\cap B_1$ by $\cap_k C_k$, with $\mathrm{diam}C_k \leq \varepsilon$, and

$$\sum_i (\mathrm{diam}C_k)^{n-2\frac{p}{p+1}} \leq \varepsilon. \tag{4.7}$$

For each k, take an $x_k \in \{u_\infty = 0\}\cap B_1\cap C_k$. Denote the open set

$$U := \cup_k B_{\mathrm{diam}C_k}(x_k).$$

U is an open neighborhood of $\{u_\infty = 0\}\cap B_1$. So in $(\{u_\infty > 0\}\cap B_1)\setminus U$, for all i large, u_i^{-p} have a uniformly positive lower bound and they converge to u_∞^{-p} uniformly. Hence

$$\lim_{i\to+\infty}\int_{(\{u_\infty>0\}\cap B_1)\setminus U} u_i^{-p} = \int_{(\{u_\infty>0\}\cap B_1)\setminus U} u_\infty^{-p}. \tag{4.8}$$

For each i and k, by (4.2),

$$\int_{B_{\mathrm{diam}C_k}(x_k)} u_i^{-p} \leq C(\mathrm{diam}C_k)^{n-2\frac{p}{p+1}}.$$

Summing in k and noting (4.7), we see

$$\int_U u_i^{-p} \leq \sum_k \int_{B_{\mathrm{diam} C_k}(x_k)} u_i^{-p} \leq C\varepsilon.$$

Combined with (4.8), we obtain

$$\int_{B_1} u_\infty^{-p} \geq \limsup_{i\to+\infty} \int_{B_1} u_i^{-p} - C\varepsilon.$$

Taking $\varepsilon \to 0$, we complete the proof. □

Corollary 4.3. *u_∞ is a solution to* (1.1) *in the distributional sense.*

Lemma 4.4. *u_i^{1-p} converges to u_∞^{1-p} in $L^1(B_1)$. u_i converges to u_∞ strongly in $H^1(B_1)$.*

Proof. Note that for any $t, s \geq 0$, $|t^{1-p} - s^{1-p}| \leq C(p)|s-t|(s^{-p} + t^{-p})$. Thus, by the previous lemma

$$\int_{B_1} |u_i^{1-p} - u_\infty^{1-p}| \leq C(p) \sup_{B_1} |u_i - u_\infty| \left(\int_{B_1} u_i^{-p} + u_\infty^{-p} \right) \leq C \sup_{B_1} |u_i - u_\infty|.$$

This converges to 0 by the uniform convergence of u_i to u_∞.

By testing the equation of u_i with $u_i \eta^2$, where $\eta \in C_0^\infty(B_2)$, we have

$$\int_{B_2} |\nabla u_i|^2 \eta^2 + u_i^{1-p} \eta^2 = \int_{B_2} u_i^2 \Delta \frac{\eta^2}{2}.$$

By the strong convergence of u_i in $L^2_{loc}(B_2)$, and the convergence of u_i^{1-p} proved above, we have

$$\lim_{i\to+\infty} \int_{B_2} |\nabla u_i|^2 \eta^2 + \int_{B_2} u_\infty^{1-p} \eta^2 = \int_{B_2} u_\infty^2 \Delta \frac{\eta^2}{2}.$$

Since $u_\infty \in H^1(B_2)$ is a weak solution of (1.1), and $u_\infty^{1-p} \in L^1_{loc}$, we also have

$$\int_{B_2} |\nabla u_\infty|^2 \eta^2 + u_\infty^{1-p} \eta^2 = \int_{B_2} u_\infty^2 \Delta \frac{\eta^2}{2}.$$

This gives

$$\lim_{i\to+\infty} \int_{B_2} |\nabla u_i|^2 = \int_{B_2} |\nabla u_\infty|^2,$$

and the strong convergence of u_i in $H^1(B_1)$. □

By this convergence, we can take limit in (1.3) for u_i to get the corresponding stationary condition for u_∞. This finishes the proof of Theorem 1.1.

5. Dimension reduction for stationary solutions

In this section we assume that u is a $2/(p+1)-$Hölder continuous, stationary solution of (1.1) in B_2, with

$$\int_{B_2} |\nabla u|^2 + u^{1-p} + u^2 = M < +\infty.$$

By the results in Section 3, u satisfies all of the estimates (4.1)-(4.4). In particular, $\{u = 0\}$ is a closed set satisfying (by Lemma 4.1)

$$H^{n-2+\frac{2}{p+1}}(\{u=0\}) = 0.$$

Assume that $u(0) = 0$, for $\lambda \to 0$, define the blow up sequence

$$u^\lambda(x) = \lambda^{-\frac{2}{p+1}} u(\lambda x).$$

By a rescaling, we see u^λ satisfies (4.1)-(4.4), for all ball $B_r(x) \subset B_{\lambda^{-1}}$. By the results established in Section 4, we can get a subsequence of $\lambda_i \to 0$, so that $u_i := u^{\lambda_i}$ converges uniformly to a u_∞ on any compact set of $\mathbb{R}^n$. We also have

(1) For each R, u_i^{-p} converges to u_∞^{-p} in $L^1(B_R)$;
(2) For each R, u_i^{1-p} converges to u_∞^{1-p} in $L^1(B_R)$;
(3) For each R, u_i converges to u_∞ in $H^1(B_R)$;
(4) u_∞ is a stationary weak solution of (1.1) in the distributional sense;
(5) u_∞ is nonzero.

To continue, we first note the following result.

Lemma 5.1. *For any $\varepsilon > 0$, if i large, $\{u_i = 0\} \cap B_1$ lies in an $\varepsilon-$neighborhood of $\{u_\infty = 0\} \cap B_1$.*

Proof. This is because u_i converges to u_∞ uniformly in any compact set $\Omega' \subset\subset \{u_\infty > 0\} \cap B_1$. Thus for i large, $u_i > 0$ in Ω'. □

Next we would like to use the monotonicity formula to explore the information of the limit u_∞.

Lemma 5.2. *The limit $\lim_{r\to 0} E(r; 0, u)$ exists and is finite.*

Proof. In view of the monotonicity of $E(r; 0, u)$, we only need to show that as $r \to 0$, $E(r; 0, u)$ has a uniform lower bound.

By Lemma 3.2, for each $r \in (0, 1)$,

$$r^{2\frac{p-1}{p+1}-n} \int_{B_r} |\nabla u|^2 + u^{1-p} \le C.$$

Next, by Theorem 2.1, $\sup_{B_r} u \le C r^{\frac{2}{p+1}}$. Thus

$$r^{2\frac{p-1}{p+1}-n-1} \int_{\partial B_r} u^2 \le C r^{2\frac{p-1}{p+1}-n-1+n-1+\frac{4}{p+1}} = C.$$

Substituting these into the first formulation of $E(r; 0, u)$, we get

$$E(r; 0, u) \ge -C.$$

□

By (3.3), for any $r \in (0, 1)$,

$$E(1; 0, u) - E(r; 0, u) = c \int_{B_1 \setminus B_r} |x|^{2\frac{p-1}{p+1}-n} \left(\frac{\partial u}{\partial r} - \frac{2}{p+1} r^{-1} u \right)^2 dx.$$

Together with the previous lemma we get

Corollary 5.3.

$$\int_{B_1} |x|^{2\frac{p-1}{p+1}-n} \left(\frac{\partial u}{\partial r} - \frac{2}{p+1} |x|^{-1} u \right)^2 dx < +\infty.$$

Lemma 5.4. *u_∞ is a homogeneous solution of (1.1) on $\mathbb{R}^n$.*

Proof. By the strong convergence of u_i in $H^1_{loc}(\mathbb{R}^n)$, for any $\eta \in (0,1)$,

$$
\begin{aligned}
& \int_{B_{\eta^{-1}}\setminus B_\eta} |x|^{2\frac{p-1}{p+1}-n}\left(\frac{\partial u_\infty}{\partial r}-\frac{2}{p+1}r^{-1}u_\infty\right)^2 dx \\
= \; & \lim_{i\to+\infty}\int_{B_{\eta^{-1}}\setminus B_\eta} |x|^{2\frac{p-1}{p+1}-n}\left(\frac{\partial u_i}{\partial r}-\frac{2}{p+1}|x|^{-1}u_i\right)^2 dx \\
= \; & \lim_{i\to+\infty}\int_{B_{\eta^{-1}\lambda_i}\setminus B_{\eta\lambda_i}} |x|^{2\frac{p-1}{p+1}-n}\left(\frac{\partial u}{\partial r}-\frac{2}{p+1}|x|^{-1}u\right)^2 dx \\
= \; & 0.
\end{aligned}
$$

The last one is guaranteed by the previous corollary.

This means for a.a. $x \in \mathbb{R}^n$,

$$\frac{\partial u_\infty}{\partial r}-\frac{2}{p+1}r^{-1}u_\infty = 0.$$

Integrating this in r, we get

$$u_\infty(x) = |x|^{\frac{2}{p+1}}u_\infty\left(\frac{x}{|x|}\right). \qquad \square$$

Define the density function (it may take value $-\infty$)

$$\Theta(x;u) := \lim_{r\to 0} E(r;x,u).$$

We have the following characterization of rupture points.

Lemma 5.5. $x \in \{u > 0\}$ *if and only if* $\Theta(x) = -\infty$.

Proof. If $u(x) = 2h > 0$, by the continuity of u, $u > h$ in a ball $B_{r_0}(x)$ and it is smooth here. Hence for $r < r_0$,

$$r^{2\frac{p-1}{p+1}-n}\int_{B_r(x)} |\nabla u|^2 + u^{1-p} \le Cr^{2\frac{p-1}{p+1}},$$

which goes to 0 as $r \to 0$.

On the other hand,

$$r^{2\frac{p-1}{p+1}-n-1}\int_{\partial B_r(x)} u^2 \ge h^2 r^{2\frac{p-1}{p+1}-2},$$

which goes to $+\infty$ as $r \to 0$. Substituting these into the first formulation of $E(r;x,u)$ we get

$$\lim_{r\to 0} E(r;x,u) = -\infty.$$

If $u(x) = 0$, the same proof of Lemma 5.2 gives

$$\Theta(x;u) = \lim_{r\to 0} E(r;x,u) \ge -C. \qquad \square$$

Lemma 5.6. $\Theta(x;u)$ *is upper semi-continuous in* x.

Proof. Because $u \in H^1(B_2)$ and $u^{1-p} \in L^1(B_2)$, by the first formulation of $E(r;x,u)$, $E(r;x,u)$ is a continuous function of x. Then since $\Theta(x)$ is the decreasing limit of this family of continuous functions, it is upper semi-continuous in x. $\square$

Lemma 5.7. *Let u be a homogeneous stationary solution of* (1.1) *on $\mathbb{R}^n$, satisfying estimates* (4.1)-(4.4) *for all balls $B_r(x)$. Then for any $x \neq 0$, $\Theta(x,u) \leq \Theta(0,u)$. Moreover, if $\Theta(x,u) = \Theta(0,u)$, u is translation invariant in the direction x, i.e. for all $t \in \mathbb{R}$,*

$$u(tx + \cdot) = u(\cdot) \text{ a.e. in } \mathbb{R}^n.$$

Proof. With the help of the estimates (4.1)-(4.4), similar to the proof of Lemma 5.2, for any $x_0 \in \mathbb{R}^n$, there exists a constant C such that

$$\lim_{r \to +\infty} E(r; x_0, u) \leq C.$$

And we can define the blowing down sequence with respect to the base point x_0,

$$u_\lambda(x) = \lambda^{\frac{4}{p-1}} u(x_0 + \lambda x) \qquad \lambda \to +\infty.$$

Since u is homogeneous with respect to 0,

$$u_\lambda(x) = u(\lambda^{-1} x_0 + x),$$

which converges to $u(x)$ as $\lambda \to +\infty$ uniformly in any compact set of $\mathbb{R}^n$. u_λ also converges strongly in $H^1_{loc}(\mathbb{R}^n)$, u_λ^{1-p} and u_λ^{-p} converges in $L^1_{loc}(\mathbb{R}^n)$. Then by the homogeneity of u and these convergence, we see

$$\begin{aligned}
\Theta(0; u) = E(1; 0, u) &= \lim_{\lambda \to +\infty} E(1; 0, u_\lambda) \\
&= \lim_{\lambda \to +\infty} E(\lambda; x_0, u) \\
&\geq \Theta(x_0; u).
\end{aligned}$$

Moreover, if $\Theta(x_0; u) = \Theta(0, u)$, the above inequality become an equality:

$$\lim_{\lambda \to +\infty} E(\lambda; x_0, u) = \Theta(x_0; u).$$

This then implies that $E(\lambda; x_0, u) \equiv \Theta(x_0; u)$ for all $\lambda > 0$. By (3.3), u is homogeneous with respect to x_0. Then for all $\lambda > 0$,

$$u(x_0 + x) = \lambda^{\frac{4}{p-1}} u(x_0 + \lambda x) = u(\lambda^{-1} x_0 + x).$$

By letting $\lambda \to +\infty$ and noting that $u(\lambda^{-1} x_0 + \cdot)$ are uniformly bounded in $C^{\frac{2}{p+1}}_{loc}(\mathbb{R}^n)$, we see

$$u(x_0 + \cdot) = u(\cdot) \text{ on } \mathbb{R}^n.$$

Because u is homogeneous with respect to 0, a direct scaling shows that $\Theta(tx_0; u) = \Theta(x_0; u)$ for all $t > 0$, so the above equality still holds if we replace x_0 by tx_0 for any $t > 0$. A change of variable shows this also holds for $t < 0$. □

To finish the proof of Theorem 1.2, we also need

Lemma 5.8. *Let u be a $2/(p+1)-$Hölder continuous, homogeneous solution of* (1.1) *in $\mathbb{R}^2$. Then $\{u = 0\} = \{0\}$.*

Here we only need the solution to be understood in the distributional sense, i.e. $u^{-p} \in L^1_{loc}(\mathbb{R}^2)$.

Proof. There exists a function $\varphi(\theta) \in C^{\frac{2}{p+1}}(\mathbb{S}^1)$ such that in the polar coordinates,

$$u(r, \theta) = r^{\frac{2}{p+1}} \varphi(\theta).$$

Then

$$\int_{B_1} u^{-p} = \int_0^1 \left(\int_{\mathbb{S}^1} \varphi(\theta)^{-p} d\theta \right) r^{-\frac{2p}{p+1}+1} dr < +\infty.$$

So

$$\int_{\mathbb{S}^1} \varphi(\theta)^{-p} d\theta < +\infty.$$

If there exists a $\theta_0 \in \mathbb{S}^1$ such that $\varphi(\theta_0) = 0$, then

$$|\varphi(\theta) - \varphi(\theta_0)| \leq C|\theta - \theta_0|^{\frac{2}{p+1}}.$$

Hence near θ_0, φ^{-p} grows like $|\theta - \theta_0|^{-\frac{2p}{p+1}}$. Since $\frac{2p}{p+1} > 1$, φ^{-p} cannot be in $L^1(\mathbb{S}^1)$. This is a contradiction and we must have $\varphi > 0$ on $\mathbb{S}^1$. $\square$

Remark 5.9. *Similar arguments show that there does not exist homogeneous solutions in* $\mathbb{R}^1$.

With these lemmas in hand we can apply the Federer's dimension reduction principle (cf. Appendix A in [13]) to deduce Theorem 1.2. For completeness we present the proof in the case of $n = 2$.

Assume there exists $x_i \in \{u = 0\} \cap B_1$, such that $x_i \to x_0$ but $x_i \neq x_0$. Take $r_i = |x - x_i|$ and define

$$u_i(x) = r_i^{-\frac{2}{p+1}} u(x_0 + r_i x).$$

After passing to a subsequence of i, we can assume that u_i converges uniformly to a $2/(p+1)-$Hölder continuous, homogeneous solution u_∞ in any compact set of $\mathbb{R}^2$. Since $z_i = (x_i - x_0)/r_i \in \mathbb{S}^1$, we can also assume that $z_i \to z_\infty \in \mathbb{S}^1$. By the uniform convergence of u_i,

$$u_\infty(z_\infty) = \lim_{i \to +\infty} u_i(z_i) = 0.$$

However, Lemma 5.8 says $u_\infty > 0$ outside the origin. This is a contradiction and $\{u = 0\} \cap B_1$ must be a discrete set.

Appendix A. A Liouville theorem

In this appendix we recall a Liouville theorem proved in [10].

Theorem A.1. *Let* $\alpha \in (0, 1)$. *Assume* $\bar{u}_\infty \geq 0$ *is a globally* $C^\alpha(\mathbb{R}^n)$ *continuous function, satisfying*

$$\bar{u}_\infty \Delta \bar{u}_\infty = 0 \quad in\ \mathbb{R}^n, \tag{A.1}$$

and that $\bar{u}_\infty$ *is stationary, i.e.*

$$\int_{\mathbb{R}^n} \frac{1}{2} |\nabla \bar{u}_\infty|^2 divY - DY(\nabla \bar{u}_\infty, \nabla \bar{u}_\infty) = 0,$$

for any vector field $Y \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$. *Then* $\bar{u}_\infty$ *is constant.*

Equation (A.1) implies that

$$\Delta \bar{u}_\infty^2 = 2|\nabla \bar{u}_\infty|^2, \tag{A.2}$$

in the distributional sense. Moreover, $\bar{u}_\infty$ is harmonic in the open set $\{\bar{u}_\infty > 0\}$. So if $\bar{u}_\infty > 0$ everywhere, it is a harmonic function on $\mathbb{R}^n$. Then because $\bar{u}_\infty$ is globally C^α, the standard Liouville theorem implies that it is constant.

In the following we assume $\{\bar{u}_\infty = 0\} \neq \emptyset$. First we present some monotonicity formulas.

Proposition A.2. *For $r > 0$ and $x \in \mathbb{R}^n$,*

$$D(r;x) := r^{2-n}\int_{B_r(x)} |\nabla \bar{u}_\infty|^2$$

is nondecreasing in r.

Proof. For a proof, see [1] Lemma 2.1. In fact by the stationary condition, we have

$$(n-2)\int_{B_r(x)} |\nabla \bar{u}_\infty|^2 = r\int_{\partial B_r(x)} |\nabla \bar{u}_\infty|^2 - 2(\frac{\partial \bar{u}_\infty}{\partial r})^2.$$

Then direct calculations give

$$\frac{d}{dr}D(r;x) = 2r^{2-n}\int_{\partial B_r(x)} \left(\frac{\partial \bar{u}_\infty}{\partial r}\right)^2 \geq 0. \tag{A.3}$$

□

Next let $H(r;x) := r^{1-n}\int_{\partial B_r} \bar{u}_\infty^2$. By (A.2), direct calculations give

$$\begin{aligned} \frac{dH}{dr} = 2r^{1-n}\int_{\partial B_r} \bar{u}_\infty \frac{\partial \bar{u}_\infty}{\partial r} &= 2r^{1-n}\int_{B_r} \bar{u}_\infty \Delta \bar{u}_\infty \\ &= \frac{2}{r}D(r). \end{aligned} \tag{A.4}$$

Then we get

Proposition A.3. (Almgren monotonicity formula.) *For $r > 0$ and $x \in \mathbb{R}^n$,*

$$N(r;x) := \frac{D(r;x)}{H(r;x)}$$

is nondecreasing in r. Moreover, if $N(r;x) \equiv d$, then

$$\bar{u}_\infty(x + ry) = r^d \bar{u}_\infty(x + y).$$

Proof. Without loss of generality, take $x = 0$.

$$\begin{aligned} \frac{d}{dr}N(r) &= \frac{H(r)\left[2r^{2-n}\int_{\partial B_r}\left(\frac{\partial \bar{u}_\infty}{\partial r}\right)^2\right] - D(r)\left(2r^{1-n}\int_{\partial B_r}\bar{u}_\infty\frac{\partial \bar{u}_\infty}{\partial r}\right)}{H(r)^2} \\ &= 2r^{3-2n}\frac{\int_{\partial B_r}\bar{u}_\infty^2\int_{\partial B_r}\left(\frac{\partial \bar{u}_\infty}{\partial r}\right)^2 - \left(\int_{\partial B_r}\bar{u}_\infty\frac{\partial \bar{u}_\infty}{\partial r}\right)^2}{H(r)^2} \\ &\geq 0. \end{aligned}$$

If $N(r) \equiv d$, for any r,

$$\int_{\partial B_r}\bar{u}_\infty^2\int_{\partial B_r}\left(\frac{\partial \bar{u}_\infty}{\partial r}\right)^2 - \left(\int_{\partial B_r}\bar{u}_\infty\frac{\partial \bar{u}_\infty}{\partial r}\right)^2 = 0.$$

By the characterization of the equality case of the Cauchy inequality, there exists a $\lambda(r)$ such that

$$\frac{\partial \bar{u}_\infty}{\partial r} = \lambda(r)\bar{u}_\infty.$$

Integrating in r we get a function $\varphi(r)$ such that

$$\bar{u}_\infty(y) = \varphi(|y|)\bar{u}_\infty(\frac{y}{|y|}).$$

Then a direct calculation shows that $\varphi(|y|) = |y|^d$. □

Proposition A.4. *If $N(r_0;x) \geq d$, then for $r > r_0$,*

$$r^{1-n-2d}\int_{\partial B_r(x)} \bar{u}_\infty^2$$

is nondecreasing in r.

Proof. Direct calculation using (A.4) shows

$$\begin{aligned} & \frac{d}{dr}\left(r^{1-n-2d}\int_{\partial B_r(x)} u^2+v^2\right) \\ = \quad & -2dr^{-n-2d}\int_{\partial B_r(x)}(u^2+v^2)+2r^{1-n-2d}\int_{B_r(x)}|\nabla u|^2+|\nabla v|^2+2u^2v^2 \\ \geq \quad & 0. \end{aligned}$$

Here we have used Proposition A.3, in particular, the fact that $N(r) \geq d$ for every $r \geq r_0$. □

Because $\bar{u}_\infty$ is globally C^α,

$$\bar{u}_\infty(x) \leq C(1+|x|^\alpha) \quad \text{in } \mathbb{R}^n.$$

Hence for any x and r large,

$$\int_{\partial B_r(x)} \bar{u}_\infty^2 \leq Cr^{n-1+2\alpha}.$$

Combining this with the previous proposition we get

$$N(r;x) \leq \alpha, \quad \text{for any } r>0, x \in \mathbb{R}^n. \tag{A.5}$$

The next result is the so called "doubling property".

Proposition A.5. *Let $x \in \{\bar{u}_\infty = 0\}$ and $R > 0$ such that $N(R;x) \leq d$, then for every $0 < r \leq R$*

$$H(r;x) \geq H(R;x)r^{2d}. \tag{A.6}$$

Proof. By (A.4), if $H(r) > 0$,

$$\frac{d}{dr}\log H(r) = \frac{2N(r)}{r} \leq \frac{2d}{r}.$$

This means $r^{-2d}H(r)$ is non-increasing in r. Consequently, $H(r) > 0$ for all $r \in (0,R)$, and (A.6) is a direct consequence of the monotonicity of $r^{-2d}H(r)$. □

Remark A.6. *By this doubling property, we can prove that $\{\bar{u}_\infty = 0\}$ has zero Lebesgue measure. In fact, more properties such as the unique continuation property can be proved by this method, see* [1].

By this doubling proeprty, if $N(R;x) \leq d < \alpha$, then for all $r \in (0,R)$,

$$H(r;x) \geq Cr^{2d}.$$

However, if $\bar{u}_\infty(x) = 0$, because $\bar{u}_\infty$ is C^α continuous,

$$H(r;x) \leq Cr^{2\alpha}.$$

If r small, this is a contradiction. In other words, $N(r;x) \geq \alpha$ for any $r > 0$.

Combining this fact with (A.5), we see for any $x \in \{\bar{u}_\infty = 0\}$ and $r > 0$, $N(r;x) \equiv \alpha$. By Proposition A.3,

$$\bar{u}_\infty(x+y) = |y|^{\frac{2}{p+1}} \bar{u}_\infty(x + \frac{y}{|y|}).$$

In particular, $\{\bar{u}_\infty = 0\}$ is a cone with respect to any point in $\{\bar{u}_\infty = 0\}$. This then implies that $\{\bar{u}_\infty = 0\}$ is a linear subspace of $\mathbb{R}^n$. Assume $\{\bar{u}_\infty = 0\} = \mathbb{R}^k$ for some $k < n$. (Note that $\bar{u}_\infty$ is nontrivial, so $\{\bar{u}_\infty = 0\}$ cannot be the whole $\mathbb{R}^n$.) If $k \leq n-2$, $\{\bar{u}_\infty = 0\}$ has zero capacity and then $\bar{u}_\infty$ is a harmonic function. Because $\bar{u}_\infty \geq 0$, by the strong maximum principle, either $\bar{u}_\infty > 0$ everywhere or $\bar{u}_\infty \equiv 0$. Both of these two lead to a contradiction.

If $k = n-1$, assume $\{\bar{u}_\infty = 0\} = \{x_1 = 0\}$. Then by the Schwarz reflection principle, $\bar{u}_\infty = c|x_1|$ for some constant $c > 0$. This again contradicts the global $\alpha-$Hölder continuity of $\bar{u}_\infty$ because $\alpha < 1$.

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