A double bubble in a ternary system with inhibitory long range interaction

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October 5, 2012

Abstract

The free energy of a ternary system with a self-organization property includes an interface energy and a longer ranging, inhibitory interaction energy. In a planar domain, if the two energies are properly balanced and two of the three constituents make up an equal but small fraction, the free energy admits a local minimizer that is shaped like a perturbed double bubble. Most difficulties in the proof of this result are related to the triple junction phenomenon that the three constituents of the ternary system meet at a point. Two techniques are developed to deal with triple junction. First one defines restricted classes of perturbed double bubbles. Each perturbed double bubble in a restricted class is obtained from a standard double bubble by a special perturbation. The two triple junction points of the standard double bubble can only move along the line connecting them, in opposite directions, and by the same distance. The second technique is the use of the so called internal variables. These variables derive from the more geometric quantities that describe perturbed double bubbles in restricted classes. The advantage of the internal variables is that they are only subject to linear constraints, and perturbed double bubbles in a restricted class represented by internal variables are elements of a Hilbert space. A local minimizer of the free energy in each restricted class is found as a fixed point of a nonlinear equation by a contraction mapping argument. The second variation at the fixed point within its restricted class is positive. This perturbed double bubble satisfies three of the four equations for critical points of the free energy. The unsolved equation is the 120 degree angle condition at triple junction points. Perform another minimization among the local minimizers from all restricted classes. A minimum of minimizers emerges and solves all the equations for critical points.

1 Introduction

Exquisitely structured patterns arise in many multi-constituent physical and biological systems as orderly outcomes of self-organization principles. Examples include morphological phases in block copolymers, animal coats, and skin pigmentation. Common in these pattern-forming systems is that a deviation from homogeneity has a strong positive feedback on its further increase. On its own, it would lead to an unlimited increase and spreading. Pattern formation requires in addition a longer ranging confinement of the locally self-enhancing process.

The simplest multi-constituent system is a binary system. Let such a system be in a domain D, which is an open and bounded set in $\mathbb{R}^n$. If the two constituent components are totally separated, D is divided into two subsets: Ω occupied by one constituent and $D\backslash\Omega$ occupied by the other constituent. These sets may or may not be connected. If the system is governed by its free energy, it is natural to postulate that the area of the interfaces separating Ω from $D\backslash\Omega$ contributes to this energy. Mathematically this quantity is termed the perimeter of Ω in D, denoted by $\mathcal{P}_D(\Omega)$ whose precise definition will be given in the next section. When Ω is bounded by a smooth boundary $\partial\Omega$, then $\mathcal{P}_D(\Omega)$ is the length of $\partial\Omega\cap D$ if $D\subset\mathbb{R}^2$ or the area of $\partial\Omega\cap D$ if $D\subset\mathbb{R}^3$. Note that $\partial\Omega$ stands for the boundary of Ω in $\mathbb{R}^n$, not the boundary of Ω in D. The latter is $\partial\Omega\cap D$.

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†Supported in part by NSF grant DMS-0907777.
‡Supported in part by an earmarked grant of RGC of Hong Kong.

However the perimeter term $\mathcal{P}_D(\Omega)$ alone is not sufficient in a pattern forming, self-organizing system. For pattern formation to take place, another longer ranging, inhibitory term is needed. One example of this comes from the Ohta-Kawasaki density functional theory for diblock copolymers [20]. There the longer range term takes the form $\int_D |(-\Delta)^{-1/2}(\chi_\Omega - \omega)|^2\,dx$. The function χ_Ω is the characteristic function of Ω: $\chi_\Omega(x) = 1$ if $x \in \Omega$ and $\chi_\Omega(x) = 0$ if $x \in D\backslash\Omega$. The constant $\omega \in (0,1)$ is prescribed to fix the size of Ω. It is required that $|\Omega| = \omega|D|$ where $|\Omega|$ is the Lebesgue measure of Ω and $|D|$ the Lebesgue measure of D. The operator $(-\Delta)^{-1/2}$ is the positive square root of the inverse of $-\Delta$ with the Neumann boundary condition.

The free energy $\mathcal{J}_B$ of the binary system combines these two terms:

$$\mathcal{J}_B(\Omega) = \frac{1}{n-1}\mathcal{P}_D(\Omega) + \frac{\gamma}{2}\int_D |(-\Delta)^{-1/2}(\chi_\Omega - \omega)|^2\,dx. \tag{1.1}$$

The constant γ is positive and functions as the second parameter of the problem after ω. It balances the short range interaction $\mathcal{P}_D(\Omega)$ with the long range interaction in $\mathcal{J}_B$. A critical point of the functional $\mathcal{J}_B$ satisfies the Euler-Lagrange equation

$$\mathcal{H}(\partial\Omega \cap D) + \gamma(-\Delta)^{-1}(\chi_\Omega - \omega) = \lambda. \tag{1.2}$$

In (1.2) $\mathcal{H}(\partial\Omega \cap D)$ is the (mean) curvature of $\partial\Omega \cap D$, and λ is a Lagrange multiplier associated with the constraint $|\Omega| = \omega|D|$. Having the coefficients $\frac{1}{n-1}$ and $\frac{\gamma}{2}$ in (1.1) helps generate a simpler looking equation (1.2). The equation (1.2) holds on the part of the boundary of Ω that is inside D, namely $\partial\Omega \cap D$. If $\partial\Omega$ intersects ∂D, then the two hyper-surfaces must meet perpendicularly. By minimizing $\mathcal{J}_B$, or more generally by solving the equation (1.2), one discovers patterns that match the ones observed in nature.

This problem is solved completely in one-dimension [24]. There are countably infinitely many solutions to (1.2) and every solution is a local minimizer of the energy functional $\mathcal{J}_B$. Among these local minimizers, depending on ω and γ, two or four of them are global minimizers. In higher dimensions several types of solutions have been found [25, 28, 30, 29, 31, 32, 21, 13, 23, 14]. But many questions remain. Even the global minimizer has not been completely identified. Nevertheless progresses have been made in [2, 36, 18].

In this work we study ternary systems. A ternary system has three constituents that occupy disjoint subsets Ω_1, Ω_2 and $\Omega_3 = D\backslash(\Omega_1 \cup \Omega_2)$ of D. As in a binary system, part of the free energy of the ternary system is the size of the interfaces separating the three domains Ω_1, Ω_2 and Ω_3. Three types of interfaces exist: $\partial\Omega_1\backslash\partial\Omega_2$, the interfaces separating Ω_1 from Ω_3; $\partial\Omega_2\backslash\partial\Omega_1$, the interfaces separating Ω_2 from Ω_3; $\partial\Omega_1 \cap \partial\Omega_2$, the interfaces separating Ω_1 from Ω_2. One can write the combined size of all the interfaces of the three types as $\frac{1}{2}(\mathcal{P}_D(\Omega_1) + \mathcal{P}_D(\Omega_2) + \mathcal{P}_D(\Omega_3))$. In $\mathcal{P}_D(\Omega_1) + \mathcal{P}_D(\Omega_2) + \mathcal{P}_D(\Omega_3)$, each of $\partial\Omega_1\backslash\partial\Omega_2$, $\partial\Omega_2\backslash\partial\Omega_1$, and $\partial\Omega_1 \cap \partial\Omega_2$ is counted twice. The half is put here to avoid double counting.

For the long range interaction we use $\int_D ((-\Delta)^{-1/2}(\chi_{\Omega_i} - \omega_i))((-\Delta)^{-1/2}(\chi_{\Omega_j} - \omega_j))\,dx$ to model the interaction between Ω_i and Ω_j. The overall free energy of the ternary system takes the form

$$\mathcal{J}_T(\Omega) = \frac{1}{2(n-1)}\sum_{i=1}^{3}\mathcal{P}_D(\Omega_i) + \sum_{i,j=1}^{2}\int_D \frac{\gamma_{ij}}{2}\Big((-\Delta)^{-1/2}(\chi_{\Omega_i} - \omega_i)\Big)\Big((-\Delta)^{-1/2}(\chi_{\Omega_j} - \omega_j)\Big)\,dx. \tag{1.3}$$

This model derives from a density functional theory for triblock copolymers proposed by Nakazawa and Ohta [19]. A more mathematical treatment of their theory can be found in [26]. The original theory is more general and allows the constituents to mix, an issue that will be addressed in the last section. Here we only consider the strong segregation limit where the three constituents are completely separated. The passage from the original theory to $\mathcal{J}_T$ relies on De Giorgi's Γ-convergence theory; see [27] for a detailed explanation.

Although experimentally a far larger number of architectures can be synthetically accessed in ternary systems like triblock copolymers than in binary systems [5, Figure 5 and the magazine's cover], mathematical study of $\mathcal{J}_T$ is still in an early stage. The cyclic lamellar phase is shown to exist in [27]; a diblock copolymer - homopolymer blend problem (a special ternary system) is investigated in [6, 38, 39]. All these papers deal with $\mathcal{J}_T$ in one-dimension.

In higher dimensions there are ternary patterns that are obviously analogous to some binary patterns, such as the lamellar and core-shell phases [5, Figure 4: a, b]. However the most interesting and mathematically challenging phenomenon of triple junction in ternary systems is not shared by binary systems. Triple junction means that the three constituents may meet at a co-dimension 2 surface. In two dimensions triple junction occurs at points, and in three dimensions it occurs at curves. Many ternary patterns include triple junction [5, Figure 4: c and Figure 5: c, d, e]. Here we investigate one particular such pattern: a double bubble, or more precisely a perturbed double bubble

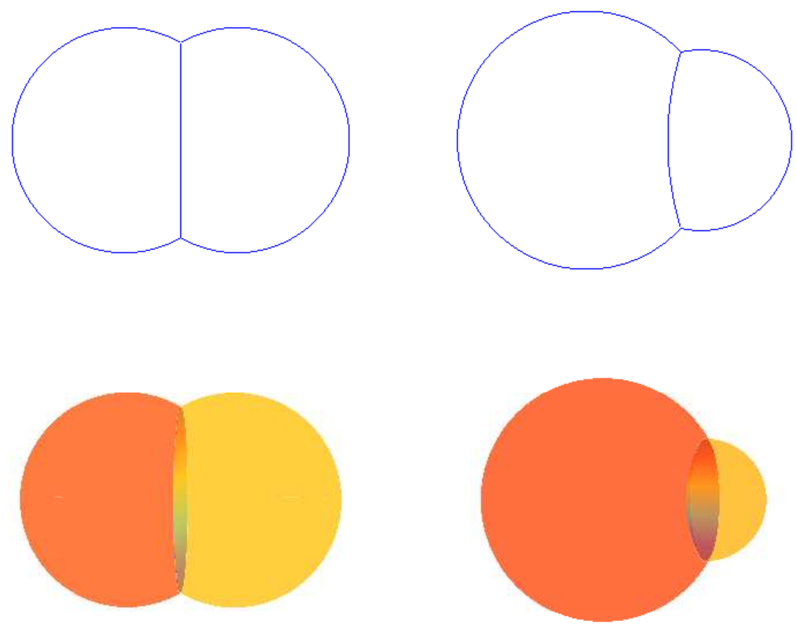

Figure 1: First row: a 2D equal area double bubble on the left and a 2D unequal area double bubble on the right; second row: a 3D equal volume double on the left and a 3D unequal volume double bubble on the right.

in two dimensions, and prove that a double bubble, properly perturbed and suitably placed, exists as a critical point of $\mathcal{J}_T$. Our proof also reveals that in some sense this critical point is a local minimizer of $\mathcal{J}_T$.

The standard double bubble, depicted in Figure 1 for $\mathbb{R}^2$ and $\mathbb{R}^3$, arises as the optimal configuration of the two component isoperimeric problem. Let $m_1 > 0$ and $m_2 > 0$. Find two disjoint sets E_1 and E_2 in $\mathbb{R}^n$ such that $|E_1| = m_1$, $|E_2| = m_2$, and the area of $\partial E_1 \cup \partial E_2$, i.e. $\frac{1}{2}(\mathcal{P}(E_1) + \mathcal{P}(E_2) + \mathcal{P}(E_3))$, where $E_3 = \mathbb{R}^n \backslash (E_1 \cup E_2)$ and $\mathcal{P}(E_i)$ is the perimeter of E_i in $\mathbb{R}^n$, is minimum. The standard double is the unique solution to this isoperimetric problem by the works of Almgren [3], Taylor [37], Foisy *et al* [10], Hutchings *et al* [12], and Reichardt [22]. Compared to the first modern proof of the standard isoperimetric problem of one component by Schwarz [35] in 1884, these results on the two component isoperimetric problem are very recent, a manifestation of the great difficulties associated with triple junction.

In $\mathbb{R}^2$ the standard double bubble $B = (B_1, B_2)$ consists of two regions B_1 and B_2 separated by an arc and bounded by two other arcs; see Figure 1, first row. In the case that the two areas m_1 and m_2 are equal, the middle arc becomes a line segment, i.e. a degenerate arc of infinite radius. Triple junction occurs at two points where the three arcs meet at 120 degree angles. The radii of the three arcs r_1, r_2 and r_0 satisfy $\frac{1}{r_0} = \frac{1}{r_2} - \frac{1}{r_1}$, where r_0 is the radius of the center arc, r_1 is the radius of the larger bubble and r_2 the radius of the smaller bubble. In $\mathbb{R}^3$ the double bubble is bounded by two spherical caps and separated by another spherical cap; when $m_1 = m_2$, the last spherical cap becomes a disc. The three spheres meet at 120 degrees at the triple junction circle, and the radii of the three spheres satisfy the same equation as in $\mathbb{R}^2$; see Figure 1, second row. In this paper we tell a story related to the equal area, two dimensional double bubble, i.e. the case of $m_1 = m_2$ and $n = 2$.

Because of the long range interaction term in (1.3), the standard double bubble is not a critical point of $\mathcal{J}_T$. Nevertheless we will prove the following result.

Main result. Let D be a bounded domain is in $\mathbb{R}^2$ and $\omega_1 = \omega_2$. If ω_1 and $|\gamma|\omega_1^{3/2}$ are sufficiently small and γ is uniformly positively definite, then the functional $\mathcal{J}_T$ has a stable critical point that is shaped approximately like a double bubble of equal area.

Here $|\gamma|$ is the norm of the matrix γ given by $\sum_{i,j=1}^{2} |\gamma_{ij}|$. The notion of the uniform positivity of γ is explained in (2.9).

In a proper sense this critical point is considered a local minimizer of $\mathcal{J}_T$, hence the claim that the solution is stable. The precise statement of this result including the range for $\omega_1 = \omega_2$ and γ is in Theorem 2.1. There is also an estimate, (9.29), that reveals how much this critical point deviates from the standard double bubble of equal area.

Without the long range term, being the minimizer of the isoperimetric problem, any standard double bubble inside D is a critical point of $\sum_{i=1}^{3} \mathcal{P}_D(\Omega_i)$, the local part of $\mathcal{J}_T$. However this point is highly degenerate, since any translation or rotation of the standard double bubble is also a critical point of $\sum_{i=1}^{3} \mathcal{P}_D(\Omega_i)$. The long range term in $\mathcal{J}_T$ removes this degeneracy. The perturbed double bubble critical point of $\mathcal{J}_T$ in our result must appear in a particular place of D with a particular direction.

The novelty in our treatment of ternary systems is the invention of restricted perturbation classes, an idea tailored to suit the triple junction phenomenon in ternary systems. It is a unique method that has no analogy in binary systems. The use of the restricted classes neatly breaks our proof of the main result into two steps. The first step solves an infinite dimensional problem by the contraction mapping argument, and the second step solves a finite dimensional problem by minimization. Another new idea is the use of internal variables in the first step. They allowed the first infinite dimensional problem to be cast as an equation for a fixed point in a Hilbert space.

The unequal area case $\omega_1 \neq \omega_2$ in two dimensions seems to be harder than the equal area case $\omega_1 = \omega_2$ for several reasons. The three dimensional version of this problem also appears to be much more difficult. We hope to deal with them in future works.

The proof of our result consists of several steps. First we fix a point ξ and a direction, specified by an angle θ, in D to set up a reference frame with the point being the center and the direction being the horizontal direction of the frame. Place a standard double bubble $B = (B_1, B_2)$ in D such that ξ is the middle point of the two centers of the two bubbles and one center points to the other center along the direction specified by θ. Consider a special class of perturbed standard bubbles. A perturbation in this class only allows the triple junction points of B to move vertically with respect to the reference frame, and the two points can only move in opposite directions by the same distance.

Next we set up a particular parametrization with three functions u_1, u_2, u_0, and two numbers A and a to describe a perturbed double bubble in the restricted class. Two polar coordinate systems are used here. Each system is centered at the center of one of the two bubbles of B. The function u_1 (and u_2 respectively) is the radius of $\partial\Omega_1 \backslash \partial\Omega_2$ ($\partial\Omega_2 \backslash \partial\Omega_1$ respectively) of a perturbed double bubble $\Omega = (\Omega_1, \Omega_2)$. The number A specifies the range of the corresponding angle. The curve $\partial\Omega_1 \cap \partial\Omega_2$ is described by the graph of u_0 in the reference frame, and a specifies the width of this graph. Geometric quantities, such as normal vectors and curvatures can all be expressed in terms of u_i, A and a.

Although u_i, A and a amply describe the picture of a perturbed double bubble, this is not a convenient setting to do analytic work. The reason is that u_i, A and a satisfy nonlinear constraints. Instead in the third step we introduce three new functions ϕ_1, ϕ_2, ϕ_0 and one new number α, termed internal variables. The original (u_i, A, a) can be transformed to (ϕ_i, α) and vice versa, but ϕ_i and α satisfy linear constraints in the form of linear boundary conditions and linear integral conditions. Consequently (ϕ_i, α) is placed in a Hilbert space, and $\mathcal{J}_T$ becomes a functional on this Hilbert space.

In step 4, we find a local minimizer (ϕ_i^*, α^*) of $\mathcal{J}_T$ in each restricted class. This is done by a contraction mapping argument and the key step here is to prove a positivity result, Lemma 7.4, on the second derivative of $\mathcal{J}_T$ at the standard double bubble.

However since $\mathcal{J}_T$ is only minimized in a restricted class specified by ξ and θ, the local minimizer (ϕ_i^*, α^*) found in Lemma 8.2 only satisfies three of the four equations for critical points of $\mathcal{J}_T$. The unsolved equation is the 120 degree angle condition at triple junction points. In step 5 we let ξ and θ vary. Note that (ϕ_i^*, α^*) depends on ξ and θ, i.e. $\phi_i^* = \phi_i^*(\cdot, \xi, \theta)$ and $\alpha^* = \alpha^*(\xi, \theta)$. It turns out that one can minimize $\mathcal{J}_T(\phi_i^*(\cdot, \xi, \theta), \alpha^*(\xi, \theta))$ with respect to (ξ, θ) and find a minimum, say at $(\vec{0}, 0)$. This particular perturbed double bubble $(\phi_i^*(\cdot, \vec{0}, 0), \alpha^*(\vec{0}, 0))$, a minimum of minimizers, is what we need. For (ξ, θ) close to $(\vec{0}, 0)$ the local minimizer $(\phi_i^*(\cdot, \xi, \theta), \alpha^*(\xi, \theta))$ in the restricted class associated with (ξ, θ) is considered a deformation of $(\phi_i^*(\cdot, \vec{0}, 0), \alpha^*(\vec{0}, 0))$. This deformation is outside the restricted class associated with $(\vec{0}, 0)$ where $(\phi_i^*(\cdot, \vec{0}, 0), \alpha^*(\vec{0}, 0))$ is. The fact that $(\vec{0}, 0)$ is a minimum of $\mathcal{J}_T(\phi_i^*(\cdot, \xi, \theta), \alpha^*(\xi, \theta))$

with respect to (ξ, θ) implies that

$$\frac{\partial \mathcal{J}_T(\phi_i^*(\cdot,\xi,\theta),\alpha^*(\xi,\theta))}{\partial \xi^l}\Big|_{(\xi,\theta)=(\vec{0},0)} = \frac{\partial \mathcal{J}_T(\phi_i^*(\cdot,\xi,\theta),\alpha^*(\xi,\theta))}{\partial \theta}\Big|_{(\xi,\theta)=(\vec{0},0)} = 0, \quad l = 1, 2, \tag{1.4}$$

where $\xi = (\xi^1, \xi^2)$. From (1.4) we prove that the 120 degree angle condition holds for $(\phi_i^*(\cdot, \vec{0}, 0), \alpha^*(\vec{0}, 0))$.

In [30] the authors proved the existence of a single bubble solution for the binary system (1.1). It is a critical point of the functional $\mathcal{J}_B$ and is shaped like a disc inside D. The double bubble problem in a ternary system is much harder, not just because of the triple junction issue. The standard double bubble is in some sense a generalization of the standard disc because the disc minimizes the perimeter among all sets of the same area, the one component isoperimetric problem. However the disc is more compatible with the long range term in $\mathcal{J}_B$ than the double bubble is with the long range term in $\mathcal{J}_T$. In the binary case, a standard disc is a solution of a related profile equation

$$\mathcal{H}(\partial E) + \gamma \mathcal{N}(E) = \lambda \tag{1.5}$$

which holds on the boundary of E. In (1.5) $\mathcal{N}$ is the Newtonian potential operator defined by $\mathcal{N}(E)(x) = \int_E \frac{1}{2\pi} \log \frac{1}{|x-y|}\, dy$ for each $x \in \mathbb{R}^2$. This profile equation may be regarded as an asymptotic limit of the equation (1.2). More results on this profile equation can be found in [33, 34]. If a standard disc is inserted into the equation (1.2), then it almost satisfies (1.2). Only a very small error occurs. Based on this fact, we proved in [30] that a small perturbation of the standard disc solves (1.2). In that result the parameters ω and γ may stay in a rather large range. In our ternary problem however, the standard double bubble is not a solution of any profile equation of $\mathcal{J}_T$. Far more delicate estimates are needed in the proof of our main result, and there exists an approximately double bubble shaped solution of the ternary problem for ω_i and γ in a much smaller parameter range.

2 Ternary system

Henceforth we consider (1.3) on a bounded and smooth open subset D of $\mathbb{R}^2$, and let ω_1 and ω_2 be two positive numbers such that $\omega_1 + \omega_2 < 1$. For two measurable subsets Ω_1 and Ω_2 of D satisfying $|\Omega_1| = \omega_1 |D|$, $|\Omega_2| = \omega_2 |D|$, and $|\Omega_1 \cap \Omega_2| = 0$, set $\Omega_3 = D \backslash (\Omega_1 \cup \Omega_2)$ and $\Omega = (\Omega_1, \Omega_2)$. The energy of the system is

$$\mathcal{J}(\Omega) = \frac{1}{2} \sum_{i=1}^{3} \mathcal{P}_D(\Omega_i) + \sum_{i,j=1}^{2} \int_D \frac{\gamma_{ij}}{2} \Big((-\Delta)^{-1/2} (\chi_{\Omega_i} - \omega_i) \Big) \Big((-\Delta)^{-1/2} (\chi_{\Omega_j} - \omega_j) \Big)\, dx. \tag{2.1}$$

The notation $\mathcal{J}_T$ in (1.3) is now simplified to $\mathcal{J}$.

In (2.1) $\mathcal{P}_D(\Omega_i)$ is the perimeter of Ω_i in D. The perimeter may be defined for any measurable subset E of D by

$$\mathcal{P}_D(E) = \sup\{ \int_E \operatorname{div} g(x)\, dx : \ g \in C_0^1(D, \mathbb{R}^2),\ |g(x)| \le 1\ \forall x \in D \} \tag{2.2}$$

where $\operatorname{div} g$ is the divergence of the C^1 vector field g on D with compact support and $|g(x)|$ stands for the Euclidean norm of the vector $g(x) \in \mathbb{R}^2$. A subset E of D has finite perimeter if and only if χ_E, the characteristic function of E, is a function of bounded variation on D. See [9] or [40] for more discussion on the notion of perimeter. For this paper it suffices to know that when E is open and its boundary is a piecewise C^1 curve, then $\mathcal{P}_D(E)$ is just the length of $\partial E \cap D$. Hence if Ω_1 and Ω_2 have piecewise C^1 boundaries, the first term in (2.1) is merely the length of the set $(\partial \Omega_1 \cup \partial \Omega_2) \cap D$.

The interaction matrix $\gamma = [\gamma_{ij}]$ is symmetric and positive definite. It balances the long range interaction strength between the three constituents, and also weighs the perimeter part of $\mathcal{J}$ against the long range part of $\mathcal{J}$. A uniform positivity condition will be imposed on all the γ's in the parameter range of interest.

A critical point $\Omega = (\Omega_1, \Omega_2)$ of $\mathcal{J}$ is a solution of the following equations:

$$\kappa_1 + \gamma_{11} I_{\Omega_1} + \gamma_{12} I_{\Omega_2} = \lambda_1 \text{ on } \partial\Omega_1 \backslash \partial\Omega_2 \tag{2.3}$$

$$\kappa_2 + \gamma_{21} I_{\Omega_1} + \gamma_{22} I_{\Omega_2} = \lambda_2 \text{ on } \partial\Omega_2 \backslash \partial\Omega_1 \tag{2.4}$$

$$\kappa_0 + (\gamma_{11} - \gamma_{12}) I_{\Omega_1} + (\gamma_{21} - \gamma_{22}) I_{\Omega_2} = \lambda_1 - \lambda_2 \text{ on } \partial\Omega_1 \cap \partial\Omega_2 \tag{2.5}$$

$$\nu_1 + \nu_2 + \nu_0 = \vec{0} \text{ at } \partial\Omega_1 \cap \partial\Omega_2 \cap \partial\Omega_3 \tag{2.6}$$

Here we assume that Ω_1 and Ω_2 do not share boundaries with D. Otherwise we need to add another condition that when the boundary of Ω_1 (or Ω_2) meets the boundary of D, it does so perpendicularly.

In (2.3)-(2.5) κ_1, κ_2, and κ_0 are the curvatures of the curves $\partial\Omega_1\backslash\partial\Omega_2$, $\partial\Omega_2\backslash\partial\Omega_1$, and $\partial\Omega_1\cap\partial\Omega_2$, respectively. These are signed curvatures defined with respect to a choice of normal vectors. On $\partial\Omega_1\backslash\partial\Omega_2$ the normal vector points inward into Ω_1. On $\partial\Omega_2\backslash\partial\Omega_1$, the normal vector points inward into Ω_2. On $\partial\Omega_1\cap\partial\Omega_2$, the normal vector points from Ω_2 towards Ω_1. If a curve bends in the direction of the normal vector, then the curvature is positive.

Also in (2.3) and (2.4) I_{Ω_1} and I_{Ω_2} are two functions on D determined from Ω_1 and Ω_2 respectively. The function I_{Ω_i} is the solution of

$$-\Delta I_{\Omega_i} = \chi_{\Omega_i} - \omega_i \ \text{ in } D, \quad \partial_n I_{\Omega_i} = 0 \ \text{ on } \partial D, \quad \int_D I_{\Omega_i}(x)\,dx = 0, \tag{2.7}$$

where $\partial_n I_{\Omega_i}$ stands for the outward normal derivative of I_{Ω_i} on ∂D. Note that the constraint $|\Omega_i| = \omega|D|$ implies that the integral of the right side of the PDE in (2.7) is zero, so the PDE together with the boundary condition is solvable. The solution is unique up to an additive constant. The last condition $\int_D I_{\Omega_i}(x)\,dx = 0$ fixes this constant and selects a particular solution. One can also write $I_{\Omega_i} = (-\Delta)^{-1}(\chi_{\Omega_i} - \omega_i)$, as the outcome of the operator $(-\Delta)^{-1}$ on $\chi_{\Omega_i} - \omega_i$. The operator $(-\Delta)^{-1/2}$ in (2.1) is the positive square root of $(-\Delta)^{-1}$.

The constants λ_1 and λ_2 are Lagrange multipliers corresponding to the constraints $|\Omega_1| = \omega_1|D|$ and $|\Omega_2| = \omega_2|D|$. They are unknown and are to be found with Ω_1 and Ω_2.

In the last equation (2.6), ν_1, ν_2, and ν_0 are the inward pointing, unit tangent vectors of the curves $\partial\Omega_1\backslash\partial\Omega_2$, $\partial\Omega_2\backslash\partial\Omega_1$, and $\partial\Omega_1\cap\partial\Omega_2$ at triple junction points. The requirement that the three unit vectors sum to zero is equivalent to the condition the three curves meet at 120 degree angles.

The first three criticality equations (2.3)-(2.5) are derived in section 6. The solution to be constructed in this paper is shown to satisfy these equations in Lemma 8.3. In (9.28) we prove that our solution also solves last criticality equation (2.6).

Since we only consider the equal area case, the area constraints $|\Omega_1| = \omega_1|D|$ and $|\Omega_2| = \omega_2|D|$ take the form

$$|\Omega_1| = |\Omega_2| = \frac{m\rho^2}{2} \ \text{ where } m = \frac{4\pi}{3} + \frac{\sqrt{3}}{2} \text{ and } \rho > 0, \quad \text{i.e. } \omega_1 = \omega_2 = \frac{m\rho^2}{2|D|}. \tag{2.8}$$

The number $\frac{m\rho^2}{2}$ is chosen because this is the area of a bubble in the equal area double bubble if the bubble radius is ρ; see Figure 2. Instead of ω_1 and ω_2, ρ now becomes one of the parameters of our problem.

The other parameter is the matrix γ. It must satisfy a uniform positivity condition. Namely, there exists $b > 0$ and we only consider positive definite matrices γ that satisfy

$$b\,\bar{\bar{\lambda}}(\gamma) \le \bar{\lambda}(\gamma) \tag{2.9}$$

where $\bar{\lambda}(\gamma)$ and $\bar{\bar{\lambda}}(\gamma)$ are the two eigenvalues of γ such that $0 < \bar{\lambda}(\gamma) \le \bar{\bar{\lambda}}(\gamma)$.

Also recall that the norm of γ is denoted by $|\gamma|$ and given by $|\gamma| = \sum_{i,j=1}^{2} |\gamma_{ij}|$. The main result of this paper is the following existence theorem.

Theorem 2.1 *Let $b \in (0,1]$. There exist $\delta > 0$ and $\sigma > 0$ depending on the domain D and b only, such that if $\rho < \delta$, $|\gamma|\rho^3 < \sigma$, and (2.9) holds with the given b, then a perturbed double bubble exists as a stable solution of the problem (2.3)-(2.6). Each of the two perturbed bubbles is bounded by a continuous curve that is C^∞ except at the two triple junction points.*

The standard double bubble is described in section 3 with an estimate of its energy. The first variation of $\mathcal{J}$ is calculated in section 4. Section 5 introduces the classes of restrictedly perturbed double bubbles, and a special way to parametrize a perturbed double bubble's boundaries. The internal variables and the internal representation are defined in section 6, and the positivity of the second variation of $\mathcal{J}$ at the standard double bubble under restricted deformations is proved in section 7. In section 8 one finds a local minimizer of $\mathcal{J}$ within each restricted class by a contraction mapping argument. Also in this section the question how much the solution in the theorem as a perturbed double bubble differs from a standard double bubble is answered in Lemma 8.2. In section 9 $\mathcal{J}$ is minimized among the local minimizers of the restricted classes. A minimum exists and it is the critical point of $\mathcal{J}$ claimed in Theorem 2.1. A few remarks are included in the last section.

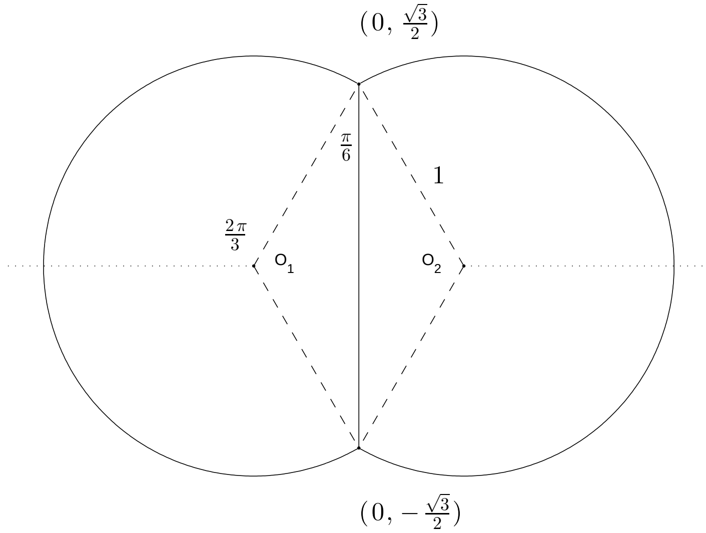

Figure 2: The equal area double bubble (E_1, E_2) where the radius of the two bubbles is 1. In each bubble the area of the sector is $\frac{2\pi}{3}$ and the area of the triangle is $\frac{\sqrt{3}}{4}$.

The perimeter part of $\mathcal{J}$ and the long range part of $\mathcal{J}$ are respectively denoted by $\mathcal{J}_S$ and $\mathcal{J}_L$, where the subscripts stand for "short range' and "long range":

$$\mathcal{J}_S(\Omega) = \frac{1}{2}\sum_{i=1}^{3} \mathcal{P}_D(\Omega_i) \text{ and } \mathcal{J}_L(\Omega) = \sum_{i,j=1}^{2} \int_D \frac{\gamma_{ij}}{2} \Big((-\Delta)^{-1/2}(\chi_{\Omega_i} - \omega_i)\Big)\Big((-\Delta)^{-1/2}(\chi_{\Omega_j} - \omega_j)\Big)\,dx. \tag{2.10}$$

For simpler notations, $\mathbb{R}^2$ is identified with the complex plane $\mathbb{C}$ and the complex multiplication is often employed. For instance we write $e^{\mathrm{i}\theta}\tilde{x}$ to denote the vector resulted from rotating $\tilde{x} \in \mathbb{R}^2$ counterclockwise by an angle θ. The O notation will be used frequently. For a quantity, say Q, that depends on ρ and/or γ, if one sees something like $Q = O(\rho^2)$, then there exists $C > 0$ independent of ρ and γ such that $|Q| \leq C\rho^2$ for all ρ and γ. If in addition to ρ and γ, Q also depends on another variable, say t, then $Q = O(\rho^2)$ always means $|Q| \leq C\rho^2$ uniformly with respect to all t.

3 Double bubble

We use $E = (E_1, E_2)$ to denote a particular equal area double bubble in $\mathbb{R}^2$. As shown in Figure 2 the set E_1 is open and is bounded by the vertical line segment $\{t\,\mathrm{i} : -\frac{\sqrt{3}}{2} \leq t \leq \frac{\sqrt{3}}{2}\}$ and the arc $\{\zeta \in \mathbb{C} : |\zeta + \frac{1}{2}| = 1, \ \mathrm{Re}(\zeta) \leq 0\}$, and E_2 is open and bounded by the same vertical line segment and the arc $\{\zeta \in \mathbb{C} : |\zeta - \frac{1}{2}| = 1, \ \mathrm{Re}(\zeta) \geq 0\}$. The three arcs meet at 120 degree angles. Each E_i is the union of a sector whose area is $\frac{2\pi}{3}$ and a triangle whose area is $\frac{\sqrt{3}}{4}$, so the area of E_i is $\frac{m}{2}$ where m is given in (2.8).

The double bubble E will be scaled by a factor $\rho > 0$, rotated by an angle $\theta \in S^1$, and translated by a vector $\xi \in \mathbb{R}^2$. An angle θ in $[0, 2\pi)$ is identified as a point on the unit circle S^1. Define

$$B_i = B_i(\rho, \xi, \theta) = \{\rho e^{\mathrm{i}\theta}\tilde{x} + \xi : \ \tilde{x} \in E_i\}, \quad B = (B_1, B_2). \tag{3.1}$$

The Green's function of $-\Delta$ on D with the Neumann boundary condition is denoted by $G = G(x, y)$. It satisfies

$$-\Delta G(\cdot, y) = \delta(\cdot - y) - \frac{1}{|D|} \text{ in } D, \quad \partial_n G(\cdot, y) = 0 \text{ on } \partial D, \quad \int_D G(x, y)\,dx = 0, \tag{3.2}$$

for every $y \in D$. Here $\partial_n G$ stands for the outward normal derivative at ∂D of G with respect to its first argument x. One can write

$$G(x, y) = \frac{1}{2\pi}\log\frac{1}{|x - y|} + R(x, y) \tag{3.3}$$

where R is the regular part of G, a smooth function on $D \times D$. It is known that

$$R(z,z) \to \infty \quad \text{as } z \to \partial D. \tag{3.4}$$

Let $\bar{\delta} > 0$ and set $D_{\bar{\delta}} = \{x \in D : \ \mathrm{dist}(x, \partial D) > \bar{\delta}\}$. Because of (3.4), we can find $\bar{\delta}$ small enough so that

$$\min_{z \in D} R(z,z) < \min_{z \in D \setminus D_{\bar{\delta}}} R(z,z). \tag{3.5}$$

Fix a $\bar{\delta}$ satisfying (3.5) throughout this paper. Next take δ such that

$$0 < 3\delta < \bar{\delta}. \tag{3.6}$$

For the moment we only assume that δ satisfies (3.6). Later more conditions on δ will be imposed.

The parameter ρ stays in the range

$$\rho \in (0, \delta). \tag{3.7}$$

Let $\bar{\bar{\delta}} = \bar{\delta} - 3\delta > 0$ and $D_{\bar{\bar{\delta}}} = \{x \in D : \ \mathrm{dis}(x, \partial D) > \bar{\bar{\delta}}\}$. If a double bubble $B(\rho, \xi, \theta) = (B_1, B_2)$ satisfies (3.7) and $\xi \in \overline{D_{\bar{\delta}}}$, then $\overline{B_1 \cup B_2} \subset \overline{D_{\bar{\bar{\delta}}}}$. Actually since The distance from the furthest points in $\overline{B_1 \cup B_2}$ to ξ is $\frac{3\rho}{2} < \frac{3\delta}{2}$, there is at least a distance of $\frac{3\delta}{2}$ from $\overline{B_1 \cup B_2}$ to the boundary of $D_{\bar{\bar{\delta}}}$. The last property is needed later when we modify $B = (B_1, B_2)$ to form a perturbed double bubble $\Omega = (\Omega_1, \Omega_2)$ and wish to keep $\overline{\Omega_1 \cup \Omega_2}$ in $\overline{D_{\bar{\bar{\delta}}}}$.

Also let $\sigma > 0$ and assume that γ satisfies

$$|\gamma| \rho^3 \in (0, \sigma). \tag{3.8}$$

The existence Theorem 2.1 will be proved for ρ and γ in the parameter range (3.7) and (3.8), and the bounds δ and σ must be sufficiently small.

Lemma 3.1 *The energy of the double bubble $B(\rho, \xi, \theta)$ is estimated as*

$$\Big| \mathcal{J}(B(\rho,\xi,\theta)) - \Big\{ \rho(2m) + \sum_{i,j=1}^{2} \frac{\gamma_{ij}}{2} \Big[\frac{\rho^4}{2\pi} \Big(\log \frac{1}{\rho} \Big) \Big(\frac{m}{2} \Big)^2 + \rho^4 \iota_{ij} + \rho^4 \Big(\frac{m}{2} \Big)^2 R(\xi, \xi) \Big] \Big\} \Big| \leq \frac{3|\gamma| \rho^5 m^2}{2} \max_{x,y \in \overline{D_{\bar{\bar{\delta}}}}} |\nabla R(x,y)|$$

where the constants ι_{ij} are given by $\iota_{ij} = \int_{E_i} \int_{E_j} \frac{1}{2\pi} \log \frac{1}{|\tilde{x} - \tilde{y}|} \, d\tilde{x} d\tilde{y}$, and ∇R denotes the gradient of $R(x,y)$ with respect to its first variable x.

Proof. The lengths of $\partial B_1 \backslash \partial B_2$, $\partial B_2 \backslash \partial B_1$, and $\partial B_1 \cap \partial B_2$ are respectively $\frac{4\pi}{3}\rho$, $\frac{4\pi}{3}\rho$ and $\sqrt{3}\rho$. Hence

$$\mathcal{J}_S(B) = \frac{1}{2} \Big(\mathcal{P}_D(B_1) + \mathcal{P}_D(B_2) + \mathcal{P}_D(B_3) \Big) = \rho \Big(\frac{8\pi}{3} + \sqrt{3} \Big) = \rho(2m). \tag{3.9}$$

To estimate $\mathcal{J}_L(B)$ note that

$$\begin{aligned} & \int_D \Big((-\Delta)^{-1/2} (\chi_{B_i(\rho,\xi,\theta)} - \frac{m\rho^2}{2|D|}) \Big) \Big((-\Delta)^{-1/2} (\chi_{B_j(\rho,\xi,\theta)} - \frac{m\rho^2}{2|D|}) \Big) \, dx \\ = \ & \int_{B_i(\rho,\xi,\theta)} \int_{B_j(\rho,\xi,\theta)} G(x,y) \, dx dy \ = \int_{B_i(\rho,\xi,\theta)} \int_{B_j(\rho,\xi,\theta)} \Big(\frac{1}{2\pi} \log \frac{1}{|x-y|} + R(x,y) \Big) \, dx dy \\ = \ & \frac{\rho^4}{2\pi} \Big(\log \frac{1}{\rho} \Big) \Big(\frac{m}{2} \Big)^2 + \rho^4 \int_{E_i} \int_{E_j} \frac{1}{2\pi} \log \frac{1}{|\tilde{x} - \tilde{y}|} \, d\tilde{x} d\tilde{y} + \rho^4 \int_{E_i} \int_{E_j} R(\rho e^{\mathrm{i}\theta} \tilde{x} + \xi, \rho e^{\mathrm{i}\theta} \tilde{y} + \xi) \, d\tilde{x} d\tilde{y}. \end{aligned} \tag{3.10}$$

For the last term note that by the symmetry $R(x,y) = R(y,x)$, there exists $\tau \in (0,1)$ such that

$$\begin{aligned} & |R(\rho e^{\mathrm{i}\theta} \tilde{x} + \xi, \rho e^{\mathrm{i}\theta} \tilde{y} + \xi) - R(\xi, \xi)| \\ & \quad = \ |\nabla R(\tau \rho e^{\mathrm{i}\theta} \tilde{x} + \xi, \tau \rho e^{\mathrm{i}\theta} \tilde{y} + \xi) \cdot (\rho e^{\mathrm{i}\theta} \tilde{x}) + \widetilde{\nabla} R(\tau \rho e^{\mathrm{i}\theta} \tilde{x} + \xi, \tau \rho e^{\mathrm{i}\theta} \tilde{y} + \xi) \cdot (\rho e^{\mathrm{i}\theta} \tilde{y})| \\ & \quad \leq \ \Big(\max_{x,y \in \overline{D_{\bar{\bar{\delta}}}}} |\nabla R(x,y)| \Big) (|\rho \tilde{x}| + |\rho \tilde{y}|) \ \leq 3\rho \max_{x,y \in \overline{D_{\bar{\bar{\delta}}}}} |\nabla R(x,y)| \end{aligned} \tag{3.11}$$

where $\widetilde{\nabla}$ denotes the gradient of $R(x,y)$ with respect to its second variable y. The lemma follows from (3.9), (3.10) and (3.11). $\square$

4 Variations

We derive the first variation of the functional $\mathcal{J}$ in this section.

Let $\mathbf{r}(t) = (x(t), y(t))$, $t \in [-1, 1]$, be a C^2 planner curve, with the length element $ds = |\mathbf{r}'|\, dt$. Let the unit tangent vector be

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}. \tag{4.1}$$

There are two choices of a unit normal vector both of which we use:

$$\mathbf{N}(t) = \mathbf{T}(t)\mathrm{i}, \quad \text{or} \quad \mathbf{N}(t) = \mathbf{T}(t)(-\mathrm{i}). \tag{4.2}$$

The signed curvature κ corresponding to $\mathbf{N}$ is

$$\kappa(t) = \begin{cases} \dfrac{\det[\mathbf{r}', \mathbf{r}'']}{|\mathbf{r}''|^3} & \text{if} \quad \mathbf{N} = \mathbf{T}\mathrm{i} \\[2ex] -\dfrac{\det[\mathbf{r}', \mathbf{r}'']}{|\mathbf{r}'|^3} & \text{if} \quad \mathbf{N} = \mathbf{T}(-\mathrm{i}) \end{cases}. \tag{4.3}$$

Under this convention,

$$\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N} \tag{4.4}$$

holds, regardless which $\mathbf{N}$ one picks. The vector $\kappa\mathbf{N}$ is called the curvature vector, which is independent of the parametrization of the curve.

The following two lemmas can be proved by direct computation.

Lemma 4.1 *Let $\mathbf{r}^\epsilon(t)$ be a deformation of $\mathbf{r}(t)$ so that, $\mathbf{r}^\epsilon(t)$ is C^2 with respect to t and C^1 with respect to ϵ, and $\mathbf{r}^0 = \mathbf{r}$. Let $\mathbf{X}$ be the infinitesimal element of $\mathbf{r}^\epsilon$: $\mathbf{X}(t) = \frac{\partial \mathbf{r}^\epsilon(t)}{\partial \epsilon}|_{\epsilon=0}$. Then*

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \int_{-1}^{1} |\mathbf{r}'_\epsilon|\, dt = \mathbf{T}\cdot\mathbf{X}\Big|_{-1}^{1} - \int_{-1}^{1} \kappa \mathbf{N}\cdot\mathbf{X}\, ds$$

where $\int_{-1}^{1} |\mathbf{r}'_\epsilon|\, dt$ is the length of $\mathbf{r}^\epsilon$.

Lemma 4.2 *Suppose that a bounded domain U is enclosed by a piecewise C^1 curve, and U^ϵ is a deformation of U with piecewise C^1 boundary. Also the deformation from ∂U to ∂U^ϵ is C^1 with respect to ϵ so the infinitesimal element $\mathbf{X}$ exists and is continuous on ∂U. Then*

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \int_{U^\epsilon} f(x)\, dx = -\int_{\partial U} f(x)\mathbf{N}\cdot\mathbf{X}\, ds$$

where $\mathbf{N}$ is the inward unit normal vector on ∂U.

We denote a perturbed double bubble by Ω, which consists of two disjoint open sets Ω_1 and Ω_2. The two sets share part of their boundaries, i.e. $\partial\Omega_1 \cap \partial\Omega_2 \neq \emptyset$. Assume that $\partial\Omega_1 \cap \partial\Omega_2$ is parametrized by $\mathbf{r}_0(t)$. The rest of the boundary of Ω_1, i.e. $\partial\Omega_1 \backslash \partial\Omega_2$, is parametrized by $\mathbf{r}_1(t)$, and the rest of the boundary of Ω_2, i.e. $\partial\Omega_2 \backslash \partial\Omega_1$, is parametrized by $\mathbf{r}_2(t)$. The argument t is in $[-1, 1]$ in all the three cases. For now we assume that the $\mathbf{r}_i$'s are C^2 vector valued functions. Since the three curves meet at two triple junction points, the conditions

$$\mathbf{r}_1(1) = \mathbf{r}_2(1) = \mathbf{r}_0(1) \quad \text{and} \quad \mathbf{r}_1(-1) = \mathbf{r}_2(-1) = \mathbf{r}_0(-1) \tag{4.5}$$

must hold. Sometimes we write $\mathbf{r}$ collectively for $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_0$, i.e. treat $\mathbf{r}$ as the union of $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_0$. Then $\mathbf{r}$ becomes a piecewise C^2 vector field on $\partial\Omega_1 \cup \partial\Omega_2$.

Let $\mathbf{N}_1$, $\mathbf{N}_2$, and $\mathbf{N}_0$ be unit normal vectors to $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_0$ respectively. We adopt the following direction convention: $\mathbf{N}_1$ points inward with respect to Ω_1, $\mathbf{N}_2$ points inward with respect to Ω_2, and $\mathbf{N}_0$ points from Ω_2 towards Ω_1, i.e. inward with respect to Ω_1 and outward with respect to Ω_2. We may also write $\mathbf{N}$ collectively for

$\mathbf{N}_1$, $\mathbf{N}_2$, and $\mathbf{N}_0$. However one must be careful that when viewed as a single vector field on $\partial\Omega_1 \cup \partial\Omega_2$, $\mathbf{N}$ is usually not single valued at triple junction points.

A deformation Ω^ϵ of Ω is a family of perturbed double bubbles parametrized by ϵ in a neighborhood of 0. The three curves $\partial\Omega_1^\epsilon \backslash \partial\Omega_2^\epsilon$, $\partial\Omega_2^\epsilon \backslash \partial\Omega_1^\epsilon$, and $\partial\Omega_1^\epsilon \cup \partial\Omega_2^\epsilon$ that enclose Ω^ϵ are parametrized respectively by $\mathbf{r}_1^\epsilon$, $\mathbf{r}_2^\epsilon$, and $\mathbf{r}_0^\epsilon$. Every $\mathbf{r}_i^\epsilon(t)$ is C^2 with respect to t and C^1 with respce to ϵ; at $\epsilon = 0$, we require that $\mathbf{r}_i^0 = \mathbf{r}_i$; $\mathbf{r}_i^\epsilon$ aslo satisfy the compatibility condition (4.5). Define

$$\mathbf{X}_i(t) = \frac{\partial \mathbf{r}_i^\epsilon(t)}{\partial \epsilon}\Big|_{\epsilon=0} \tag{4.6}$$

which is the infinitesimal element of the deformation $\mathbf{r}_i^\epsilon$. Again we write $\mathbf{X}$ for $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{X}_0$. Because $\mathbf{r}_j^\epsilon$ satisfy (4.5), unlike $\mathbf{N}$ the vector field $\mathbf{X}$ on $\partial\Omega_1 \cup \partial\Omega_2$ remains single valued at the triple junction points.

We proceed to find $\frac{d\mathcal{J}(\Omega^\epsilon)}{d\epsilon}|_{\epsilon=0}$. Recall I_{Ω_i} from (2.7) which can be written as

$$I_{\Omega_i}(x) = \int_{\Omega_i} G(x,y)\,dy, \quad i = 1,2, \tag{4.7}$$

in terms of the Green's function. Then the product rule of differentiation implies that

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \int_{\Omega_i^\epsilon}\int_{\Omega_j^\epsilon} G(x,y)\,dx\,dy = \frac{d}{d\epsilon}\Big|_{\epsilon=0}\int_{\Omega_i^\epsilon} I_{\Omega_j}(x)\,dx + \frac{d}{d\epsilon}\Big|_{\epsilon=0}\int_{\Omega_j^\epsilon} I_{\Omega_i}(x)\,dx. \tag{4.8}$$

However Lemma 4.2 shows

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\int_{\Omega_i^\epsilon} I_{\Omega_j}(x)\,dx = \begin{cases} -\int_{\partial\Omega_1\backslash\partial\Omega_2} I_{\Omega_j}\mathbf{N}_1\cdot\mathbf{X}_1\,ds - \int_{\partial\Omega_1\cap\partial\Omega_2} I_{\Omega_j}\mathbf{N}_0\cdot\mathbf{X}_0\,ds, & i=1 \\ -\int_{\partial\Omega_2\backslash\partial\Omega_1} I_{\Omega_j}\mathbf{N}_2\cdot\mathbf{X}_2\,ds + \int_{\partial\Omega_1\cap\partial\Omega_2} I_{\Omega_j}\mathbf{N}_0\cdot\mathbf{X}_0\,ds, & i=2 \end{cases}. \tag{4.9}$$

Therefore

$$\begin{aligned} &\frac{d}{d\epsilon}\Big|_{\epsilon=0}\int_{\Omega_i^\epsilon}\int_{\Omega_j^\epsilon} G(x,y)\,dx\,dy \\ &= \begin{cases} -2\int_{\partial\Omega_1\backslash\partial\Omega_2} I_{\Omega_1}\mathbf{N}_1\cdot\mathbf{X}_1\,ds - 2\int_{\partial\Omega_1\cap\partial\Omega_2} I_{\Omega_1}\mathbf{N}_0\cdot\mathbf{X}_0\,ds, \quad i=j=1 \\ -2\int_{\partial\Omega_2\backslash\partial\Omega_1} I_{\Omega_2}\mathbf{N}_2\cdot\mathbf{X}_2\,ds + 2\int_{\partial\Omega_1\cap\partial\Omega_2} I_{\Omega_2}\mathbf{N}_0\cdot\mathbf{X}_0\,ds, \quad i=j=2 \\ -\int_{\partial\Omega_1\backslash\partial\Omega_2} I_{\Omega_2}\mathbf{N}_1\cdot\mathbf{X}_1\,ds - \int_{\partial\Omega_2\backslash\partial\Omega_1} I_{\Omega_1}\mathbf{N}_2\cdot\mathbf{X}_2\,ds - \int_{\partial\Omega_1\cap\partial\Omega_2}(I_{\Omega_2}-I_{\Omega_1})\mathbf{N}_0\cdot\mathbf{X}_0\,ds, \quad i=1,\ j=2 \end{cases} \end{aligned} \tag{4.10}$$

Hence

$$\begin{aligned} &\frac{d}{d\epsilon}\Big|_{\epsilon=0}\sum_{i,j=1}^{2}\frac{\gamma_{ij}}{2}\int_{\Omega_i^\epsilon}\int_{\Omega_j^\epsilon} G(x,y)\,dx\,dy \\ &= -\int_{\partial\Omega_1\backslash\partial\Omega_2}(\gamma_{11}I_{\Omega_1}+\gamma_{12}I_{\Omega_2})\mathbf{N}_1\cdot\mathbf{X}_1\,ds - \int_{\partial\Omega_2\backslash\partial\Omega_1}(\gamma_{21}I_{\Omega_1}+\gamma_{22}I_{\Omega_2})\mathbf{N}_2\cdot\mathbf{X}_2\,ds \\ &\quad -\int_{\partial\Omega_1\cap\partial\Omega_2}[(\gamma_{11}-\gamma_{12})I_{\Omega_1}+(\gamma_{21}-\gamma_{22})I_{\Omega_2}]\mathbf{N}_0\cdot\mathbf{X}_0\,ds \end{aligned} \tag{4.11}$$

Combining (4.11) with Lemma 4.1 we obtain the following.

Lemma 4.3 *Let Ω^ϵ be a deformation of a perturbed double bubble Ω as described earlier. The three curves $\partial\Omega_1^\epsilon\backslash\partial\Omega_2^\epsilon$, $\partial\Omega_2^\epsilon\backslash\partial\Omega_1^\epsilon$ and $\partial\Omega_1^\epsilon\cap\partial\Omega_2^\epsilon$, are parametrized by $\mathbf{r}_1^\epsilon(t)$, $\mathbf{r}_2^\epsilon(t)$ and $\mathbf{r}_0^\epsilon(t)$ respectively, which are C^2 with respect to t and*

C^1 *with respect to* ϵ*, and satisfy (4.5). Then*

$$\begin{aligned}\frac{d\mathcal{J}(\Omega^\epsilon)}{d\epsilon}\Big|_{\epsilon=0} &= (\mathbf{T}_1+\mathbf{T}_2+\mathbf{T}_0)\cdot\mathbf{X}\Big|_{-1}^{1} - \int_{\partial\Omega_1\backslash\partial\Omega_2}(\kappa_1+\gamma_{11}I_{\Omega_1}+\gamma_{12}I_{\Omega_2})\mathbf{N}_1\cdot\mathbf{X}_1\,ds \\ &\quad - \int_{\partial\Omega_2\backslash\partial\Omega_1}(\kappa_2+\gamma_{21}I_{\Omega_1}+\gamma_{22}I_{\Omega_2})\mathbf{N}_2\cdot\mathbf{X}_2\,ds \\ &\quad - \int_{\partial\Omega_1\cap\partial\Omega_2}(\kappa_0+(\gamma_{11}-\gamma_{12})I_{\Omega_1}+(\gamma_{21}-\gamma_{22})I_{\Omega_2})\mathbf{N}_0\cdot\mathbf{X}_0\,ds\end{aligned}$$

where κ_i *and* $\mathbf{N}_i$ *are respectively the curvature and the normal vector of* $\mathbf{r}_i$ *conforming to the direction convention, and* $\mathbf{X}$ *is the infinitesimal element of the deformation given in (4.6).*

If Ω is a critical point of the functional $\mathcal{J}$, then the quantity on the right side of Lemma 4.3 equals 0 for the infinitesimal element $\mathbf{X}$ of any permissible deformation $\mathbf{r}^\epsilon$. A deformation is permissible if the area of each Ω_i^ϵ remains unchanged under the deformation. Lemma 4.2 shows that

$$\frac{d|\Omega_i^\epsilon|}{d\epsilon}\Big|_{\epsilon=0} = \begin{cases} -\int_{\partial\Omega_1\backslash\partial\Omega_2}\mathbf{N}_1\cdot\mathbf{X}_1\,ds - \int_{\partial\Omega_1\cap\partial\Omega_2}\mathbf{N}_0\cdot\mathbf{X}_0\,ds, & i=1 \\ -\int_{\partial\Omega_2\backslash\partial\Omega_1}\mathbf{N}_2\cdot\mathbf{X}_2\,ds + \int_{\partial\Omega_1\cap\partial\Omega_2}\mathbf{N}_0\cdot\mathbf{X}_0\,ds, & i=2\end{cases}. \tag{4.12}$$

Hence for a permissible deformation, the two lines on the right side of (4.12) must vanish.

5 Restricted perturbations

Given a double bubble $B(\rho,\xi,\theta)$ as in (3.1) we associate a reference frame. The point ξ becomes the origin of the new frame whose horizontal axis points from the center of $B_1(\rho,\xi,\theta)$ to the center of $B_2(\rho,\xi,\theta)$. The vertical axis is obtained by rotating the horizontal axis 90 degrees counterclockwise. This new coordinate system is termed the (ξ,θ)-frame. The centers of the left and right bubbles of $B(\rho,\xi,\theta)$ are $O_1=(-\frac{\rho}{2},0)$ and $O_2=(\frac{\rho}{2},0)$ respectively, in the (ξ,θ)-frame, and the triple junction points are $(0,\frac{\sqrt{3}}{2}\rho)$ and $(0,-\frac{\sqrt{3}}{2}\rho)$. Henceforward $B(\rho,\xi,\theta)$ stands for the double bubble in the (ξ,θ)-frame. Under this frame we describe perturbed double bubbles. However we only consider a restricted class of perturbed double bubbles depicted in Figure 3 at this point. To form a restrictedly perturbed double bubble, the upper and lower triple junction points of $B(\rho,\xi,\theta)$ are allowed to move to new positions P^+ and P^- only vertically and only by the same distance in opposite directions; namely there is $a>0$ such that in the (ξ,θ)-frame $P^+=(0,a)$ and $P^-=(0,-a)$. Here a is close to $\frac{\sqrt{3}}{2}\rho$. Between now and the end of section 8 we will only consider restrictedly perturbed double bubbles in a fixed (ξ,θ)-frame. Deformations of a restrictedly perturbed double bubble will also lie in the same restricted class. In section 9 deformations outside the restricted class will be explored.

In general curves are described by parametrization up to diffeomorphism. Say the three curves of a perturbed double bubble Ω in the restricted class are parametrized by $\mathbf{r}_1(t)$, $\mathbf{r}_2(t)$ and $\mathbf{r}_0(t)$. Then any reparametrization of each $\mathbf{r}_i$ will give the same curve. To eliminate this freedom, we set up a particular way of parametrization. Introduce three functions u_1, u_2, and u_0, all defined on $[-1,1]$, so that in the (ξ,θ)-frame the three curves on the left, right and center, of the perturbed double bubble Ω are parametrized respectively by

$$\mathbf{r}_1(t)=u_1(t)e^{\mathrm{i}(\pi-At)}+O_1,\quad \mathbf{r}_2(t)=u_2(t)e^{\mathrm{i}At}+O_2,\quad \mathbf{r}_0(t)=(u_0(t),at). \tag{5.1}$$

A polar coordinate system centered at O_1 with the angle starting from the negative horizontal direction of the (ξ,θ)-frame describes the curve $\mathbf{r}_1$. In this polar coordinate system the radius and angle of P^+ are $u_1(1)$ and A; the radius and angle of P^- are $u_1(-1)$ and $-A$. For $\mathbf{r}_2$ another polar coordinate system centered at O_2 with the angle starting from the positive horizontal direction of the (ξ,θ)-frame is adopted, P^+ is described by the radius $u_2(1)$ and the angle A, and P^- by $u_2(-1)$ and $-A$ accordingly. The center curve $\mathbf{r}_0$ is given as a graph, up to a factor a, in the (ξ,θ)-frame, with the vertical direction representing the independent variable and the horizontal direction representing the dependent variable.

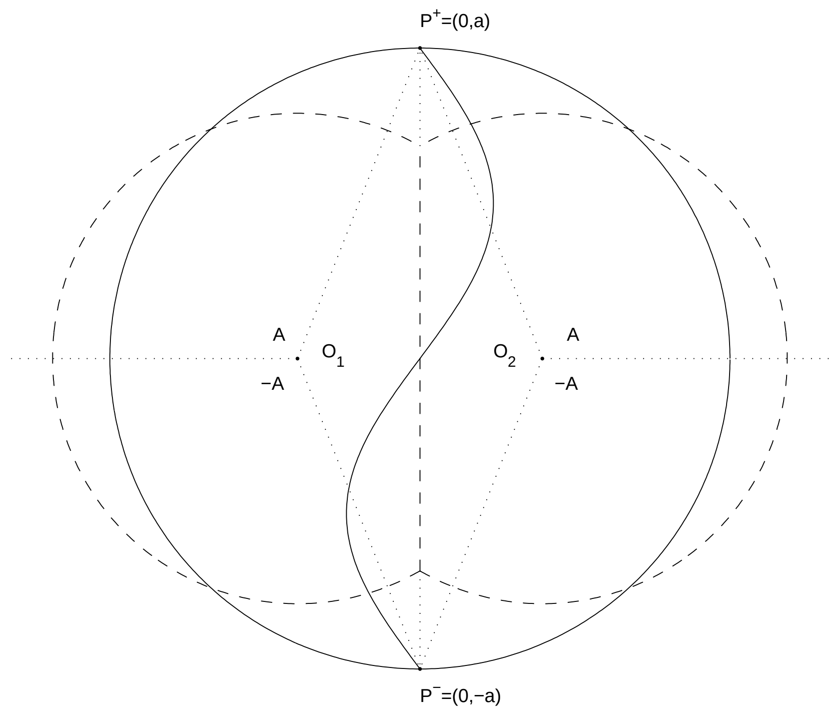

Figure 3: A standard double bubble is enclosed by the dashed curves in a (ξ, θ)-frame; the solid curves bound a restrictedly perturbed double bubble; also plotted are the centers O_1 and O_2, triple junction points P^+ and P^-, and the height a of P^+ and the angle A; the triangles $\triangle O_1P^+P^-$ and $\triangle O_2P^+P^-$ are mentioned in section 6.

From now on Ω will denote a perturbed double bubble in the (ξ, θ)-frame. Obviously that Ω can be described by (5.1) means more restrictions. As a graph, the center curve $\partial\Omega_1 \cap \partial\Omega_2$ must intersect every horizontal line whose height is between $-a$ and a (with respect to the (ξ, θ)-frame) exactly once. The left curve $\partial\Omega_1\backslash\partial\Omega_2$ must be a graph in the polar coordinate system centered at O_1; namely each ray between $-A$ and A must intersect $\partial\Omega_1\backslash\partial\Omega_2$ precisely once. A similar condition holds for the right curve $\partial\Omega_2\backslash\partial\Omega_1$.

The two numbers A and a in (5.1) are related:

$$A = \pi - \arctan\left(\frac{2a}{\rho}\right), \quad \text{where } A \in \left(\frac{\pi}{2}, \pi\right) \text{ and } a \in (0, \infty). \tag{5.2}$$

At the two triple junction points,

$$u_1(1) = u_1(-1) = u_2(1) = u_2(-1) = \sqrt{a^2 + \left(\frac{\rho}{2}\right)^2}, \quad u_0(1) = u_0(-1) = 0. \tag{5.3}$$

To maintain the proper shape of a perturbed double bubble with two triple junction points, u_i, A and a need to be close to the corresponding variables of the standard double bubble. The value of (u_1, u_2, u_0, A, a) for the standard double bubble $B(\rho, \xi, \theta)$ is $(\rho, \rho, 0, \frac{2\pi}{3}, \frac{\sqrt{3}}{2}\rho)$. Let $\tilde{c} > 0$ such

$$\|u_i - \rho\|_{C^1} \le \tilde{c}\rho, \; i = 1, 2, \; \|u_0\|_{C^1} \le \tilde{c}\rho, \; |A - \frac{2\pi}{3}| \le \tilde{c}, \; |a - \frac{\sqrt{3}}{2}\rho| \le \tilde{c}\rho. \tag{5.4}$$

By choosing $\tilde{c}$ suitably small, we are guaranteed two sets sharing part of their boundaries . The use of the C^1 norm in (5.4) makes the three curves meet at the two triple junction points properly. Moreover, according to the remark following (3.7), with a small $\tilde{c}$, $\overline{\Omega_1 \cup \Omega_2}$ always stays inside $\overline{D_{\frac{\delta}{3}}}$, a compact subset of the domain D.

In summary the class of restricted perturbations only include perturbed double bubbles described by (5.1) subject to (5.2), (5.3) and (5.4), plus the constraints (2.8). A member in this class is termed a restrictedly perturbed double bubble, or just a perturbed double bubble if no possibility of ambiguity exists. Despite the many conditions, a member in the restricted class may deviate quite a bit from the standard double bubble as seen in Figure 3.

The length element of the curve $\mathbf{r}_i$ is

$$ds = |\mathbf{r}_i'|\ dt = \begin{cases} \sqrt{(u_i')^2 + A^2 u_i^2}\, dt, & i = 1, 2 \\ \\ \sqrt{(u_0')^2 + a^2}\, dt, & i = 0 \end{cases}, \tag{5.5}$$

and the unit tangent vectors and the unit normal vectors are

$$\mathbf{T}_i = \begin{cases} \dfrac{u_1' e^{\mathrm{i}(\pi - At)} + A u_1 e^{\mathrm{i}(\pi - At)}(-\mathrm{i})}{\sqrt{(u_1')^2 + A^2 u_1^2}}, & i = 1 \\ \\ \dfrac{u_2' e^{\mathrm{i}At} + A u_2 e^{\mathrm{i}At}\mathrm{i}}{\sqrt{(u_2)'^2 + A^2 u^2}}, & i = 2 \\ \\ \dfrac{(u_0', a)}{\sqrt{(u_0')^2 + a^2}}, & i = 0 \end{cases}, \quad \mathbf{N}_i = \begin{cases} \mathbf{T}_1(t)(-\mathrm{i}) = \dfrac{-A u_1 e^{\mathrm{i}(\pi - At)} + u_1' e^{\mathrm{i}(\pi - At)}(-\mathrm{i})}{\sqrt{(u_1')^2 + A^2 u_1^2}} \\ \\ \mathbf{T}_2(t)\mathrm{i} = \dfrac{-A u_2 e^{\mathrm{i}At} + u_2' e^{\mathrm{i}At}\mathrm{i}}{\sqrt{(u_2')^2 + A^2 u_2^2}} \\ \\ \mathbf{T}_0(t)\mathrm{i} = \dfrac{(-a, u_0')}{\sqrt{(u_0')^2 + a^2}} \end{cases}. \tag{5.6}$$

The signed curvatures are

$$\kappa_i = \begin{cases} \dfrac{-A u_i u_i'' + 2A(u_i')^2 + A^3 u_i^2}{((u_i')^2 + A^2 u_i^2)^{3/2}}, & i = 1, 2 \\ \\ \dfrac{-a u_0''}{((u_0')^2 + a^2)^{3/2}}, & i = 0 \end{cases}. \tag{5.7}$$

For a deformation $\mathbf{r}_i^\epsilon(t)$ of $\mathbf{r}_i(t)$, we set

$$u_i^0(t) = u_i(t),\ \frac{\partial u_i^\epsilon(t)}{\partial \epsilon}\Big|_{\epsilon=0} = v_i(t),\ A(0) = A,\ A'(0) = A', \text{ and } \mathbf{X}_i(t) = \frac{\partial \mathbf{r}_i^\epsilon(t)}{\partial \epsilon}\Big|_{\epsilon=0}. \tag{5.8}$$

More precisely

$$\mathbf{r}_i^\epsilon = \begin{cases} u_1^\epsilon e^{\mathrm{i}(\pi - A(\epsilon)t)} + O_1, & i = 1 \\ \\ u_2^\epsilon e^{\mathrm{i}A(\epsilon)t} + O_2, & i = 2 \\ \\ (u_0^\epsilon, a(\epsilon)t), & i = 0 \end{cases}, \quad \mathbf{X}_i = \begin{cases} v_1 e^{\mathrm{i}(\pi - At)} + u_1 A' t e^{\mathrm{i}(\pi - At)}(-\mathrm{i}), & i = 1 \\ \\ v_2 e^{\mathrm{i}At} + u_2 A' t e^{\mathrm{i}At}\mathrm{i}, & i = 2 \\ \\ (v_0, a' t), & i = 0 \end{cases}. \tag{5.9}$$

6 Internal variables

While capturing the geometric picture of a restrictedly perturbed double bubble, the variables u_i, A, and a are not very convenient for analytic techniques, such as the contraction mapping theorem, because (5.2), (5.3) and (2.8) are nonlinear constraints. We introduce a new set of variables to describe perturbed double bubbles. These so called internal variables will be elements in a Hilbert space, so that our problem becomes a nonlinear equation between Hilbert spaces, which can also be formulated in a fixed point form.

Let $\Omega = (\Omega_1, \Omega_2)$ be a restrictedly perturbed double bubble. Figure 3 shows that the area of Ω_1 (and also of Ω_2) is the sum of the areas of three regions: a triangle, a sector, and a strip. The triangle is formed from the two triple junction points P^+, P^- and the center O_1 of Ω_1; the sector is the part of Ω_1 described by points whose angle in the polar coordinates is between $-A$ and A; the strip is the (signed) region bounded by the line segment connecting P^+ and P^- and the curve $\mathbf{r}_0$. Hence the area of Ω_1 and the area of Ω_2 are written as

$$|\Omega_1| = \frac{a\rho}{2} + \int_{-1}^{1} \frac{A}{2} u_1^2\, dt + \int_{-1}^{1} a u_0\, dt = \int_{-1}^{1} \Big(\frac{a\rho}{4} + \frac{A}{2} u_1^2\Big)\, dt + \int_{-1}^{1} a u_0\, dt, \tag{6.1}$$

$$|\Omega_2| = \frac{a\rho}{2} + \int_{-1}^{1} \frac{A}{2} u_2^2\, ds - \int_{-1}^{1} a u_0\, ds = \int_{-1}^{1} \Big(\frac{a\rho}{4} + \frac{A}{2} u_2^2\Big)\, dt - \int_{-1}^{1} a u_0\, dt. \tag{6.2}$$

Inspired by (6.1) and (6.2), introduce ϕ_1, ϕ_2, and ϕ_0 such that

$$\frac{a\rho}{4}+\frac{A}{2}u_1^2=\frac{m\rho^2}{4}+\phi_1,\quad \frac{a\rho}{4}+\frac{A}{2}u_2^2=\frac{m\rho^2}{4}+\phi_2,\quad au_0=\phi_0 \tag{6.3}$$

where m is given in (2.8). Also introduce $\alpha\in\mathbb{R}$ such that

$$\frac{a\rho}{4}+\frac{A}{2}\Big(\frac{\rho^2}{4}+a^2\Big)=\frac{m\rho^2}{4}+\alpha. \tag{6.4}$$

By the equation (5.2) that $A=\pi-\arctan(\frac{2a}{\rho})$ and (6.4) we view both A and a as functions of α: $A=A(\alpha)$ and $a=a(\alpha)$. Our choice of α leads to the linear boundary condition

$$\phi_1(\pm1)=\phi_2(\pm1)=\alpha \ \text{ and } \ \phi_0(\pm1)=0. \tag{6.5}$$

With these new variables, we find that

$$|\Omega_1|=\frac{m\rho^2}{2}+\int_{-1}^1\phi_1\,dt+\int_{-1}^1\phi_0\,dt,\quad |\Omega_2|=\frac{m\rho^2}{2}+\int_{-1}^1\phi_2\,dt-\int_{-1}^1\phi_0\,dt. \tag{6.6}$$

The area constraints (2.8) become

$$\int_{-1}^1\phi_1\,dt+\int_{-1}^1\phi_0\,dt=0,\quad \int_{-1}^1\phi_2\,dt-\int_{-1}^1\phi_0\,dt=0, \tag{6.7}$$

again linear conditions.

Henceforth we use the ϕ_i's and α as our primary variables, called internal variables. Collectively write ϕ for (ϕ_1,ϕ_2,ϕ_0), and use (ϕ,α) to represent a restrictedly perturbed double bubble. In terms of internal variables the standard double bubble $B(\rho,\xi,\theta)$ is represented by $(0,0)$. This way of representing double bubbles and perturbed double bubbles is termed the internal representation. The previous functions u_i and the numbers A and a can be derived from ϕ_i and α. Of course this transformation between the two sets of variables can only be done for (ϕ,α) close to $(0,0)$, i.e. $(\phi(t),\alpha)$ must be uniformly (with respect to t) within a certain distance of order ρ^2 from $(0,0)$.

Because of the linear conditions (6.5) and (6.7) on (ϕ,α), it is very easy to define deformations within the restricted class in this setting. Let a perturbed double bubble be parametrized by $\mathbf{r}_i$ with the original variables u_i, A and a. Transform them to the internal variables ϕ_i and α. For $\phi_i\in C^1[-1,1]$ and $\alpha\in\mathbb{R}$ satisfying (6.5) and (6.7), let $\psi_i\in C^1[-1,1]$, $\beta\in\mathbb{R}$, satisfying (6.5) and (6.7) as well, and $\epsilon\in\mathbb{R}$. Then for ϵ sufficiently close to 0, $(\phi^\epsilon,\alpha^\epsilon)=(\phi,\alpha)+\epsilon(\psi,\beta)$ also describes a perturbed double bubble. Transforming them back to the original variables u_i^ϵ, A^ϵ, and a^ϵ, we obtain a deformation $\mathbf{r}_i^\epsilon$ of $\mathbf{r}_i$. Let $\mathbf{X}_i=\frac{\partial\mathbf{r}_i^\epsilon}{\partial\epsilon}|_{\epsilon=0}$ be the infinitesimal element of the deformation $\mathbf{r}_i^\epsilon$. Calculations show that in terms of the internal variables

$$-\mathbf{N}_i\cdot\mathbf{X}_i\,ds=(\psi_i+e_i(\phi_i,\alpha)\beta)\,dt,\quad i=1,2,0. \tag{6.8}$$

where e_1, e_2 and e_0 act on (ϕ_1,α), (ϕ_2,α), and (ϕ_0,α) respectively as follows.

$$e_i(\phi_i,\alpha)=-\frac{\rho}{4}\frac{da}{d\alpha}\ -\frac{1}{A}\Big(\frac{m\rho^2}{4}-\frac{a\rho}{4}+\phi_i+t\phi_i'\Big)\frac{dA}{d\alpha},\quad i=1,2;\quad e_0(\phi_0,\alpha)=-\frac{1}{a}\Big(\phi_0+t\phi_0'\Big)\frac{da}{d\alpha}. \tag{6.9}$$

Implicit differentiation from (5.2) and (6.4) gives that

$$\frac{da}{d\alpha}=\frac{1}{aA},\quad \frac{dA}{d\alpha}=-\Big(\frac{2\rho}{\rho^2+4a^2}\Big)\frac{1}{aA}. \tag{6.10}$$

Using the internal variables we derive a necessary and sufficient condition for $-\mathbf{N}\cdot\mathbf{X}\,ds$ to be associated with a deformation $\mathbf{r}_i^\epsilon$.

Lemma 6.1 *Let $\mathbf{r}_i$ parametrize a perturbed double bubble via u_i, A and a.*

1. *Suppose that $\mathbf{r}_i^\epsilon(t)$ is a deformation of $\mathbf{r}_i$ within the restricted class and is C^1 with respect to both t and ϵ. If $-\mathbf{N}_i\cdot\mathbf{X}_i\,ds=f_i(t)\,dt$, then the following two properties hold:*

(a) $\int_{-1}^{1} f_1\,dt + \int_{-1}^{1} f_0\,dt = \int_{-1}^{1} f_2\,dt - \int_{-1}^{1} f_0\,dt = 0$;

(b) there exists $k \in \mathbb{R}$ such that

$$\begin{pmatrix} f_1(1) \\ f_2(1) \\ f_0(1) \\ f_1(-1) \\ f_2(-1) \\ f_0(-1) \end{pmatrix} = k \begin{pmatrix} aA - u_1'(1)\cos A \\ aA - u_2'(1)\cos A \\ -u_0'(1) \\ aA + u_1'(-1)\cos A \\ aA + u_2'(-1)\cos A \\ u_0'(-1) \end{pmatrix}.$$

2. *Conversely if $\mathbf{r}_i(t)$ with (u_i, A, a) give a C^2 parametrization of a perturbed double bubble and $f_i \in C^1[-1,1]$, $i = 1,2,0$, satisfy the two conditions in part 1, then there is a deformation $\mathbf{r}_i^\epsilon$ of $\mathbf{r}_i$, C^1 with respect to t and C^∞ with respect to ϵ, within the restricted class such that $-\mathbf{N}_i \cdot \mathbf{X}_i\,ds = f_i(t)\,dt$ for $i = 1,2,0$, where $\mathbf{X}_i = \frac{\partial \mathbf{r}_i^\epsilon}{\partial \epsilon}|_{\epsilon=0}$.*

Proof. To show part 1, note that (a) follows from the area constraints (2.8) and the formula (4.12). For (b), note that deformations in the restricted class satisfy that $\mathbf{X}_1(1) = \mathbf{X}_2(1) = \mathbf{X}_0(1) = (0,k)$ and $\mathbf{X}_1(-1) = \mathbf{X}_2(-1) = \mathbf{X}_0(-1) = (0,-k)$ for some $k \in \mathbb{R}$. Since $\mathbf{X}_0(1) = (v_0(1), a')$ by (5.8), $v_0(1) = 0$ and $k = a'$. Also $-\mathbf{N}_0 \cdot \mathbf{X}_0\,ds = (av_0 - tu_0'a')\,dt$ by (5.8). Hence $f_0(1) = -u_0'(1)k$. Similarly $\mathbf{X}_1(1) = v_1(1)e^{\mathrm{i}(\pi-A)} + u_1(1)A'e^{\mathrm{i}(\pi-A)}(-\mathrm{i}) = (0,k)$ by (5.8) implies that $v_1(1) = \frac{ak}{u_1(1)}$. Then $-\mathbf{N}_1 \cdot \mathbf{X}_1\,ds = (Au_1v_1 - u_1'u_1A't)\,dt$ by (5.6) shows that $f(1) = (aA - u_1'(1)\cos A)k$. Other equations in (b) are derived in the same way.

To prove part 2, transform u_i, A, and a to the new variables ϕ_i and α. Set

$$\beta = aAk \quad \text{and} \quad \psi_i = f_i - e_i(\phi_i, \alpha)\beta, \quad i = 1,2,0, \tag{6.11}$$

where k is given in condition (b). By (6.3), (6.4), (6.9), and (b), we can show that $\psi_0(1) = \psi_0(-1) = 0$ and $\psi_1(1) = \psi_1(-1) = \psi_2(1) = \psi_2(-1) = \beta$, i.e. (ψ, β) satisfies the boundary condition (6.5). By (6.4) and (6.5), we see that

$$\int_{-1}^{1} e_i(\phi, \alpha)\,dt = 0, \quad i = 1,2,0. \tag{6.12}$$

Then condition (a) implies that the ψ_i's satisfy the constraints (6.7). Consider $(\phi^\epsilon, \alpha^\epsilon) = (\phi, \alpha) + \epsilon(\psi, \beta)$ as an internal representation and transform it to $(u_i^\epsilon, A^\epsilon, a^\epsilon)$ and consequently $\mathbf{r}_i^\epsilon$ to be a restricted deformation of $\mathbf{r}_i$. Then (6.8) and (6.11) show that $-\mathbf{N}_i \cdot \mathbf{X}_i\,ds = f_i(t)\,dt$. □

In the case of a perfect double bubble, i.e. $(\phi, \alpha) = (0,0)$, it follows from (6.9) that

$$e_i(0,0) = 0, \quad i = 1,2,0. \tag{6.13}$$

Hence in this case f_i in Lemma 6.1 is just ψ_i.

The functional $\mathcal{J}$ is now considered a functional of (ϕ, α). To specify the domain of $\mathcal{J}$ let

$$\begin{aligned} \mathcal{Y} \;=\; & \{(\phi,\alpha) \in H^1(-1,1) \times H^1(-1,1) \times H^1(-1,1) \times \mathbb{R} : \\ & \phi_1(\pm 1) - \alpha = \phi_2(\pm 1) - \alpha = \phi_0(\pm 1) = 0,\ \int_{-1}^{1} \phi_1\,dt + \int_{-1}^{1} \phi_0\,dt = \int_{-1}^{1} \phi_2\,dt - \int_{-1}^{1} \phi_0\,dt = 0\}. \end{aligned} \tag{6.14}$$

The space $\mathcal{Y}$ is equipped with the norm $\|\cdot\|_{\mathcal{Y}}$ given by

$$\|(\phi,\alpha)\|_{\mathcal{Y}}^2 = \|\phi_1\|_{H^1}^2 + \|\phi_2\|_{H^1}^2 + \|\phi_0\|_{H^1}^2 + \alpha^2. \tag{6.15}$$

In (6.15) $\|\cdot\|_{H^1}$ denotes the norm of the Sobolev space $H^1(-1,1)$; namely $\|f\|_{H^1}^2 = \int_{-1}^{1} [(f')^2 + f^2]\,dt$. Since the transformation between (ϕ, α) and (u, a, A) is valid only if (ϕ, α) is in a neighborhood of $(0,0)$ of order ρ^2, there exists $\bar{c} > 0$ such that the domain of $\mathcal{J}$ is the closed ball of radius $\bar{c}\rho^2$ centered at $(0,0)$ in $\mathcal{Y}$:

$$\mathcal{D}(\mathcal{J}) = \{(\phi,\alpha) \in \mathcal{Y} :\ \|(\phi,\alpha)\|_{\mathcal{Y}} \le \bar{c}\rho^2\}. \tag{6.16}$$

With $\bar{c}$ being sufficiently small, the variables u_i, A, and a corresponding to $(\phi, \alpha) \in \mathcal{D}(\mathcal{J})$ meet the requirement (5.4).

In addition to $\mathcal{Y}$, two more spaces are needed:

$$\begin{aligned}
\mathcal{X} &= \{(\phi,\alpha) \in H^2(-1,1)\times H^2(-1,1)\times H^2(-1,1)\times \mathbb{R}: \\
&\quad \phi_1(\pm 1)-\alpha = \phi_2(\pm 1)-\alpha = \phi_0(\pm 1) = 0,\ \int_{-1}^1 \phi_1\,dt + \int_{-1}^1 \phi_0\,dt = \int_{-1}^1 \phi_2\,dt - \int_{-1}^1 \phi_0\,dt = 0\} \qquad (6.17)\\
\mathcal{Z} &= \{(\psi,\beta) \in L^2(-1,1)\times L^2(-1,1)\times L^2(-1,1)\times \mathbb{R}:\ \int_{-1}^1 \psi_1\,dt + \int_{-1}^1 \psi_0\,dt = \int_{-1}^1 \psi_2\,dt - \int_{-1}^1 \psi_0\,dt = 0\}.
\end{aligned}$$

Note that

$$\mathcal{X} \subset \mathcal{Y} \subset \mathcal{Z} \subset L^2(-1,1)\times L^2(-1,1)\times L^2(-1,1)\times \mathbb{R}. \qquad (6.18)$$

Define the inner product on $L^2(-1,1)\times L^2(-1,1)\times L^2(-1,1)\times \mathbb{R}$ by

$$\langle(\psi,\beta),(\tilde\psi,\tilde\beta)\rangle = \sum_{i=1,2,0}\int_{-1}^1 \psi_i\tilde\psi_i\,dt + \beta\tilde\beta. \qquad (6.19)$$

Then $L^2(-1,1)\times L^2(-1,1)\times L^2(-1,1)\times \mathbb{R}$ becomes a Hilbert space and $\mathcal{Z}$ a closed subspace. The norm in $\mathcal{Z}$ inherited from $L^2(-1,1)\times L^2(-1,1)\times L^2(-1,1)\times \mathbb{R}$ is denoted by $\|\cdot\|_{\mathcal{Z}}$ given by

$$\|(\psi,\beta)\|_{\mathcal{Z}}^2 = \int_{-1}^1 \psi_1^2\,dt + \int_{-1}^1 \psi_2^2\,dt + \int_{-1}^1 \psi_0^2\,dt + \beta^2. \qquad (6.20)$$

Denote the orthogonal projection of $L^2(-1,1)\times L^2(-1,1)\times L^2(-1,1)\times \mathbb{R}$ onto $\mathcal{Z}$ by Π. Namely

$$\begin{aligned}
\Pi(\psi,\beta) &= (\psi_1,\psi_2,\psi_0,\beta) - \Big(\frac{1}{3}\int_{-1}^1 \psi_1\,dt + \frac{1}{6}\int_{-1}^1 \psi_2\,dt + \frac{1}{6}\int_{-1}^1 \psi_0\,dt\Big)(1,0,1,0)\\
&\quad -\Big(\frac{1}{6}\int_{-1}^1 \psi_1\,dt + \frac{1}{3}\int_{-1}^1 \psi_2\,dt - \frac{1}{6}\int_{-1}^1 \psi_0\,dt\Big)(0,1,-1,0). \qquad (6.21)
\end{aligned}$$

In $\mathcal{X}$ we use the norm

$$\|(\phi,\alpha)\|_{\mathcal{X}}^2 = \|\phi_1\|_{H^2}^2 + \|\phi_2\|_{H^2}^2 + \|\phi_0\|_{H^2}^2 + \alpha^2, \qquad (6.22)$$

where $\|\cdot\|_{H^2}$ is the norm of the Sobolev space $H^2(-1,1)$: $\|f\|_{H^2}^2 = \int_{-1}^1 [(f'')^2 + (f')^2 + f^2]\,dt$. Note that both $\mathcal{X}$ and $\mathcal{Y}$ are also Hilbert spaces under their respective norms.

We now introduce the gradient of $\mathcal{J}$, which is an operator $\mathcal{S}$ from a neighborhood of $(0,0)$ in $\mathcal{X}$ to $\mathcal{Z}$ such that

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{J}((\phi,\alpha)+\epsilon(\psi,\beta)) = \langle \mathcal{S}(\phi,\alpha),(\psi,\beta)\rangle \qquad (6.23)$$

for all $(\psi,\beta)\in\mathcal{X}$. Any local minimizer (ϕ,α) of $\mathcal{J}$ in a restricted class is a solution of the equation $\mathcal{S}(\phi,\alpha) = (0,0)$. In section 8 we will find such a local minimizer by solving the equation. The domain of $\mathcal{S}$ is taken to be

$$\mathcal{D}(\mathcal{S}) = \{(\phi,\alpha)\in\mathcal{X}:\ \|(\phi,\alpha)\|_{\mathcal{X}} \le \bar c\rho^2\} \qquad (6.24)$$

where $\bar c$ in (6.24) is the same as the $\bar c$ in (6.16). Consequently $\mathcal{D}(\mathcal{S}) \subset \mathcal{D}(\mathcal{J})$. This nonlinear operator is written as the sum of two operators,

$$\mathcal{S} = \mathcal{S}_S + \mathcal{S}_L, \qquad (6.25)$$

where $\mathcal{S}_S$ and $\mathcal{S}_L$ correspond to the two parts $\mathcal{J}_S$ and $\mathcal{J}_L$ of $\mathcal{J}$ given in (2.10).

To find $\mathcal{S}_S$ we express the length of each curve in terms of ϕ_i and α:

$$\mathcal{J}_S(\phi,\alpha) = \int_{-1}^1 L_1(\phi_1',\phi_1,\alpha)\,dt + \int_{-1}^1 L_2(\phi_2',\phi_2,\alpha)\,dt, + \int_{-1}^1 L_0(\phi_0',\alpha)\,dt \qquad (6.26)$$

where the three integrals are the length of $\partial\Omega_1\backslash\partial\Omega_2$, $\partial\Omega_2\backslash\partial\Omega_1$, and $\partial\Omega_1\cap\partial\Omega_2$ respectively. The L_i's are given by

$$L_i(\phi_i',\phi_i,\alpha) = \sqrt{\frac{(\phi_i')^2}{2A(\phi_i+\frac{m\rho^2}{4}-\frac{a\rho}{4})} + 2A(\phi_i + \frac{m\rho^2}{4} - \frac{a\rho}{4})},\quad i=1,2,\quad \text{and}\quad L_0(\phi_0',\alpha) = \sqrt{\frac{(\phi_0')^2}{a^2} + a^2}. \qquad (6.27)$$

In these formulas A and a are functions of α. Regard curvature as an operator on ϕ_i and α, and let

$$\kappa_i(\phi_i,\alpha) = -\frac{\partial}{\partial t}\Big(\frac{\partial L_i(\phi_i',\phi_i,\alpha)}{\partial \phi_i'}\Big) + \frac{\partial L_i(\phi_i',\phi_i,\alpha)}{\partial \phi_i}, \quad i=1,2, \quad \text{and} \quad \kappa_0(\phi_0,\alpha) = -\frac{\partial}{\partial t}\Big(\frac{\partial L_0(\phi_0',\alpha)}{\partial \phi_0'}\Big). \tag{6.28}$$

Define another operator from $\mathcal{D}(\mathcal{S})$ to $\mathbb{R}$ by

$$\kappa_s(\phi,\alpha) = \sum_{i=1}^{2} \frac{\partial L_i(\phi_i',\phi_i,\alpha)}{\partial \phi_i'}\Big|_{-1}^{1} + \sum_{i=1}^{2}\int_{-1}^{1} \frac{\partial L_i(\phi_i',\phi_i,\alpha)}{\partial \alpha}\, dt + \int_{-1}^{1} \frac{\partial L_0(\phi_0',\alpha)}{\partial \alpha}\, dt. \tag{6.29}$$

Now set $\mathcal{S}_S$ to be

$$\mathcal{S}_S(\phi,\alpha) = \Pi \begin{pmatrix} \kappa_1(\phi_1,\alpha) \\ \kappa_2(\phi_2,\alpha) \\ \kappa_0(\phi_0,\alpha) \\ \kappa_s(\phi,\alpha) \end{pmatrix}. \tag{6.30}$$

This operator is the gradient of $\mathcal{J}_S$ in the sense that for every $(\phi,\alpha)\in\mathcal{D}(\mathcal{S})$ and $(\psi,\beta)\in\mathcal{X}$

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{J}_S((\phi,\alpha)+\epsilon(\psi,\beta)) = \langle \mathcal{S}_S(\phi,\alpha),(\psi,\beta)\rangle. \tag{6.31}$$

For $\mathcal{S}_L$ we have

$$\mathcal{S}_L(\phi,\alpha) = \Pi \begin{pmatrix} \gamma_{11} I_{\Omega_1} + \gamma_{12} I_{\Omega_2} \\ \gamma_{21} I_{\Omega_1} + \gamma_{22} I_{\Omega_2} \\ (\gamma_{11}-\gamma_{12}) I_{\Omega_1} + (\gamma_{21}-\gamma_{22}) I_{\Omega_2} \\ h(\phi,\alpha) \end{pmatrix}. \tag{6.32}$$

A remark regarding the I_{Ω_i}'s in (6.32) is in order. Recall that each I_{Ω_i}, $i=1,2$, is a function on D given in (2.7), and the set Ω_i is represented by the internal variables ϕ_i, ϕ_0 and α for $i=1,2$. The I_{Ω_i}'s ($i=1,2$) in the first three components on the right side of (6.32) are now considered as the outcomes of the operators

$$I_{ij}: \ (\phi_i,\phi_0,\alpha) \to I_{\Omega_i}(e^{\mathrm{i}\theta}\mathbf{r}_j(t)+\xi), \quad i=1,2, \ j=1,2,0. \tag{6.33}$$

where $j=1,2,0$ corresponds to the first, second, and third component in (6.32) respectively.

The last component h in (6.32) is a scalar valued operator given by

$$h(\phi,\alpha) = \int_{-1}^{1} (\gamma_{11} I_{\Omega_1} + \gamma_{12} I_{\Omega_2}) e_1\, dt + \int_{-1}^{1} (\gamma_{21} I_{\Omega_1} + \gamma_{22} I_{\Omega_2}) e_2\, dt + \int_{-1}^{1} [(\gamma_{11}-\gamma_{12}) I_{\Omega_1} + (\gamma_{21}-\gamma_{22}) I_{\Omega_2}] e_0\, dt. \tag{6.34}$$

Consequently by (4.11) and (6.8),

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{J}_L((\phi,\alpha)+\epsilon(\psi,\beta)) = \langle \mathcal{S}_L(\phi,\alpha),(\psi,\beta)\rangle. \tag{6.35}$$

By (6.30) and (6.32), one obtains the expression of the operator $\mathcal{S}$:

$$\mathcal{S}(\phi,\alpha) = \Pi \begin{pmatrix} \kappa_1(\phi_1,\alpha) + \gamma_{11} I_{\Omega_1} + \gamma_{12} I_{\Omega_2} \\ \kappa_2(\phi_2,\alpha) + \gamma_{21} I_{\Omega_1} + \gamma_{22} I_{\Omega_2} \\ \kappa_0(\phi_0,\alpha) + (\gamma_{11}-\gamma_{12}) I_{\Omega_1} + (\gamma_{21}-\gamma_{22}) I_{\Omega_2} \\ \kappa_s(\phi,\alpha) + h(\phi,\alpha) \end{pmatrix}. \tag{6.36}$$

Lemma 6.2 *It holds uniformly with respect to t that*

$$\mathcal{S}(0,0) = \begin{pmatrix} O(|\gamma|\rho^2) \\ O(|\gamma|\rho^2) \\ O(|\gamma|\rho^2) \\ 0 \end{pmatrix}.$$

Consequently there exists $\widetilde{C}>0$ such that $\|\mathcal{S}(0,0)\|_{\mathcal{Z}} \le \widetilde{C}|\gamma|\rho^2$.

Proof. Calculations show that

$$\kappa_i(0,0) = \frac{1}{\rho} \quad (i=1,2), \quad \kappa_0(0,0) = 0, \quad \kappa_s(0,0) = 0 \tag{6.37}$$

which follow from

$$\frac{\partial L_i(0,0,0)}{\partial \phi_i'} = 0, \quad \frac{\partial L_i(0,0,0)}{\partial \phi_i} = \frac{1}{\rho}, \quad \frac{\partial L_i(0,0,0)}{\partial \alpha} = -\frac{\sqrt{3}}{2\pi\rho}, \quad i=1,2; \quad \frac{\partial L_0(0,0)}{\partial \phi_0'} = 0, \quad \frac{\partial L_0(0,0)}{\partial \alpha} = \frac{\sqrt{3}}{\pi\rho} \tag{6.38}$$

with the help of (6.10). Consequently

$$\mathcal{S}_S(0,0) = \Pi\Big(\frac{1}{\rho}, \frac{1}{\rho}, 0, 0\Big) = (0,0,0,0). \tag{6.39}$$

Because of (6.13),

$$h(0,0) = 0. \tag{6.40}$$

When Ω_i becomes $B_i(\rho,\xi,\theta)$, for every $x \in \overline{B_1 \cup B_2}$

$$I_{B_i(\rho,\xi,\theta)}(e^{\mathrm{i}\theta}x+\xi) = \int_{B_i(\rho,\xi,\theta)} \frac{1}{2\pi} \log \frac{1}{|x-y|}\, dy + \int_{B_i(\rho,\xi,\theta)} R(e^{\mathrm{i}\theta}x+\xi, e^{\mathrm{i}\theta}y+\xi)\, dy = \frac{m\rho^2}{4\pi} \log\frac{1}{\rho} + O(\rho^2) \tag{6.41}$$

holds uniformly with respect to such x. Note that here B_1 and B_2 are considered sets under the (ξ,θ)-frame so x and y are in this frame, but the arguments of R are still in the original coordinate system of $\mathbb{R}^2$; hence the composition in $R(e^{\mathrm{i}\theta}x+\xi, e^{\mathrm{i}\theta}y+\xi)$. Therefore, uniformly with respect to t,

$$\mathcal{S}_L(0,0) = \Pi \begin{pmatrix} (\gamma_{11}+\gamma_{12})\frac{m\rho^2}{4\pi}\log\frac{1}{\rho} + O(|\gamma|\rho^2) \\ (\gamma_{21}+\gamma_{22})\frac{m\rho^2}{4\pi}\log\frac{1}{\rho} + O(|\gamma|\rho^2) \\ (\gamma_{11}-\gamma_{22})\frac{m\rho^2}{4\pi}\log\frac{1}{\rho} + O(|\gamma|\rho^2) \\ 0 \end{pmatrix} = \begin{pmatrix} O(|\gamma|\rho^2) \\ O(|\gamma|\rho^2) \\ O(|\gamma|\rho^2) \\ 0 \end{pmatrix}. \tag{6.42}$$

The lemma follows from (6.39) and (6.42). □

7 Positivity

In this section we study the linear operator $\mathcal{S}'(0,0):\ \mathcal{X} \to \mathcal{Z}$, and show that $\mathcal{S}'(0,0)$ is positive definite and invertible when $|\gamma|\rho^3$ is sufficiently small. A few simple estimates regarding functions in $H_0^1(-1,1) = \{f \in H^1(-1,1):\ f(\pm 1) = 0\}$ are needed.

Lemma 7.1 *1. For all $f \in H_0^1(-1,1)$, $\int_{-1}^1 (f')^2\,dt \ge (\frac{\pi}{2})^2 \int_{-1}^1 f^2\,dt$.*

2. For all $f \in H_0^1(-1,1)$, $\int_{-1}^1 (f')^2\,dt \ge \frac{3}{2}(\int_{-1}^1 f\,dt)^2$.

3. Let $\mu > 0$. Then for all $f \in H_0^1(-1,1)$, $\int_{-1}^1 (f')^2\,dt + \mu(\int_{-1}^1 f\,dt)^2 \ge S_\mu \int_{-1}^1 f^2\,dt$, where S_μ is the smallest positive solution of $\frac{\tan\sqrt{S}}{\sqrt{S}} = 1 - \frac{S}{2\mu}$ if $\mu < \frac{\pi^2}{2}$, and $S_\mu = \pi^2$ if $\mu \ge \frac{\pi^2}{2}$.

Proof. For part 1 we minimize $\int_{-1}^1 |f'|^2\,dt$ among $f \in H_0^1(-1,1)$ subject to the constraint $\int_{-1}^1 f^2\,dt = 1$. The minimizer exists by the standard argument and is a solution of the eigenvalue problem $-f'' = Sf$ where S is the principal eigenvalue. The solution is $S = (\frac{\pi}{2})^2$ and, up to normalization, $f(t) = \cos\frac{\pi}{2}t$. This S is also the best constant in the desired inequality, achieved by $\cos\frac{\pi}{2}t$.

To prove part 2, we minimize $\int_{-1}^1 (f')^2\,dt$ among all $f \in H_0^1(-1,1)$ subject to the constraint $\int_{-1}^1 f\,dt = 1$. The minimizer exits and is a solution of $-f'' = S$ for some $S \in \mathbb{R}$. Then the solution is $f(t) = \frac{S}{2}(1-t^2)$ and consequently $1 = \int_{-1}^1 \frac{S}{2}(1-t^2) = \frac{2S}{3}$. Hence $S = \frac{3}{2}$, $f(t) = \frac{3}{4}(1-t^2)$, and $\int_{-1}^1 |f'|^2\,dt = \frac{3}{2}$.

For part 3 we minimize $\int_{-1}^1 (f')^2\,dt + \mu(\int_{-1}^1 f\,dt)^2$ among $f \in H_0^1(-1,1)$ under the constraint $\int_{-1}^1 f^2\,dt = 1$. The minimizer exists and satisfies the integro-differential equation $-f'' + \mu\int_{-1}^1 f\,dt = Sf$. In this eigenvalue problem S is the principal eigenvalue. For f to be a non-trivial solution, S is necessarily positive. This can be seen by multiplying the equation by f and integrating on $(-1,1)$. Let $h = \mu\int_{-1}^1 f\,dt$. Then $f(t) = c_1\cos\sqrt{S}t + c_2\sin\sqrt{S}t + \frac{h}{S}$. Therefore $h = \mu\int_{-1}^1 f\,dt = \mu(\frac{2\sin\sqrt{S}}{\sqrt{S}}c_1 + \frac{2}{S}h)$, which is coupled to the boundary conditions $c_1\cos\sqrt{S} \pm c_2\sin\sqrt{S} + \frac{h}{S} = 0$. They form a system of three linear homogeneous equations for c_1, c_2 and h. Its determinant must be 0 for a non-trivial solution f to exist, i.e.

$$\det\begin{bmatrix} \frac{2\sin\sqrt{S}}{\sqrt{S}} & 0 & \frac{2}{S} - \frac{1}{\mu} \\ \cos\sqrt{S} & \sin\sqrt{S} & \frac{1}{S} \\ \cos\sqrt{S} & -\sin\sqrt{S} & \frac{1}{S} \end{bmatrix} = 0.$$

There are two possibilities: $\sin\sqrt{S} = 0$ and $\sin\sqrt{S} \neq 0$. In the first case $S = (n\pi)^2$, $n = 1, 2, 3, \ldots$ In the second case S must be a positive solution of the equation $\frac{\tan\sqrt{S}}{\sqrt{S}} = 1 - \frac{S}{2\mu}$. Since only the principal eigenvalue is considered, we just compare the smallest possible S from the first case, which is π^2, with the smallest positive solution of the equation from the second case. As μ increases from 0 to ∞, the smallest positive solution of the second case increases from $(\frac{\pi}{2})^2$ to x_*^2 where $x_* \approx 4.4934$ is the smallest positive solution of $\tan x = x$. Note that $x_* > \pi$. So which of π^2 and the smallest solution of the second case is smaller depends on μ. Because $S = \pi^2$ is the smallest positive solution of $\frac{\tan\sqrt{S}}{\sqrt{S}} = 1 - \frac{S}{2\mu}$ when $\mu = \frac{\pi^2}{2}$, S_μ is the smallest positive solution of $\frac{\tan\sqrt{S}}{\sqrt{S}} = 1 - \frac{S}{2\mu}$ if $\mu < \frac{\pi^2}{2}$, and $S_\mu = \pi^2$ if $\mu \geq \frac{\pi^2}{2}$. □

The derivative of the first part of $\mathcal{S}$ is studied in the next lemma. The lemma is also valuable for the equal area, two component isoperimetric problem. See the last section for a discussion on this point.

Lemma 7.2 *There exists a universal constant $d > 0$ such that*

$$\langle \mathcal{S}_S'(0,0)(\psi,\beta), (\psi,\beta)\rangle \geq \frac{2d}{\rho^3}\|(\psi,\beta)\|_{\mathcal{Y}}^2$$

for all $(\psi,\beta) \in \mathcal{X}$.

Proof. Define some constants:

$$l^{11} = \frac{\partial^2 L_i(0,0,0)}{\partial(\phi_i')^2},\ l^{00} = \frac{\partial^2 L_i(0,0,0)}{\partial\phi_i^2},\ l^{ss} = \frac{\partial^2 L_i(0,0,0)}{\partial\alpha^2},\quad i = 1,2,$$

$$l^{10} = \frac{\partial^2 L_i(0,0,0)}{\partial\phi_i'\partial\phi_i},\ l^{1s} = \frac{\partial^2 L_i(0,0,0)}{\partial\phi_i'\partial\alpha},\ l^{0s} = \frac{\partial^2 L_i(0,0,0)}{\partial\phi_i\partial\alpha},\quad i = 1,2,$$

$$l_0^{11} = \frac{\partial^2 L_0(0,0)}{\partial(\phi_0')^2},\ l_0^{ss} = \frac{\partial^2 L_0(0,0)}{\partial\alpha^2},\ l_0^{1s} = \frac{\partial^2 L_0(0,0)}{\partial\phi_0'\partial\alpha}. \tag{7.1}$$

After some lengthy calculations, we find that

$$l^{11} = \frac{27}{8\pi^3\rho^3},\ l^{00} = -\frac{3}{2\pi\rho^3},\ l^{ss} = \frac{14\pi\sqrt{3} - 9}{8\pi^3\rho^3},\ l^{10} = l^{1s} = l^{0s} = 0,\ l_0^{11} = \frac{8\sqrt{3}}{9\rho^3},\ l_0^{ss} = -\frac{8\pi\sqrt{3} - 9}{4\pi^3\rho^3},\ l_0^{1s} = 0. \tag{7.2}$$

The linearized operators of κ_i at $(0,0)$ are

$$\kappa_i'(0,0):\ (\psi_i,\beta) \to -l^{11}\psi_i'' + l^{00}\psi_i,\quad i = 1,2,\quad \kappa_0'(0,0):\ (\psi_0,\beta) \to -l_0^{11}\psi_0''. \tag{7.3}$$

For $\kappa_s'(0,0)$ calculations show that the linearized operator is

$$\kappa_s'(0,0):\ (\psi,\beta) \to \sum_{i=1}^2 l^{11}\psi_i'|_{-1}^1 + (4l^{ss} + 2l_0^{ss})\beta. \tag{7.4}$$

Hence

$$\mathcal{S}'_S(0,0) : \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_0 \\ \beta \end{pmatrix} \to \Pi \begin{pmatrix} -l^{11}\psi_1'' + l^{00}\psi_1 \\ -l^{11}\psi_2'' + l^{00}\psi_2 \\ -l_0^{11}\psi_0'' \\ l^{11}(\psi_1' + \psi_2')|_{-1}^{1} + (4l^{ss} + 2l_0^{ss})\beta \end{pmatrix}. \tag{7.5}$$

Define a quadratic form

$$\begin{aligned}\mathcal{B}(\psi,\beta) &= \langle \mathcal{S}'_S(0,0)(\psi,\beta),(\psi,\beta)\rangle - 2d\rho^{-3}\|(\psi,\beta)\|^2_{\mathcal{Y}} \\ &= (l^{11} - 2d\rho^{-3})\int_{-1}^{1}((\psi_1')^2 + (\psi_2')^2)\,dt + (l^{00} - 2d\rho^{-3})\int_{-1}^{1}(\psi_1^2 + \psi_2^2)\,dt \\ &\quad + (l_0^{11} - 2d\rho^{-3})\int_{-1}^{1}(\psi_0')^2\,dt - 2d\rho^{-3}\int_{-1}^{1}\psi_0^2\,dt + (4l^{ss} + 2l_0^{ss} - 2d\rho^{-3})\beta^2 \end{aligned} \tag{7.6}$$

where d is a small positive number, independent of ρ, to be specified later. By Lemma 7.1 part 1, we have

$$\begin{aligned}\mathcal{B}(\psi,\beta) &\geq (l^{11} - 2d\rho^{-3})\int_{-1}^{1}((\psi_1')^2 + (\psi_2')^2)\,dt + (l^{00} - 2d\rho^{-3})\int_{-1}^{1}(\psi_1^2 + \psi_2^2)\,dt \\ &\quad + \Big(l_0^{11} - 2d\rho^{-3} - 2d\Big(\frac{2}{\pi}\Big)^2\rho^{-3}\Big)\int_{-1}^{1}(\psi_0')^2\,dt + (4l^{ss} + 2l_0^{ss} - 2d\rho^{-3})\beta^2. \end{aligned} \tag{7.7}$$

At this point we impose our first condition on d: it must be small enough so that

$$l^{11} - 2d\rho^{-3} > 0 \;\text{ and }\; l_0^{11} - 2d\rho^{-3} - 2d\Big(\frac{2}{\pi}\Big)^2\rho^{-3} > 0. \tag{7.8}$$

Now we hold ψ_1, ψ_2 and β fixed and minimize the right side of (7.7) with respect to ψ_0 subject to the constraints $\int_{-1}^1 \psi_1\,dt + \int_{-1}^1 \psi_0\,dt = 0$ and $\int_{-1}^1 \psi_2\,dt - \int_{-1}^1 \psi_0\,dt = 0$. In other words we minimize the right side of (7.7) among $\psi_0 \in H_0^1(-1,1)$ with the fixed value of $\int_{-1}^1 \psi_0\,dt$. Then part 2 of Lemma 7.1 implies that

$$\begin{aligned}\mathcal{B}(\psi,\beta) &\geq (l^{11} - 2d\rho^{-3})\int_{-1}^{1}((\psi_1')^2 + (\psi_2')^2)\,dt + (l^{00} - 2d\rho^{-3})\int_{-1}^{1}(\psi_1^2 + \psi_2^2)\,dt \\ &\quad + \frac{3}{2}\Big(l_0^{11} - 2d\rho^{-3} - 2d\Big(\frac{2}{\pi}\Big)^2\rho^{-3}\Big)\Big(\int_{-1}^{1}\psi_0\,dt\Big)^2 + (4l^{ss} + 2l_0^{ss} - 2d\rho^{-3})\beta^2. \end{aligned} \tag{7.9}$$

Define two more quadratic forms:

$$\begin{aligned}\mathcal{B}_1(\psi_1,\beta) &= (l^{11} - 2d\rho^{-3})\int_{-1}^{1}(\psi_1')^2\,dt + (l^{00} - 2d\rho^{-3})\int_{-1}^{1}\psi_1^2\,dt \\ &\quad + \frac{3}{4}\Big(l_0^{11} - 2d\rho^{-3} - 2d\Big(\frac{2}{\pi}\Big)^2\rho^{-3}\Big)\Big(\int_{-1}^{1}\psi_1\,dt\Big)^2 + (2l^{ss} + l_0^{ss} - d\rho^{-3})\beta^2 \end{aligned} \tag{7.10}$$

$$\begin{aligned}\mathcal{B}_2(\psi_2,\beta) &= (l^{11} - 2d\rho^{-3})\int_{-1}^{1}(\psi_2')^2\,dt + (l^{00} - 2d\rho^{-3})\int_{-1}^{1}\psi_2^2\,dt \\ &\quad + \frac{3}{4}\Big(l_0^{11} - 2d\rho^{-3} - 2d\Big(\frac{2}{\pi}\Big)^2\rho^{-3}\Big)\Big(\int_{-1}^{1}\psi_2\,dt\Big)^2 + (2l^{ss} + l_0^{ss} - d\rho^{-3})\beta^2. \end{aligned} \tag{7.11}$$

By the constraints on ψ_1, ψ_2 and ψ_0, we deduce, following (7.9), that

$$\mathcal{B}(\psi,\beta) \geq \mathcal{B}_1(\psi_1,\beta) + \mathcal{B}_2(\psi_2,\beta). \tag{7.12}$$

Introduce $g_1 \in H_0^1(-1,1)$ so that $\psi_1(t) = g_1(t) + \beta$. Then

$$\begin{aligned}\mathcal{B}_1(\psi_1,\beta) &= (l^{11} - 2d\rho^{-3})\int_{-1}^{1}(g_1')^2\,dt + (l^{00} - 2d\rho^{-3})\int_{-1}^{1}g_1^2\,dt + \frac{3}{4}\Big(l_0^{11} - 2d\rho^{-3} - 2d\Big(\frac{2}{\pi}\Big)^2\rho^{-3}\Big)\Big(\int_{-1}^{1}g_1\,dt\Big)^2 \\ &\quad + \Big[2(l^{00} - 2d\rho^{-3}) + 3\Big(l_0^{11} - 2d\rho^{-3} - 2d\Big(\frac{2}{\pi}\Big)^2\rho^{-3}\Big)\Big]\beta\int_{-1}^{1}g_1\,dt \\ &\quad + \Big[2(l^{00} - 2d\rho^{-3}) + 3\Big(l_0^{11} - 2d\rho^{-3} - 2d\Big(\frac{2}{\pi}\Big)^2\rho^{-3}\Big) + 2l^{ss} + l_0^{ss} - d\rho^{-3}\Big]\beta^2. \end{aligned} \tag{7.13}$$

To apply Lemma 7.1 part 3, we will choose a proper $\mu > 0$ so that

$$S_\mu = \frac{-l^{00} + 2d\rho^{-3}}{l^{11} - 2d\rho^{-3}}. \tag{7.14}$$

Note that if d were 0 then S_μ would be $\frac{-l^{00}}{l^{11}} = \frac{4\pi^2}{9}$. Since $\frac{4\pi^2}{9} \in ((\frac{\pi}{2})^2, \pi^2)$, we can make d small so that

$$S_\mu \in \Big(\Big(\frac{\pi}{2}\Big)^2, \pi^2\Big). \tag{7.15}$$

According to Lemma 7.1 part 3, S_μ has to be the smallest positive solution of $\frac{\tan(\sqrt{S})}{\sqrt{S}} = 1 - \frac{S}{2\mu}$. Therefore by taking

$$\mu = \frac{S_\mu}{2\Big(1 - \frac{\tan\sqrt{S_\mu}}{\sqrt{S_\mu}}\Big)} \tag{7.16}$$

we achieve (7.14). With this choice of μ the first three terms in (7.13) give

$$\begin{aligned}(l^{11} - 2d\rho^{-3}) \int_{-1}^{1} (g_1')^2\, dt + (l^{00} - 2d\rho^{-3}) \int_{-1}^{1} g_1^2\, dt + \frac{3}{4}\Big(l_0^{11} - 2d\rho^{-3} - 2d\Big(\frac{2}{\pi}\Big)^2 \rho^{-3}\Big)(\int_{-1}^{1} g_1\, dt)^2 \\ \geq \Big[\frac{3}{4}\Big(l_0^{11} - 2d\rho^{-3} - 2d\Big(\frac{2}{\pi}\Big)^2 \rho^{-3}\Big) - \mu(l^{11} - 2d\rho^{-3})\Big](\int_{-1}^{1} g_1\, dt)^2\end{aligned} \tag{7.17}$$

and consequently (7.13) becomes

$$\begin{aligned}\mathcal{B}_1(\psi_1, \beta) \quad \geq \quad & \Big[\frac{3}{4}\Big(l_0^{11} - 2d\rho^{-3} - 2d\Big(\frac{2}{\pi}\Big)^2 \rho^{-3}\Big) - \mu(l^{11} - 2d\rho^{-3})\Big](\int_{-1}^{1} g_1\, dt)^2 \\ & + \Big[2(l^{00} - 2d\rho^{-3}) + 3\Big(l_0^{11} - 2d\rho^{-3} - 2d\Big(\frac{2}{\pi}\Big)^2 \rho^{-3}\Big)\Big]\beta \int_{-1}^{1} g_1\, dt \\ & + \Big[2(l^{00} - 2d\rho^{-3}) + 3\Big(l_0^{11} - 2d\rho^{-3} - 2d\Big(\frac{2}{\pi}\Big)^2 \rho^{-3}\Big) + 2l^{ss} + l_0^{ss} - d\rho^{-3}\Big]\beta^2.\end{aligned} \tag{7.18}$$

To check the sign of the coefficient of $(\int_{-1}^{1} g_1\, dt)^2$ on the right side of (7.18), replace d by 0 in S_μ of (7.14) so that S_μ becomes $\frac{-l^{00}}{l^{11}} = \frac{4\pi^2}{9}$, μ of (7.16) becomes

$$\frac{\frac{4\pi^2}{9}}{2\Big(1 - \frac{\tan\frac{2\pi}{3}}{\frac{2\pi}{3}}\Big)}, \tag{7.19}$$

and the coefficient of $(\int_{-1}^{1} g_1\, dt)^2$ becomes

$$\frac{3}{4} l_0^{11} - \frac{\frac{4\pi^2}{9}}{2\Big(1 - \frac{\tan\frac{2\pi}{3}}{\frac{2\pi}{3}}\Big)} l^{11} = (1.0240...)\rho^{-3}. \tag{7.20}$$

We add another condition on d: it must be sufficiently small so that the coefficient of $(\int_{-1}^{1} g_1\, dt)^2$ stays positive, i.e.

$$\frac{3}{4}\Big(l_0^{11} - 2d\rho^{-3} - 2d\Big(\frac{2}{\pi}\Big)^2 \rho^{-3}\Big) - \mu(l^{11} - 2d\rho^{-3}) > 0. \tag{7.21}$$

Completing the square of the right side of (7.18) we obtain

$$\begin{aligned}\mathcal{B}_1(\psi_1, \beta) \quad \geq \quad & \Big[- \frac{\Big(2(l^{00} - 2d\rho^{-3}) + 3\Big(l_0^{11} - 2d\rho^{-3} - 2d\Big(\frac{2}{\pi}\Big)^2 \rho^{-3}\Big)\Big)^2}{4\Big(\frac{3}{4}\Big(l_0^{11} - 2d\rho^{-3} - 2d\Big(\frac{2}{\pi}\Big)^2 \rho^{-3}\Big) - \mu(l^{11} - 2d\rho^{-3})\Big)} \\ & + 2(l^{00} - 2d\rho^{-3}) + 3\Big(l_0^{11} - 2d\rho^{-3} - 2d\Big(\frac{2}{\pi}\Big)^2 \rho^{-3}\Big) + 2l^{ss} + l_0^{ss} - d\rho^{-3}\Big]\beta^2\end{aligned} \tag{7.22}$$

To check the sign of the quantity in the brackets, we again set d to be 0 and replace μ by (7.19). Then this quantity becomes

$$-\frac{(2l^{00}+3l_0^{11})^2}{4\Big(\frac{3}{4}l_0^{11}-\frac{\frac{4\pi^2}{9}}{2\Big(1-\frac{\tan\frac{2\pi}{3}}{\frac{2\pi}{3}}\Big)}l^{11}\Big)}+2l^{00}+3l_0^{11}+2l^{ss}+l_0^{ss}=(0.6499...)\rho^{-3}. \tag{7.23}$$

Therefore we choose d so small that the quantity in the brackets of (7.22) is positive, in addition to the requirements (7.8), (7.15) and (7.21). This shows that when d is sufficiently small $\mathcal{B}_1(\psi_1,\beta)\geq 0$ for all $\psi_1\in H^1(-1,1)$ and $\beta\in\mathbb{R}$ such that $\psi_1(\pm 1)=\beta$.

Similarly $\mathcal{B}_2(\psi_2,\beta)\geq 0$. Then by (7.12) we obtain that $\mathcal{B}(\psi,\beta)\geq 0$ from which the lemma follows. □

Lemma 7.3 *There exists $\check{C}>0$ depending on D only such that*

$$\|\mathcal{S}_L'(0,0)(\psi,\beta)\|_{\mathcal{Z}}\leq \check{C}|\gamma|\|(\psi,\beta)\|_{\mathcal{Z}}$$

for all $(\psi,\beta)\in\mathcal{X}$.

Proof. Recall that for $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_0$ parametrizing the boundaries of the perturbed double bubble Ω as in (5.1) with $(\phi,\alpha)\in\mathcal{X}$ being its internal variables, the terms I_{Ω_1} and I_{Ω_2} in the first, second, and third components of (6.32) (corresponding to $j=1,2,0$) are the outcomes of the operators I_{ij} given in (6.33).

To compute the Fréchet derivatives of I_{ij}, deform (ϕ,α) to $(\phi,\alpha)+\epsilon(\psi,\beta)$ and denote the corresponding deformation of $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_0$ by $\mathbf{r}_1^\epsilon$, $\mathbf{r}_2^\epsilon$ and $\mathbf{r}_0^\epsilon$ respectively. Then for $i=1,2$ and $j=1,2,0$,

$$I_{ij}'(\phi,\alpha):\ (\psi,\beta)\to\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}\int_{\Omega_i^\epsilon}G(e^{\mathrm{i}\theta}\mathbf{r}_j(t)+\xi,e^{\mathrm{i}\theta}y+\xi)\,dy+\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}\int_{\Omega_i}G(e^{\mathrm{i}\theta}\mathbf{r}_j^\epsilon(t)+\xi,e^{\mathrm{i}\theta}y+\xi)\,dy. \tag{7.24}$$

Apply Lemma 4.2 to the first term on the left side of (7.24) with $\Omega=B$ whose boundaries are parametrized by

$$\mathbf{r}_1(t)=\rho e^{\mathrm{i}(\pi-\frac{2\pi t}{3})}+O_1,\quad \mathbf{r}_2(t)=\rho e^{\mathrm{i}\frac{2\pi t}{3}}+O_2,\quad \mathbf{r}_0(t)=\Big(0,\frac{\sqrt{3}}{2}\rho\,t\Big) \tag{7.25}$$

to obtain

$$\begin{aligned}&\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}\int_{\Omega_i^\epsilon}G(e^{\mathrm{i}\theta}\mathbf{r}_j(t)+\xi,e^{\mathrm{i}\theta}y+\xi)\,dy\\ &=\begin{cases}-\int_{\partial B_1\setminus\partial B_2}G(e^{\mathrm{i}\theta}\mathbf{r}_j(t)+\xi,e^{\mathrm{i}\theta}\mathbf{r}_1(\tau)+\xi)\mathbf{N}_1\cdot\mathbf{X}\,ds(\tau)-\int_{\partial B_1\cap\partial B_2}G(e^{\mathrm{i}\theta}\mathbf{r}_j(t)+\xi,e^{\mathrm{i}\theta}\mathbf{r}_0(\tau)+\xi)\mathbf{N}_0\cdot\mathbf{X}\,ds(\tau)\\ -\int_{\partial B_2\setminus\partial B_1}G(e^{\mathrm{i}\theta}\mathbf{r}_j(t)+\xi,e^{\mathrm{i}\theta}\mathbf{r}_2(\tau)+\xi)\mathbf{N}_2\cdot\mathbf{X}\,ds(\tau)+\int_{\partial B_1\cap\partial B_2}G(e^{\mathrm{i}\theta}\mathbf{r}_j(t)+\xi,e^{\mathrm{i}\theta}\mathbf{r}_0(\tau)+\xi)\mathbf{N}_0\cdot\mathbf{X}\,ds(\tau)\end{cases}\end{aligned} \tag{7.26}$$

where the first line holds if $i=1$ and the second holds if $i=2$. We calculated earlier that $-\mathbf{N}_l\cdot\mathbf{X}\,ds(\tau)=(\psi_l+e_l(\phi_l,\alpha)\beta)\,d\tau$. Since $e_l(0,0)=0$ at $(\phi,\alpha)=(0,0)$ by (6.13),

$$\begin{aligned}&\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}\int_{\Omega_i^\epsilon}G(e^{\mathrm{i}\theta}\mathbf{r}_j(t)+\xi,e^{\mathrm{i}\theta}y+\xi)\,dy\\ &=\begin{cases}\int_{\partial B_1\setminus\partial B_2}G(e^{\mathrm{i}\theta}\mathbf{r}_j(t)+\xi,e^{\mathrm{i}\theta}\mathbf{r}_1(\tau)+\xi)\psi_1(\tau)\,d\tau+\int_{\partial B_1\cap\partial B_2}G(e^{\mathrm{i}\theta}\mathbf{r}_j(t)+\xi,e^{\mathrm{i}\theta}\mathbf{r}_0(\tau)+\xi)\psi_0(\tau)\,d\tau\\ \int_{\partial B_2\setminus\partial B_1}G(e^{\mathrm{i}\theta}\mathbf{r}_j(t)+\xi,e^{\mathrm{i}\theta}\mathbf{r}_2(\tau)+\xi)\psi_2(\tau)\,d\tau-\int_{\partial B_1\cap\partial B_2}G(e^{\mathrm{i}\theta}\mathbf{r}_j(t)+\xi,e^{\mathrm{i}\theta}\mathbf{r}_0(\tau)+\xi)\psi_0(\tau)\,d\tau\end{cases}\end{aligned} \tag{7.27}$$

To estimate the right side of (7.27) we write G as the sum of the fundamental solution and the regular part, and treat the two parts separately. First

$$\begin{aligned}\int_{-1}^1\frac{1}{2\pi}\log\frac{1}{|\mathbf{r}_j(t)-\mathbf{r}_l(\tau)|}\psi_l(\tau)\,d\tau &= \frac{1}{2\pi}\log\frac{1}{\rho}\int_{-1}^1\psi_l(\tau)\,d\tau+\int_{-1}^1\frac{1}{2\pi}\log\frac{1}{|\rho^{-1}\mathbf{r}_j(t)-\rho^{-1}\mathbf{r}_l(\tau)|}\psi_l(\tau)\,d\tau\\ &= \frac{1}{2\pi}\log\frac{1}{\rho}\int_{-1}^1\psi_l(\tau)\,d\tau+O(1)\|\psi_l\|_{L^2}\end{aligned} \tag{7.28}$$

holds uniformly with respect to t. Next for the regular part it suffices to note that

$$\int_{-1}^{1} R(e^{\mathrm{i}\theta}\mathbf{r}_j(t)+\xi, e^{\mathrm{i}\theta}\mathbf{r}_l(\tau)+\xi)\,\psi_l(\tau)d\tau = O(1)\|\psi_l\|_{L^2} \tag{7.29}$$

uniformly with respect to t. By (6.17), (7.28) and (7.29) we deduce that

$$\begin{aligned}
&\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}\int_{\Omega_i^\epsilon} G(e^{\mathrm{i}\theta}\mathbf{r}_j(t)+\xi, e^{\mathrm{i}\theta}y+\xi)\,dy \\
&= \begin{cases} \dfrac{1}{2\pi}\log\dfrac{1}{\rho}\displaystyle\int_{-1}^{1}\psi_1(\tau)\,d\tau + \dfrac{1}{2\pi}\log\dfrac{1}{\rho}\displaystyle\int_{-1}^{1}\psi_0(\tau)\,d\tau + O(1)(\|\psi_1\|_{L^2}+\|\psi_0\|_{L^2}), & i=1 \\ \dfrac{1}{2\pi}\log\dfrac{1}{\rho}\displaystyle\int_{-1}^{1}\psi_2(\tau)\,d\tau - \dfrac{1}{2\pi}\log\dfrac{1}{\rho}\displaystyle\int_{-1}^{1}\psi_0(\tau)\,d\tau + O(1)(\|\psi_2\|_{L^2}+\|\psi_0\|_{L^2}), & i=2 \end{cases} \qquad (7.30)\\
&= O(1)(\|\psi_1\|_{L^2}+\|\psi_2\|_{L^2}+\|\psi_0\|_{L^2}) \qquad (7.31)
\end{aligned}$$

holds uniformly with respect to t.

The second part on the right side of (7.24), for $(\phi,\alpha)=(0,0)$, is written as

$$\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}\int_{B_i} G(\epsilon^{\mathrm{i}\theta}\mathbf{r}_j^\epsilon(t)+\xi, e^{\mathrm{i}\theta}y+\xi)\,dy = \int_{B_i}\nabla G(e^{\mathrm{i}\theta}\mathbf{r}_j(t)+\xi, e^{\mathrm{i}\theta}y+\xi)\cdot e^{\mathrm{i}\theta}\mathbf{X}_j(t)\,dy \tag{7.32}$$

where ∇G stands for the gradient of G with respect to its first argument, and $\mathbf{X}_j(t) = \frac{\partial \mathbf{r}_j^\epsilon}{\partial\epsilon}\Big|_{\epsilon=0}$. Clearly

$$\int_{B_i} |\nabla G(e^{\mathrm{i}\theta}\mathbf{r}_j(t)+\xi, e^{\mathrm{i}\theta}y+\xi)|\,dy = O(\rho) \tag{7.33}$$

holds uniformly with respect to t. Calculations show that

$$\mathbf{X}_j(t) = \begin{cases} \dfrac{3\psi_1}{2\pi\rho}e^{\mathrm{i}(\pi-\frac{2\pi}{3}t)} + \Big(-\dfrac{\sqrt{3}t\beta}{2\pi\rho}\Big)e^{\mathrm{i}(\pi-\frac{2\pi}{3}t)}(-\mathrm{i}), & j=1 \\ \dfrac{3\psi_2}{2\pi\rho}e^{\mathrm{i}\frac{2\pi}{3}t} + \Big(-\dfrac{\sqrt{3}t\beta}{2\pi\rho}\Big)e^{\mathrm{i}\frac{2\pi}{3}t}\mathrm{i}, & j=2 \\ \Big(\dfrac{2\psi_0}{\sqrt{3}\rho}-\dfrac{2\beta}{\pi\rho}, \dfrac{\sqrt{3}t\beta}{\pi\rho}\Big), & j=0 \end{cases}. \tag{7.34}$$

Hence

$$\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}\int_{B_i} G(e^{\mathrm{i}\theta}\mathbf{r}_j^\epsilon(t)+\xi, e^{\mathrm{i}\theta}y+\xi)\,dy = O(1)(\|\psi_j\|_{L^2}+|\beta|) \tag{7.35}$$

holds uniformly with respect to t.

By (7.31) and (7.35) we find that

$$\|(I'_{ij}(0,0))(\psi,\beta)\|_{L^2} = O(1)\|(\psi,\beta)\|_{\mathcal{Z}} \tag{7.36}$$

for the I_{Ω_i} terms in the first three components of (6.32).

Finally consider $h(\phi,\alpha)$, the last component of $\mathcal{S}_L(\phi,\alpha)$. Since $e_i(0,0)=0$ for $i=1,2,0$,

$$\begin{aligned} h'(0,0)(\psi,\beta) \;=\; & \int_{-1}^{1}(\gamma_{11}I_{B_1}+\gamma_{12}I_{B_2})e_1'(0,0)(\psi_1,\beta)\,dt + \int_{-1}^{1}(\gamma_{21}I_{B_1}+\gamma_{22}I_{B_2})e_2'(0,0)(\psi_2,\beta)\,dt \\ & + \int_{-1}^{1}((\gamma_{11}-\gamma_{12})I_{B_1}+(\gamma_{21}-\gamma_{22})I_{B_2})e_0'(0,0)(\psi_0,\beta)\,dt. \end{aligned} \tag{7.37}$$

Note that (6.12) implies that

$$\int_{-1}^{1} e_i'(0,0)(\psi,\beta) = 0. \tag{7.38}$$

By (6.9) we can write

$$e_i'(0,0)(\psi,\beta) = p_i(\psi_i + t\psi_i') + q_i\beta = \frac{\partial(p_i t\psi_i + q_i t\beta)}{\partial t} \tag{7.39}$$

where p_i and q_i are constants that depend on ρ only. Then (7.38) implies that

$$p_i t\psi_i + q_i t\beta\Big|_{-1}^{1} = 0. \tag{7.40}$$

Calculations from (6.9) show that

$$p_i = O(\rho^{-2}) \text{ and } q_i = O(\rho^{-2}). \tag{7.41}$$

Consider a general term in (7.37):

$$\begin{aligned}\int_{-1}^{1} I_{B_i} e_j'(0,0)(\psi_j,\beta)\,dt &= \int_{-1}^{1} I_{B_i}(e^{\mathrm{i}\theta}\mathbf{r}_j(t)+\xi)\frac{\partial(p_j t\psi_j + q_j t\beta)}{\partial t}\,dt \\ &= I_{B_i}(e^{\mathrm{i}\theta}\mathbf{r}_j(t)+\xi)(p_1 t\psi_j + q_j t\beta)\Big|_{-1}^{1} - \int_{-1}^{1}\frac{\partial I_{B_i}(\mathbf{r}_j(t))}{\partial t}(p_j t\psi_j + q_j t\beta)\,dt. \end{aligned} \tag{7.42}$$

The two terms on the last line are estimated below.

Since

$$I_{B_i}(e^{\mathrm{i}\theta}\mathbf{r}_j(t)+\xi) = \frac{m\rho^2}{4\pi}\log\frac{1}{\rho} + O(\rho^2) \tag{7.43}$$

holds uniformly with respect to t, by (7.40) and (7.41)

$$I_{B_i}(e^{\mathrm{i}\theta}\mathbf{r}_j(t)+\xi)(p_j t\psi + q_j t\beta)\Big|_{-1}^{1} = O(1)|\beta|. \tag{7.44}$$

For the second term note that by (7.25)

$$\frac{\partial I_{B_i}(e^{\mathrm{i}\theta}\mathbf{r}_j(t)+\xi)}{\partial t} = \int_{B_i}\nabla G(e^{\mathrm{i}\theta}\mathbf{r}_j(t)+\xi,y)\cdot e^{\mathrm{i}\theta}\mathbf{r}_j'(t)\,dy = O(\rho^2) \tag{7.45}$$

holds uniformly with respect to t. Then

$$\int_{-1}^{1}\frac{\partial I_{B_i}(e^{\mathrm{i}\theta}\mathbf{r}_j(t)+\xi)}{\partial t}(p_j t\psi_1 + q_j t\beta)\,dt = O(1)(\|\psi_j\|_{L^2} + |\beta|). \tag{7.46}$$

By (7.42), (7.44) and (7.46) we deduce that

$$\int_{-1}^{1} I_{B_i} e_j'(0,0)(\psi_j,\beta)\,dt = O(1)(\|\psi_j\|_{L^2} + |\beta|). \tag{7.47}$$

Hence one estimates (7.37) and deduces that

$$h'(0,0)(\psi,\beta) = O(1)\ |\gamma|\ \|(\psi,\beta)\|_{\mathcal{Z}}. \tag{7.48}$$

Because of (7.36) and (7.48), there exists $\check{C} > 0$ such that

$$\|\mathcal{S}_L'(0,0)(\psi,\beta)\|_{\mathcal{Z}} \le \check{C}|\gamma|\ \|(\psi,\beta)\|_{\mathcal{Z}} \tag{7.49}$$

for all $(\psi,\beta)\in\mathcal{X}$. □

While $\mathcal{S}_S'(0,0)$ is an unbounded self-adjoint operator on $\mathcal{Z}$ with a dense domain $\mathcal{X}\subset\mathcal{Z}$, Lemma 7.3 shows that $\mathcal{S}_L'(0,0)$ may be extended to a bounded self-adjoint operator on $\mathcal{Z}$.

Lemma 7.4 *There exist $d > 0$ and $\sigma > 0$ such that when $|\gamma|\rho^3 < \sigma$,*

$$\langle\mathcal{S}'(0,0)(\psi,\beta),(\psi,\beta)\rangle \ge \frac{d}{\rho^3}\|(\psi,\beta)\|_{\mathcal{Y}}^2$$

for all $(\psi,\beta)\in\mathcal{X}$.

Proof. Let d be the positive number given in Lemma 7.2 and $\sigma = \frac{d}{\check{C}}$ where $\check{C}$ comes from Lemma 7.3. Then Lemma 7.3 shows that for $|\gamma|\rho^3 < \sigma$,

$$\|\mathcal{S}'_L(0,0)(\psi,\beta)\|_{\mathcal{Z}} \le \check{C}|\gamma|\|(\psi,\beta)\|_{\mathcal{Z}} \le \frac{\check{C}\sigma}{\rho^3}\|(\psi,\beta)\|_{\mathcal{Z}} = \frac{d}{\rho^3}\|(\psi,\beta)\|_{\mathcal{Z}} \tag{7.50}$$

for all $(\psi,\beta) \in \mathcal{X}$. By Lemma 7.2 and (7.50)

$$\begin{aligned} \langle \mathcal{S}'(0,0)(\psi,\beta),(\psi,\beta)\rangle &= \langle \mathcal{S}'_S(0,0)(\psi,\beta),(\psi,\beta)\rangle + \langle \mathcal{S}'_L(0,0)(\psi,\beta),(\psi,\beta)\rangle \\ &\ge \frac{2d}{\rho^3}\|(\psi,\beta)\|^2_{\mathcal{Y}} - \frac{d}{\rho^3}\|(\psi,\beta)\|^2_{\mathcal{Z}} \ \ge \ \frac{d}{\rho^3}\|(\psi,\beta)\|^2_{\mathcal{Y}} \end{aligned}$$

for all $(\psi,\beta) \in \mathcal{X}$. □

A consequence of the positivity of $\mathcal{S}'(0,0)$ is its invertibility.

Lemma 7.5 1. *There exists $\tilde{d} > 0$ such that when $|\gamma|\rho^3 < \sigma$ where σ is given in Lemma 7.4, $\|\mathcal{S}'(0,0)(\psi,\beta)\|_{\mathcal{Z}} \ge \frac{\tilde{d}}{\rho^3}\|(\psi,\beta)\|_{\mathcal{X}}$ holds for all $(\psi,\beta) \in \mathcal{X}$.*

2. *The linear map $\mathcal{S}'(0,0)$ is one-to-one and onto from $\mathcal{X}$ to $\mathcal{Z}$, and $\|(\mathcal{S}'(0,0))^{-1}\| \le \frac{\rho^3}{\tilde{d}}$ where $\|(\mathcal{S}'(0,0))^{-1}\|$ is the operator norm of $(\mathcal{S}'(0,0))^{-1}$.*

Proof. By Lemma 7.4 it is easy to see that if $|\gamma|\rho^3 < \sigma$, then for all $(\psi,\beta) \in \mathcal{X}$

$$\|(\psi,\beta)\|_{\mathcal{Z}} \le \frac{\rho^3}{d}\|\mathcal{S}'(0,0)(\psi,\beta)\|_{\mathcal{Z}}. \tag{7.51}$$

The first part of Lemma 7.5 asserts that the $\mathcal{Z}$-norm of (ψ,β) on the left side of (7.51) can be strengthened to the stronger $\mathcal{X}$-norm, if d is replaced by a possibly smaller $\tilde{d}$.

If part 1 is false, then there exist γ_n, ρ_n, and $(\psi_n,\beta_n) \in \mathcal{X}$ such that $|\gamma_n|\rho_n^3 < \sigma$, $\|(\psi_n,\beta_n)\|_{\mathcal{X}} = 1$ and with $\rho = \rho_n$ and $\gamma = \gamma_n$ in $\mathcal{S}$,

$$\|\rho_n^3\mathcal{S}'(0,0)(\psi_n,\beta_n)\|_{\mathcal{Z}} \to 0, \quad \text{as } n \to \infty. \tag{7.52}$$

By (7.51),

$$\|(\psi_n,\beta_n)\|_{\mathcal{Z}} \to 0. \tag{7.53}$$

Moreover due to the compactness of the embedding $H^2(-1,1) \to C^1[-1,1]$ and $\|(\psi_n,\beta_n)\|_{\mathcal{X}} = 1$, $\|\psi_{n.i}\|_{C^1} \to 0$ and in particular

$$\psi'_{n,i}(\pm 1) \to 0, \ i = 1,2,0, \text{ as } n \to \infty. \tag{7.54}$$

Since $\mathcal{S}'(0,0) = \mathcal{S}'_S(0,0) + \mathcal{S}'_L(0,0)$, and (7.50) and (7.53) imply that

$$\|\rho_n^3\mathcal{S}'_L(0,0)(\psi_n,\beta_n)\|_{\mathcal{Z}} \to 0, \tag{7.55}$$

we derive from (7.52) and (7.55) that

$$\|\rho_n^3\mathcal{S}'_S(0,0)(\psi_n,\beta_n)\|_{\mathcal{Z}} \to 0. \tag{7.56}$$

By (7.5) write

$$\rho_n^3\mathcal{S}'_S(0,0)(\psi_n,\beta_n) = \Pi\begin{pmatrix} -\rho_n^3 l^{11}\psi''_{n,1} \\ -\rho_n^3 l^{11}\psi''_{n,2} \\ -\rho_n^3 l_0^{11}\psi''_{n,0} \\ 0 \end{pmatrix} + \Pi\begin{pmatrix} \rho_n^3 l^{00}\psi_{n,1} \\ \rho_n^3 l^{00}\psi_{n,2} \\ 0 \\ \rho_n^3 l^{11}(\psi'_{n,1}+\psi'_{n,2})|_{-1}^1 + \rho_n^3(4l^{ss}+2l_0^{ss})\beta_n \end{pmatrix}. \tag{7.57}$$

Here $\rho_n^3 l^{11}$, $\rho_n^3 l^{00}$, $\rho_n^3 l^{ss}$, $\rho_n^3 l_0^{11}$, and $\rho_n^3 l_0^{ss}$ are all constants independent of ρ_n. By (7.53) and (7.54) we find that

$$\left\|\Pi\begin{pmatrix} \rho_n^3 l^{00}\psi_{n,1} \\ \rho_n^3 l^{00}\psi_{n,2} \\ 0 \\ \rho_n^3 l^{11}(\psi'_{n,1}+\psi'_{n,2})|_{-1}^1 + \rho_n^3(4l^{ss}+2l_0^{ss})\beta_n \end{pmatrix}\right\|_{\mathcal{Z}} \to 0. \tag{7.58}$$

Then (7.56), (7.57) and (7.58) give that

$$\left\| \Pi \begin{pmatrix} -\rho_n^3 l^{11} \psi''_{n,1} \\ -\rho_n^3 l^{11} \psi''_{n,2} \\ -\rho_n^3 l_0^{11} \psi''_{n,0} \\ 0 \end{pmatrix} \right\|_{\mathcal{Z}} \to 0. \tag{7.59}$$

By the definition of Π, (6.21),

$$\Pi \begin{pmatrix} -\rho_n^3 l^{11} \psi''_{n,1} \\ -\rho_n^3 l^{11} \psi''_{n,2} \\ -\rho_n^3 l_0^{11} \psi''_{n,0} \\ 0 \end{pmatrix} = \begin{pmatrix} -\rho_n^3 l^{11} \psi''_{n,1} \\ -\rho_n^3 l^{11} \psi''_{n,2} \\ -\rho_n^3 l_0^{11} \psi''_{n,0} \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{\rho_n^3 l^{11}}{3} \psi'_{n,1} \Big|_{-1}^{1} + \frac{\rho_n^3 l^{11}}{6} \psi'_{n,2} \Big|_{-1}^{1} + \frac{\rho_n^3 l_0^{11}}{6} \psi'_{n,0} \Big|_{-1}^{1} \\ \frac{\rho_n^3 l^{11}}{6} \psi'_{n,1} \Big|_{-1}^{1} + \frac{\rho_n^3 l^{11}}{3} \psi'_{n,2} \Big|_{-1}^{1} - \frac{\rho_n^3 l_0^{11}}{6} \psi'_{n,0} \Big|_{-1}^{1} \\ \frac{\rho_n^3 l^{11}}{6} \psi'_{n,1} \Big|_{-1}^{1} - \frac{\rho_n^3 l^{11}}{6} \psi'_{n,2} \Big|_{-1}^{1} + \frac{\rho_n^3 l_0^{11}}{3} \psi'_{n,0} \Big|_{-1}^{1} \\ 0 \end{pmatrix}. \tag{7.60}$$

Moreover (7.54) implies that

$$\begin{pmatrix} \frac{\rho_n^3 l^{11}}{3} \psi'_{n,1} \Big|_{-1}^{1} + \frac{\rho_n^3 l^{11}}{6} \psi'_{n,2} \Big|_{-1}^{1} + \frac{\rho_n^3 l_0^{11}}{6} \psi'_{n,0} \Big|_{-1}^{1} \\ \frac{\rho_n^3 l^{11}}{6} \psi'_{n,1} \Big|_{-1}^{1} + \frac{\rho_n^3 l^{11}}{3} \psi'_{n,2} \Big|_{-1}^{1} - \frac{\rho_n^3 l_0^{11}}{6} \psi'_{n,0} \Big|_{-1}^{1} \\ \frac{\rho_n^3 l^{11}}{6} \psi'_{n,1} \Big|_{-1}^{1} - \frac{\rho_n^3 l^{11}}{6} \psi'_{n,2} \Big|_{-1}^{1} + \frac{\rho_n^3 l_0^{11}}{3} \psi'_{n,0} \Big|_{-1}^{1} \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^4. \tag{7.61}$$

Therefore by (7.59), (7.60) and (7.61),

$$\|\psi''_{n,i}\|_{L^2} \to 0, \quad i = 1, 2, 0, \quad \text{as } n \to \infty. \tag{7.62}$$

From (7.53) and (7.62) we deduce that $\|(\psi_n, \beta_n)\|_{\mathcal{X}} \to 0$, a contradiction to our assumption at the beginning that $\|(\psi_n, \beta_n)\|_{\mathcal{X}} = 1$.

For part 2, it suffices to show that $\mathcal{S}'(0,0)$ is onto. First note that by the standard theory of second order linear differential equations, $\mathcal{S}'(0,0)$ is an unbounded self-adjoint operator on $\mathcal{Z}$ with the domain $\mathcal{X} \subset \mathcal{Z}$. Second if $(\tilde{\psi}, \tilde{\beta}) \in \mathcal{Z}$ is perpendicular to the range of $\mathcal{S}'(0,0)$, i.e. $\langle \mathcal{S}'(0,0)(\psi,\beta), (\tilde{\psi}, \tilde{\beta}) \rangle = 0$ for all $(\psi, \beta) \in \mathcal{X}$, then the self-adjointness of $\mathcal{S}'(0,0)$ implies that $(\tilde{\psi}, \tilde{\beta}) \in \mathcal{X}$ and $\mathcal{S}'(0,0)(\tilde{\psi}, \tilde{\beta}) = 0$. By (7.51), $(\tilde{\psi}, \tilde{\beta})$ is zero. Hence the range of $\mathcal{S}'(0,0)$ is dense in $\mathcal{Z}$. Finally (7.51) implies that the range of $\mathcal{S}'(0,0)$ is a closed subset of $\mathcal{Z}$. Therefore $\mathcal{S}'(0,0)$ is onto. □

8 Restricted minimizers

Before solving the equation

$$\mathcal{S}(\phi, \alpha) = (0, 0), \tag{8.1}$$

we need an estimate on the second derivative of $\mathcal{S}$. The proof of the next lemma, which is skipped, is straight forward estimation, similar to the proof of [30, Lemma 3.2].

Lemma 8.1 *There exists $\widehat{C} > 0$ such that for all $(\phi, \alpha) \in \mathcal{D}(\mathcal{S})$,*

$$\|\mathcal{S}''(\phi,\alpha)((\psi,\beta),(\tilde{\psi},\tilde{\beta}))\|_{\mathcal{Z}} \le \widehat{C}(\rho^{-5} + |\gamma|\rho^{-2}) \|(\psi,\beta)\|_{\mathcal{X}} \|(\tilde{\psi},\tilde{\beta})\|_{\mathcal{X}}$$

holds for all (ψ, β), $(\tilde{\psi}, \tilde{\beta}) \in \mathcal{X}$.

The equation (8.1) is solved for (ϕ, α) near $(0, 0)$ in each restricted class of perturbed double bubbles associated with a (ξ, θ)-frame.

Lemma 8.2 *There exists $\sigma > 0$ such that (8.1) admits a solution (ϕ^*, α^*) satisfying $\|(\phi^*, \alpha^*)\|_{\mathcal{X}} \le \frac{2\widetilde{C}|\gamma|\rho^5}{\tilde{d}}$, provided $|\gamma|\rho^3 < \sigma$.*

Proof. For $(\phi, \alpha) \in \mathcal{D}(\mathcal{S})$ write

$$\mathcal{S}(\phi, \alpha) = \mathcal{S}(0,0) + \mathcal{S}'(0,0)(\phi, \alpha) + \mathcal{R}(\phi, \alpha) \tag{8.2}$$

where $\mathcal{R}(\phi, \alpha)$ is a higher order term defined by (8.2). Define an operator $\mathcal{T}$ from $\mathcal{D}(\mathcal{S}) \subset \mathcal{X}$ into $\mathcal{X}$ by

$$\mathcal{T}(\phi, \alpha) = -(\mathcal{S}'(0,0))^{-1}(\mathcal{S}(0,0) + \mathcal{R}(\phi, \alpha)), \tag{8.3}$$

and re-write the equation $\mathcal{S}(\phi, \alpha) = 0$ as a fixed point problem $\mathcal{T}(\phi, \alpha) = (\phi, \alpha)$.

Let $c \in (0, \bar{c}]$, where $\bar{c}$ is given in (6.24), and define a closed ball $\mathcal{W} = \{(\phi, \alpha) \in \mathcal{X} : \ \|(\phi, \alpha)\|_{\mathcal{X}} \le c\rho^2\} \subset \mathcal{D}(\mathcal{S})$. For $(\phi, \alpha) \in \mathcal{W}$,

$$\|\mathcal{R}(\phi, \alpha)\|_{\mathcal{Z}} \le \frac{1}{2} \sup_{\tau \in (0,1)} \|\mathcal{S}''(\tau(\phi, \alpha))((\phi, \alpha), (\phi, \alpha))\|_{\mathcal{Z}} \le \frac{\widehat{C}(\rho^{-5} + |\gamma|\rho^{-2})}{2} \|(\phi, \alpha)\|_{\mathcal{X}}^2 \tag{8.4}$$

by Lemma 8.1. Then by Lemmas 6.2 and 7.5

$$\begin{aligned} \|\mathcal{T}(\phi, \alpha)\|_{\mathcal{X}} &\le \|(\mathcal{S}'(0,0))^{-1}\| (\|\mathcal{S}(0,0)\|_{\mathcal{Z}} + \|\mathcal{R}(\phi, \alpha)\|_{\mathcal{Z}}) \\ &\le \frac{\rho^3}{\tilde{d}} \Big(\widetilde{C}|\gamma|\rho^2 + \frac{\widehat{C}(\rho^{-5} + |\gamma|\rho^{-2})}{2} (c\rho^2)^2 \Big) \\ &\le \frac{\widetilde{C}\sigma}{\tilde{d}} \rho^2 + \frac{\widehat{C} + \widehat{C}\sigma}{2\tilde{d}} c^2 \rho^2. \end{aligned} \tag{8.5}$$

Let $(\psi, \beta) \in \mathcal{W}$. Consider

$$\begin{aligned} \|\mathcal{T}(\phi, \alpha) - \mathcal{T}(\psi, \beta)\|_{\mathcal{X}} &\le \|(\mathcal{S}'(0,0))^{-1}\| \ \|\mathcal{R}(\phi, \alpha) - \mathcal{R}(\psi, \beta)\|_{\mathcal{Z}} \\ &\le \frac{\rho^3}{\tilde{d}} \|\mathcal{S}(\phi, \alpha) - \mathcal{S}(\psi, \beta) - \mathcal{S}'(0,0)((\phi, \alpha) - (\psi, \beta))\|_{\mathcal{Z}} \\ &\le \frac{\rho^3}{\tilde{d}} \|\mathcal{S}(\phi, \alpha) - \mathcal{S}(\psi, \beta) - \mathcal{S}'(\psi, \beta)((\phi, \alpha) - (\psi, \beta))\|_{\mathcal{Z}} \\ &\quad + \frac{\rho^3}{\tilde{d}} \|(\mathcal{S}'(\psi, \beta) - \mathcal{S}'(0,0))((\phi, \alpha) - (\psi, \beta))\|_{\mathcal{Z}} \\ &\le \frac{\rho^3}{2\tilde{d}} \sup_{\tau \in (0,1)} \|\mathcal{S}''((\psi, \beta) + \tau((\phi, \alpha) - (\psi, \beta)))\| \ \|(\phi, \alpha) - (\psi, \beta)\|_{\mathcal{X}}^2 \\ &\quad + \frac{\rho^3}{\tilde{d}} \sup_{\tau \in (0,1)} \|\mathcal{S}''(\tau(\psi, \beta))\| \ \|(\psi, \beta)\|_{\mathcal{X}} \|(\phi, \alpha) - (\psi, \beta)\|_{\mathcal{X}} \\ &\le \frac{\rho^3 \widehat{C}(\rho^{-5} + |\gamma|\rho^{-2})}{\tilde{d}} \Big(c\rho^2 + c\rho^2 \Big) \|(\phi, \alpha) - (\psi, \beta)\|_{\mathcal{X}} \\ &\le \frac{2\widehat{C}(1 + \sigma)c}{\tilde{d}} \|(\phi, \alpha) - (\psi, \beta)\|_{\mathcal{X}}. \end{aligned} \tag{8.6}$$

Take

$$c = \min\Big\{ \frac{\tilde{d}}{6\widehat{C}}, \ \bar{c} \Big\}. \tag{8.7}$$

Let σ be small enough so that Lemma 7.5 holds, and moreover

$$\sigma \le \min\Big\{ 1, \ \frac{\tilde{d}c}{2\widetilde{C}} \Big\}. \tag{8.8}$$

It follows from (8.5) and (8.6) that

$$\|\mathcal{T}(\phi, \alpha)\|_{\mathcal{X}} \le c\rho^2 \ \text{ and } \ \|\mathcal{T}(\phi, \alpha) - \mathcal{T}(\psi, \beta)\|_{\mathcal{X}} \le \frac{2}{3} \|(\phi, \alpha) - (\psi, \beta)\|_{\mathcal{X}} \tag{8.9}$$

for all $(\phi, \alpha),\ (\psi, \beta) \in \mathcal{W}$. The contraction mapping theorem shows that $\mathcal{T}$ has a fixed point in $\mathcal{W}$. This fixed point is denoted by (ϕ^*, α^*).

To prove the estimate of (ϕ^*, α^*), revisit the equation $(\phi, \alpha) = \mathcal{T}(\phi, \alpha)$, satisfied by (ϕ^*, α^*), and derive from (8.4) that

$$\begin{aligned} \|(\phi^*, \alpha^*)\|_{\mathcal{X}} &\leq \|(\mathcal{S}'(0,0))^{-1}\|(\|\mathcal{S}(0,0)\|_{\mathcal{Z}} + \|\mathcal{R}(\phi^*, \alpha^*)\|_{\mathcal{Z}}) \\ &\leq \frac{\rho^3}{\tilde{d}}\Big(\widetilde{C}|\gamma|\rho^2 + \frac{\widehat{C}(\rho^{-5} + |\gamma|\rho^{-2})}{2}\|(\phi^*, \alpha^*)\|_{\mathcal{X}}^2\Big). \end{aligned}$$

Rewrite the above as

$$\Big(1 - \frac{\widehat{C}(\rho^{-2} + |\gamma|\rho)}{2\tilde{d}}\|(\phi^*, \alpha^*)\|_{\mathcal{X}}\Big)\|(\phi^*, \alpha^*)\|_{\mathcal{X}} \leq \frac{\widetilde{C}|\gamma|\rho^5}{\tilde{d}}. \tag{8.10}$$

In (8.10) estimate

$$\frac{\widehat{C}(\rho^{-2} + |\gamma|\rho)}{2\tilde{d}}\|(\phi^*, \alpha^*)\|_{\mathcal{X}} \leq \frac{\widehat{C}(\rho^{-2} + |\gamma|\rho)}{2\tilde{d}}(c\rho^2) \leq \frac{\widehat{C}c(1+\sigma)}{2\tilde{d}} \leq \frac{1}{6} \tag{8.11}$$

by (8.7) and (8.8). The estimate of (ϕ^*, α^*) follows from (8.10). □

Three of the four equations (2.3)-(2.6) are satisfied by (ϕ^*, α^*).

Lemma 8.3 *The solution (ϕ^*, α^*) from Lemma 8.2 solves the following equations:*

$$\begin{aligned} \kappa_1 + \gamma_{11} I_{\Omega_1} + \gamma_{12} I_{\Omega_2} &= \lambda_1 \quad \text{on } \partial\Omega_1 \backslash \partial\Omega_2 \\ \kappa_2 + \gamma_{21} I_{\Omega_1} + \gamma_{22} I_{\Omega_2} &= \lambda_2 \quad \text{on } \partial\Omega_2 \backslash \partial\Omega_1 \\ \kappa_0 + (\gamma_{11} - \gamma_{12}) I_{\Omega_1} + (\gamma_{21} - \gamma_{22}) I_{\Omega_2} &= \lambda_1 - \lambda_2 \quad \text{on } \partial\Omega_1 \cap \partial\Omega_2 \\ (\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_0)(t) \cdot (0, t)\Big|_{-1}^{1} &= 0. \end{aligned}$$

Proof. The first three equations of the lemma clearly follow from (6.21), (6.36) and the first three equations of $\mathcal{S}(\phi, \alpha) = (0, 0)$. The fourth equation of $\mathcal{S}(\phi, \alpha) = (0, 0)$ states that

$$\kappa_s(\phi, \alpha) + h(\phi, \alpha) = 0. \tag{8.12}$$

The terms in (6.29) that make up κ_s are simplified as follows. For $i = 0$,

$$\begin{aligned} \int_{-1}^{1} \frac{\partial L_0(\phi_0', \alpha)}{\partial \alpha}\, dt &= \int_{-1}^{1} \Big[\frac{\frac{\phi_0'}{a}}{\sqrt{\frac{(\phi_0')^2}{a^2} + a^2}}\Big(-\frac{\phi_0'}{a^3 A}\Big) + \frac{a}{\sqrt{\frac{(\phi_0')^2}{a^2} + a^2}}\Big(\frac{1}{aA}\Big)\Big]\, dt \\ &= \Big[\frac{\frac{\phi_0'}{a}}{\sqrt{\frac{(\phi_0')^2}{a^2} + a^2}}\Big(-\frac{\phi_0}{a^3 A}\Big) + \frac{a}{\sqrt{\frac{(\phi_0')^2}{a^2} + a^2}}\Big(\frac{t}{aA}\Big)\Big]_{-1}^{1} \\ &\quad - \int_{-1}^{1} \Big[\frac{\partial}{\partial t}\Big(\frac{\frac{\phi_0'}{a}}{\sqrt{\frac{(\phi_0')^2}{a^2} + a^2}}\Big)\Big(-\frac{\phi_0}{a^3 A}\Big) + \frac{\partial}{\partial t}\Big(\frac{a}{\sqrt{\frac{(\phi_0')^2}{a^2} + a^2}}\Big)\Big(\frac{t}{aA}\Big)\Big]\, dt \\ &= \mathbf{T}_0 \cdot \Big(0, \frac{t}{aA}\Big)\Big|_{-1}^{1} + \int_{-1}^{1} \kappa_0(\phi_0, \alpha) e_0(\phi_0, \alpha)\, dt; \end{aligned} \tag{8.13}$$

similarly for $i = 1, 2$,

$$\frac{\partial L_i(\phi_i', \phi_i, \alpha)}{\partial \phi_i'}\Big|_{-1}^{1} + \int_{-1}^{1} \frac{\partial L_i(\phi_i', \phi_i, \alpha)}{\partial \alpha}\, dt = \mathbf{T}_i \cdot \Big(0, \frac{t}{aA}\Big)\Big|_{-1}^{1} + \int_{-1}^{1} \kappa_i(\phi_i, \alpha) e_i(\phi_i, \alpha)\, dt. \tag{8.14}$$

They turn (8.12) to

$$\int_{-1}^{1} \Big\{(\kappa_1 + \gamma_{11} I_{\Omega_1} + \gamma_{12} I_{\Omega_2}) e_1 + (\kappa_2 + \gamma_{21} I_{\Omega_1} + \gamma_{22} I_{\Omega_2}) e_2 + [\kappa_0 + (\gamma_{11} - \gamma_{12}) I_{\Omega_1} + (\gamma_{21} - \gamma_{22}) I_{\Omega_2}] e_0\Big\}\, dt$$

$$+\mathbf{T}_1 \cdot \Big(0, \frac{t}{aA}\Big)\Big|_{-1}^{1} + \mathbf{T}_2 \cdot \Big(0, \frac{t}{aA}\Big)\Big|_{-1}^{1} + \mathbf{T}_0 \cdot \Big(0, \frac{t}{aA}\Big)\Big|_{-1}^{1} = 0. \tag{8.15}$$

The first three equations in the lemma show that the integral term in (8.15) vanishes. Hence the last equation of the lemma holds. □

The first three equations satisfied by (ϕ^*, α^*) in the lemma are just the equations (2.3)-(2.5). However the fourth equation in the lemma does not imply the 120 degree angle condition (2.6). For most $(\xi, \theta) \in \overline{D_{\tilde{\delta}}} \times S^1$ at which the reference frame is set and the restricted class of perturbed double bubbles specified, the corresponding fixed point (ϕ^*, α^*) from Lemma 8.2 does not satisfy (2.6). In the next section we will find a particular (ξ, θ) whose corresponding (ϕ^*, α^*) does satisfy (2.6).

The first part of the next lemma reveals that the solution found in Lemma 8.2 is a local minimizer of $\mathcal{J}$ in the restricted class under the $\mathcal{Y}$-norm. The second part of the lemma will be used later when we study the dependence of (ϕ^*, α^*) on (ξ, θ).

Lemma 8.4 *1. There exist $\hat{d} > 0$ and $\sigma > 0$ such that if $|\gamma|\rho^3 < \sigma$, then the solution (ϕ^*, α^*) found in Lemma 8.2 satisfies $\langle \mathcal{S}'(\phi^*, \alpha^*)(\psi, \beta), (\psi, \beta)\rangle \geq \frac{\hat{d}}{\rho^3}\|(\psi, \beta)\|_{\mathcal{Y}}^2$ for all $(\psi, \beta) \in \mathcal{X}$.*

2. There exist $\check{d} > 0$ and $\sigma > 0$ such that if $|\gamma|\rho^3 < \sigma$, then the solution (ϕ^*, α^*) satisfies $\|\mathcal{S}'(\phi^*, \alpha^*)(\psi, \beta)\|_{\mathcal{Z}} \geq \frac{\check{d}}{\rho^3}\|(\psi, \beta)\|_{\mathcal{X}}$ for all $(\psi, \beta) \in \mathcal{X}$.

Proof. There exists $\tilde{\tau} \in (0, 1)$ such that

$$\langle \mathcal{S}'(\phi^*, \alpha^*)(\psi, \beta), (\psi, \beta)\rangle = \langle \mathcal{S}'(0, 0)(\psi, \beta), (\psi, \beta)\rangle + \langle \mathcal{S}''(\tilde{\tau}(\phi^*, \alpha^*))((\phi^*, \alpha^*), (\psi, \beta)), (\psi, \beta)\rangle.$$

Similar to Lemma 8.1, one can show that for all $(\phi, \alpha) \in \mathcal{D}(\mathcal{S})$,

$$|\langle \mathcal{S}''(\phi, \alpha)((\phi^*, \alpha^*), (\psi, \beta)), (\psi, \beta)\rangle| \leq \widehat{C}(\rho^{-5} + |\gamma|\rho^{-2})\|(\phi^*, \alpha^*)\|_{\mathcal{X}}\|(\psi, \beta)\|_{\mathcal{Y}}^2. \tag{8.16}$$

See [30, Lemma 4.1] for the proof of a similar formula. Consequently by Lemmas 7.4 and 8.2

$$\begin{aligned}\langle \mathcal{S}'(\phi^*, \alpha^*)(\psi, \beta), (\psi, \beta)\rangle &\geq \frac{d}{\rho^3}\|(\psi, \beta)\|_{\mathcal{Y}}^2 - \widehat{C}(\rho^{-5} + |\gamma|\rho^{-2})\frac{2\widetilde{C}|\gamma|\rho^5}{\tilde{d}}\|(\psi, \beta)\|_{\mathcal{Y}}^2 \\ &\geq \frac{1}{\rho^3}\Big(d - \frac{2\widehat{C}\widetilde{C}(\sigma + \sigma^2)}{\tilde{d}}\Big)\|(\psi, \beta)\|_{\mathcal{Y}}^2 \;\geq\; \frac{d}{2\rho^3}\|(\psi, \beta)\|_{\mathcal{Y}}^2\end{aligned}$$

if σ is sufficiently small. The first part follows if $\hat{d} = \frac{d}{2}$.

By Lemmas 7.5, 8.1 and 8.2,

$$\begin{aligned}\|\mathcal{S}'(\phi^*, \alpha^*)(\psi, \beta)\|_{\mathcal{Z}} &\geq \|\mathcal{S}'(0, 0)(\psi, \beta)\|_{\mathcal{Z}} - \sup_{\tau \in (0,1)} \|\mathcal{S}''(\tau(\phi^*, \alpha^*))((\phi^*, \alpha^*), (\psi, \beta))\|_{\mathcal{Z}} \\ &\geq \frac{\tilde{d}}{\rho^3}\|(\psi, \beta)\|_{\mathcal{X}} - \widehat{C}(\rho^{-5} + |\gamma|\rho^{-2})\|(\phi^*, \alpha^*)\|_{\mathcal{X}}\|(\psi, \beta)\|_{\mathcal{X}} \\ &\geq \Big(\frac{\tilde{d}}{\rho^3} - \widehat{C}(\rho^{-5} + |\gamma|\rho^{-2})\frac{2\widetilde{C}|\gamma|\rho^5}{\tilde{d}}\Big)\|(\psi, \beta)\|_{\mathcal{Z}} \\ &\geq \frac{1}{\rho^3}\Big(\tilde{d} - \frac{2\widehat{C}\widetilde{C}(\sigma + \sigma^2)}{\tilde{d}}\Big)\|(\psi, \beta)\|_{\mathcal{Z}} \;\geq\; \frac{\tilde{d}}{2\rho^3}\|(\psi, \beta)\|_{\mathcal{Z}}\end{aligned}$$

if σ is sufficiently small. Part 2 follows if $\check{d} = \frac{\tilde{d}}{2}$. □

An estimate on the difference between the energy of (ϕ^*, α^*) and the energy of the exact double bubble $B(\rho, \xi, \theta)$ is given below.

Lemma 8.5 *If σ is small, then $|\mathcal{J}(\phi^*, \alpha^*) - \mathcal{J}(0, 0)| \leq |\gamma|\rho^4\Big(\frac{\widetilde{C}^2}{\tilde{d}}|\gamma|\rho^3 + \frac{10\widehat{C}\widetilde{C}^3}{3\tilde{d}^3}(|\gamma|\rho^3)^2 + \frac{10\widehat{C}\widetilde{C}^3}{3\tilde{d}^3}(|\gamma|\rho^3)^3\Big)$.*

Proof. Expanding $\mathcal{J}(\phi^*,\alpha^*)$ yields

$$\mathcal{J}(\phi^*,\alpha^*) = \mathcal{J}(0,0) + \langle \mathcal{S}(0,0),(\phi^*,\alpha^*)\rangle + \frac{1}{2}\langle \mathcal{S}'(0,0)(\phi^*,\alpha^*),(\phi^*,\alpha^*)\rangle + \frac{1}{6}\langle \mathcal{S}''(\tilde{\tau}(\phi^*,\alpha^*))((\phi^*,\alpha^*),(\phi^*,\alpha^*)),(\phi^*,\alpha^*)\rangle \tag{8.17}$$

for some $\tilde{\tau} \in (0,1)$. Also expanding $\mathcal{S}(\phi^*,\alpha^*)$ gives

$$\|\mathcal{S}(\phi^*,\alpha^*) - \mathcal{S}(0,0) - \mathcal{S}'(0,0)(\phi^*,\alpha^*)\|_{\mathcal{Z}} \le \sup_{\tau\in(0,1)} \frac{1}{2}\|\mathcal{S}''(\tau(\phi^*,\alpha^*))((\phi^*,\alpha^*),(\phi^*,\alpha^*))\|_{\mathcal{Z}}. \tag{8.18}$$

Since $\mathcal{S}(\phi^*,\alpha^*)=0$, the above shows that

$$\|\mathcal{S}(0,0) + \mathcal{S}'(0,0)(\phi^*,\alpha^*)\|_{\mathcal{Z}} \le \sup_{\tau\in(0,1)} \frac{1}{2}\|\mathcal{S}''(\tau(\phi^*,\alpha^*))((\phi^*,\alpha^*),(\phi^*,\alpha^*))\|_{\mathcal{Z}},$$

which implies that

$$|\langle \mathcal{S}(0,0),(\phi^*,\alpha^*)\rangle + \langle \mathcal{S}'(0,0)(\phi^*,\alpha^*),(\phi^*,\alpha^*)\rangle| \le \Big(\frac{1}{2}\sup_{\tau\in(0,1)} \|\mathcal{S}''(\tau(\phi^*,\alpha^*))((\phi^*,\alpha^*),(\phi^*,\alpha^*))\|_{\mathcal{Z}}\Big)\|(\phi^*,\alpha^*)\|_{\mathcal{X}}. \tag{8.19}$$

It follows from (8.17) and (8.19) that

$$\Big|\mathcal{J}(\phi^*,\alpha^*) - \mathcal{J}(0,0) - \frac{1}{2}\langle \mathcal{S}(0,0),(\phi^*,\alpha^*)\rangle\Big| \le \Big(\frac{5}{12}\sup_{\tau\in(0,1)} \|\mathcal{S}''(\tau(\phi^*,\alpha^*))((\phi^*,\alpha^*),(\phi^*,\alpha^*))\|_{\mathcal{Z}}\Big)\|(\phi^*,\alpha^*)\|_{\mathcal{X}}. \tag{8.20}$$

Lemmas 6.2, 8.1 and 8.2 show that

$$\begin{aligned} |\mathcal{J}(\phi^*,\alpha^*) - \mathcal{J}(0,0)| &\le \frac{1}{2}|\langle \mathcal{S}(0,0),(\phi^*,\alpha^*)\rangle| + \Big(\frac{5}{12}\sup_{\tau\in(0,1)} \|\mathcal{S}''(\tau(\phi^*,\alpha^*))((\phi^*,\alpha^*),(\phi^*,\alpha^*))\|_{\mathcal{Z}}\Big)\|(\phi^*,\alpha^*)\|_{\mathcal{X}} \\ &\le \frac{1}{2}(\widetilde{C}|\gamma|\rho^2)\frac{2\widetilde{C}|\gamma|\rho^5}{\tilde{d}} + \frac{5}{12}\widehat{C}(\rho^{-5}+|\gamma|\rho^{-2})\Big(\frac{2\widetilde{C}|\gamma|\rho^5}{\tilde{d}}\Big)^3 \\ &= |\gamma|\rho^4\Big(\frac{\widetilde{C}^2}{\tilde{d}}|\gamma|\rho^3 + \frac{10\widehat{C}\widetilde{C}^3}{3\tilde{d}^3}(|\gamma|\rho^3)^2 + \frac{10\widehat{C}\widetilde{C}^3}{3\tilde{d}^3}(|\gamma|\rho^3)^3\Big) \end{aligned} \tag{8.21}$$

which proves the lemma. □

9 Minimum of minimizers

The solution (ϕ^*,α^*) of (8.1) found in Lemma 8.2 depends on ξ and θ. To emphasize this dependence, write $\phi^* = \phi^*(\cdot,\xi,\theta)$ and $\alpha^* = \alpha^*(\xi,\theta)$. The perfect double bubble $B(\rho,\xi,\theta)$ whose internal representation is $(0,0)$ also depends on ξ and θ. Now let ξ vary in $\overline{D_{\bar{\delta}}}$ and θ vary in S^1 and set

$$J(\xi,\theta) = \mathcal{J}(\phi^*(\cdot,\xi,\theta),\alpha^*(\xi,\theta)) \text{ and } \bar{J}(\xi,\theta) = \mathcal{J}(B(\rho,\xi,\theta)). \tag{9.1}$$

Both J and $\bar{J}$ are treated as functions of $(\xi,\theta) \in \overline{D_{\bar{\delta}}}\times S^1$. Note that $\mathcal{J}(B(\rho,\xi,\theta))$ here is the same as $\mathcal{J}(0,0)$ in Lemma 8.5. Before now we did not emphasize the dependence of the perfect double bubble represented by $(0,0)$ on ξ and θ.

Lemma 9.1 *When δ and σ are sufficiently small, the function J defined on $\overline{D_{\bar{\delta}}}\times S^1$ attains a minimum in $D_{\bar{\delta}}\times S^1$, the interior of $\overline{D_{\bar{\delta}}}\times S^1$. Every minimum of J on $\overline{D_{\bar{\delta}}}\times S^1$ must be in $D_{\bar{\delta}}\times S^1$.*

Proof. Let $(\xi,\theta)\in \partial D_{\bar{\delta}}\times S^1$ and $(\tilde{\xi},\tilde{\theta})\in D_{\bar{\delta}}\times S^1$, with $\tilde{\xi}$ being a minimum of $R(z,z)$ in D, i.e. $R(\tilde{\xi},\tilde{\xi}) = \min_{z\in D} R(z,z)$. Recall that by (3.5) every minimum of $R(z,z)$ in D must be in $D_{\bar{\delta}}$. By Lemma 8.5,

$$J(\xi,\theta) - J(\tilde{\xi},\tilde{\theta}) \ge \bar{J}(\xi,\theta) - \bar{J}(\tilde{\xi},\tilde{\theta}) - 2|\gamma|\rho^4\Big(\frac{\widetilde{C}^2\sigma}{\tilde{d}} + \frac{10\widehat{C}\widetilde{C}^3\sigma^2}{3\tilde{d}^3} + \frac{10\widehat{C}\widetilde{C}^3\sigma^3}{3\tilde{d}^3}\Big). \tag{9.2}$$

Lemma 3.1 shows that

$$\bar{J}(\xi,\theta)-\bar{J}(\tilde{\xi},\tilde{\theta}) \geq \frac{(\sum_{i,j=1}^{2}\gamma_{ij})\rho^4 m^2}{8}(R(\xi,\xi)-R(\tilde{\xi},\tilde{\xi}))-3|\gamma|\rho^5 m^2 \max_{x,y\in\overline{D_{\bar{\delta}}}}|\nabla R(x,y)| \tag{9.3}$$

The condition (2.9) gives that

$$\Big(\sum_{i,j=1}^{2}\gamma_{ij}\Big) \geq \frac{b|\gamma|}{4}. \tag{9.4}$$

Then (9.2), (9.3) and (9.4) show that

$$\begin{aligned} & J(\xi,\theta)-J(\tilde{\xi},\tilde{\theta}) \\ \geq\ & \frac{b|\gamma|\rho^4 m^2}{32}(R(\xi,\xi)-R(\tilde{\xi},\tilde{\xi}))-\Big[2|\gamma|\rho^4\Big(\frac{\widetilde{C}^2\sigma}{\tilde{d}}+\frac{10\widehat{C}\widetilde{C}^3\sigma^2}{3\tilde{d}^3}+\frac{10\widehat{C}\widetilde{C}^3\sigma^3}{3\tilde{d}^3}\Big)+3|\gamma|\rho^5 m^2\max_{x,y\in\overline{D_{\bar{\delta}}}}|\nabla R(x,y)|\Big] \\ \geq\ & |\gamma|\rho^4\Big\{\frac{bm^2}{32}(R(\xi,\xi)-R(\tilde{\xi},\tilde{\xi}))-\Big(\frac{2\widetilde{C}^2\sigma}{\tilde{d}}+\frac{20\widehat{C}\widetilde{C}^3\sigma^2}{3\tilde{d}^3}+\frac{20\widehat{C}\widetilde{C}^3\sigma^3}{3\tilde{d}^3}+3\delta m^2\max_{x,y\in\overline{D_{\bar{\delta}}}}|\nabla R(x,y)|\Big)\Big\}. \end{aligned} \tag{9.5}$$

Because of (3.5), if σ and δ are sufficiently small, then

$$J(\xi,\theta)-J(\tilde{\xi},\tilde{\theta})>0 \tag{9.6}$$

for all $(\xi,\theta)\in\partial D_{\bar{\delta}}\times S^1$ and $(\tilde{\xi},\tilde{\theta})\in D_{\bar{\delta}}\times S^1$, with $\tilde{\xi}$ being a minimum of $R(z,z)$. Therefore any minimum of J on $\overline{D_{\bar{\delta}}}\times S^1$ must be in $D_{\bar{\delta}}\times S^1$, the interior of $\overline{D_{\bar{\delta}}}\times S^1$. □

Note that this is the first time after (3.6) that δ is required to be small. It is also the first time that the condition (2.9) is used. Only from this moment on, δ and σ become dependent on b.

The dependence of $(\phi^*,\alpha^*)=(\phi^*(t,\xi,\theta),\alpha^*(\xi,\theta))$ on $\xi=(\xi^1,\xi^2)$, and θ is investigated in the next lemma.

Lemma 9.2 *When σ is sufficiently small, $\|\frac{\partial(\phi^*,\alpha^*)}{\partial\xi^l}\|_{\mathcal{X}}=O(|\gamma|\rho^5)$, $l=1,2$, and $\|\frac{\partial(\phi^*,\alpha^*)}{\partial\theta}\|_{\mathcal{X}}=O(|\gamma|\rho^6)$ uniformly with respect to all $(\xi,\theta)\in\overline{D_{\bar{\delta}}}\times S^1$.*

Proof. The equation (8.1) is now written as

$$\mathcal{S}(\phi,\alpha,\xi,\theta)=0, \tag{9.7}$$

with the operator $\mathcal{S}$ acting as

$$\mathcal{S}:\ (\phi,\alpha)\times(\xi,\theta)\to\mathcal{S}(\phi,\alpha,\xi,\theta). \tag{9.8}$$

Estimate $\frac{D\mathcal{S}(\phi,\alpha,\xi,\theta)}{D\xi^l}$ and $\frac{D\mathcal{S}(\phi,\alpha,\xi,\theta)}{D\theta}$, the Fréchet derivatives of $\mathcal{S}$ with respect to ξ^l and θ respectively. In $\mathcal{S}$, only the parts involving I_{Ω_i} depend on ξ and θ. And in

$$I_{\Omega_i}=\int_{\Omega_i}G(e^{\mathrm{i}\theta}\mathbf{r}_j(t)+\xi,e^{\mathrm{i}\theta}y+\xi)\,dy=\int_{\Omega_i}\frac{1}{2\pi}\log\frac{1}{|\mathbf{r}_j(t)-y|}dy+\int_{\Omega_i}R(e^{\mathrm{i}\theta}\mathbf{r}_j(t)+\xi,e^{\mathrm{i}\theta}y+\xi)\,dy$$

ξ and θ appears in the regular part R of G. Then clearly

$$\frac{\partial I_{\Omega_i}}{\partial\xi^l}=O(\rho^2)\ \text{ and }\ \frac{\partial I_{\Omega_i}}{\partial\theta}=O(\rho^3) \tag{9.9}$$

hold uniformly with respect to t, ξ, and θ. Consequently

$$\Big\|\frac{D\mathcal{S}(\phi,\alpha,\xi,\theta)}{D\xi^l}\Big\|=O(|\gamma|\rho^2)\ \text{ and }\ \Big\|\frac{D\mathcal{S}(\phi,\alpha,\xi,\theta)}{D\theta}\Big\|=O(|\gamma|\rho^3). \tag{9.10}$$

Here the Fréchet derivatives are operators from $\mathbb{R}$ to $\mathcal{Z}$ and the above are estimates on the norms of these operators. On the other hand Lemma 8.4 part 2 shows that at $(\phi^*(\cdot,\xi,\theta),\alpha^*(\xi,\theta))$, the solution found in Lemma 8.2,

$$\Big\|\Big(\frac{D\mathcal{S}(\phi^*,\alpha^*,\xi,\theta)}{D(\phi,\alpha)}\Big)^{-1}\Big\|\leq\frac{\rho^3}{\check{d}} \tag{9.11}$$

if σ is small. Note that $\frac{D\mathcal{S}(\phi^*,\alpha^*,\xi,\theta)}{D(\phi,\alpha)}$ here is the same as $\mathcal{S}'(\phi^*,\alpha^*)$ in Lemma 8.4. The implicit function theorem reveals that when σ is small enough,

$$\Big\|\frac{D(\phi^*,\alpha^*)}{D\xi^l}\Big\| = O(|\gamma|\rho^5) \ \text{ and } \ \Big\|\frac{D(\phi^*,\alpha^*)}{D\theta}\Big\| = O(|\gamma|\rho^6). \tag{9.12}$$

Since

$$\Big\|\frac{D(\phi^*,\alpha^*)}{D\xi^l}\Big\| = \Big\|\frac{\partial(\phi^*,\alpha^*)}{\partial\xi^l}\Big\|_{\mathcal{X}} \ \text{ and } \Big\|\frac{D(\phi^*,\alpha^*)}{D\theta}\Big\| = \Big\|\frac{\partial(\phi^*,\alpha^*)}{\partial\theta}\Big\|_{\mathcal{X}} \tag{9.13}$$

the lemma follows. □

Finally we complete the proof of the main theorem.

Proof of Theorem 2.1. Let the three curves of the fixed point $(\phi^*(\cdot,\xi,\theta),\alpha^*(\xi,\theta))$ found in Lemma 8.2 be parametrized by $\mathbf{r}_i^*(t,\xi,\theta)$ in the (ξ,θ)-frame as in (5.1). Without the loss of generality we assume that $J(\xi,\theta)$ given in (9.1) is minimized at $(\vec{0},0)$. For (ξ,θ) near $(\vec{0},0)$ we express the curves $\mathbf{r}_i^*(t,\xi,\theta)$ in the $(\vec{0},0)$-frame as $\tilde{\mathbf{r}}_i(t,\xi,\theta)$. The two parametrizations are related by

$$\tilde{\mathbf{r}}_i(t,\xi,\theta) = e^{\mathrm{i}\theta}\mathbf{r}_i^*(t,\xi,\theta)+\xi. \tag{9.14}$$

One views $\tilde{\mathbf{r}}_i(t,\xi,\theta)$ as a three parameter family of deformations of $\mathbf{r}_i^*(t,\vec{0},0)$. If $(\xi,\theta)=(\epsilon,0,0)$, then it is approximately a horizontal deformation whose infinitesimal element is

$$\mathbf{X}^H(t) = \frac{\partial\tilde{\mathbf{r}}_i(t,\xi,\theta)}{\partial\xi^1}\Big|_{(\xi,\theta)=(\vec{0},0)} = (1,0) + \frac{\partial\mathbf{r}_i^*(t,\xi,\theta)}{\partial\xi^1}\Big|_{(\xi,\theta)=(\vec{0},0)}; \tag{9.15}$$

if $(\xi,\theta)=(0,\epsilon,0)$, then it is nearly a vertical deformation whose infinitesimal element is

$$\mathbf{X}^V(t) = \frac{\partial\tilde{\mathbf{r}}_i(t,\xi,\theta)}{\partial\xi^2}\Big|_{(\xi,\theta)=(\vec{0},0)} = (0,1) + \frac{\partial\mathbf{r}_i^*(t,\xi,\theta)}{\partial\xi^2}\Big|_{(\xi,\theta)=(\vec{0},0)}; \tag{9.16}$$

and if $(\xi,\theta)=(0,0,\epsilon)$, then it is almost a rotational deformation whose infinitesimal element is

$$\mathbf{X}^R(t) = \frac{\partial\tilde{\mathbf{r}}_i(t,\xi,\theta)}{\partial\theta}\Big|_{(\xi,\theta)=(\vec{0},0)} = \mathrm{i}\,\mathbf{r}_i^*(t,\vec{0},0) + \frac{\partial\mathbf{r}_i^*(t,\xi,\theta)}{\partial\theta}\Big|_{(\xi,\theta)=(\vec{0},0)}. \tag{9.17}$$

Note that these three deformations are no longer in the restricted class.

By Lemma 9.2, since $(\vec{0},0)$ is an interior minimum of J,

$$\frac{\partial J(\xi,\theta)}{\partial\xi^1}\Big|_{(\xi,\theta)=(\vec{0},0)} = \frac{\partial J(\xi,\theta)}{\partial\xi^2}\Big|_{(\xi,\theta)=(\vec{0},0)} = \frac{\partial J(\xi,\theta)}{\partial\theta}\Big|_{(\xi,\theta)=(\vec{0},0)} = 0. \tag{9.18}$$

On the other hand Lemma 4.3, which holds for both restricted deformations and non-restricted deformations, shows that $\frac{\partial J(\xi,\theta)}{\partial\xi^1}\Big|_{(\xi,\theta)=(\vec{0},0)}$, $\frac{\partial J(\xi,\theta)}{\partial\xi^2}\Big|_{(\xi,\theta)=(\vec{0},0)}$, and $\frac{\partial J(\xi,\theta)}{\partial\theta}\Big|_{(\xi,\theta)=(\vec{0},0)}$ are equal to

$$\begin{aligned}(\mathbf{T}_1+\mathbf{T}_2+\mathbf{T}_0)\cdot\mathbf{X}\Big|_{-1}^{1} &- \int_{\partial\Omega_1\setminus\partial\Omega_2}(\kappa_1+\gamma_{11}I_{\Omega_1}+\gamma_{12}I_{\Omega_2})\mathbf{N}_1\cdot\mathbf{X}\,ds - \int_{\partial\Omega_2\setminus\partial\Omega_1}(\kappa_2+\gamma_{21}I_{\Omega_1}+\gamma_{22}I_{\Omega_2})\mathbf{N}_2\cdot\mathbf{X}\,ds \\ &- \int_{\partial\Omega_1\cap\partial\Omega_2}(\kappa_0+(\gamma_{11}-\gamma_{12})I_{\Omega_1}+(\gamma_{21}-\gamma_{22})I_{\Omega_2})\mathbf{N}_0\cdot\mathbf{X}\,ds\end{aligned} \tag{9.19}$$

with $\mathbf{X}$ being $\mathbf{X}^H$, $\mathbf{X}^V$, and $\mathbf{X}^R$ respectively. In (9.19) $\mathbf{T}_i$ and $\mathbf{N}_i$ are the tangent and normal vectors of the curves $\mathbf{r}_i^*(t,\vec{0},0)$. But these curves satisfy the first three equations of Lemma 8.3. Hence the integrals in (9.19) vanish and the following equations hold.

$$(\mathbf{T}_1+\mathbf{T}_2+\mathbf{T}_0)\cdot\mathbf{X}^H\Big|_{-1}^{1} = 0, \tag{9.20}$$

$$(\mathbf{T}_1+\mathbf{T}_2+\mathbf{T}_0)\cdot\mathbf{X}^V\Big|_{-1}^{1} = 0, \tag{9.21}$$

$$(\mathbf{T}_1+\mathbf{T}_2+\mathbf{T}_0)\cdot\mathbf{X}^R\Big|_{-1}^{1} = 0. \tag{9.22}$$

The last equation in Lemma 8.3 says that

$$(\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_0) \cdot \mathbf{X}^S \Big|_{-1}^{1} \quad = \quad 0 \tag{9.23}$$

where

$$\mathbf{X}^S(1) = (0, 1) \ \text{ and } \mathbf{X}^S(-1) = (0, -1). \tag{9.24}$$

Unlike $\mathbf{X}^H$, $\mathbf{X}^V$, and $\mathbf{X}^R$, this $\mathbf{X}^S$ is the infinitesimal element of a restricted deformation. It stretches the middle curve $\mathbf{r}_0^*$ connecting the two triple junction pionts P^+ and P^- of $(\phi^*(\cdot, \vec{0}, 0), \alpha^*(\vec{0}, 0))$. Under this deformation P^+ moves up and P^- moves down in the $(\vec{0}, 0)$-frame. The equations (9.20)-(9.23) form a four by four linear homogeneous system for the two components of the vector $(\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_0)(1)$ and the two components of the vector $(\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_0)(-1)$. The coefficients of the matrix are the components of $\mathbf{X}^H(\pm 1)$, $\mathbf{X}^V(\pm 1)$, $\mathbf{X}^R(\pm 1)$, and $\mathbf{X}^S(\pm 1)$. To estimate these coefficients, note that by Lemma 9.2 and the transformations (6.3) and (6.4),

$$\Big| \frac{\partial \mathbf{r}_i^*(\pm 1, \xi, \theta)}{\partial \xi^1} \Big|_{(\vec{0},0)} \Big| = O(|\gamma|\rho^4), \ \Big| \frac{\partial \mathbf{r}_i^*(\pm 1, \xi, \theta)}{\partial \xi^2} \Big|_{(\vec{0},0)} \Big| = O(|\gamma|\rho^4), \ \Big| \frac{\partial \mathbf{r}_i^*(\pm 1, \xi, \theta)}{\partial \theta} \Big|_{(\vec{0},0)} \Big| = O(|\gamma|\rho^5). \tag{9.25}$$

It follows that

$$|\mathbf{X}^H(\pm 1) - (1, 0)| = O(|\gamma|\rho^4), \quad |\mathbf{X}^V(\pm 1) - (0, 1)| = O(|\gamma|\rho^4), \quad |\mathbf{X}^R(\pm 1) - (\mp a, 0)| = O(|\gamma|\rho^5). \tag{9.26}$$

By (9.15)-(9.17) and (9.26) the linear system is written as

$$\begin{bmatrix} 1 + O(|\gamma|\rho^4) & O(|\gamma|\rho^4) & -1 + O(|\gamma|\rho^4) & O(|\gamma|\rho^4) \\ O(|\gamma|\rho^4) & 1 + O(|\gamma|\rho^4) & O(|\gamma|\rho^4) & -1 + O(|\gamma|\rho^4) \\ -a + O(|\gamma|\rho^5) & O(|\gamma|\rho^5) & -a + O(|\gamma|\rho^5) & O(|\gamma|\rho^5) \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} (\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_0)(1) \\ (\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_0)(-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{9.27}$$

Since the matrix on the left side is non-singular when δ and σ are small,

$$(\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_0)(1) = (\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_0)(-1) = \vec{0}. \tag{9.28}$$

In (2.6) the ν_i's are the unit inward tangential vectors at the triple junction points, so $\nu_i = -\mathbf{T}_i$ at the point P^+ and $\nu_i = \mathbf{T}_i$ at P^-. Hence (9.28) implies (2.6).

According to Lemma 8.2 the solution $(\phi^*(\cdot, \vec{0}, 0), \alpha^*(\vec{0}, 0))$ is found in the space $\mathcal{X}$, so the functions $\phi_i^*(\cdot, 0, 0)$ are in $H^2(-1, 1)$. The standard boot-strapping argument applied to the second order integro-differential equations (2.3)-(2.5) shows that the $\phi_i^*(\cdot, 0, 0)$'s are all C^∞. By the transformation (6.3) we conclude that the two bubbles of the solution are enclosed by continuous curves that are C^∞ except at the triple junction points.

Our assertion that the solution $(\phi^*(\cdot, \vec{0}, 0), \alpha^*(\vec{0}, 0))$ is stable is interpreted by its local minimization property. Recall that the solution $(\phi^*(\cdot, \vec{0}, 0), \alpha^*(\vec{0}, 0))$ is found in two steps. First for each $(\xi, \theta) \in \overline{D_{\bar{\delta}}} \times S^1$, a fixed point $(\phi^*(\cdot, \xi, \theta), \alpha^*(\xi, \theta))$ is constructed in a restricted class of perturbed double bubbles. This fixed point is shown to be a local minimizer of $\mathcal{J}$ in the restricted class in Lemma 8.4 part 1. In the second step $\mathcal{J}$ is minimized among the $(\phi^*(\cdot, \xi, \theta), \alpha^*(\xi, \theta))$'s where (ξ, θ) ranges over $\overline{D_{\bar{\delta}}} \times S^1$, and $(\phi^*(\cdot, \vec{0}, 0), \alpha^*(\vec{0}, 0))$ emerges as a minimum. As a minimum of local minimizers from restricted classes, $(\phi^*(\cdot, \vec{0}, 0), \alpha^*(\vec{0}, 0))$ is a local minimizer of $\mathcal{J}$ in a neighborhood of both restrictedly perturbed double bubbles and non-restrictedly perturbed double bubbles; hence our claim that $(\phi^*(\cdot, \vec{0}, 0), \alpha^*(\vec{0}, 0))$ is stable.

How much does this solution of a perturbed double bubble resemble an exact double bubble? One may consider the ratio of $\|(\phi^*(\cdot, \vec{0}, 0), \alpha^*(\vec{0}, 0))\|_{\mathcal{X}}$ and the area $m\rho^2$. Both ϕ^* and α^* have the "dimension" of area, as seen from (6.3) and (6.4), so the ratio is a good "dimensionless" quantity. By Lemma 8.2,

$$\frac{\|(\phi^*(\cdot, \vec{0}, 0), \alpha^*(\vec{0}, 0))\|_{\mathcal{X}}}{m\rho^2} \leq \frac{2\widetilde{C}|\gamma|\rho^3}{\tilde{d}m} \leq \frac{2\widetilde{C}\sigma}{\tilde{d}m}. \tag{9.29}$$

Therefore the smaller $|\gamma|\rho^3$ is, the closer the solution is to an exact double bubble. The bound σ on $|\gamma|\rho^3$ is also a bound on the deviation of the solution from a standard double bubble. $\square$

10 Discussion

Lemma 7.2 is interesting in its own right. It addresses an issue in the two component, equal area isoperimetric problem. It shows that in the restricted class the perimeter functional $\mathcal{J}_S$ has a positive second variation at the standard double bubble. In other words, the smallest eigenvalue of the linearized problem $\mathcal{S}'_S(0,0)$ is strictly positive.

The location of the double bubble solution found in this work can be ascertained from the proof of Lemma 9.1. Denote the minimum of J on $\overline{D_{\bar\delta}}$ by $(\widetilde{\xi}(\rho,\gamma),\widetilde{\theta}(\rho,\gamma))$ to emphasize its dependence on ρ and γ, (in the last section for simplicity we assumed that this point is $(\vec{0},0)$). If $(\widetilde{\xi}(\rho,\gamma),\widetilde{\theta}(\rho,\gamma)) \to (\xi^\circ,\theta^\circ)$ as $\rho\to 0$ and $|\gamma|\rho^3\to 0$ possibly along a subsequence, then

$$R(\xi^\circ,\xi^\circ)=\min_{z\in D} R(z,z); \tag{10.1}$$

namely that if ρ and $|\gamma|\rho^3$ are small, then the approximate double bubble solution is situated close to a minimum point of the function $R(z,z)$. However the direction of this perturbed double bubble cannot be determined from the argument in Lemma 9.1. A much more delicate study is needed to determine what θ° is. A somewhat similar question for binary systems is the determination of the direction of a oval shaped solution to (1.2) in [32].

Neither the binary system (1.1) nor the ternary system (1.3) allows different constituents to mix. This is a simplification in the strong segregation limit. The original density functional theories in [20, 19] do not have this limitation. There one uses density fields, i.e. functions on D, instead of subsets of D, to describe the concentrations of the constituents. In the binary case there is a function u defined on D which represents the concentration of one constituent; the function $1-u$ gives the concentration of the other constituent. The free energy of the binary system takes the form

$$\mathcal{I}_B(u)=\int_D \Big[\frac{\epsilon^2}{2}|\nabla u|^2+W_B(u)+\frac{\epsilon\gamma}{2}\Big((-\Delta)^{-1/2}(u-\omega)\Big)^2\Big]\,dx. \tag{10.2}$$

In (10.2) ϵ is a positive parameter; γ and ω are the same as the ones in (1.1); $W_B(u)$ is a double-well potential with two minimum points at 0 and 1, such as $W_B(u)=u^2(1-u)^2$; u satisfies the constraint $\int_D u(x)\,dx=\omega|D|$. For the ternary system one has two density functions u_1 and u_2 on D that, together with $1-u_1-u_2$, give the concentrations of the three constituents respectively. The free energy is given by

$$\mathcal{I}_T(u)=\int_D \Big[\frac{\epsilon^2}{2}(|\nabla u_1|^2+|\nabla u_2|^2)+W_T(u)+\sum_{i,j=1}^{2}\frac{\epsilon\gamma_{ij}}{2}\Big((-\Delta)^{-1/2}(u_i-\omega_i)\Big)\Big((-\Delta)^{-1/2}(u_j-\omega_j)\Big)\Big]\,dx \tag{10.3}$$

where $u=(u_1,u_2)$, W_T is a triple-well potential with three minimum points at $(1,0)$, $(0,1)$ and $(0,0)$, like $W_T(u)=((u_1-1)^2+u_2^2)(u_1^2+(u_2-1)^2)(u_1^2+u_2^2)$, and u_1 and u_2 satisfy the constraints $\int_D u_1(x)\,dx=\omega_1|D|$ and $\int_D u_2(x)\,dx=\omega_2|D|$. One can interpret $\mathcal{J}_B$ and $\mathcal{J}_T$ of (1.1) and (1.3) as limits of $\epsilon^{-1}\mathcal{I}_B$ and $\epsilon^{-1}\mathcal{I}_T$ as $\epsilon\to 0$. The convergence may be formulated under the framework of the Γ-convergence theory developed in [8, 17, 16, 4]; see [24, 27] for more details. There are also activator-inhibitor type reaction diffusion PDE systems, such as the Gierer-Meinhardt system [11], that can be reduced to problems like $\mathcal{J}_B$; see [32].

Conversely by a result of Kohn and Sternberg [15] one can show that if there is an isolated local minimizer of $\mathcal{J}_B$ (or $\mathcal{J}_T$ respectively), then for sufficient small ϵ there is a local minimizer of $\mathcal{I}_B$ (or $\mathcal{I}_T$) near the local minimizer of $\mathcal{J}_B$ (or $\mathcal{I}_T$). Unfortunately the local minimality concept in this result is defined with respect to the rather weak $L^1(D)$ norm; namely the distance between two subsets E and F of D is $\|\chi_E-\chi_F\|_{L^1}$. Nevertheless there is a theorem by Acerbi, Fusco and Morini [1] regarding $\mathcal{J}_B$, which states that if a critical point of $\mathcal{J}_B$ has a positive second variation (see [7] for the formula of the second variation), then it is always a local minimizer under the L^1 norm. Hopefully a similar property holds for $\mathcal{J}_T$, so one can construct local minimizers of $\mathcal{I}_T$ from critical points of $\mathcal{J}_T$ that have positive second variations. According to Lemma 8.4 the critical point found in Theorem 2.1 already has positive second variation with respect to restricted deformations. If the point $(\xi^\circ,\theta^\circ)\in D_{\bar\delta}\times S^1$ given before (10.1) has some "non-degeneracy" property, then we believe that the critical point should also have positive second variation with respect to deformations outside the restricted class.

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