LAYERED SOLUTIONS FOR A FRACTIONAL INHOMOGENEOUS ALLEN-CAHN EQUATION

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Abstract. We consider the problem

$$\varepsilon^{2s}(-\partial_{xx})^s\tilde{u}(\tilde{x}) - V(\tilde{x})\tilde{u}(\tilde{x})(1-\tilde{u}^2(\tilde{x})) = 0 \quad \text{in } \mathbb{R},$$

where $(-\partial_{xx})^s$ denotes the usual fractional Laplace operator, $\varepsilon > 0$ is a small parameter and the smooth bounded function V satisfies $\inf_{\tilde{x}\in\mathbb{R}} V(\tilde{x}) > 0$. For $s \in (\frac{1}{2}, 1)$, we prove the existence of separate multi-layered solutions for any small ε, where the layers are located near any non-degenerate local maximal points and non-degenerate local minimal points of function V. We also prove the existence of clustering-layered solutions, and these clustering layers appear within a very small neighborhood of a local maximum point of V.

Contents

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1. Introduction

We consider the following fractional inhomogeneous Allen-Cahn equation

$$\varepsilon^{2s}(-\partial_{xx})^s \tilde{u} - V(\tilde{x})\tilde{u}(1-\tilde{u}^2) = 0 \quad \tilde{x} \in \mathbb{R}, \tag{1}$$

where $(-\partial_{xx})^s,\ s \in (0,1)$, denotes the usual fractional Laplace operator, a Fourier multiplier of symbol $|\xi|^{2s}$. Here $\varepsilon > 0$ is a small parameter and the bounded smooth function V satisfies $\inf_{\tilde{x}\in\mathbb{R}} V(\tilde{x}) > 0$. We investigate the existence of layer solutions to (1) by applying a Lyapunov Schmidt reduction method. We call layer solution an heteroclinic connection for equation (1). This method has been applied in [6] to construct concentrating standing waves for the fractional nonlinear Schrödinger equation.

For the case $s = 1$, it is shown in [14] that the corresponding problem in a bounded interval of (1)

$$\varepsilon^2 u'' + V(x)u(1-u^2) = 0 \quad x \in (0,1),\ u'(0) = u'(1) = 0,$$

has interior layer solutions, and any layer solution can have its layers (namely its zeros) only near two endpoints of the interval, the local minimum points and local maximum points of $V(x)$. Furthermore, there appears at most one zero near each local minimum point of V. Subsequently, in [9], the authors extended this result to the two space dimension case considering

$$\varepsilon^2 \Delta u + V(x)u(1-u^2) = 0 \quad \text{in } \Omega,\ \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \ \partial\Omega, \tag{2}$$

and introduced a weighted arclength $\int_\Gamma V^{\frac{1}{2}} \mathrm{d}s$. The authors proved that (2) has an interior layer solution and this layer appears near a non-degenerate closed geodesic curve relative to the weighted arclength $\int_\Gamma V^{\frac{1}{2}} \mathrm{d}s$. Existence of layer solutions and clustering layer solution of (2) in general dimension Euclidean spaces and Riemannian manifolds were also obtained in [15], [16], [10], [11]. The case $V \equiv 1$ of the equation in (2) corresponds to the standard Allen-Cahn equation(see [1])

$$\varepsilon^2 \Delta u + u(1-u^2) = 0 \text{ in } \Omega.$$

We now come back to our problem (1) scaling the variables as $\tilde{x} = \varepsilon x,\ \tilde{u}(\tilde{x}) = \tilde{u}(\varepsilon x) := u(x)$. Therefore, equation (1) writes

$$(-\partial_{xx})^s u(x) - V(\varepsilon x)F(u(x)) = 0 \quad x \in \mathbb{R}, \tag{3}$$

where

$$F(u) := u(1-u^2).$$

Note that F is an odd function. We will find a solution to (1) if we may construct a solution to (3).

Denote w the unique solution of

$$(-\partial_{xx})^s w - F(w(x)) = 0,\ w(0) = 0,\ w(\pm\infty) = \pm 1. \tag{4}$$

The previous heteroclinic connection w has been proved to exist and to be unique in [2]. We now describe our main results.

Theorem 1.1. *Let $s \in (\frac{1}{2}, 1)$ and let $\Lambda_i \subset \mathbb{R},\ i = 1, \dots, m,\ m \geq 1$, be disjoint bounded open interval. Set $\Lambda = \Lambda_1 \times \cdots \times \Lambda_m$. Assume that the function*

$$\Upsilon(\xi_1, \dots, \xi_m) = \sum_{i=1}^{m} V^{\theta}(\xi_i),\ \theta = 1 - \frac{1}{2s} > 0$$

has a stable critical point situation in Λ in the following sense: there exists $\delta_0 > 0$ such that for any $g \in C^1(\bar{\Lambda})$ with $\|g\|_{L^\infty(\Lambda)} + \|\nabla g\|_{L^\infty(\Lambda)} < \delta_0$, there is a $\xi_g \in \Lambda$ such that $\nabla\Upsilon(\xi_g) + \nabla g(\xi_g) = 0$.

Then for all sufficiently small ε, (1) has a solution of the form

$$\tilde{u}(\tilde{x}) = \sum_{i=1}^{m} (-1)^{i-1} w\left(V(\xi_i^\varepsilon)^{\frac{1}{2s}} \frac{\tilde{x} - \xi_i^\varepsilon}{\varepsilon}\right) + \frac{(-1)^{m-1} - 1}{2} + o(1), \tag{5}$$

where $\xi^\varepsilon = (\xi_1^\varepsilon, \dots, \xi_m^\varepsilon) \in \Lambda$ and $\nabla\Upsilon(\xi^\varepsilon) \to 0$ as $\varepsilon \to 0$.

Corollary 1.1. *Let $s \in (\frac{1}{2}, 1)$ and let $\xi_1^0, \dots, \xi_m^0$ be m non-degenerate critical points of V, namely*

$$V'(\xi_i^0) = 0,\ V''(\xi_i^0) \neq 0,\ \forall\ i = 1, \dots, m.$$

Then (1) possesses a layer solution of the form (5) with $\xi_i^\varepsilon \to \xi_i^0$.

In Corollary 1.1 multi-layered solutions are constructed in "separate" non-degenerate local maximum or local minimum points of the potential V. These layers(zero points of solutions) are well separated. We will also obtain so-called clustering-layered solutions in the next theorem, and these layers appear within a very small neighborhood of a local maximum point of V.

Theorem 1.2. *Let $s \in (\frac{1}{2}, 1)$ and τ be a positive constant satisfying $\tau < \frac{2(2s-1)}{2s+1}$. Let $\bar{\xi}$ be a local maximum point of V, namely there exists a bounded open interval I such that*

$$\bar{x} \in I, \quad V(\bar{x}) = \max_{x \in I} V(x) > V(z),\ \forall z \in I \backslash \{\bar{x}\}.$$

Then for any $m \geq 1$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, (1) has a solution of the form (5), where these layers satisfy $\xi_i^\varepsilon \to \bar{x}$ as

$\varepsilon \to 0$. *Moreover*

$$\min_{1\le i\le m-1}\left|\frac{\xi_i^\varepsilon - \xi_{i+1}^\varepsilon}{\varepsilon}\right| > C\varepsilon^{-\frac{\tau}{2s-1}} \to \infty, \qquad as\ \ \varepsilon \to 0. \tag{6}$$

Furthermore, if $\bar{x}$ *is non-degenerate, namely* $V''(\bar{x}) < 0$, *then*

$$|\xi_i^\varepsilon - \bar{x}| \le C\varepsilon^{\frac{\tau}{2}},\ \ i = 1, \dots, m. \tag{7}$$

Note that the condition $\tau < \frac{2(2s-1)}{2s+1}$ in Theorem 1.2 yields

$$\frac{\tau}{2} < 1 - \frac{\tau}{2s-1},$$

which is a necessary condition to make that both (6) and (7) hold true. In other words, if $\frac{\tau}{2} \ge 1 - \frac{\tau}{2s-1}$, it is impossible that (1) possesses a solution of the form (5) satisfying (6) and (7).

For convenience, we shall assume that the non-degenerate local maximum point $\bar{x}$ of V is the origin.

2. Preliminaries

We first introduce the fractional Sobolev space $H^s(\mathbb{R}^n)$ as the space of functions $\phi \in L^2(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} (1 + |\lambda|^{2s})|\hat{\phi}(\lambda)|^2 d\lambda < +\infty,$$

where ˆ denotes the usual Fourier transform. The fractional Laplacian (in $\mathbb{R}^n$) $(-\Delta)^s\phi$ of a function $\phi \in H^s(\mathbb{R}^n)$ is defined in terms of its Fourier transform (in the space of tempered distributions) by the relation

$$\widehat{(-\Delta)^s}\phi(\lambda) = |\lambda|^{2s}\hat{\phi}(\lambda).$$

The fractional Laplace operator $(-\Delta)^s$ can also be defined as a Dirichlet-to-Neumann map for a so-called s-harmonic extension problem (see [5]). Given a function ϕ, the solution $\tilde{\phi}$ of the following problem

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla\tilde{\phi}) = 0 & \text{in } \mathbb{R}^{n+1}_+ = \{(x,y) : x \in \mathbb{R}^n, y > 0\}, \\ \tilde{\phi}(x,0) = \phi(x) & \text{on } \mathbb{R}^n \end{cases}$$

is called the s-harmonic extension of ϕ. One has

$$\tilde{\phi}(x,y) = \int_{\mathbb{R}^n} p_s(x-z,y)\phi(z)dz,$$

where $p_s(x,y)$ is the s-Poisson kernel

$$p_s(x,y) = C_{n,s}\frac{y^{2s}}{(|x|^2 + |y|^2)^{\frac{n+2s}{2}}},$$

and $C_{n,s}$ is the constant makes $\int_{\mathbb{R}^n} p_s(x,y)dx = 1$. Under suitable regularity, the authors in [5] proved that

$$(-\Delta)^s\phi(x) = -\lim_{y\to 0+} y^{2s}\partial_y\tilde{\phi}(x,y).$$

For the linear problem

$$(-\Delta)^s\varphi + D(x)\varphi = g \quad \text{ in } \mathbb{R}^n, \tag{8}$$

where D is a bounded potential, we need to use the following results in [6].

Proposition 2.1. *([6]) Let D be a continuous function, such that for m points $\xi_i, i = 1,\ldots,m$ and $B = \cup_{i=1}^m B_R(\xi_i)$ we have*

$$\inf_{x\in\mathbb{R}^n\setminus B} D(x) > 0.$$

Then, given any number $\frac{n}{2} < \mu < n+2s$, there exists $C = C(\mu,m,R)$ such that for any $\varphi \in H^s \cap L^\infty(\mathbb{R}^n)$ and g with $\|\rho^{-1}g\|_{L^\infty(\mathbb{R}^n)} < +\infty$ that satisfy (8), one has the following estimate

$$\|\rho^{-1}\varphi\|_{L^\infty(\mathbb{R}^n)} \le C[\|\varphi\|_{L^\infty(B)} + \|\rho^{-1}g\|_{L^\infty(\mathbb{R}^n)}].$$

Here

$$\rho(x) = \sum_{i=1}^m \frac{1}{(1+|x-\frac{\xi_i}{\varepsilon}|)^\mu}.$$

Furthermore, if $\inf_{x\in\mathbb{R}^n} D(x) > 0$ holds true, then

$$\|\rho^{-1}\varphi\|_{L^\infty(\mathbb{R}^n)} \le C\|\rho^{-1}g\|_{L^\infty(\mathbb{R}^n)}.$$

Particularly, if $D(x) \equiv d > 0$ holds true, then (6) has a unique solution $\varphi = T_d[g]$ and it satisfies the Hölder estimate

$$\sup_{x\neq y}\frac{|\varphi(x)-\varphi(y)|}{|x-y|^\alpha} \le C\|g\|_{L^\infty(\mathbb{R}^n)}$$

where $\alpha = \min\{1,2s\}$.

Note that the results in Proposition 2.1 hold true for all $s \in (0,1)$ and general dimensions n. In what follows, we let $s \in (\frac{1}{2},1)$ and $n = 1$.

3. Formulation of the problem: the ansatz

The existence of the solution w to (4) has been proven in [2] and additionally one has the following asymptotics (see also [2]): there exist constants $0 < C_1 < C_2$ such that the solution $w(x)$ of (4) satisfies

$$\frac{C_1}{|x|^{2s}} \le |1-w^2(x)| \le \frac{C_2}{|x|^{2s}},\ |x| > 1 \tag{9}$$

$$\frac{C_1}{|x|^{1+2s}} \leq w'(x) \leq \frac{C_2}{|x|^{1+2s}}, \ |x| > 1. \tag{10}$$

Note that for fixed constant $\lambda > 0$, $w_\lambda(x) := w(\lambda^{\frac{1}{2s}} x)$ satisfies

$$(-\partial_{xx})^s w_\lambda(x) - \lambda F(w_\lambda(x)) = 0 \quad x \in \mathbb{R}.$$

For points $\xi_i \in \mathbb{R}(i = 1, \ldots, m)$, we let

$$w_i(x) := (-1)^{i-1} w_{V(\xi_i)} \left(x - \frac{\xi_i}{\varepsilon} \right).$$

Then $w_i(x)$ satisfies

$$(-\partial_{xx})^s w_i(x) - V(\xi_i) F(w_i(x)) = 0 \quad x \in \mathbb{R}. \tag{11}$$

Given numbers $M > 0$ large and $\delta > 0$, we define the configuration space U as

$$U = \left\{ \xi = (\xi_1, \ldots, \xi_m) : \min_{1 \leq i \leq m-1} \left| \frac{\xi_i - \xi_{i+1}}{\varepsilon} \right| \geq M, \ \max_{1 \leq i \leq m} |\xi_i| \leq \delta \right\}. \tag{12}$$

We construct the approximate solution

$$W_\xi(x) := \sum_{i=1}^m w_i(x) + \frac{(-1)^{m-1} - 1}{2}.$$

With this definition we have that $W_\xi(x) \approx w_i(x)$ for values of x close to $\frac{\xi_i}{\varepsilon}$.

We construct a solution u of (3) of the form

$$u(x) = W_\xi(x) + \phi(x),$$

where $\phi \in H^s(\mathbb{R})$ is a small function. Now (3) can be expanded as

$$(-\partial_{xx})^s \phi(x) - V(\varepsilon x)(1 - 3W_\xi^2(x))\phi(x) = E + N(\phi) \quad x \in \mathbb{R}, \tag{13}$$

where

$$E = V(\varepsilon x) F(W_\xi) - \sum_{i=1}^m V(\xi_i) F(w_i), \tag{14}$$

$$N(\phi) = -V(\varepsilon x)[3W_\xi(x)\phi^2 + \phi^3]. \tag{15}$$

We would like to invert the operator $(-\partial_{xx})^s - V(\varepsilon x)(1 - 3W_\xi^2)$ in equation (13) to obtain a fixed point equation for ϕ. However, the operator

$$L_\xi \phi := (-\partial_{xx})^s - V(\varepsilon x)(1 - 3W_\xi^2(x))$$

may have a kernel, near the kernel

$$\text{Span}\{w_1'(x), \ w_2'(x), \ \ldots, \ w_m'(x)\}.$$

Hence, rather than solving problem (13) directly, we shall first solve the following projected problem

$$L_\xi \phi = E + N(\phi) + \sum_{i=1}^{m} c_i w_i' \quad x \in \mathbb{R}, \tag{16}$$

$$\int_{\mathbb{R}} \phi w_i'(x) dx = 0, \; i = 1, \ldots, m. \tag{17}$$

4. Linear theory

In this section we consider the corresponding linear problem

$$L_\xi \phi = h(x) + \sum_{i=1}^{m} c_i w_i'(x) \quad x \in \mathbb{R}, \tag{18}$$

$$\int_{\mathbb{R}} \phi w_i'(x) dx = 0, \; i = 1, \ldots, m. \tag{19}$$

Note that the coefficients c_i are uniquely determined in terms of ϕ and h when ε is sufficiently small. Indeed, we have

$$\sum_{i=1}^{m} c_i \int_{\mathbb{R}} w_i' w_j' dx = \int_{\mathbb{R}} w_j' [(-\partial_{xx})^s \phi - V(\varepsilon x)(1 - 3W_\xi^2(x))\phi - h] dx.$$

Since

$$\int_{\mathbb{R}} w_j' (-\partial_{xx})^s \phi dx = \int_{\mathbb{R}} \phi (-\partial_{xx})^s w_j' dx = \int_{\mathbb{R}} \phi V(\xi_j)[1 - 3w_j^2] w_j' dx, \tag{20}$$

we have

$$\begin{aligned} &\sum_{i=1}^{m} c_i \int_{\mathbb{R}} w_i' w_j' dx \\ &= \int_{\mathbb{R}} w_j' \{ [V(\xi_j)(1 - 3w_j^2) - V(\varepsilon x)(1 - 3W_\xi^2)]\phi - h \} dx \qquad (21) \\ &= \int_{\mathbb{R}} w_j' \{ [(V(\xi_j) - V(\varepsilon x))(1 - 3w_j^2) - 3V(\varepsilon x)(W_\xi^2 - w_j^2)]\phi - h \} dx. \end{aligned}$$

It is easy to see that

$$\int_{\mathbb{R}} w_i' w_j' dx = \beta_j \delta_{ij} + \mathrm{O}(M^{-1-2s})$$

where the numbers $\beta_j > 0$ are independent of ε and M is large. Hence the matrix of linear system (21) for $c_i (i = 1, \ldots, m)$ is diagonally dominant for small ε, hence system (21) is uniquely solvable.

For the right hand side terms of (21) we have

$$\left|\int_{\mathbb{R}} w_j'[(V(\xi_j)-V(\varepsilon x))(1-3w_j^2)-3V(\varepsilon x)(W_\xi^2-w_j^2)]\phi dx\right|$$
$$\leq C[\varepsilon+M^{-2s}]\|\phi\|_{L^2(\mathbb{R})}.$$

Hence we obtain the following lemma.

Lemma 4.1. *The numbers c_i in (18) satisfy*

$$c_i=\frac{-1}{\beta_i}\int_{\mathbb{R}} w_i' h dx+\theta_i,$$

where

$$|\theta_i|\leq C[\varepsilon+M^{-2s}]\|\phi\|_{L^2(\mathbb{R})}.$$

The main task of this section is to establish the following proposition.

Proposition 4.1. *Given $m\geq 1$, $\frac{1}{2}<\mu<1+2s$, there exist positive numbers $M_0,\varepsilon_0,\delta_0$ such that for any points $\xi_1,\dots,\xi_m$ and any ε with*

$$\min_{1\leq i\leq m-1}\left|\frac{\xi_i-\xi_{i+1}}{\varepsilon}\right|\geq M_0,\quad 0<\varepsilon<\varepsilon_0,\quad \max_i|\xi_i|\leq\delta_0$$

there exists a unique solution $\phi=T[h]$ of (18)-(19) that defines a linear operator of h, provided that

$$\|\rho^{-1}h\|_{L^\infty(\mathbb{R})}<+\infty,\quad \rho(x)=\sum_{i=1}^m\frac{1}{(1+|x-\frac{\xi_i}{\varepsilon}|)^\mu}.$$

Moreover

$$\|\rho^{-1}\phi\|_{L^\infty(\mathbb{R})}\leq C\|\rho^{-1}h\|_{L^\infty(\mathbb{R})}.$$

To prove this result we need to establish the following several lemmas. We have the following nondegeneracy lemma.

Lemma 4.2. *The only bounded solution to*

$$(-\partial_{xx})^s\phi-\hat{\lambda}[1-3w_{\hat{\lambda}}^2]\phi=0,\ |\phi|\leq 1$$

is $cw_{\hat{\lambda}}'$.

Proof. For $s=\frac{1}{2}$, this has been proved in [7]. It is easy to see that the same proof works exactly in the case of $s>\frac{1}{2}$. In fact for $s>\frac{1}{2}$, $w_{\hat{\lambda}}'$ works as a super-solution and hence one can prove that $|\phi|\leq\frac{1}{|x|^{1+2s}}$ for $|x|>1$. Then we let $\phi=w_{\hat{\lambda}}'\psi$. Integrating by parts we then obtain that $\psi\equiv Constant$. We omit the details.

□

Lemma 4.3. *Under the conditions of Proposition 4.1, there exists $C > 0$ such that for any solutions of (18)-(19) with $\|\rho^{-1}\phi\|_{L^\infty(\mathbb{R})} < \infty$ we have the apriori estimate*

$$\|\rho^{-1}\phi\|_{L^\infty(\mathbb{R})} \le C\|\rho^{-1}h\|_{L^\infty(\mathbb{R})}.$$

Proof. We argue by contradiction: namely there exist sequences $\varepsilon_n \to 0$, $\xi_{in}, i = 1, \dots, m$, with

$$\min_{1\le i\le m-1} \left|\frac{\xi_{in} - \xi_{i+1,n}}{\varepsilon_n}\right| \to \infty$$

and ϕ_n, h_n satisfying (18)-(19) such that

$$\|\rho_n^{-1}\phi_n\|_{L^\infty(\mathbb{R})} = 1, \quad \|\rho_n^{-1}h_n\|_{L^\infty(\mathbb{R})} \to 0, \tag{22}$$

where

$$\rho_n(x) = \sum_{i=1}^{m} \frac{1}{(1 + |x - \frac{\xi_{in}}{\varepsilon_n}|)^\mu}$$

We claim that for any fixed $R > 0$ we have that

$$\sum_{i=1}^{m} \|\phi_n\|_{L^\infty(B_R(\xi_{in}/\varepsilon_n))} \to 0. \tag{23}$$

Indeed, assume that for a fixed j we have that $\|\phi_n\|_{L^\infty(B_R(\xi_{jn}/\varepsilon_n))} \ge \gamma > 0$. We set $\hat\phi_n(x) = \phi_n(x + \frac{\xi_{jn}}{\varepsilon_n})$. We also assume that $\lambda_{jn} = V(\xi_{jn}) \to \hat\lambda > 0$. One has

$$(-\partial_{xx})^s\hat\phi_n(x) - V(\xi_{jn}+\varepsilon_n x)\{1-3[(-1)^{j-1}w_{\lambda_{jn}}(x)+\theta_n(x)]^2\}\hat\phi_n(x) = \hat h_n(x),$$

where

$$\hat h_n(x) = h_n(x + \frac{\xi_{jn}}{\varepsilon_n}) + \sum_{i=1}^{m} c_{in}(-1)^{i-1}w'_{\lambda_{in}}(\frac{\xi_{jn} - \xi_{in}}{\varepsilon_n} + x).$$

We observe that $\hat h_n(x) \to 0$ uniformly on bounded closed intervals. From the uniform Hölder estimates in Proposition 2.1, we also obtain equicontinuity of the sequence $\hat\phi_n$. Thus, passing to a subsequence, we may assume that $\hat\phi_n$ converges, uniformly on bounded closed intervals, to a bounded function $\hat\phi$ which satisfies $\|\hat\phi\|_{L^\infty(B_R(0))} \ge \gamma$ and

$$(-\partial_{xx})^s\hat\phi - \hat\lambda[1 - 3w_{\hat\lambda}^2]\hat\phi = 0, \tag{24}$$

$$\int_{\mathbb{R}} \hat\phi w'_{\hat\lambda} dx = 0. \tag{25}$$

Combining (24), (25) and the nondegeneracy of the solution w to (4) obtained in Lemma 4.2 we know that $\hat\phi = 0$, which contradicts with the fact $\|\hat\phi\|_{L^\infty(B_R(0))} \ge \gamma$. Formula (23) and the apriori estimate in

Proposition 2.1 give that $\|\rho_n^{-1}\phi_n\|_{L^\infty(\mathbb{R})} \to 0$, which contradicts with (22). $\square$

In order to construct a solution to problem (18)-(19), we first establish a solution to a simpler problem

$$(-\partial_{xx})^s\phi(x) + 2V(\varepsilon x)\phi = h(x) + \sum_{i=1}^{m} c_i w_i'(x), \tag{26}$$

$$\int_{\mathbb{R}} \phi w_i'(x)dx = 0,\ i = 1, \ldots, m. \tag{27}$$

Lemma 4.4. *For any h with $\|\rho^{-1}h\|_{L^\infty(\mathbb{R})} < \infty$, there exists a unique solution of (26)-(27), $\phi = Q[h] \in H^s(\mathbb{R})$. Moreover*

$$\|\rho^{-1}Q[h]\|_{L^\infty(\mathbb{R})} \le C\|\rho^{-1}h\|_{L^\infty(\mathbb{R})}. \tag{28}$$

Proof. Let H be the closure of the set of all functions in $C_c^\infty(\overline{\mathbb{R}^2_+})$ under the norm

$$\|\tilde{\phi}\|_H^2 := \int_{\mathbb{R}^2_+} |\nabla\tilde{\phi}|^2 y^{1-2s} dxdy + \int_{\mathbb{R}} 2V(\varepsilon x)\phi^2 dx < +\infty,$$

where $\tilde{\phi}$ is the s-harmonic extension of ϕ. Furthermore we define a closed subspace X of H as

$$X = \left\{ \tilde{\phi} \in H : \int_{\mathbb{R}} \phi w_i' dx = 0, \forall i = 1, \ldots, m \right\}.$$

Then, given $h \in L^2$, we consider the problem of finding a $\tilde{\phi} \in X$ such that

$$\langle\tilde{\phi}, \tilde{\psi}\rangle := \int_{\mathbb{R}^2_+} \nabla\tilde{\phi}\cdot\nabla\tilde{\psi} y^{1-2s} dxdy + \int_{\mathbb{R}} 2V(\varepsilon x)\phi\psi dx = \int_{\mathbb{R}} h\psi dx,\ \forall\tilde{\psi} \in X. \tag{29}$$

We observe that $\langle\cdot,\cdot\rangle$ defines an inner product in X. Then Riesz's theorem yields existence and uniqueness of a solution to (26)-(27). Moreover we have

$$\|\phi\|_{L^2(\mathbb{R})} \le C\|h\|_{L^2(\mathbb{R})}.$$

Next we check that this produces a solution in strong sense. Let W be the space spanned by the functions w_i'. We denote by $P[h]$ the $L^2(\mathbb{R})$ orthogonal projection of h onto W and by $\tilde{P}[h]$ its s-harmonic extension. Then for each $\tilde{\eta} \in H$, we know that $\tilde{\psi} := \tilde{\eta} - \tilde{P}[\eta] \in X$.

Substituting this $\tilde{\psi}$ into (29) we obtain

$$\begin{aligned}&\int_{\mathbb{R}^2_+} \nabla\tilde{\phi}\cdot\nabla\tilde{\eta}y^{1-2s}dxdy + \int_{\mathbb{R}} 2V\phi\eta dx\\ &= \int_{\mathbb{R}} h\eta dx + \int_{\mathbb{R}}(2V\phi - h)P[\eta]dx + \int_{\mathbb{R}}\phi(-\partial_{xx})^s P[\eta]dx,\end{aligned}$$

where we used the relation

$$\int_{\mathbb{R}^2_+} \nabla\tilde{\phi}\cdot\nabla\tilde{P}[\eta]y^{1-2s}dxdy = \int_{\mathbb{R}}\phi(-\partial_{xx})^s P[\eta]dx.$$

For $\eta\in L^2(\mathbb{R})$ we consider the functional

$$z_\phi(\eta) := \int_{\mathbb{R}}\phi(-\partial_{xx})^s P[\eta]dx.$$

We have

$$\begin{aligned}|z_\phi(\eta)| \quad &= \left|\int_{\mathbb{R}}\eta P[(-\partial_{xx})^s\phi]dx\right| \le C\|\eta\|_2\|P[(-\partial_{xx})^s\phi]\|_2\\ &\le C\|\eta\|_2\sum_{i=1}^m\left|\int_{\mathbb{R}} w_i'(-\partial_{xx})^s\phi dx\right| \le C\|\phi\|_2\|\eta\|_2,\end{aligned}$$

where in the last inequality we have used (29). Hence there exists an $e(\phi)\in L^2(\mathbb{R})$ such that

$$z_\phi(\eta) = \int_{\mathbb{R}} e(\phi)\eta dx.$$

If ϕ was a priori known to be in $H^s(\mathbb{R})$ we would have precise formula of $e(\phi)$

$$e(\phi) = P[(-\partial_{xx})^s\phi].$$

Since P is a self-adjoint operator in $L^2(\mathbb{R})$ we then have that

$$\int_{\mathbb{R}^2_+} \nabla\tilde{\phi}\cdot\nabla\tilde{\eta}y^{1-2s}dxdy + \int_{\mathbb{R}} 2V\phi\eta dx = \int_{\mathbb{R}}\bar{h}\eta dx,$$

where

$$\bar{h} = h + P[2V\phi - h] + P[(-\partial_{xx})^s\phi].$$

Since $\bar{h}\in L^2(\mathbb{R})$, it follows that $\phi\in H^s(\mathbb{R})$ and it satisfies

$$(-\partial_{xx})^s\phi(x) + 2V(\varepsilon x)\phi - h(x) = P[(-\partial_{xx})^s\phi + 2V\phi - h]\in W,$$

hence equations (26)-(27) are satisfied.

Now we prove (28). We have

$$\|\rho^{-1}P[(-\partial_{xx})^s\phi + 2V\phi - h]\|_\infty \le C[\|\phi\|_2 + \|h\|_2] \le C\|h\|_2 \le C\|\rho^{-1}h\|_\infty,$$

where we used the condition $\frac{1}{2} < \mu < 1 + 2s$. This and Proposition 2.1 show the desired estimate. □

Proof of Proposition 4.1. Let $\mathbb{B}$ be the Banach space

$$\mathbb{B} := \{\phi \in C(\mathbb{R}) : \|\phi\|_{\mathbb{B}} := \|\rho^{-1}\phi\|_{L^\infty(\mathbb{R})} < \infty\}. \tag{30}$$

Problem (18)-(19) can be written as the fixed point problem

$$\phi - Q[3V(\varepsilon x)(1 - W_\xi^2(x))\phi] = Q[h], \ \phi \in \mathbb{B}. \tag{31}$$

We claim that

$$A[\phi] := Q[V(\varepsilon x)(1 - W_\xi^2(x))\phi]$$

defines a compact operator in $\mathbb{B}$. Indeed, assume that $\{\phi_n\}$ is a bounded sequence in $\mathbb{B}$. It is easy to see that for some $\alpha > 0$ the estimate holds true

$$|V(\varepsilon x)(1 - W_\xi^2(x))\phi_n| \leq C\|\phi_n\|_{\mathbb{B}}\rho^{1+\alpha},$$

namely

$$\rho^{-(1+\alpha)}|V(\varepsilon x)(1 - W_\xi^2(x))\phi_n| \leq C\|\phi_n\|_{\mathbb{B}}.$$

It follows that $g_n := A[\phi_n]$ satisfies

$$\begin{aligned}
|\rho^{-1}g_n| &= |\rho^{-1}Q[V(\varepsilon x)(1 - W_\xi^2(x))\phi]| \\
&\leq C\|\rho^{-1}V(\varepsilon x)(1 - W_\xi^2(x))\phi\|_\infty \\
&= C\rho^\alpha\rho^{-\alpha}\|\rho^{-1}V(\varepsilon x)(1 - W_\xi^2(x))\phi\|_\infty \\
&\leq C\rho^\alpha\|\rho^{-(1+\alpha)}V(\varepsilon x)(1 - W_\xi^2(x))\phi\|_\infty \leq C\rho^\alpha.
\end{aligned}$$

Besides since $g_n = T_d[(d-V)g_n + h_n]$, we use Hölder estimate in Proposition 2.1 to get that for some $\beta > 0$

$$\sup_{x \neq y} \frac{|g_n(x) - g_n(y)|}{|x - y|^\beta} \leq C.$$

Arzela's theorem gives the existence of a subsequence of g_n which we label the same way, that converges uniformly to a continuous function g with

$$|\rho^{-1}g| \leq C\rho^\alpha.$$

Let $R > 0$ be a large number. Then we have

$$\|\rho^{-1}(g_n - g)\|_{L^\infty(\mathbb{R})} \leq \|\rho^{-1}(g_n - g)\|_{L^\infty(B_R(0))} + C\max_{|x|>R}\rho^\alpha(x).$$

Since

$$\max_{|x|>R}\rho^\alpha(x) \to 0 \quad \text{ as } R \to \infty,$$

we deduce that $\|g_n - g\|_{\mathbb{B}} \to 0$, and the claim is proved.

Now, the apriori estimate Lemma 4.3 tell us that for $h = 0$, (31) has only the trivial solution. Fredholm's alternative gives the desired result in this proposition. $\square$

Let us write the solution $\phi = T_\xi[h]$ to emphasize the dependence of the operator T on ξ. In the rest of this section, we obtain the differentiability of $\phi = T_\xi[h]$ with respect to ξ.

Lemma 4.5. *The map $\xi \mapsto T_\xi$ is continuously differentiable, and for some $C > 0$, one has*

$$\|\partial_{\xi_i} T_\xi[h]\|_{\mathbb{B}} \leq \frac{C}{\varepsilon}\|h\|_{\mathbb{B}}, \quad \forall\ i = 1, \dots, m,$$

for all ξ satisfying constraints (12).

The argument of this lemma is rather similar to that of Lemma 4.4 in [6], we omit it.

5. Solving the nonlinear intermediate problem

In this section we will apply contraction mapping principle to solve nonlinear problem (16)-(17).

We first make an estimate of the error E in the norm $\|\cdot\|_{\mathbb{B}}$. Recall that

$$E = V(\varepsilon x)F(W_\xi) - \sum_{i=1}^{m} V(\xi_i)F(w_i).$$

We rewrite it as

$$E = V(\varepsilon x)\left[F(W_\xi) - \sum_{i=1}^{m} F(w_i)\right] + \sum_{i=1}^{m}[V(\varepsilon x) - V(\xi_i)]F(w_i).$$

Here we need to take $\mu \in (\frac{1}{2}, s)$. Let

$$M := \min_{1\leq i\leq m-1}\left|\frac{\xi_i - \xi_{i+1}}{\varepsilon}\right| \gg 1.$$

The second term in E can be easily estimated as

$$\left|\rho^{-1}(x)\sum_{i=1}^{m}[V(\varepsilon x) - V(\xi_i)]F(w_i)\right| \leq C\varepsilon^s.$$

To estimate the interaction term (the first term) in E, we divide the $\mathbb{R}$ into the m sub-intervals

$$I_j := \{x \in \mathbb{R} : |w_j(x)| \leq |w_i(x)|,\ \forall\ i \neq j,\ 1 \leq i \leq m\}.$$

For $x \in I_j$, we have

$$\begin{aligned}
&\left| V(\varepsilon x)\left[F(W_\xi) - \sum_{i=1}^m F(w_i)\right]\right| \\
&\le C\sum_{i\neq j}\frac{1}{|x-\frac{\xi_i}{\varepsilon}|^{2s}} \le C\sum_{i\neq j}\frac{1}{|\frac{\xi_i-\xi_j}{\varepsilon}|^{2s}} \\
&\le C\frac{1}{1+|x-\frac{\xi_j}{\varepsilon}|^{\mu}}\sum_{i\neq j}\frac{1}{|\frac{\xi_i-\xi_j}{\varepsilon}|^{2s-\mu}} \\
&\le C\rho(x)M^{\mu-2s} \le C\rho(x)M^{-s}.
\end{aligned}$$

Therefore we obtain that

$$\|E\|_{\mathbb{B}} \le C[\varepsilon^s + M^{-s}]. \tag{32}$$

Similarly, we can obtain

$$\|\partial_\xi E\|_{\mathbb{B}} \le \frac{C}{\varepsilon}[\varepsilon^s + M^{-s}]. \tag{33}$$

We denote

$$\kappa := C[\varepsilon^s + M^{-s}].$$

We have the following result.

Lemma 5.1. *Assume that $\|E\|_{\mathbb{B}}$ is sufficiently small, then (16)-(17) possesses a unique small solution $\phi = \Phi(\xi)$ with*

$$\|\Phi(\xi)\|_{\mathbb{B}} \le C\|E\|_{\mathbb{B}}.$$

Moreover the map $\xi \mapsto \Phi(\xi)$ is of class C^1, and for some $C > 0$

$$\|\partial_\xi \Phi(\xi)\|_{\mathbb{B}} \le C\left[\frac{1}{\varepsilon}\|E\|_{\mathbb{B}} + \|\partial_\xi E\|_{\mathbb{B}}\right] \tag{34}$$

for all ξ satisfying constraints (12).

Proof. Problem (16)-(17) can be written as the fixed point problem

$$\phi = T_\xi(E + N(\phi)) =: K_\xi(\phi), \quad \phi \in \mathbb{B}. \tag{35}$$

Let

$$Z = \{\phi \in \mathbb{B} : \|\phi(\xi)\|_{\mathbb{B}} \le \sigma\}.$$

If $\phi \in Z$, then it is easy to see that

$$\|N(\phi)\|_{\mathbb{B}} \le C\|\phi\|_{\mathbb{B}}^2.$$

Hence

$$\|K_\xi(\phi)\|_{\mathbb{B}} \le C_0[\|E\|_{\mathbb{B}} + \sigma^2].$$

Choosing

$$\sigma = 2C_0\|E\|_{\mathbb{B}},$$

we have

$$\|K_\xi(\phi)\|_{\mathbb{B}} \le C_0[\frac{\sigma}{2C_0} + \sigma^2] \le \sigma,$$

which means that $K_\xi(Z) \subset Z$.

We observe that

$$|N(\phi_1) - N(\phi_2)| \le C[|\phi_1| + |\phi_2|]|\phi_1 - \phi_2|,$$

which yields that

$$\|N(\phi_1) - N(\phi_2)\|_{\mathbb{B}} \le C\sigma\|\phi_1 - \phi_2\|_{\mathbb{B}}.$$

Hence

$$\|K_\xi(\phi_1) - K_\xi(\phi_2)\|_{\mathbb{B}} \le C\sigma\|\phi_1 - \phi_2\|_{\mathbb{B}}.$$

Reducing σ if necessary, we obtain that K_ξ is a contraction mapping and hence has a unique solution of problem (35) in Z. We denote it as $\phi = \Phi(\xi)$.

Next we prove that Φ is C^1 with respect to ξ. Denote

$$G(\phi, \xi) := \phi - T_\xi(E + N(\phi)).$$

Let $\phi_0 = \Phi(\xi_0)$, then $G(\phi_0, \xi_0) = 0$. We have

$$\partial_\phi G(\phi, \xi)[\psi] = \psi - T_\xi(N'(\phi)\psi),$$

where $N'(\phi) = -V(\varepsilon x)[6W_\xi(x)\phi + 3\phi^2]$. Hence

$$\|N'(\phi)\psi\|_{\mathbb{B}} \le C\sigma\|\psi\|_{\mathbb{B}}.$$

Then, if σ is sufficiently small, we have that $\partial_\phi G(\phi_0, \xi_0)$ is an invertible operator with uniformly bounded inverse. Besides

$$\partial_\xi G(\phi, \xi) = (\partial_\xi T_\xi)(E + N(\phi)) + T_\xi(\partial_\xi E + \partial_\xi N(\phi)).$$

Both partial derivatives are continuous in their arguments. The implicit function theorem applies in a small neighborhood of (ϕ_0, ξ_0) to give existence and uniqueness of a function $\phi = \phi(\xi)$ with $\phi_0 = \phi(\xi_0)$ defined near ξ_0. Besides $\phi(\xi)$ is of class C^1. However, by uniqueness, we must have $\phi(\xi) = \Phi(\xi)$.

Finally we note that

$$\partial_\xi N(\phi) = -3V(\varepsilon x)\partial_\xi W_\xi(x)\phi^2,$$

so

$$\|\partial_\xi N(\Phi(\xi))\|_{\mathbb{B}} \le \frac{C}{\varepsilon}\|\Phi(\xi)\|^2_{\mathbb{B}} \le \frac{C}{\varepsilon}\|E\|^2_{\mathbb{B}}. \tag{36}$$

Since

$$\partial_\xi \Phi(\xi) = -\frac{1}{\partial_\phi G(\Phi(\xi), \xi)}[(\partial_\xi T_\xi)(E + N(\Phi(\xi))) + T_\xi(\partial_\xi E + \partial_\xi N(\Phi(\xi)))],$$

from this, (36) and Lemma 4.5, we obtain (34).

□

6. The variational reduced problem

Recalling that we have obtained the existence of a unique solution $u = W_\xi(x) + \Phi(\xi)$ of the problem (16)-(17). Namely, if we denote this solution as $u = u_\xi$, we have

$$(-\partial_{xx})^s u_\xi - V(\varepsilon x)(u_\xi - u_\xi^3) = \sum_{i=1}^m c_i w_i'. \tag{37}$$

Then, in order to prove Theorem 1.1, we need to verify that the coefficients $c_i (i = 1, \dots, m)$ are equal to zero, by choosing an appropriate point $\xi = (\xi_1, \dots, \xi_m)$.

Problem (3) corresponds to an energy functional

$$J_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}} u(x)(-\partial_{xx})^s u(x)dx + \frac{1}{4}\int_{\mathbb{R}} V(\varepsilon x)(1 - u^2(x))^2 dx.$$

Note that $J_\varepsilon(u)$ is well-defined, since $s > \frac{1}{2}$. We denote

$$\mathcal{J}(\xi) := J_\varepsilon(u_\varepsilon) = J_\varepsilon(W_\xi(x) + \Phi(\xi)).$$

We will first establish expansions of the energy $\mathcal{J}(\xi)$.

Lemma 6.1. *Assume that the number M^{-1} in the definition of U in (12) is taken so small that*

$$\|E\|_{\mathbb{B}} + \varepsilon\|\partial_\xi E\|_{\mathbb{B}} \leq \kappa \ll 1.$$

Then

$$\mathcal{J}(\xi) = J_\varepsilon(W_\xi(x)) + O(\kappa^2) \tag{38}$$

and

$$\partial_\xi \mathcal{J}(\xi) = \partial_\xi J_\varepsilon(W_\xi(x)) + O(\frac{\kappa^2}{\varepsilon}) \tag{39}$$

uniformly on points ξ in U.

Proof. Since

$$\mathcal{J}(\xi) = \frac{1}{2}\int_{\mathbb{R}} u_\xi(x)(-\partial_{xx})^s u_\xi(x)dx + \frac{1}{4}\int_{\mathbb{R}} V(\varepsilon x)(1 - u_\xi^2(x))^2 dx,$$

we can expand

$$\begin{aligned} \mathcal{J}(\xi) = \quad & J_\varepsilon(W_\xi) + \frac{1}{2}\int_{\mathbb{R}} \Phi(-\partial_{xx})^s \Phi dx \\ & + \int_{\mathbb{R}} \Phi[(-\partial_{xx})^s W_\xi - V(\varepsilon x) W_\xi(1 - W_\xi^2(x))]dx \\ & + \frac{1}{4}\int_{\mathbb{R}} V(\varepsilon x)\{[u_\xi^4 - W_\xi^4 - 4W_\xi^3\Phi] - 2[u_\xi^2 - W_\xi^2 - 2W_\xi\Phi]\}dx. \end{aligned} \tag{40}$$

In view of $\|E\|_{\mathbb{B}} \le \kappa$ then $\|\Phi\|_{\mathbb{B}} \le C\kappa$, and from equation (16) we also have $\|(-\partial_{xx})^s\Phi\|_{\mathbb{B}} \le C\kappa$. Hence

$$\int_{\mathbb{R}} \Phi(-\partial_{xx})^s \Phi dx \le C\kappa^2 \int_{\mathbb{R}} \rho^2 dx \le C\kappa^2 \tag{41}$$

and

$$\begin{aligned}&\int_{\mathbb{R}} V(\varepsilon x)\{[u_\xi^4 - W_\xi^4 - 4W_\xi^3\Phi] - 2[u_\xi^2 - W_\xi^2 - 2W_\xi\Phi]\}dx \qquad (42)\\ &= \int_{\mathbb{R}} V\{[(W_\xi+\Phi)^4 - W_\xi^4 - 4W_\xi^3\Phi] - 2[(W_\xi+\Phi)^2 - W_\xi^2 - 2W_\xi\Phi]\}\\ &\le C\kappa^2 \int_{\mathbb{R}} \rho^2 dx \le C\kappa^2,\end{aligned}$$

where we have used the definition of $\rho(x) = \sum_{i=1}^m \frac{1}{(1+|x-\frac{\xi_i}{\varepsilon}|)^\mu}$ and the fact that $\mu > \frac{1}{2}$.

Note that $(-\partial_{xx})^s W_\xi - V(\varepsilon x)W_\xi(1-W_\xi^2(x)) = E$, so

$$\int_{\mathbb{R}} \Phi[(-\partial_{xx})^s W_\xi - V(\varepsilon x)W_\xi(1-W_\xi^2(x))]dx \le C\kappa^2 \int_{\mathbb{R}} \rho^2 dx \le C\kappa^2. \tag{43}$$

From (40)-(43), we obtain (38).

Differentiating (40) with respect to ξ_j, we have

$$\begin{aligned}\partial_{\xi_j}\mathcal{J}(\xi) = \quad & \partial_{\xi_j} J_\varepsilon(W_\xi) + \int_{\mathbb{R}} [\partial_{\xi_j}\Phi(-\partial_{xx})^s\Phi dx + \int_{\mathbb{R}} [E\partial_{\xi_j}\Phi + \Phi\partial_{\xi_j}E]dx\\ &+ \int_{\mathbb{R}} V(\varepsilon x)\{[(W_\xi+\Phi)^3 - W_\xi^3 - 3W_\xi^3\Phi]\partial_{\xi_j}W_\xi \qquad (44)\\ &+[(W_\xi+\Phi)^3 - W_\xi^3]\partial_{\xi_j}\Phi + \Phi\partial_{\xi_j}\Phi\}dx.\end{aligned}$$

We have that

$$\int_{\mathbb{R}} V(\varepsilon x)[(W_\xi+\Phi)^3 - W_\xi^3 - 3W_\xi^3\Phi]\partial_{\xi_j}W_\xi dx \le \frac{C}{\varepsilon}\kappa^2.$$

From this, (44), (34) and the condition $\|E\|_{\mathbb{B}} + \varepsilon\|\partial_\xi E\|_{\mathbb{B}} \le \kappa$, we can obtain (39). □

Next we estimate $J_\varepsilon(W_\xi)$ and $\partial_\xi J_\varepsilon(W_\xi)$. We begin with the simpler case $m=1$. Note that the condition $\|E\|_{\mathbb{B}} + \varepsilon\|\partial_\xi E\|_{\mathbb{B}} \le \kappa$ is always true. Now

$$W_\xi(x) = w_\lambda(x - \frac{\xi}{\varepsilon}), \quad \lambda = V(\xi).$$

It is easy to see that

$$J_\varepsilon(W_\xi) = J^\lambda(w_\lambda) + \frac{1}{4}\int_{\mathbb{R}} [V(\xi+\varepsilon x) - V(\xi)](1-w_\lambda^2(x))^2 dx, \tag{45}$$

where
$$J^\lambda(v)=\frac{1}{2}\int_{\mathbb{R}} v(-\partial_{xx})^s v dx+\frac{\lambda}{4}\int_{\mathbb{R}}(1-v^2)^2dx.$$
Note that
$$J^\lambda(w_\lambda)=\lambda^{1-\frac{1}{2s}}J^1(w). \tag{46}$$
Indeed, recalling that $w_\lambda(x)=w(\lambda^{\frac{1}{2s}}x)$ satisfies the equation
$$(-\partial_{xx})^s w_\lambda-\lambda(w_\lambda-w_\lambda^3)=0 \ \text{ in } \ \mathbb{R},$$
where $w=w_1$ is the unique solution of
$$(-\partial_{xx})^s w-(w-w^3)=0, \ w(0)=0, \ w(\pm\infty)=\pm 1.$$
Then, after a change of variables we obtain (46).

We claim that
$$\int_{\mathbb{R}}[V(\xi+\varepsilon x)-V(\xi)](1-w_\lambda^2(x))^2dx=\mathrm{O}(\varepsilon^{2s}). \tag{47}$$
Indeed, for any large number ζ with $\zeta<\varepsilon^{-1}$, we have
$$\begin{aligned}
\int_{\mathbb{R}} & [V(\xi+\varepsilon x)-V(\xi)](1-w_\lambda^2(x))^2dx\\
= & \int_{\mathbb{R}}[V(\varepsilon x+\xi)-V(\xi)](1-w^2(V(\xi)^{\frac{1}{2s}}x))^2dx\\
= & \int_{|x|>\varepsilon^{-1}}[V(\varepsilon x+\xi)-V(\xi)](1-w^2(V(\xi)^{\frac{1}{2s}}x))^2dx\\
& +\int_{|x|<\zeta}[V(\varepsilon x+\xi)-V(\xi)-\varepsilon V'(\xi)x](1-w^2(V(\xi)^{\frac{1}{2s}}x))^2dx\\
& +\int_{\zeta<|x|<\varepsilon^{-1}}[V(\varepsilon x+\xi)-V(\xi)-\varepsilon V'(\xi)x](1-w^2(V(\xi)^{\frac{1}{2s}}x))^2dx\\
\le & C[\varepsilon^{4s-1}+\max_{x\in(-\zeta,\zeta)}|V(\varepsilon x+\xi)-V(\xi)-\varepsilon V'(\xi)x|]\\
& +C\int_\zeta^{\varepsilon^{-1}}\varepsilon^2x^2x^{-4s}dx.
\end{aligned}$$
In view of
$$\varepsilon^2\int_\zeta^{\varepsilon^{-1}}x^{2-4s}dx\le\begin{cases} C\varepsilon^2, & s>\frac{3}{4},\\ C\varepsilon^2\ln\frac{1}{\varepsilon}, & s=\frac{3}{4},\\ C\varepsilon^{4s-1}, & \frac{1}{2}<s<\frac{3}{4},\end{cases}$$
we have
$$\left|\int_{\mathbb{R}}[V(\xi+\varepsilon x)-V(\xi)](1-w_\lambda^2)^2dx\right|\le C[\varepsilon^{4s-1}+\varepsilon^2\zeta^2+\varepsilon^2\ln\frac{1}{\varepsilon}].$$
Choosing $\zeta=\varepsilon^{s-1}$, we obtain (47), where we used the fact $\frac{1}{2}<s<1$.

We claim

$$\partial_\xi \int_{\mathbb{R}} [V(\xi+\varepsilon x)-V(\xi)](1-w_\lambda^2(x))^2 dx = \mathrm{O}(\varepsilon^{2s-1}). \tag{48}$$

Indeed

$$\begin{aligned}
&\partial_\xi \int_{\mathbb{R}} [V(\xi+\varepsilon x)-V(\xi)](1-w_\lambda^2(x))^2 dx \\
&= \int_{\mathbb{R}} [\partial_\xi V(\xi+\varepsilon x)-\partial_\xi V(\xi)](1-w_\lambda^2(x))^2 dx \\
&-\frac{2}{s} V^{\frac{1}{2s}-1}(\xi)V'(\xi)\int_{\mathbb{R}} [V(\xi+\varepsilon x)-V(\xi)]x(1-w_\lambda^2(x))w'(V^{\frac{1}{2s}}x)dx.
\end{aligned}$$

From the proof of (47), we know that

$$\int_{\mathbb{R}} [\partial_\xi V(\xi+\varepsilon x)-\partial_\xi V(\xi)](1-w_\lambda^2(x))^2 dx = \mathrm{O}(\varepsilon^{2s}).$$

For the other term, we have

$$\begin{aligned}
&\int_{\mathbb{R}} [V(\xi+\varepsilon x)-V(\xi)]x(1-w_\lambda^2(x))w'(V^{\frac{1}{2s}}x)dx \\
&= \left\{\int_{|x|>\varepsilon^{-1}} + \int_{|x|<\zeta} + \int_{\zeta<|x|<\varepsilon^{-1}}\right\}[V(\xi+\varepsilon x)-V(\xi)]x(1-w_\lambda^2(x))w'(V^{\frac{1}{2s}}x) \\
&\le C[\varepsilon^{4s-1} + \max_{x\in(-\zeta,\zeta)}|V(\varepsilon x+\xi)-V(\xi)|] + C\int_\zeta^{\varepsilon^{-1}} \varepsilon x^2 x^{-4s-1}dx \\
&\le C[\varepsilon^{4s-1}+\varepsilon\zeta+\varepsilon],
\end{aligned}$$

where in the last inequality we have used the fact $s>\frac{1}{2}$. Choosing $\zeta=\varepsilon^{2s-2}$ and noting that $\frac{1}{2}<s<1$, we obtain (48).

Hence, from Lemma 6.1, the definition of κ and (45)-(48), we obtain the following lemma.

Lemma 6.2. *Let $c_* = J^1(w)$ and $m=1$. Then the following expansions hold true*

$$\mathcal{J}(\xi) = c_* V(\xi)^{1-\frac{1}{2s}} + O(\varepsilon^{2s}+M^{-2s}),$$
$$\partial_\xi \mathcal{J}(\xi) = c_*\partial_\xi[V(\xi)^{1-\frac{1}{2s}}] + \frac{1}{\varepsilon}O(\varepsilon^{2s}+M^{-2s}).$$

For the general case $m>1$, without loss of generality, we may assume that $\xi_1\le\xi_2\le\cdots\le\xi_m$. Taking $\min_{1\le i\le m-1}\left|\frac{\xi_i-\xi_{i+1}}{\varepsilon}\right|\ge M\gg 1$, we know that $\|E\|_{\mathbb{B}}\le C\kappa$ also holds true. Hence from Lemma 6.1, we have also

$$\mathcal{J}(\xi) = J_\varepsilon(W_\xi(x)) + \mathrm{O}(\kappa^2), \quad \partial_\xi\mathcal{J}(\xi) = \partial_\xi J_\varepsilon(W_\xi(x)) + \mathrm{O}(\frac{\kappa^2}{\varepsilon}). \tag{49}$$

For each $i(i = 1, \ldots, m-1)$, we denote the unique number in $(\frac{\xi_i}{\varepsilon}, \frac{\xi_{i+1}}{\varepsilon})$ as ζ_i such that $|w_i(\zeta_i)| = |w_{i+1}(\zeta_i)|$. From the properties of the potential function V, we know that there exists $\sigma_i \in (0,1)$, independent of ε, such that

$$\zeta_i = \frac{\xi_i}{\varepsilon} + \sigma_i \frac{\xi_{i+1} - \xi_i}{\varepsilon}, \quad i = 1, \ldots, m-1.$$

We have the following lemma.

Lemma 6.3. *The following expansions hold true*

$$\mathcal{J}(\xi) = c_* \sum_{i=1}^{m} V(\xi_i)^{1-\frac{1}{2s}} - \sum_{i=1}^{m-1} \frac{\tilde{c}_i + o(1)}{|\frac{\xi_{i+1}-\xi_i}{\varepsilon}|^{2s-1}} + O(\varepsilon^{2s} + M^{-2s}), \tag{50}$$

$$\partial_\xi \mathcal{J}(\xi) = c_* \partial_\xi \left[\sum_{i=1}^{m} V(\xi_i)^{1-\frac{1}{2s}} \right] - \sum_{i=1}^{m-1} \frac{\tilde{c}_i + o(1)}{|\frac{\xi_{i+1}-\xi_i}{\varepsilon}|^{2s-1}} + \frac{1}{\varepsilon} O(\varepsilon^{2s} + M^{-2s}). \tag{51}$$

Here

$$\tilde{c}_i = \frac{c_i \sigma_i^{1-2s} + c_{i+1}(1-\sigma_i)^{1-2s} + o(1)}{2s-1}, \quad i = 1, \ldots, m-1,$$

where $c_i > 0$ is a constant between C_1 and C_2, which are given in (9)-(10).

Proof. It suffices to expand $J_\varepsilon(W_\xi(x))$. We have

$$\begin{aligned}
& J_\varepsilon(W_\xi(x)) \qquad (52)\\
= & \frac{1}{2}\int_{\mathbb{R}} W_\xi(-\partial_{xx})^s W_\xi dx + \frac{1}{4}\int_{\mathbb{R}} V(\varepsilon x)(1-W_\xi^2)^2 dx\\
= & \frac{1}{2}\sum_{j=1}^m \int_{I_j} \left[\sum_{i<j}(w_i(x)-(-1)^{i-1}) + w_j(x) + \sum_{i>j}(w_i(x)+(-1)^{i-1})\right]\\
& \times(-\partial_{xx})^s \left[\sum_{i<j}(w_i(x)-(-1)^{i-1}) + w_j(x) + \sum_{i>j}(w_i(x)+(-1)^{i-1})\right] dx\\
& + \sum_{j=1}^m \int_{I_j} \frac{V(\varepsilon x)}{4}\left\{1-\left[\sum_{i<j}(w_i-(-1)^{i-1}) + w_j + \sum_{i>j}(w_i+(-1)^{i-1})\right]^2\right\}^2 dx\\
= & \sum_{j=1}^m \int_{I_j} \frac{1}{2} w_j(-\partial_{xx})^s w_j + \frac{V(\varepsilon x)}{4}(1-w_j^2)^2 dx\\
& + \sum_{j=1}^m \int_{I_j} \left[\sum_{i<j}(w_i(x)-(-1)^{i-1}) + \sum_{i>j}(w_i(x)+(-1)^{i-1})\right](-\partial_{xx})^s w_j(x)\\
& + \frac{1}{2}\sum_{j=1}^m \int_{I_j} (W_\xi - w_j)(-\partial_{xx})^s (W_\xi - w_j) dx\\
& + \sum_{j=1}^m \int_{I_j} \frac{V(\varepsilon x)}{4}\{[1-W_\xi^2]^2 - [1-w_j^2]^2\} dx + \mathrm{O}(M^{-2s})\\
= & \sum_{j=1}^m \int_{I_j} \frac{1}{2} w_j(-\partial_{xx})^s w_j + \frac{V(\varepsilon x)}{4}(1-w_j^2)^2 dx + \mathrm{O}(M^{-2s}).
\end{aligned}$$

Note that

$$\sum_{j=1}^m \int_{I_j} \frac{V(\varepsilon x)}{4}(1-w_j^2)^2 dx = \sum_{j=1}^m \int_{\mathbb{R}} \frac{V(\varepsilon x)}{4}(1-w_j^2)^2 dx + \mathrm{O}(M^{1-4s}). \quad (53)$$

Besides

$$\sum_{j=1}^m \int_{I_j} \frac{1}{2} w_j(-\partial_{xx})^s w_j = \sum_{j=1}^m \int_{\mathbb{R}} \frac{1}{2} w_j(-\partial_{xx})^s w_j - \sum_{j=1}^m \int_{\mathbb{R}\setminus I_j} \frac{1}{2} w_j(-\partial_{xx})^s w_j.$$

We claim that

$$\sum_{j=1}^m \int_{\mathbb{R}\setminus I_j} \frac{1}{2} w_j(-\partial_{xx})^s w_j dx = \sum_{j=1}^{m-1} \frac{\tilde{c}_j + \mathrm{o}(1)}{|\frac{\xi_{j+1}-\xi_j}{\varepsilon}|^{2s-1}}. \quad (54)$$

Indeed, one has

$$\begin{aligned}\int_{\mathbb{R}\setminus I_j}\frac{1}{2}w_j(-\partial_{xx})^s w_j dx &= \int_{\mathbb{R}\setminus I_j}\frac{V(\xi_j)}{2}w_j^2(1-w_j^2)dx \qquad (55)\\ &= \int_{\mathbb{R}\setminus I_j}\frac{V(\xi_j)}{2}w^2\left(V^{\frac{1}{2s}}(\xi_j)\left(x-\frac{\xi_j}{\varepsilon}\right)\right)\left[1-w^2\left(V^{\frac{1}{2s}}(\xi_j)\left(x-\frac{\xi_j}{\varepsilon}\right)\right)\right]dx.\end{aligned}$$

For $x\in\mathbb{R}\setminus I_j$, we have

$$w^2(V^{\frac{1}{2s}}(\xi_j)(x-\frac{\xi_j}{\varepsilon}))\sim 1,$$

$$\frac{C_1}{V(\xi_j)|x-\frac{\xi_j}{\varepsilon}|^{2s}}\le 1-w^2(V^{\frac{1}{2s}}(\xi_j)(x-\frac{\xi_j}{\varepsilon}))\le\frac{C_2}{V(\xi_j)|x-\frac{\xi_j}{\varepsilon}|^{2s}}. \qquad (56)$$

Recall that

$$\zeta_j=\frac{\xi_j}{\varepsilon}+\sigma_j\frac{\xi_{j+1}-\xi_j}{\varepsilon},\quad j=1,\dots,m-1.$$

Hence for $2\le j\le m-1$, we have

$$\begin{aligned}&\int_{\mathbb{R}\setminus I_j}\frac{V(\xi_j)}{2}w^2(V^{\frac{1}{2s}}(\xi_j)(x-\frac{\xi_j}{\varepsilon}))[1-w^2(V^{\frac{1}{2s}}(\xi_j)(x-\frac{\xi_j}{\varepsilon}))]dx\\ =\;&\int_{-\infty}^{\zeta_{j-1}}\frac{V(\xi_j)}{2}w^2(V^{\frac{1}{2s}}(\xi_j)(x-\frac{\xi_j}{\varepsilon}))[1-w^2(V^{\frac{1}{2s}}(\xi_j)(x-\frac{\xi_j}{\varepsilon}))]dx \qquad (57)\\ &+\int_{\zeta_j}^{+\infty}\frac{V(\xi_j)}{2}w^2(V^{\frac{1}{2s}}(\xi_j)(x-\frac{\xi_j}{\varepsilon}))[1-w^2(V^{\frac{1}{2s}}(\xi_j)(x-\frac{\xi_j}{\varepsilon}))]dx\\ =\;&\frac{c_j+\mathrm{o}(1)}{2s-1}\left[(1-\sigma_{j-1})^{1-2s}\left|\frac{\xi_j-\xi_{j-1}}{\varepsilon}\right|^{1-2s}+\sigma_j^{1-2s}\left|\frac{\xi_{j+1}-\xi_j}{\varepsilon}\right|^{1-2s}\right],\end{aligned}$$

where $c_j>0$ is a constant between C_1 and C_2, which are given in (56). Formula (57) also holds true for $j=1$ and $j=m$, the only difference is that the right hand side term is respectively replaced by

$$\frac{c_1+\mathrm{o}(1)}{2s-1}(1-\sigma_1)^{1-2s}\left|\frac{\xi_2-\xi_1}{\varepsilon}\right|^{1-2s},\quad\frac{c_m+\mathrm{o}(1)}{2s-1}\sigma_{m-1}^{1-2s}\left|\frac{\xi_m-\xi_{m-1}}{\varepsilon}\right|^{1-2s}. \qquad (58)$$

From (55), (57)-(58), we obtain (54).

Then, from (52)-(54) we have

$$\begin{aligned}J_\varepsilon(W_\xi(x))=\;&\sum_{j=1}^m\int_{\mathbb{R}}\frac{1}{2}w_j(-\partial_{xx})^s w_j+\frac{V(\varepsilon x)}{4}(1-w_j^2)^2dx\\ &-\sum_{j=1}^{m-1}\frac{\tilde{c}_j+\mathrm{o}(1)}{|\frac{\xi_{j+1}-\xi_j}{\varepsilon}|^{2s-1}}+\mathrm{O}(M^{-2s}).\end{aligned}$$

By the same argument for the case of $m=1$, we know

$$\int_{\mathbb{R}} \frac{1}{2} w_j(-\partial_{xx})^s w_j + \frac{V(\varepsilon x)}{4}(1-w_j^2)^2 dx = c_* V(\xi_j)^{1-\frac{1}{2s}} + \mathrm{O}(\varepsilon^{2s}).$$

Hence

$$J_\varepsilon(W_\xi(x)) = c_* \sum_{j=1}^{m} V(\xi_j)^{1-\frac{1}{2s}} - \sum_{j=1}^{m-1} \frac{\tilde{c}_j + \mathrm{o}(1)}{|\frac{\xi_{j+1}-\xi_j}{\varepsilon}|^{2s-1}} + \mathrm{O}(M^{-2s} + \varepsilon^{2s}).$$

This and (49) yield (50).

Similarly, we can obtain (51). We omit the precise argument.

□

In the rest of this section, we establish the following variational result.

Lemma 6.4. *$c := (c_1, \dots, c_m) = 0$ if and only if $\partial_\xi \mathcal{J}(\xi) = 0$.*

Proof. We have

$$\begin{aligned}
\partial_{\xi_j} \mathcal{J}(\xi) &= \int_{\mathbb{R}^2_+} \nabla \tilde{u}_\xi \cdot \nabla(\partial_{\xi_j} \tilde{u}_\xi) y^{1-2s} dx dy - \int_{\mathbb{R}} V(\varepsilon x) u_\xi (1-u_\xi^2) \partial_{\xi_j} u_\xi dx \\
&= \int_{\mathbb{R}} [(-\partial_{xx})^s u_\xi - V(\varepsilon x) u_\xi(x)(1-u_\xi^2(x))] \partial_{\xi_j} u_\xi(x) dx \qquad (59) \\
&= \sum_{i=1}^{m} c_i w_i' \partial_{\xi_j} u_\xi(x),
\end{aligned}$$

where $\tilde{u}_\xi$ is the s-harmonic extension of $u_\xi = W_\xi(x) + \Phi(\xi)$. Note that

$$\partial_{\xi_j} u_\xi(x) = -\frac{1}{\varepsilon} w_j' + \mathrm{O}(1) + \partial_{\xi_j} \Phi(\xi),$$

and, from Lemma 5.1, we have

$$\|\partial_{\xi_j} \Phi(\xi)\|_{\mathbb{B}} \leq C[\frac{1}{\varepsilon} \|E\|_{\mathbb{B}} + \|\partial_\xi E\|_{\mathbb{B}}].$$

From $\int_{\mathbb{R}} w_i' w_j' dx = \beta_j \delta_{ij} + \mathrm{O}(M^{-1-2s})$ and (32)-(33), we know that, for small ε and $M \gg 1$, the matrix of linear system (59) for c_i is diagonally dominant. This shows that $(c_1, \dots, c_m) = 0$ if and only if $\partial_\xi \mathcal{J}(\xi) = 0$. □

7. The proof of Theorem 1.1

In this section, we will complete the proof of Theorem 1.1.

Proof of Theorem 1.1. By the definition of configuration space U (12), we can choose $M \sim \varepsilon^{-1}$ and achieve that $\Lambda \subset U$. Then we obtain

$$\|E\|_{\mathbb{B}} + \varepsilon \|\partial_\xi E\|_{\mathbb{B}} \leq C\varepsilon^s.$$

By Lemma 6.3 we have

$$\mathcal{J}(\xi) - c_*\Upsilon(\xi) = \mathrm{o}(1), \quad \nabla\mathcal{J}(\xi) - c_*\nabla\Upsilon(\xi) = \mathrm{o}(1)$$

uniformly in $\xi \in \Lambda$ as $\varepsilon \to 0$, where the function Υ is defined in Theorem 1.1. We choose $\mathcal{J}(\xi) - c_*\Upsilon(\xi)$ as the function g in Theorem 1.1. Then, by the assumption on Υ, we know that for all sufficiently small ε there exists a $\xi^\varepsilon \in \Lambda$ such that $\nabla\mathcal{J}(\xi^\varepsilon) = 0$. Now applying Lemma 6.4, we obtain the result of this theorem.

□

8. The proof of Theorem 1.2

Let I be as in Theorem 1.2 and $\bar{c} > 0$ be a small number. Set

$$\Lambda_\varepsilon = \left\{ \xi = (\xi_1, \dots, \xi_m) \in I \times \cdots \times I : \sum_{i=1}^{m-1} \left| \frac{\xi_i - \xi_{i+1}}{\varepsilon} \right|^{1-2s} < \bar{c}\varepsilon^\tau \right\},$$

where $0 < \tau < \frac{2(2s-1)}{2s+1}$. Note that $\frac{2(2s-1)}{2s+1} < 2s - 1 < 2s$, since $s > \frac{1}{2}$.

Similarly we construct a solution with the form

$$u(x) = W_\xi(x) + \phi(x).$$

Repeating the argument of Theorem 1.1, we can prove that the corresponding projected problem possess a unique solution $\phi = \Phi(\xi)$ with

$$\|\Phi(\xi)\|_{\mathbb{B}} \le C\|E\|_{\mathbb{B}} \le C[\varepsilon^s + M^{-s}].$$

Note that now $M = \min_{1\le i\le m-1} \left|\frac{\xi_i - \xi_{i+1}}{\varepsilon}\right| \ge C\varepsilon^{-\frac{\tau}{2s-1}}$, instead of $M \sim \varepsilon^{-1}$ as in the proof of Theorem 1.1. We also can obtain variational Lemma 6.4, and the same energy expansion as (50) for sufficiently small ε as follows

$$\mathcal{J}(\xi) = c_* \sum_{i=1}^{m} V(\xi_i)^{1-\frac{1}{2s}} - \sum_{i=1}^{m-1} \frac{\tilde{c}_i + \mathrm{o}(1)}{|\frac{\xi_{i+1}-\xi_i}{\varepsilon}|^{2s-1}} + \mathrm{O}(\varepsilon^{2s} + M^{-2s}), \quad (60)$$

where $\tilde{c}_i > 0$.

To prove Theorem 1.2, applying Lemma 6.4, we know that the only task rest is to obtain the following result.

Lemma 8.1. *For ε sufficiently small, the following maximizing problem*

$$\max\{\mathcal{J}(\xi) : \xi \in \overline{\Lambda}_\varepsilon\}$$

has a solution $\xi^\varepsilon \in \Lambda_\varepsilon$.

Proof. We will borrow the idea in Proposition 4.2 [13] to prove this lemma.

Since $\mathcal{J}(\xi)$ is continuous in ξ, the maximizing problem has a solution. Let $\xi^\varepsilon \in \overline{\Lambda}_\varepsilon$ be a maximum point of $\mathcal{J}(\xi)$.

We claim that $\xi^\varepsilon \in \Lambda_\varepsilon$. We prove this by energy comparison.

We first establish a lower bound for $\mathcal{J}(\xi^\varepsilon)$. Recall that $\tau < \frac{2(2s-1)}{2s+1}$, which guarantees that $\frac{\tau}{2s-1} < \frac{2-\tau}{2}$. Hence we may choose $\sigma \in (\frac{\tau}{2s-1}, \frac{2-\tau}{2})$, which implies

$$\sigma(2s-1) > \tau, \quad 2(1-\sigma) > \tau. \tag{61}$$

The condition $\tau > 0$ makes that $\frac{2-\tau}{2} < 1$, and so $\sigma < 1$. Set $\xi_i^0 = \varepsilon^{1-\sigma}(i - \frac{m+1}{2})$. Clearly $\xi_i^0 \in I$. Moreover

$$\left|\frac{\xi_{i+1}^0 - \xi_i^0}{\varepsilon}\right|^{1-2s} \leq C\varepsilon^{\sigma(2s-1)} < \bar{c}\varepsilon^\tau.$$

So $\xi^0 = (\xi_1^0, \ldots, \xi_m^0) \in \Lambda_\varepsilon$.

Since $V'(0) = 0$, we have the Taylor's expansion

$$V(\xi_i^0) = V(0) + \mathrm{O}(\varepsilon^{2(1-\sigma)}).$$

Hence from (60) we obtain

$$\begin{aligned}\mathcal{J}(\xi^\varepsilon) &= \max_{\xi \in \overline{\Lambda}_\varepsilon} \mathcal{J}(\xi) \geq \mathcal{J}(\xi^0)\\ &\geq mc_* V(0)^{1-\frac{1}{2s}} - C(\varepsilon^{\sigma(2s-1)} + \varepsilon^{2(1-\sigma)} + \varepsilon^{2s} + M^{-2s})\\ &\geq mc_* V(0)^{1-\frac{1}{2s}} - C(\varepsilon^{\sigma(2s-1)} + \varepsilon^{2(1-\sigma)} + \varepsilon^{2s} + \varepsilon^{\frac{2s}{2s-1}\tau}),\end{aligned}$$

where in the last inequality we used $M \geq C\varepsilon^{-\frac{\tau}{2s-1}}$. Hence

$$\begin{aligned}&c_* \sum_{i=1}^m V(\xi_i^\varepsilon)^{1-\frac{1}{2s}} - \sum_{i=1}^{m-1} \frac{\tilde{c}_i + \mathrm{o}(1)}{|\frac{\xi_{i+1}^\varepsilon - \xi_i^\varepsilon}{\varepsilon}|^{2s-1}}\\ &\geq mc_* V(0)^{1-\frac{1}{2s}} - C(\varepsilon^{\sigma(2s-1)} + \varepsilon^{2(1-\sigma)} + \varepsilon^{2s} + \varepsilon^{\frac{2s}{2s-1}\tau}).\end{aligned} \tag{62}$$

From the previous analysis in this section, we know that four exponentials of the corresponding powers of ε in the right hand side of (62) all larger than τ. From (62) we can deduce that $\xi^\varepsilon \in \Lambda_\varepsilon$. Indeed, suppose not, then by the definition of Λ_ε there are two possible case. The first case is that one of the ξ_i^ε is an endpoint of I. Then by condition $V(0) = \max_{x\in I} V(x) > V(z),\ \forall z \in I\backslash\{0\}$, we know that there exists $\beta_1 > 0$ such that $V(\xi_i^\varepsilon) < V(0) - \beta_1$, so

$$c_* \sum_{i=1}^m V(\xi_i^\varepsilon)^{1-\frac{1}{2s}} \leq mc_* V(0)^{1-\frac{1}{2s}} - \beta_2$$

for some $\beta_2 > 0$. This contradicts with (62). The other case is that $\sum_{i=1}^{m-1}\left|\frac{\xi_i^\varepsilon-\xi_{i+1}^\varepsilon}{\varepsilon}\right|^{1-2s} = \bar{c}\varepsilon^\tau$. Then

$$c_*\sum_{i=1}^{m} V(\xi_i^\varepsilon)^{1-\frac{1}{2s}} - \sum_{i=1}^{m-1}\frac{\tilde{c}_i+\mathrm{o}(1)}{|\frac{\xi_{i+1}^\varepsilon-\xi_i^\varepsilon}{\varepsilon}|^{2s-1}} \le mc_*V(0)^{1-\frac{1}{2s}} - \bar{c}[\min_{1\le i\le m-1}\tilde{c}_i+\mathrm{o}(1)]\varepsilon^\tau,$$

which contradicts with (62) again.

Hence $\xi^\varepsilon \in \Lambda_\varepsilon$. □

Proof of Theorem 1.2. Combining Lemmas 6.4 and 8.1, we see that (1) possesses a solution of the form (5). From the argument of Lemma 8.1, we know that

$$\sum_{i=1}^{m-1}\left|\frac{\xi_i^\varepsilon-\xi_{i+1}^\varepsilon}{\varepsilon}\right|^{1-2s} = \mathrm{o}(\varepsilon^\tau),$$

which gives (6). We also know that $V(\xi_i^\varepsilon) - \max_{x\in I} V(x) = V(\xi_i^\varepsilon) - V(0) = \mathrm{o}(1),\ i = 1,\dots,m$.

Next we prove (7). Suppose not, then by the Taylor's expansion, due to $V''(0) < 0$, there exists some i such that

$$V(\xi_i^\varepsilon) < V(0) - C\varepsilon^\tau,$$

where $C > 0$ is a constant. Hence applying Taylor's expansion again, we have

$$V(\xi_i^\varepsilon)^{1-\frac{1}{2s}} < V(0)^{1-\frac{1}{2s}} - C\varepsilon^\tau,$$

which yields

$$\begin{aligned} c_*\sum_{i=1}^{m} V(\xi_i^\varepsilon)^{1-\frac{1}{2s}} - \sum_{i=1}^{m-1}\frac{\tilde{c}_i+\mathrm{o}(1)}{|\frac{\xi_{i+1}^\varepsilon-\xi_i^\varepsilon}{\varepsilon}|^{2s-1}} &< c_*\sum_{i=1}^{m} V(\xi_i^\varepsilon)^{1-\frac{1}{2s}} \\ &< mc_*V(0)^{1-\frac{1}{2s}} - C\varepsilon^\tau. \end{aligned}$$

This contradicts with (62), and so (7) holds true.

The proof of Theorem 1.2 is complete.

□

9. Open Questions

This paper initiates the study of effect of inhomogeneity in fractional Allen-Can equations. We pose several challenging questions in line with the standard $s = 1$ case.

- Are results stated in this paper true even when $s = \frac{1}{2}$? In view of the results of [3]-[4], we turn to believe so. $s = \frac{1}{2}$ is the borderline case.

- What happens when $0 < s < \frac{1}{2}$? It is expected that nonlocal interactions and nonlocal mean curvature will come into effect.
- What about higher dimensional concentrations (on geodesics, minimal surfaces)? Again there should be a dramatic difference between $s \geq \frac{1}{2}$ and $s < \frac{1}{2}$.

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