FAST AND SLOW DECAY SOLUTIONS FOR SUPERCRITICAL ELLIPTIC PROBLEMS IN EXTERIOR DOMAINS

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ABSTRACT. We consider the elliptic problem $\Delta u + u^p = 0$, $u > 0$ in an exterior domain, $\Omega = \mathbb{R}^N \setminus \mathcal{D}$ under zero Dirichlet and vanishing conditions, where $\mathcal{D}$ is smooth and bounded in $\mathbb{R}^N$, $N \geq 3$, and p is *supercritical*, namely $p > \frac{N+2}{N-2}$. We prove that this problem has infinitely many solutions with *slow decay* $O(|x|^{-\frac{2}{p-1}})$ at infinity. In addition, a solution with *fast decay* $O(|x|^{2-N})$ exists if p is close enough from above to the critical exponent.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

A basic model of nonlinear elliptic boundary problem is the Lane-Emden-Fowler equation,

$$\Delta u + u^p = 0\,,\ u > 0 \quad \text{in } \Omega\,, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega\,, \tag{1.2}$$

where Ω is a domain with smooth boundary in $\mathbb{R}^N$ and $p > 1$. Introduced in the mid 19th century by Lane, an astrophysicist, the role of this and related equations has been broad outside and inside mathematics. While simple looking, the structure of the solution set of this problem may be surprisingly complex. Much has been learned over the last decades, particularly thanks to the development of techniques from the calculus of variations, see [18], but many basic issues remain far from understood. Among those, *solvability above criticality* is a paradigm of the difficulties arising in solving nonlinear elliptic PDEs. An intriguing characteristic of this problem is the role played by the critical exponent $p = \frac{N+2}{N-2}$ in the solvability question. When Ω is *bounded* and $1 < p < \frac{N+2}{N-2}$, compactness of Sobolev's embedding yields a solution as a minimizer of the variational problem

$$\inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^{p+1}\right)^{\frac{2}{p+1}}}. \tag{1.3}$$

When $p \geq \frac{N+2}{N-2}$, compactness is lost, and this minimization procedure fails, as existence does in general: Pohozaev [17] discovered in 1965 that no solution exists if the domain is strictly star-shaped. In 1975, Kazdan and Warner [12] observed that in strong contrast, if Ω is an annulus, $\Omega = \{a < |x| < b\}$, compactness holds for any $p > 1$ within the class of radial functions, and a solution can again be found variationally, regardless the value of p. Solvability for critical and supercritical values of p is thus strongly dependent on special characteristics of the domain under consideration. The critical case $p = \frac{N+2}{N-2}$ can still be handled by variational arguments, since the loss of compactness of Sobolev's embedding is well-understood. In the classical paper [2], Brezis and Nirenberg proved that for $p = \frac{N+2}{N-2}$ that compactness of minimizing sequences in problem (1.3), and hence solvability, is

restored by the addition of suitable linear terms in the equation. Coron [4] and Bahri and Coron [1] established the deep relation between topology and solvability of (1.1)-(1.2) when $p = \frac{N+2}{N-2}$: solvability holds whenever Ω has a non-trivial topology. Nontrivial topology does not suffice for solvability for large supercritical exponents, as shown by an example in [15].

Except for results in domains involving symmetries or exponents close to critical, see for instance [7, 8, 10, 14, 16], solvability of (1.1)-(1.2) in the supercritical case has been a widely open matter, particularly since variational machinery no longer applies, at least in its naturally adapted way for subcritical or critical problems.

In this paper we shall concentrate in Problem (1.1)-(1.2) for exponents p above critical in a special class of domains with nontrivial topology, *exterior domains*, continuing a study initiated in [5]. Let $\mathcal{D}$ be a bounded open set with smooth boundary, such that $\Omega = \mathbb{R}^N \setminus \bar{\mathcal{D}}$ is connected. We consider the problem of finding classical solutions of the problem

$$\Delta u + u^p = 0\,,\ u > 0 \quad \text{in } \mathbb{R}^N \setminus \bar{\mathcal{D}}\,, \tag{1.4}$$

$$u = 0 \ \text{ on } \partial\mathcal{D}\,, \qquad \lim_{|x|\to+\infty} u(x) = 0 \tag{1.5}$$

where $p > \frac{N+2}{N-2}$. The supercritical case is meaningful in this problem since Pohozaev's identity does not pose obstructions for its solvability. To fix ideas, let us consider the simple case $\mathcal{D} = B(0,1)$ and look for radially symmetric solutions to the problem $u = u(r)$, $r = |x|$. The equation

$$\Delta u + u^p = 0 \tag{1.6}$$

then corresponds to the ODE

$$u'' + \frac{N-1}{r}\,u' + u^p = 0\,. \tag{1.7}$$

This equation can be analyzed through phase plane analysis after a transformation introduced by Fowler [9] in 1931: $v(s) = r^{\frac{2}{p-1}}u(r)$, $r = e^s$, which transforms equation (1.7) into the autonomous ODE

$$v'' + \alpha v' - \beta v + v^p = 0 \tag{1.8}$$

where

$$\alpha = N - 2 - \frac{4}{p-1}, \quad \beta = \frac{2}{p-1}\left(N - 2 - \frac{2}{p-1}\right). \tag{1.9}$$

Since α and β are positive for $p > \frac{N+2}{N-2}$, the Hamiltonian energy

$$E(v) = \frac{1}{2}\dot{v}^2 + \frac{1}{p+1}v^{p+1} - \frac{\beta}{2}v^2$$

strictly decreases along trajectories. Using this it is easy to see the existence of a heteroclinic orbit which connects the equilibria $(0,0)$ and $(\beta^{\frac{1}{p-1}},0)$ in the phase plane (v,v'). These equilibria correspond respectively to a saddle point and an attractor. A solution $v(s)$ of (1.8) corresponding to this orbit satisfies $v(-\infty) = 0$, $v(+\infty) = \beta^{\frac{1}{p-1}}$ and $w(r) = r^{-\frac{2}{p-1}}v(\log r)$ solves (1.7) and is bounded at $r = 0$. Then all radial solutions of (1.6) defined in all $\mathbb{R}^N$ have the form

$$w_\lambda(x) := \lambda^{\frac{2}{p-1}}w(\lambda|x|), \quad \lambda > 0. \tag{1.10}$$

We denote in what follows by $w(x)$ the unique positive radial solution

$$\Delta w + w^p = 0 \quad \text{in } \mathbb{R}^N, \quad w(0) = 1. \tag{1.11}$$

Coming back to the analysis for (1.8), we see in phase plane (v, v') the presence of a continuum of orbits that begin on the axis $v = 0$ as close to the equilibrium $(0, 0)$ as we please, which eventually end in the attractor $(\beta^{\frac{1}{p-1}}, 0)$. If $v(s)$ is a solution associated to one of these orbits, then a suitable translation makes it defined in $[0, \infty)$ with $v(0) = 0$. Its associated $u(r)$ then satisfies $u(1) = 0$ and represents a positive solution of problem (1.4)-(1.5) with $\mathcal{D} = B(0, 1)$. The closer the starting point of the orbit is taken from $(0, 0)$, the smaller the associated $v(s)$ gets on compact subsets of $(0, \infty)$, at the same time getting close to the heteroclinic, more precisely the solution $u(|x|)$ is close to some w_λ for small $\lambda > 0$. The solutions u built this way are small in their entire domain and all have the uniform *slow decay*

$$u(x) = \beta^{\frac{1}{p-1}} |x|^{-\frac{2}{p-1}} (1 + o(1)) \quad \text{as } |x| \to \infty,$$

with β given by (1.9). This analysis establishes the existence of a one-parameter, asymptotically vanishing continuum of radial solutions of problem (1.4)-(1.5) with $\mathcal{D} = B(0, 1)$ with slow decay.

We establish in Theorem 1 below that the above mentioned phenomenon is *very robust*. In fact, we have, for arbitrary domain $\mathcal{D}$ the existence of this continuum of slow decay solutions, in particular proving that the supercritical exterior problem (1.4)-(1.5) is *always solvable*.

Theorem 1. *For any $p > \frac{N+2}{N-2}$ there is a continuum of solutions u_λ, $\lambda > 0$, to Problem* (1.4)-(1.5)*, such that*

$$u_\lambda(x) = \beta^{\frac{1}{p-1}} |x|^{-\frac{2}{p-1}} (1 + o(1)) \quad \textit{as } |x| \to \infty \tag{1.12}$$

and $u_\lambda(x) \to 0$ as $\lambda \to 0$, uniformly in $\mathbb{R}^N \setminus \mathcal{D}$.

This result has been proven in [5] when $N \geq 4$ and $p > \frac{N+1}{N-3}$. The above explained analysis of the radial case, makes it natural to seek for a solution u_λ in the form of a small perturbation of w_λ. This naturally leads to construct an inverse of the linearized operator $\Delta + p w_\lambda^{p-1}$ in $\mathbb{R}^N \setminus \mathcal{D}$ under Dirichlet boundary conditions. Since w_λ is small on bounded sets for small λ, such an inverse can be found as a small perturbation of an inverse of this operator in entire $\mathbb{R}^N$. By scaling, it suffices to carry out that analysis for $\lambda = 1$. This inverse indeed exists for $p > \frac{N+1}{N-3}$ and this is the basis of the proof in [5]. However, if $\frac{N+2}{N-2} < p \leq \frac{N+1}{N-3}$ the linearized operator is not surjective, having a range orthogonal to the generators of translations.

We will prove that a further adjustment of the location of the origin, taking as a first approximation $\lambda^{\frac{2}{p-1}} w(\lambda x + \xi)$ and then choosing ξ, indeed produces, after adding a lower order correction, a family of solutions as predicted in Theorem 1. In summary, the structure difference between the cases $p > \frac{N+1}{N-3}$ and $\frac{N+2}{N-2} < p \leq \frac{N+1}{N-3}$ is that in the former case, the solutions found constitute $(N+1)-$paramters family parametrized by a small scaling parameter and a point in $\mathbb{R}^N$, while in the latter it is an one-parameter family only dependent on the small scaling value λ.

The analysis in [5] has a strong resemblance with that in [13] in the construction of singular solutions with prescribed singularities for $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ in bounded

domains. At the radial level, supercritical and subcritical in this range are completely dual: In equation (1.8) β remains positive but α becomes negative. The effect of this is basically to make the phase portraits equivalent, just with arrows inverted in the orbits, with obvious dual consequences. For instance, the inner-subcritical problem in a ball has a classical solution, which in the phase diagram is represented by the unstable manifold of $(0,0)$. Correspondingly, in the supercritical case, the orbit representing the stable manifold of $(0,0)$ corresponds to the unique solution w_* to the exterior problem with *fast decay*, namely w_* satisfies

$$\Delta w_* + w_*^p = 0\,,\ w_* > 0 \quad \text{in } \mathbb{R}^N \setminus \bar{B}_1(0)\,, \tag{1.13}$$

$$w_* = 0 \ \text{ on } \partial B_1(0)\,, \quad \limsup_{|x|\to+\infty} |x|^{2-N} w_*(x) < +\infty\,. \tag{1.14}$$

Since the general inner-subcritical problem always has a solution, obtained by the minimization problem (1.3), it is natural to ask whether existence of a fast decay solution remains true for the domain $\mathbb{R}^N \setminus \bar{\mathcal{D}}$. This may be in general a difficult question which we are able to answer for supercritical powers sufficiently close to critical.

Theorem 2. *There exists a number $p_0 > \frac{N+2}{N-2}$ such that for any $\frac{N+2}{N-2} < p < p_0$, problem (1.4)-(1.5) has a fast decay solution u, $u(x) = O(|x|^{2-N})$ as $|x| \to +\infty$.*

The idea in the proof of Theorem 2 is to consider as an initial approximation the function $\lambda^{\frac{N-2}{2}} w_{**}(\lambda x + \xi)$ where

$$w_{**}(r) = \left(\frac{1}{1 + c_N r^2}\right)^{\frac{N-2}{2}} \tag{1.15}$$

is the unique positive radial solution of the problem

$$\Delta w_{**} + w_{**}^{\frac{N+2}{N-2}} = 0 \quad \text{in } \mathbb{R}^N\,, \quad w_{**}(0) = 1.$$

These scalings will constitute good approximations for small λ if p is sufficiently close to $\frac{N+2}{N-2}$. We prove then that *adjusting both ξ and λ,* produces a solution as desired after addition of a lower order term.

2. The set up for Theorem 1

In what follows of this paper we will assume $\frac{N+2}{N-2} < p \le \frac{N+1}{N-3}$ since the case $p > \frac{N+1}{N-3}$ has already been covered in [5].

By the change of variables

$$\tilde{u}(x) := \lambda^{-\frac{2}{p-1}} u(\frac{x-\xi}{\lambda})$$

and the maximum principle, problem (1.4)-(1.5) becomes equivalent to

$$\Delta \tilde{u} + |\tilde{u}|^p = 0\,,\ \tilde{u} \not\equiv 0 \quad \text{in } \mathbb{R}^N \setminus \bar{\mathcal{D}}_{\lambda,\xi}\,, \tag{2.1}$$

$$\tilde{u} = 0 \ \text{ on } \partial\mathcal{D}_{\lambda,\xi}\,, \quad \lim_{|x|\to+\infty} \tilde{u}(x) = 0 \tag{2.2}$$

where $\lambda > 0$ is small and $\mathcal{D}_{\lambda,\xi}$ is the shrinking domain

$$\mathcal{D}_{\lambda,\xi} = \{\lambda x + \xi \ / \ x \in \mathcal{D}\}.$$

We want to consider the function $w(x)$ in (1.11) as an approximation of a solution of this problem. We need of course a correction so that the boundary condition is satisfied. Thus we let $\varphi_\lambda(x)$ be the unique solution of the problem

$$\Delta\varphi_\lambda = 0 \text{ in } \mathbb{R}^N \setminus \mathcal{D}_{\lambda,\xi}, \quad \varphi_\lambda(x) = w(x) \text{ on } \partial\mathcal{D}_{\lambda,\xi}, \quad \lim_{|x|\to+\infty} \varphi_\lambda(x) = 0 \tag{2.3}$$

and consider $w - \varphi_\lambda$ as a first approximation to a solution of problem (2.1)-(2.2). It is easy to see that

$$\varphi_\lambda(x) = (w(\xi) + O(\lambda))\varphi_0\Big(\frac{x-\xi}{\lambda}\Big) \quad \forall x \in \mathbb{R}^N \setminus \mathcal{D}_{\lambda,\xi} \tag{2.4}$$

where φ_0 is the unique solution of

$$\Delta\varphi_0 = 0 \text{ in } \mathbb{R}^N \setminus \mathcal{D}, \quad \varphi_0(x) = 1 \text{ on } \partial\mathcal{D}, \quad \lim_{|x|\to+\infty} \varphi_0(x) = 0. \tag{2.5}$$

We also note that

$$\lim_{|x|\to+\infty} |x|^{N-2}\varphi_0(x) = f_0 := \frac{1}{(N-2)|S^{N-1}|}\int_{\mathbb{R}^N\setminus\mathcal{D}} |\nabla\varphi_0|^2, \tag{2.6}$$

which in particular implies

$$|\varphi_\lambda(x)| \le C\lambda^{N-2}|x-\xi|^{2-N} \quad \text{for all } x \in \mathbb{R}^N \setminus \mathcal{D}_{\lambda,\xi}.$$

The number $\int_{\mathbb{R}^N\setminus\mathcal{D}} |\nabla\varphi_0|^2$ is by definition the *Newtonian capacity of* $\mathcal{D}$. The latter estimate tells us in particular that the correction is small compared with w as soon as we get away from ξ. Thus we look for a solution to problem (2.1)-(2.2) of the form

$$\tilde{u} = w - \varphi_\lambda + \phi,$$

which yields the following equation for ϕ

$$\begin{cases} \Delta\phi + pw^{p-1}\phi = N(\phi) + E_\lambda \quad \text{in } \mathbb{R}^N \setminus \bar{\mathcal{D}}_{\lambda,\xi}, \\ \\ \phi = 0 \quad \text{on } \partial\mathcal{D}_{\lambda,\xi}, \quad \lim_{|x|\to+\infty} \phi(x) = 0\,, \end{cases} \tag{2.7}$$

where

$$E_\lambda = pw^{p-1}\varphi_\lambda, \quad N(\phi) = -|w+\phi-\varphi_\lambda|^p + w^p + pw^{p-1}\phi - pw^{p-1}\varphi_\lambda. \tag{2.8}$$

Thus a solution of problem (2.7) for which ϕ is small compared with $w - \varphi_\lambda$ yields one of (1.4)-(1.5) as predicted by Theorem 1.

Problem (2.7) may not be solvable in the required range for p unless ξ is chosen in a very special way. Regardless of the value of ξ, we consider instead the following *projected problem*,

$$\begin{cases} \Delta\phi + pw^{p-1}\phi = N(\phi) + E_\lambda + \sum_{i=1}^N c_i Z_i \text{ in } \mathbb{R}^N \setminus \bar{\mathcal{D}}_{\lambda,\xi}\,, \\ \\ \phi = 0 \quad \text{on } \partial\mathcal{D}_{\lambda,\xi}, \quad \lim_{|x|\to+\infty} \phi(x) = 0, \end{cases} \tag{2.9}$$

where the c_i's are constants, which are part of the unknown, and

$$Z_i(x) = \frac{\partial w}{\partial x_i}(x)\,, \quad i = 1, \dots, N.$$

Through an application of the Banach fixed point theorem in a suitable L^∞ weighted space, we shall prove in §5 that (2.9) is indeed solvable, within a class of ϕ's which

are small compared with w, in the form $\phi = \phi(\lambda, \xi)$, $c_i = c_i(\lambda, \xi)$ where the dependence on the parameters is continuous. We then obtain a solution of problem (2.7) if

$$c_i(\lambda, \xi) = 0 \quad \text{for all } i = 1, \dots, N.$$

We will show in §6 that for each sufficiently small λ there is indeed a point ξ such that this system of equations is satisfied.

The use of contraction mapping principle in §5 for solving problem (2.9) is based on the construction of a bounded (right) inverse for the linear problem

$$\begin{cases} \Delta\phi + pw^{p-1}\phi = h + \sum_{i=1}^{N} c_i Z_i & \text{in } \mathbb{R}^N \setminus \bar{\mathcal{D}}_{\lambda,\xi} \\ \lim_{|x|\to+\infty} \phi(x) = 0, \quad \phi = 0 \quad \text{on } \partial\mathcal{D}_{\lambda,\xi}, \end{cases} \tag{2.10}$$

for norms on functions ϕ and h defined on $\mathbb{R}^N \setminus \bar{\mathcal{D}}_{\lambda,\xi}$ given as follows. We consider, for a given number σ with

$$0 < \sigma < N - 2,$$

the norms

$$\|\phi\|_{*,\xi} = \sup_{|x-\xi|\le 1} |x-\xi|^{\sigma} |\phi(x)| + \sup_{|x-\xi|\ge 1} |x-\xi|^{\frac{2}{p-1}} |\phi(x)| \tag{2.11}$$

$$\|h\|_{**,\xi} = \sup_{|x-\xi|\le 1} |x-\xi|^{2+\sigma} |h(x)| + \sup_{|x-\xi|\ge 1} |x-\xi|^{2+\frac{2}{p-1}} |h(x)|\,. \tag{2.12}$$

We have the validity of the following result.

Proposition 2.1. *Assume $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$. Let us consider a number $\Lambda > 0$. Then there exist constants C and λ_0 such that for any $|\xi| \le \Lambda$ and any $0 < \lambda < \lambda_0$ the following holds: For any h with $\|h\|_{**,\xi} < \infty$, there exists a solution of problem* (2.10)

$$(\phi, c_1, \dots, c_N) = \mathcal{T}_\lambda(h)$$

which defines a linear operator of h, such that

$$\|\phi\|_{*,\xi} + \max_{1\le i\le N} |c_i| \le C\|h\|_{**,\xi}.$$

This is proven in §4, on the basis of the analysis of the same problem in the entire space, carried out in §3.

If $p = \frac{N+1}{N-3}$ the proof Theorem 1 is based on a result similar to Proposition 2.1 but for slightly different norms, see Remark 6.1.

A very similar scheme is followed for the proof of Theorem 2, having as its basic cell the function w_{**} in (1.15) rather than w in (1.11). In this case, the relevant projected problem must also involve the generator of dilations, and both the point ξ and the number λ must be determined as functions of the *small parameter* given by the difference $p - \frac{N+2}{N-2}$. This is done in §7.

3. The operator $\Delta + pw^{p-1}$ in $\mathbb{R}^N$

We will keep the notation of the previous section. In particular we assume $N \geq 3$,

$$\frac{N+2}{N-2} < p < \frac{N+1}{N-3}, \quad 0 < \sigma < N-2$$

and consider the norms in (2.11), (2.12) now for functions defined in entire $\mathbb{R}^N$. We consider the version of problem (2.7) in entire space,

$$\begin{cases} \Delta\phi + pw^{p-1}\phi = h + \sum_{i=1}^{N} c_i Z_i & \text{in } \mathbb{R}^N \\ \lim_{|x|\to+\infty} \phi(x) = 0. \end{cases} \tag{3.1}$$

The main result in this section is

Proposition 3.1. *Let us consider a number $\Lambda > 0$. Then there exists a $C > 0$ such that for any $|\xi| \leq \Lambda$ the following holds: For any h with $\|h\|_{**,\xi} < \infty$, there exists a solution of problem* (3.1)

$$(\phi, c_1, \ldots, c_N) = T(h)$$

which defines a linear operator of h, such that

$$\|\phi\|_{*,\xi} + \max_{1\leq i\leq N} |c_i| \leq C\|h\|_{**,\xi}. \tag{3.2}$$

We observe that the numbers c_i above are explicit functions of h. Indeed, since $p < \frac{N+1}{N-3}$, we see that if ϕ solves (3.1) with the bound (3.2) then two integrations by parts against $Z_i = w_{x_i}$ yield

$$c_i = -\frac{\int_{\mathbb{R}^N} hZ_i}{\int_{\mathbb{R}^N} |Z_i|^2}. \tag{3.3}$$

Observe that these quantities are well defined since $\|h\|_{**,\xi} < +\infty$ and $p < \frac{N+1}{N-3}$.

To prove the above result we consider first the situation $\xi = 0$. We denote the corresponding norms simply by $\| \ \|_*$ and $\| \ \|_{**}$. Although the proposition in this case is proven in [5, 6] we summarize the main points of the argument.

By virtue of formula (3.3), it suffices to construct the solution ϕ to problem (3.1) for h with

$$\int_{\mathbb{R}^N} hZ_i = 0 \quad \text{for all } i = 1, \ldots, N, \tag{3.4}$$

so that all numbers c_i are automatically zero.

Let Θ_k, $k \geq 0$ be the eigenfunctions of the Laplace-Beltrami operator $-\Delta_{S^{N-1}}$ on the sphere S^{N-1} with eigenvalues λ_k repeated according to their multiplicity, normalized so that they constitute an othonormal system in $L^2(S^{N-1})$. We let Θ_0 be a positive constant, associated to the eigenvalue 0 and Θ_i, $1 \leq i \leq N$ is an appropriate multiple of $\frac{x_i}{|x|}$ which has eigenvalue $\lambda_i = N-1$, $1 \leq i \leq N$. We write h as

$$h(x) = \sum_{k=0}^{\infty} h_k(r)\Theta_k(\theta), \quad r > 0, \, \theta \in S^{N-1} \tag{3.5}$$

and look for a solution ϕ to (3.1) in the form

$$\phi(x) = \sum_{k=0}^{\infty} \phi_k(r)\Theta_k(\theta)\,.$$

Then

$$\phi_k'' + \frac{N-1}{r}\phi_k' + \left(pw^{p-1} - \frac{\lambda_k}{r^2}\right)\phi_k = h_k, \quad \text{for all } r > 0, \text{ for all } k \geq 0. \tag{3.6}$$

Equation (3.6) can be solved for each k separately:

- If $k = 0$ and $p > \frac{N+2}{N-2}$ then equation (3.6) has a solution ϕ_0 which depends linearly on h_0 and satisfies
$$\|\phi_0\|_* \leq C\|h_0\|_{**}. \tag{3.7}$$
- If $N \geq 3$ and $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$ ($p > \frac{N+2}{N-2}$ if $N = 3$), $\|h\|_{**} < +\infty$ and
$$\int_0^{\infty} h_1(r)w'(r)r^{N-1}\,dr = 0 \tag{3.8}$$
then (3.6) has a solution ϕ_1 depending linearly on h_1 and satisfying
$$\|\phi_1\|_* \leq C\|h_1\|_{**}. \tag{3.9}$$
- Let $k \geq 2$ and $p > \frac{N+2}{N-2}$. If $\|h_k\|_{**} < \infty$ equation (3.6) has a unique solution ϕ_k with $\|\phi_k\|_* < \infty$ and there exists $C_k > 0$ such that
$$\|\phi_k\|_* \leq C_k\|h_k\|_{**}. \tag{3.10}$$

For the case $k = 0$ this solution is defined using the variation of parameters formula

$$\phi_0(r) := z_{1,0}(r)\int_1^r z_{2,0}h_0 s^{N-1}\,ds - z_{2,0}(r)\int_0^r z_{1,0}h_0 s^{N-1}\,ds,$$

where $z_{1,0}$, $z_{2,0}$ are two special linearly independent solutions to (3.6) with $k = 0$ and $h_0 = 0$. More precisely, we take $z_{1,0} = rw' + \frac{2}{p-1}w$ and $z_{2,0}$ a linearly independent solution. Linearization shows that $z_{j,0}(r) = O(r^{-\frac{N-2}{2}})$ as $r \to +\infty$, $j = 1, 2$, while $z_{2,0}(r) \sim r^{2-N}$ near $r = 0$. Using this definition of ϕ_0, we easily get estimate (3.7).

When $k = 1$, we have that the positive function $z_1 := -w'(r)$ solves (3.6) with $k = 1$ and $h_1 = 0$. Using this, we then define $\phi_1(r)$ as

$$\phi_1(r) = -z_1(r)\int_1^r z_1(s)^{-2}s^{1-N}ds\int_0^s z_1(\tau)h_1(\tau)\tau^{N-1}\,d\tau\,. \tag{3.11}$$

Using this formula and the fact that $\int_0^{\infty} z_1(\tau)h_1(\tau)\tau^{N-1}\,d\tau = 0$, estimate (3.9) is readily found to hold true.

The case $k \geq 2$ is simpler because the operator satisfies the maximum principle since the function z_1 above is a positive supersolution for the operator corresponding to any such k.

The previous construction and (3.7), (3.9)and (3.10) imply that given an integer $m > 0$, if $\|h\|_{**} < \infty$ satisfies (3.4) and $h_k \equiv 0$ $\forall k \geq m$ then there exists a solution ϕ to (3.1) that depends linearly with respect to h and moreover

$$\|\phi\|_* \leq C_m\|h\|_{**}$$

where C_m may depend only on m. Then it is possible to show that C_m can be chosen independently of m using a blow up argument that has been used before in [3, 5, 6, 13].

The above steps then yield:

Lemma 3.1. *There exists a number $C > 0$ such that the for any h with $\|h\|_{**} < \infty$, there exists a solution of problem* (3.1)

$$(\phi, c_1, \dots, c_N) = T(h)$$

which defines a linear operator of h, such that

$$\|\phi\|_* + \max_{1\le i\le N} |c_i| \le C\|h\|_{**}. \tag{3.12}$$

The numbers c_i are given by formula (3.3).

Proof of Proposition 3.1. Let η be a smooth cut-off function such that

$$\eta(x) = 0 \text{ for all } |x-\xi| \le \delta, \quad \eta(x) = 1 \text{ for all } |x-\xi| \ge 2\delta,$$

where $\delta > 0$ is small. Then solve

$$-\Delta\phi_2 + pw^{p-1}(1-\eta)\phi_2 = (1-\eta)h \quad \text{in } \mathbb{R}^N, \lim_{|x|\to+\infty} \phi_2(x) = 0.$$

Remark that for $\delta > 0$ sufficiently small but fixed the operator $-\Delta - pw^{p-1}(1-\eta)$ is coercive and hence there exists a solution to this problem. Moreover we have

$$|\phi_2(x)| \le C\|h\|_{**,\xi}\,|x-\xi|^{-\sigma} \quad \text{for all } |x-\xi| \le 1, \tag{3.13}$$

$$|\phi_2(x)| \le C\|h\|_{**,\xi}\,(1+|x|)^{2-N} \quad \text{for all } |x-\xi| \ge 1. \tag{3.14}$$

By Lemma 3.1 the equation

$$\Delta\phi_1 + pw^{p-1}\phi_1 = -pw^{p-1}\eta\phi_2 + \eta h + \sum_{i=1}^N c_i Z_i \quad \text{in } \mathbb{R}^N, \lim_{|x|\to+\infty} \phi_1(x) = 0, \tag{3.15}$$

has a solution provided the right hand side has finite $\|\ \|_{**}$ norm. But since $\eta\phi_2 = 0$ for $|x-\xi| \le \delta$ we see, using (3.13) and (3.14), that

$$\|w^{p-1}\eta\phi_2\|_{**} \le C\|h\|_{**,\xi}.$$

Thus by Lemma 3.1

$$\|\phi_1\|_* + \sum_{i=1}^N |c_i| \le C\|h\|_{**,\xi}. \tag{3.16}$$

This estimate implies that

$$|\phi_1(x)| \le C\|h\|_{**,\xi} \quad \text{for all } |x| = \delta. \tag{3.17}$$

Since the right hand side of (3.15) is bounded, from (3.17) and (3.15), using standard estimates for elliptic equations, we deduce that

$$\|\phi_1\|_{L^\infty(B_\delta)} \le C\|h\|_{**,\xi}. \tag{3.18}$$

Define $\phi = \phi_1 + \phi_2$, which is a solution to (3.1). Then from (3.13), (3.14), (3.16) and (3.18) we see that (3.12) holds. This finishes the proof. □

4. The proof of Proposition 2.1

We will use the result of the previous section in order to prove Proposition 2.1.

We shall fix $\Lambda > 0$ large and work with $|\xi| \le \Lambda$. Again the estimates will depend on ξ only through Λ. As mentioned above, we assume that $0 \in \mathcal{D}$. Let $0 < R_0 < R_1$ be fixed such that $3R_0 < R_1$ and $\mathcal{D} \subset B_{R_0}$. Let $\rho \in C^\infty(\mathbb{R}^N)$, $0 \le \rho \le 1$ be such that

$$\rho(x) = 0 \quad \text{for } |x| \le 1, \quad \rho(x) = 1 \quad \text{for } |x| \ge 2$$

and set

$$\eta_\lambda(x) = \rho\Big(\frac{x-\xi}{\lambda R_0}\Big), \quad \zeta_\lambda(x) = \rho\Big(\frac{x-\xi}{\lambda R_1}\Big).$$

We look for a solution to (2.10) of the form

$$\phi = \eta_\lambda \varphi + \psi.$$

We need then to solve the system of equations

$$\begin{cases} \Delta\varphi + pw^{p-1}\varphi = -pw^{p-1}\zeta_\lambda\psi + \zeta_\lambda h + \displaystyle\sum_{i=1}^N c_i \zeta_\lambda Z_i \quad \text{in } \mathbb{R}^N \\ \lim\limits_{|x|\to+\infty} \varphi(x) = 0 \end{cases} \tag{4.1}$$

$$\begin{cases} \Delta\psi + p(1-\zeta_\lambda)w^{p-1}\psi = -2\nabla\eta_\lambda\nabla\varphi - \varphi\Delta\eta_\lambda + (1-\zeta_\lambda)h \\ \qquad\qquad + \displaystyle\sum_{i=1}^N c_i(1-\zeta_\lambda)Z_i \quad \text{in } \mathbb{R}^N \setminus \bar{\mathcal{D}}_{\lambda,\xi} \\ \psi = 0 \quad \text{on } \partial\mathcal{D}_{\lambda,\xi}, \quad \lim\limits_{|x|\to+\infty} \psi(x) = 0. \end{cases} \tag{4.2}$$

where φ, ψ are the unknowns. We assume $\|h\|_{**,\xi} < \infty$. Let

$$E_\lambda = B_{2\lambda R_0}(\xi) \setminus B_{\lambda R_0}(\xi)$$

and consider the Banach space

$$X = \{(\varphi, c_1, \dots, c_N) \,/\, \varphi : \mathbb{R}^N \to \mathbb{R} \text{ is Lipschitz continuous in } E_\lambda \text{ with } \|\varphi\|_{*,\xi} < \infty \text{ and } c_i \in \mathbb{R},\ 1 \le i \le N\}$$

with the norm

$$\|(\varphi, c_1, \dots, c_N)\|_X = \|\varphi\|_{*,\xi} + \lambda^{1+\sigma}\|\nabla\varphi\|_{L^\infty(E_\lambda)} + \sum_{i=1}^N |c_i|.$$

Given $(\varphi, c_1, \dots, c_N) \in X$ we first note that (4.2) has a solution for suitably small λ because $\|p(1-\zeta_\lambda)w^{p-1}\|_{L^{N/2}(\mathbb{R}^N\setminus\mathcal{D}_{\lambda,\xi})} \to 0$ as $\lambda \to 0$. Let $\psi(\varphi, c_1, \dots, c_N)$ denote this solution, which is clearly linear in φ. Then $\zeta_\lambda\psi$ is well defined in $\mathbb{R}^N$ and $|\psi| \le \frac{C}{|x|^{N-2}}$ for large $|x|$, which implies that the right hand side of (4.1) has a finite $\|\ \|_{**,\xi}$ norm. Then by Proposition 3.1 equation (4.1) has a solution $(\bar\varphi, \bar c_1, \dots, \bar c_N)$ such that $\|\bar\varphi\|_{*,\xi} < +\infty$. Set $F(\varphi, c_1, \dots, c_N) = (\bar\varphi, \bar c_1, \dots, \bar c_N)$.

Proposition 2.1 will be proved by showing that F has a fixed point in X.

For $(\varphi, c_1, \dots, c_N) \in X$ we will first establish a pointwise estimate for the solution $\psi(\varphi, c_1, \dots, c_N)$ of (4.2), namely

$$|\psi(x)| \le C\lambda^{N-2-\sigma}(\|h\|_{**,\xi} + \|(\varphi, c_1, \dots, c_N)\|_X)|x-\xi|^{2-N} \quad \text{for all } x \in \mathbb{R}^N \setminus \mathcal{D}_{\lambda,\xi}. \tag{4.3}$$

Indeed, let $\tilde{\psi}(z) = \psi(\xi + \lambda z)$, $z \in \mathbb{R}^N \setminus \mathcal{D}$. Then

$$\begin{cases} \Delta\tilde{\psi} + p\lambda^2(1-\rho(z/R_1))w^{p-1}(\xi+\delta z)\tilde{\psi} \quad = g \quad \text{in } \mathbb{R}^N \setminus \mathcal{D} \\ \lim\limits_{|z|\to+\infty} \tilde{\psi}(x) = 0 \end{cases} \tag{4.4}$$

where

$$g = -2\frac{\lambda}{R_0}\nabla\rho(z/R_0)\nabla\varphi(\xi+\delta z) - \frac{1}{R_0^2}\Delta\rho(z/R_0)\varphi(\xi+\delta z) + \lambda^2(1-\rho(z/R_1)h(\xi+\delta z) + \lambda^2\sum_{i=1}^N c_i(1-\rho(z/R_1)Z_i(\xi+\delta z).$$

Then the support of g is contained in the ball B_{2R_1} and we can estimate for all $z \in \mathbb{R}^N \setminus \mathcal{D}$, $|z| \le 2R_1$:

$$2\frac{\lambda}{R_0}|\nabla\rho(z/R_0)\nabla\varphi(\xi+\delta z)| \le C\lambda^{-\sigma}\|(\varphi, c_1, \dots, c_N)|_X \tag{4.5}$$

$$|\Delta\rho(z/R_0)\,\varphi(\xi+\lambda z)| \le C\lambda^{-\sigma}\|(\varphi, c_1, \dots, c_N)\|_X|z|^{-\sigma} \tag{4.6}$$

$$\lambda^2|(1-\rho(z/R_1)h(\xi+\lambda z)| \le \lambda^{-\sigma}\|h\|_{**,\xi}|z|^{-2-\sigma} \tag{4.7}$$

$$\lambda^2\sum_{i=1}^N |c_i(1-\rho(z/R_1)Z_i(\xi+\lambda z)| \le C\lambda^2\|(\varphi, c_1, \dots, c_N)\|_X. \tag{4.8}$$

Since $0 \in \mathcal{D}$ we see from (4.5)–(4.8) that

$$|g(z)| \le C\lambda^{-\sigma}(\|\varphi\|_X + \|h\|_{**,\xi})\chi_{B_{2R_1}}.$$

This estimate and equation (4.4) then yield

$$|\tilde{\psi}(z)| \le C(\|h\|_{**,\xi} + \|\varphi\|_X)\lambda^{-\sigma}|z|^{-N+2} \quad \text{for all } z \in \mathbb{R}^N \setminus \mathcal{D}$$

which implies (4.3).

Let $(\varphi, c_1, \dots, c_N) \in X$, $\psi = \psi(\varphi, c_1, \dots, c_N)$ be the solution to (4.2) and $(\bar{\varphi}, \bar{c}_1, \dots, \bar{c}_N) = F(\varphi, c_1, \dots, c_N)$. By Proposition 3.1 we have

$$\|\bar{\varphi}\|_{*,\xi} + \sum_{i=1}^N |\bar{c}_i| \le C(\|pw^{p-1}\zeta_\lambda\psi\|_{**,\xi} + \|\zeta_\lambda h\|_{**,\xi}). \tag{4.9}$$

Using (4.3) we estimate $\|w^{p-1}\zeta_\lambda\psi\|_{**,\xi}$. We have

$$\begin{aligned}\sup_{|x-\xi|\le 1}|x-\xi|^{2+\sigma}w^{p-1}\zeta_\lambda|\psi| &\le C\lambda^{N-2-\sigma}(\|h\|_{**,\xi} + \|(\varphi, c_1, \dots, c_N)\|_X) \\ &\quad \cdot \sup_{\lambda R_1 \le |x-\xi| \le 1}|x-\xi|^{4-N+\sigma} \\ &\le C\lambda^\gamma(\|h\|_{**,\xi} + \|(\varphi, c_1, \dots, c_N)\|_X)\end{aligned} \tag{4.10}$$

where

$$\gamma = \min(2, N-2-\sigma) > 0.$$

On the other hand

$$\sup_{|x-\xi|\geq 1} |x-\xi|^{2+\frac{2}{p-1}}\zeta_\lambda|\psi| \leq C\lambda^{N-2-\sigma}(\|h\|_{**,\xi} + \|(\varphi, c_1, \ldots, c_N)\|_X) \cdot \sup_{|x-\xi|\geq 1} |x-\xi|^{2+\frac{2}{p-1}-N}$$
$$\leq C\lambda^{N-2-\sigma}(\|h\|_{**,\xi} + \|(\varphi, c_1, \ldots, c_N)\|_X). \qquad (4.11)$$

We see from (4.10) and (4.11) that

$$\|w^{p-1}\zeta_\lambda\psi\|_{**,\xi} \leq C\lambda^\gamma(\|h\|_{**,\xi} + \|(\varphi, c_1, \ldots, c_N)\|_X). \qquad (4.12)$$

Since $\|\zeta_\lambda h\|_{**,\xi} \leq \|h\|_{**,\xi}$ from (4.9) and (4.12) we deduce

$$\|\bar{\varphi}\|_{*,\xi} \leq C(\lambda^\gamma\|(\varphi, c_1, \ldots, c_N)\|_X + \|h\|_{**,\xi}).$$

But from elliptic estimates we can prove

$$\sup_{E_\lambda} |\nabla\bar{\varphi}| \leq C\lambda^{-1-\sigma}\|\bar{\varphi}\|_{*,\xi}$$

and hence

$$\|F(\varphi, c_1, \ldots, c_N)\|_X \leq C(\lambda^\gamma\|(\varphi, c_1, \ldots, c_N)\|_X + \|h\|_{**,\xi}).$$

Since F is affine, this estimate shows that F has a unique fixed point $(\varphi, c_1, \ldots, c_N)$ in X for $\lambda > 0$ suitably small, and that this fixed point satisfies

$$\|(\varphi, c_1, \ldots, c_N)\|_X \leq C\|h\|_{**,\xi}.$$

□

Finally we make a remark on how to recognize when $c_i = 0$ in equation (2.10).

Lemma 4.1. *Assume* $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$. *There is* $\varepsilon_0 > 0$ *small such that if* $\lambda < \varepsilon_0$ *and* ϕ *is a solution to* (2.10) *such that* $\|\phi\|_{*,\xi} < +\infty$, $\|h\|_{**,\xi} < +\infty$, *then* $c_i = 0$ *for all* $1 \leq i \leq N$ *if and only if*

$$\int_{\partial\mathcal{D}_{\lambda,\xi}} \frac{\partial\phi}{\partial n}\frac{\partial w}{\partial x_i} + \int_{\mathbb{R}^N\setminus\mathcal{D}_{\lambda,\xi}} h\frac{\partial w}{\partial x_i} = 0 \quad \text{for all } 1 \leq i \leq N.$$

Proof. Since $\frac{\partial w}{\partial x_j}$ satisfies the linear homogeneous equation in $\mathbb{R}^N$, multiplying (2.10) by $\frac{\partial w}{\partial x_j}$ and integrating by parts in $B_R(0)\setminus\mathcal{D}_{\lambda,\xi}$, where R is large, yields

$$\int_{\partial(B_R(0)\setminus\mathcal{D}_{\lambda,\xi})} \left(\frac{\partial\phi}{\partial n}\frac{\partial w}{\partial x_j} - \phi\frac{\partial}{\partial n}\frac{\partial w}{\partial x_j}\right) = \int_{B_R(0)\setminus\mathcal{D}_{\lambda,\xi}} \left(h + \sum_{i=1}^N c_i Z_i\right)\frac{\partial w}{\partial x_j}. \qquad (4.13)$$

Since $\|\phi\|_{**,\xi} < +\infty$ we have

$$|\phi(x)| \leq C|x|^{-\frac{2}{p-1}} \text{ for all } |x| \geq R'$$

and elliptic estimates show that

$$|\nabla\phi(x)| \leq C|x|^{-\frac{2}{p-1}-1} \text{ for all } |x| \geq R'$$

where $R' > 0$ is a large fixed number. Thus

$$\left|\frac{\partial\phi}{\partial n}\frac{\partial w}{\partial x_j} - \phi\frac{\partial}{\partial n}\frac{\partial w}{\partial x_j}\right| \leq C|x|^{-\frac{4}{p-1}-2} \text{ for all } |x| \geq R'$$

and hence, since $p < \frac{N+1}{N-3}$

$$\lim_{R\to+\infty} \int_{\partial B_R(0)} \left(\frac{\partial \phi}{\partial n} \frac{\partial w}{\partial x_j} - \phi \frac{\partial}{\partial n} \frac{\partial w}{\partial x_j} \right) = 0.$$

Letting $R \to +\infty$ in (4.13) yields

$$\sum_{i=1}^{N} c_i \int_{\mathbb{R}^N \setminus \mathcal{D}_{\lambda,\xi}} Z_i \frac{\partial w}{\partial x_j} = - \int_{\mathbb{R}^N \setminus \mathcal{D}_{\lambda,\xi}} h \frac{\partial w}{\partial x_j} - \int_{\partial \mathcal{D}_{\lambda,\xi}} \frac{\partial \phi}{\partial n} \frac{\partial w}{\partial x_j}.$$

For $\lambda > 0$ sufficiently small the matrix with entries $\int_{\mathbb{R}^N \setminus \mathcal{D}_{\lambda,\xi}} Z_i \frac{\partial w}{\partial x_j}$ is close to $\int_{\mathbb{R}^N} Z_i \frac{\partial w}{\partial x_j}$ which is invertible. This implies the desired conclusion. □

5. The nonlinear projected problem (2.9)

Lemma 5.1. *Let $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$ and $\Lambda > 0$. Then there are positive numbers λ_0, C such that for $|\xi| < \Lambda$ and $0 < \lambda < \lambda_0$ there exist $\phi_\lambda(\xi)$, $c_1(\lambda,\xi), \ldots, c_N(\lambda,\xi)$ solution to problem (2.9) such that*

$$\|\phi_\lambda(\xi)\|_{*,\xi} + \max_{1\le i\le N} |c_i(\lambda,\xi)| \le C\lambda^\nu \quad \text{for all } 0 < \lambda < \lambda_0\,, \quad |\xi| < \Lambda, \tag{5.1}$$

where

$$\nu = \min(2+\sigma, N-2).$$

Proof.

Claim. For any fixed $0 < \sigma \le N-2$ we have

$$\|E_\lambda\|_{**,\xi} \le C\lambda^{\min(\sigma+2,N-2)}. \tag{5.2}$$

We assume $0 \in \mathcal{D}$ and let $\delta > 0$ be such that $B_\delta(0) \subset \mathcal{D}$. Then

$$\begin{aligned} &\sup_{|x-\xi|\le 1,\, x\notin \mathcal{D}_{\lambda,\xi}} |x-\xi|^{2+\sigma} \varphi_\lambda(x) w^{p-1}(x) \\ &\qquad \le C\|w^{p-1}\|_{L^\infty} \lambda^{N-2} \sup_{\delta\lambda\le|x-\xi|\le 1} |x-\xi|^{2+\sigma-(N-2)} \\ &\qquad \le C\lambda^{\min(\sigma+2,N-2)}. \end{aligned} \tag{5.3}$$

Also

$$\begin{aligned} \sup_{|x-\xi|\ge 1} |x-\xi|^{2+\frac{2}{p-1}} \varphi_\lambda(x) w^{p-1}(x) &\le C\lambda^{N-2} \sup_{|x-\xi|\ge 1} |x-\xi|^{\frac{2}{p-1}-(N-2)} \\ &\le C\lambda^{N-2} \end{aligned} \tag{5.4}$$

and collecting (5.3) and (5.4) yields (5.2).

Claim. Next, we estimate $\|N(\phi)\|_{**,\xi}$. We shall show that for any fixed $0 < \sigma \le \min(2, \frac{2}{p-1})$ and for $\|\phi\|_{*,\xi} \le 1$ we have

$$\|N(\phi)\|_{**,\xi} \le C(\,\|\phi\|^2_{*,\xi} + \|\phi\|^p_{*,\xi} + \lambda^{\min(\sigma+2,N-2)}). \tag{5.5}$$

Case $p \ge 2$. Assuming $0 < \sigma \le \frac{2}{p-1}$ and using

$$|N(\phi)| \le Cw^{p-2}(|\phi|^2 + |\varphi_\lambda|^2) + C(|\phi|^p + |\varphi_\lambda|^p) \tag{5.6}$$

we have

$$\sup_{|x-\xi|\le 1} |x-\xi|^{2+\sigma}|\phi|^2 \le C\|\phi\|^2_{*,\xi} \sup_{\delta\lambda\le|x-\xi|\le 1} |x-\xi|^{2-\sigma} \le C\|\phi\|^2_{*,\xi} \tag{5.7}$$

$$\begin{aligned}\sup_{|x-\xi|\le 1} |x-\xi|^{2+\sigma}|\phi|^p &\le C\|\phi\|^p_{*,\xi} \sup_{\delta\lambda\le|x-\xi|\le 1} |x-\xi|^{2-\sigma(p-1)} \le C\|\phi\|^p_{*,\xi} \\ &\le C\|\phi\|^2_{*,\xi},\end{aligned} \tag{5.8}$$

since we work with $\|\phi\|_{*,\xi} \le 1$. Similarly to the calculation in (5.3)

$$\sup_{|x-\xi|\le 1} |x-\xi|^{2+\sigma}|\varphi_\lambda|^2 \le C\lambda^{\min(\sigma+2,N-2)} \tag{5.9}$$

$$\sup_{|x-\xi|\le 1} |x-\xi|^{2+\sigma}|\varphi_\lambda|^p \le C\lambda^{\min(\sigma+2,N-2)}. \tag{5.10}$$

The inequalities (5.6)–(5.10) yield, for $p \ge 2$, $0 < \sigma \le \frac{2}{p-1}$ and $\|\phi\|_{*,\xi} \le 1$,

$$\sup_{|x-\xi|\le 1} |x-\xi|^{2+\sigma}|N(\phi)| \le C(\|\phi\|^2_{*,\xi} + \lambda^{\min(\sigma+2,N-2)}). \tag{5.11}$$

Now we consider $|x-\xi| \ge 1$. By the definition of $\|\ \|_{*,\xi}$ and the assumption $\|\phi\|_{*,\xi} \le 1$ we have that

$$|\phi(x)| \le w(x) \quad \text{for all } |x-\xi| \ge 1.$$

Also, for $\lambda > 0$ small

$$\varphi_\lambda(x) \le C\lambda^{N-2}|x-\xi|^{2-N} \le Cw(x) \quad \text{for all } x \in \mathbb{R}^N \setminus \mathcal{D}_{\lambda,\xi}. \tag{5.12}$$

Thus instead of (5.6) we can estimate $N(\phi)$ by

$$|N(\phi)| \le Cw^{p-2}(\phi^2 + \varphi_\lambda^2).$$

Using this inequality and the estimate $w(x) \le C(1+|x|)^{-\frac{2}{p-1}}$ we obtain

$$\sup_{|x-\xi|\ge 1} |x-\xi|^{2+\frac{2}{p-1}} w^{p-2}|\phi|^2 \le C\|\phi\|^2_{*,\xi} \tag{5.13}$$

and

$$\sup_{|x-\xi|\ge 1} |x-\xi|^{2+\frac{2}{p-1}} w^{p-2}|\varphi_\lambda|^2 \le C\lambda^{2(N-2)}. \tag{5.14}$$

Thus, (5.13), (5.14) yield

$$\sup_{|x-\xi|\ge 1} |x-\xi|^{2+\sigma}|N(\phi)| \le C(\|\phi\|^2_{*,\xi} + \lambda^{\min(\sigma+2,N-2)}),$$

and this estimate together with (5.11) prove (5.5) in the case $p \ge 2$.

Case $1 < p < 2$**.** For $0 < \sigma \le 2$ a similar calculation using

$$|N(\phi)| \le C(|\phi|^p + |\varphi_\lambda|^p)$$

implies

$$\sup_{|x-\xi|\le 1} |x-\xi|^{2+\sigma}|N(\phi)| \le C(\|\phi\|^p_{*,\xi} + \lambda^{\min(\sigma+2,N-2)}). \tag{5.15}$$

To estimate $|x-\xi|^{2+\sigma}|N(\phi)|$ for $|x-\xi| \ge 1$ we write

$$-N(\phi) = |w+\phi-\varphi_\lambda|^p - w^p - pw^{p-1}(\phi-\varphi_\lambda) = N_1 + N_2 + pw^{p-1}\varphi_\lambda \tag{5.16}$$

where

$$N_1 = |w+\phi-\varphi_\lambda|^p - |w+\phi|^p, \quad N_2 = |w+\phi|^p - w^p - pw^{p-1}\phi. \tag{5.17}$$

We note that since we assume $\|\phi\|_{*,\xi} \le 1$ we have $|\phi(x)| \le Cw(x)$ for $|x-\xi| \ge 1$ which, together with (5.12) means that we can estimate

$$|N_1| = \big|\, |w+\phi-\varphi_\lambda|^p - |w+\phi|^p \,\big| \le Cw^{p-1}\varphi_\lambda.$$

Then

$$\begin{aligned}\sup_{|x-\xi|\ge 1} |x-\xi|^{2+\frac{2}{p-1}}|N_1| &= \sup_{|x-\xi|\ge 1} |x-\xi|^{2+\frac{2}{p-1}}\big|\, |w+\phi-\varphi_\lambda|^p - |w+\phi|^p \,\big| \\ &\le C \sup_{|x-\xi|\ge 1} |x-\xi|^{2+\frac{2}{p-1}} w^{p-1}\varphi_\lambda \le C\lambda^{\min(\sigma+2,N-2)} \qquad (5.18)\end{aligned}$$

as (5.4) shows. Next we can estimate N_2 as follows

$$\begin{aligned}\sup_{|x-\xi|\ge 1} |x-\xi|^{2+\frac{2}{p-1}}|N_2| &= \sup_{|x-\xi|\ge 1} |x-\xi|^{2+\frac{2}{p-1}}|\, |w+\phi|^p - w^p - pw^{p-1}\phi| \\ &\le C \sup_{|x-\xi|\ge 1} |x-\xi|^{2+\frac{2}{p-1}}|\phi|^p \qquad (5.19)\\ &\le C\|\phi\|^p_{*,\xi}. \qquad (5.20)\end{aligned}$$

Thus, by (5.17)–(5.20) and (5.4) for the last term in (5.16) we deduce

$$\sup_{|x-\xi|\ge 1} |x-\xi|^{2+\frac{2}{p-1}}|N(\phi)| \le C(\|\phi\|^p_{*,\xi} + \lambda^{\min(\sigma+2,N-2)}).$$

This inequality and (5.15) prove (5.5) in the case $1 < p < 2$.

Fixed point argument. We fix $0 < \sigma \le \min(2, \frac{2}{p-1})$ and define for small $\rho > 0$

$$\mathcal{F} = \{\phi : \mathbb{R}^N \setminus \mathcal{D}_{\lambda,\xi} \to \mathbb{R} \mid \|\phi\|_{*,\xi} \le \rho\}$$

and the operator $\bar\phi = \mathcal{A}(\phi)$ where $\bar\phi, c_1, \dots, c_N$ is the solution of Proposition 2.1 to

$$\begin{cases} \Delta\bar\phi + pw^{p-1}\bar\phi = N(\phi) + E_\lambda + \sum_{i=1}^N c_i Z_i & \text{in } \mathbb{R}^N \setminus \mathcal{D}_{\lambda,\xi} \\ \bar\phi = 0 \quad \text{on } \partial\mathcal{D}_{\lambda,\xi}, \qquad \lim_{|x|\to+\infty} |\bar\phi(x)| = 0, \end{cases}$$

where N, E_λ are given by (2.8).

We prove that $\mathcal{A}$ has a fixed point in $\mathcal{F}$. From Proposition 2.1 we have the estimate,

$$\|\mathcal{A}(\phi)\|_{*,\xi} \le C(\|N(\phi)\|_{**,\xi} + \|E_\lambda\|_{**,\xi})$$

and by (5.2) and (5.5)

$$\|\mathcal{A}(\phi)\|_{*,\xi} \le C(\,\|\phi\|^2_{*,\xi} + \|\phi\|^p_{*,\xi} + \lambda^{\min(2+\sigma,N-2)}) \le C(\rho^2 + \rho^p + \lambda^{\min(2+\sigma,N-2)}) \le \rho$$

if $\rho > 0$ is fixed suitably small and then one considers $\lambda \to 0$. This proves $\mathcal{A}(\mathcal{F}) \subset \mathcal{F}$.

Now we show that $\mathcal{A}$ is a contraction mapping in $\mathcal{F}$. Let us take ϕ_1, ϕ_2 in $\mathcal{F}$. Then

$$\|\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)\|_{*,\xi} \le C\,\|N(\phi_1) - N(\phi_2)\|_{**,\xi}\,. \qquad (5.21)$$

Write

$$N(\phi_1) - N(\phi_2) = D_{\bar\phi}N(\bar\phi)(\phi_1 - \phi_2)$$

where $\bar\phi$ lies in the segment joining ϕ_1 and ϕ_2. Then, for $|x-\xi| \le 1$,

$$|x-\xi|^{2+\sigma}\,|\,N(\phi_1) - N(\phi_2)\,| \le |x-\xi|^2|\,D_{\bar\phi}N(\bar\phi)\,|\,\|\phi_1-\phi_2\|_{*,\xi},$$

while, for $|x-\xi| \geq 1$,

$$|x-\xi|^{2+\frac{2}{p-1}} \, |\, N(\phi_1) - N(\phi_2)\,| \;\leq\; |x-\xi|^2 \, |D_{\bar\phi} N(\bar\phi)\,| \, \|\phi_1 - \phi_2\|_{*,\xi}\,.$$

Then we have

$$\|\, N(\phi_1) - N(\phi_2)\,\|_{**,\xi} \;\leq\; C \sup_{x\in\mathbb{R}^N\setminus\mathcal{D}_{\lambda,\xi}} (|x|^2 |D_{\bar\phi} N(\bar\phi)|)\, \|\, \phi_1 - \phi_2\,\|_{*,\xi}\,. \tag{5.22}$$

Directly from the definition of N, we compute

$$D_{\bar\phi} N(\phi) = -p\left[(w+\bar\phi-\varphi_\lambda)^{p-1} - w^{p-1}\right].$$

If $p \geq 2$ and $0 < \sigma \leq \frac{2}{p-1}$ we can use $|D_\phi N(\bar\phi)| \leq C(w^{p-2}(|\bar\phi|+\varphi_\lambda) + |\bar\phi|^{p-1} + \varphi_\lambda^{p-1})$ to estimate

$$\begin{aligned}
\sup_{|x-\xi|\leq 1} |x-\xi|^2\, |D_\phi N(\bar\phi)| &\leq C|x-\xi|^2 (w^{p-2}(|\bar\phi(x)|+\varphi_\lambda) + |\bar\phi(x)|^{p-1} + \varphi_\lambda^{p-1}) \\
&\leq C(\|\phi_1\|_{*,\xi} + \|\phi_2\|_{*,\xi} + \lambda^{\min(2,N-2)}) \\
&\leq C(\rho + \lambda^{\min(2,N-2)}).
\end{aligned} \tag{5.23}$$

In the region $|x-\xi| \geq 1$ we can use $|D_\phi N(\bar\phi)| \leq C w^{p-2}(|\bar\phi| + \varphi_\lambda)$ and we obtain

$$\sup_{|x-\xi|\geq 1} |x-\xi|^2\, |D_\phi N(\bar\phi)| \leq C(\rho + \lambda^{\min(2,N-2)}). \tag{5.24}$$

Similarly, if $1 < p < 2$ and $0 < \sigma \leq \frac{2}{p-1}$ then for all $x \in \mathbb{R}^N \setminus \mathcal{D}_{\lambda,\xi}$

$$\begin{aligned}
|x|^2 |\, D_{\bar\phi} N(\bar\phi)| &\leq C|x|^2 (|\bar\phi(x)|^{p-1} + \varphi_\lambda^{p-1}) \\
&\leq C\lambda^{-2}(\|\phi_1\|_{*,0}^{p-1} + \|\phi_2\|_{*,0}^{p-1} + \lambda^2) \leq C(\rho^{p-1} + \lambda^2).
\end{aligned} \tag{5.25}$$

Estimates (5.23)–(5.25) show that

$$\sup_{x\in\mathbb{R}^N\setminus\mathcal{D}_{\lambda,\xi}} (|x|^2 |D_\phi N(\bar\phi)|) \leq C(\rho + \rho^{p-1} + \lambda^{\min(2,N-2)}). \tag{5.26}$$

Gathering relations (5.21), (5.22) and (5.26) we conclude that $\mathcal{A}$ is a contraction mapping in $\mathcal{F}$ provided $\rho > 0$ is fixed suitably small, and hence it has unique fixed point in this set.

Claim. Let $\phi_\lambda \in \mathcal{F}$ denote the fixed point of $\mathcal{A}$ found in the previous step. For any fixed $0 < \sigma < N-2$ we have

$$\|\phi_\lambda\|_{*,\xi,\sigma} \leq C\lambda^{\min(2+\sigma,N-2)} \tag{5.27}$$

where for convenience, we emphasize the dependence on σ in the notation of the norm $\|\ \|_{*,\xi}$.

From the previous step we see that $\|\phi_\lambda\|_{*,\xi,\sigma} \leq C\lambda^{\min(2+\sigma,N-2)}$ for $\sigma > 0$ small. Actually we will fix $0 < \sigma < \frac{2}{p}$ for the rest of the proof. In order to improve the estimate of the fixed point ϕ_λ we need to estimate better $N(\phi_\lambda)$. First we observe that ϕ_λ is uniformly bounded. Indeed, the function $u_\lambda = w - \varphi_\lambda + \phi_\lambda$ solves

$$\begin{cases}
\Delta u_\lambda + u_\lambda^p = \displaystyle\sum_{i=1}^N c_i(\lambda,\xi) Z_i \quad \text{in } \mathbb{R}^N \setminus \bar{\mathcal{D}}_{\lambda,\xi} \\
\displaystyle\lim_{|x|\to+\infty} u_\lambda(x) = 0,\ u_\lambda = 0 \quad \text{on } \partial\mathcal{D}_{\lambda,\xi}.
\end{cases} \tag{5.28}$$

For x with $|x-\xi|=1$ $u_\lambda(x)$ remains bounded because $|\phi_\lambda(x)| \le C$ for $|x-\xi|=1$. Then a uniform upper bound for u_λ follows from (5.28) and by observing that $\|u_\lambda^p\|_{L^q(B_1(\xi)\setminus\mathcal{D}_{\lambda,\xi})}$ remains bounded as $\lambda \to 0$ for $q > \frac{N}{2}$. In fact

$$\int_{B_1(\xi)\setminus\mathcal{D}_{\lambda,\xi}} u_\lambda^{pq} \le C\int_{B_1} w^{pq} + |\phi_\lambda|^{pq} \le C + C\int_{B_1(\xi)\setminus\mathcal{D}_{\lambda,\xi}} |x|^{-\sigma pq}\,dx \le C$$

for some $q > \frac{N}{2}$ if we choose $\sigma < \frac{2}{p}$, as we have done . Hence

$$|u_\lambda(x)| \le C \quad \text{for all } |x-\xi| \le 1. \tag{5.29}$$

It follows from (5.29) that

$$|\phi_\lambda(x)| \le C \quad \text{for all } x \in \mathbb{R}^N \setminus \mathcal{D}_{\lambda,\xi}. \tag{5.30}$$

We shall estimate $\|\phi_\lambda\|_{*,\xi,\theta}$ for a $\theta > \sigma$. Since ϕ_λ is a fixed point of $\mathcal{A}$, if $0 < \theta < N-2$ we have, by (5.2)

$$\begin{aligned}\|\phi_\lambda\|_{*,\xi,\theta} = \|\mathcal{A}(\phi_\lambda)\|_{*,\xi,\theta} &\le C(\|N(\phi_\lambda)\|_{**,\xi,\theta} + \|E_\lambda\|_{**,\xi,\theta}) \\ &\le C\|N(\phi_\lambda)\|_{**,\xi,\theta} + C\lambda^{\min(2+\theta,N-2)}.\end{aligned} \tag{5.31}$$

Since ϕ_λ is uniformly bounded, when $p \ge 2$

$$|N(\phi_\lambda)| \le C(|\phi_\lambda|^2 + \varphi_\lambda^2). \tag{5.32}$$

Take $0 < \theta < N-2$ such that $2+\theta \ge 2\sigma$. Then by (5.27) we have

$$\begin{aligned}\sup_{\delta\lambda \le |x-\xi| \le 1} |x-\xi|^{2+\theta}|\phi_\lambda(x)| &\le C\|\phi_\lambda\|_{*,\xi,\sigma}^2 \sup_{\lambda \le |x-\xi| \le 1} |x-\xi|^{2+\theta-2\sigma} \\ &\le C\lambda^{2\min(2+\sigma,N-2)}.\end{aligned} \tag{5.33}$$

On the other hand

$$\sup_{|x-\xi|\ge 1} |x-\xi|^{2+\frac{2}{p-1}}|N(\phi_\lambda(x))| \le C\|\phi_\lambda\|_{*,\xi,\sigma}^2 \le C\lambda^{2\min(2+\sigma,N-2)}. \tag{5.34}$$

Thus, from (5.32)–(5.34), (5.9) and (5.14) we see that

$$\|N(\phi_\lambda)\|_{**,\xi,\theta} \le C\lambda^{2\min(2+\sigma,N-2)}.$$

This and (5.31) imply

$$\|\phi_\lambda\|_{*,\xi,\theta} \le C\lambda^{\min(2+\theta,2(2+\sigma),N-2)}.$$

provided $0 < \theta < N-2$, $\theta \ge 2\sigma - 2$. Repeating this argument a finite number of times we deduce the validity of (5.1) in the case $p \ge 2$.

If $p < 2$ instead of (5.32), using

$$|N(\phi_\lambda)| \le C|\phi_\lambda|^p$$

we obtain

$$\|N(\phi_\lambda)\|_{**,\xi,\theta} \le C\lambda^{\min(2+\theta,p(2+\sigma),N-2)}$$

and the same argument as before yields the conclusion. □

6. The proof of Theorem 1

We will present in what follows the detailed proof in under the assumption

$$\frac{N+2}{N-2} < p < \frac{N+1}{N-3}.$$

For the case $p = \frac{N+1}{N-3}$ see Remark 6.1.

We have found a solution $\phi_\lambda(\xi)$, $c_1(\lambda,\xi),\dots,c_N(\lambda,\xi)$ to (2.9). By Lemma 4.1 the solution constructed satisfies for all $1 \le j \le N$:

$$\int_{\mathbb{R}^N\setminus(\mathcal{D}_{\lambda,\xi})} \left(E_\lambda + N(\phi_\lambda) + \sum_{i=1}^N c_i Z_i\right)\frac{\partial w}{\partial x_j} + \int_{\partial\mathcal{D}_{\lambda,\xi}} \frac{\partial\phi_\lambda}{\partial n}\frac{\partial w}{\partial x_j} = 0$$

Thus, for all λ small, we need to find $\xi = \xi_\lambda$ so that $c_i = 0$, $1 \le i \le N$, that is

$$\int_{\mathbb{R}^N\setminus(\mathcal{D}_{\lambda,\xi})} (E_\lambda + N(\phi_\lambda))\frac{\partial w}{\partial x_j} + \int_{\partial\mathcal{D}_{\lambda,\xi}} \frac{\partial\phi_\lambda}{\partial n}\frac{\partial w}{\partial x_j} = 0 \quad \forall 1 \le j \le N. \tag{6.1}$$

Let us define

$$G_j(\xi) := \int_{\mathbb{R}^N\setminus(\mathcal{D}_{\lambda,\xi})} \Big(E_\lambda + N(\phi_\lambda)\Big)\frac{\partial w}{\partial x_j} + \int_{\partial\mathcal{D}_{\lambda,\xi}} \frac{\partial\phi_\lambda}{\partial n}\frac{\partial w}{\partial x_j}. \tag{6.2}$$

The functions G_j are continuous, as it follows from local uniqueness, the fixed point characterization of ϕ_λ and elliptic estimates. We claim that

$$G_j(\xi) = f_0\lambda^{N-2}\int_{\mathbb{R}^N} |x-\xi|^{-(N-2)} w(x)^{p-1}\frac{\partial w}{\partial x_j}(x) + o(\lambda^{N-2}) \tag{6.3}$$

uniformly for ξ on compact sets of $\mathbb{R}^N$. This fact follows observing first that

$$\int_{\mathbb{R}^N\setminus(\mathcal{D}_{\lambda,\xi})} \left|N(\phi_\lambda)\frac{\partial w}{\partial x_j}\right| = o(\lambda^{N-2}) \quad \text{as } \lambda \to 0 \tag{6.4}$$

uniformly for ξ on compact sets of $\mathbb{R}^N$. Indeed,

$$\int_{\mathbb{R}^N\setminus(\mathcal{D}_{\lambda,\xi})} \left|N(\phi_\lambda)\frac{\partial w}{\partial x_j}\right| = \int_{B_1(\xi)\setminus(\mathcal{D}_{\lambda,\xi})} \dots + \int_{\mathbb{R}^N\setminus B_1(\xi)} \dots.$$

In the case $p \ge 2$, by (5.1), we have for $\sigma < N/2$

$$\int_{B_1(\xi)\setminus(\mathcal{D}_{\lambda,\xi})} \left|N(\phi_\lambda)\frac{\partial w}{\partial x_j}\right| \le \|\phi_\lambda\|^2_{*,\xi}\int_{B_1(\xi)\setminus(\mathcal{D}_{\lambda,\xi})} |x-\xi|^{-2\sigma} \le C\lambda^{2\min(2+\sigma,N-2)}$$

and recalling that $|N(\phi_\lambda)| \le Cw^{p-2}|\phi_\lambda|^2$

$$\int_{\mathbb{R}^N\setminus B_1(\xi)} \left|N(\phi_\lambda)\frac{\partial w}{\partial x_j}\right| \le C\|\phi_\lambda\|^2_{*,\xi}\int_{\mathbb{R}^N\setminus B_1(\xi)} |x-\xi|^{-\frac{4}{p-1}-3} \le C\lambda^{2\min(2+\sigma,N-2)}.$$

Choosing $\frac{N-2}{2} < \sigma < \min(N-2, N/2)$ we obtain (6.4) in the case $p \ge 2$.

Similarly, if $p < 2$ we have for $0 < \sigma < N/p$

$$\int_{\mathbb{R}^N} \left|N(\phi_\lambda)\frac{\partial w}{\partial w_j}\right| = O(\lambda^{p\min(2+\sigma,N-2)}) \quad \text{as } \lambda \to 0,$$

and taking $(N-2)/p < \sigma < \min(N-2, N/p)$ we still obtain (6.4).

Next we need to estimate the boundary integral of (6.2). We claim that

$$\left|\frac{\partial\phi_\lambda}{\partial n}(x)\right| = O(\lambda^{\min(1,N-3-\sigma)}) \quad \text{uniformly for } x \in \partial\mathcal{D}_{\lambda,\xi}. \tag{6.5}$$

Let

$$\tilde{\phi}_\lambda(z) = \phi_\lambda(\xi + \lambda z) \quad \text{for all } z \in \mathbb{R}^N \setminus \mathcal{D}.$$

Note that by (5.1), for $0 < \sigma < N-2$

$$|\tilde{\phi}_\lambda(z)| \le \|\phi_\lambda\|_{*,\xi} \lambda^{-\sigma} |z|^{-\sigma} \le C\lambda^{\min(2,N-2-\sigma)} |z|^{-\sigma} \quad \text{for all } |z| \le \frac{1}{\lambda}.$$

Moreover we have already observed that ϕ_λ is uniformly bounded (c.f. (5.30)) and this implies, using (2.9) that $|\Delta\phi_\lambda| \le C$ in $\mathbb{R}^N \setminus \mathcal{D}_{\lambda,\xi}$. It follows that $\tilde{\phi}_\lambda$ satisfies

$$|\Delta\tilde{\phi}_\lambda| \le C\lambda^2 \quad \text{in } \mathbb{R}^N \setminus \mathcal{D}.$$

By elliptic estimates

$$\sup_{\partial\mathcal{D}} |\nabla\tilde{\phi}_\lambda| \le C\lambda^{\min(2,N-2-\sigma)},$$

which proves (6.5). Using this inequality we derive

$$\int_{\partial\mathcal{D}_{\lambda,\xi}} \frac{\partial\phi_\lambda}{\partial n} \frac{\partial w}{\partial x_j} = O(\lambda^{\min(N,2(N-2)-\sigma)}).$$

This fact together with (6.4) prove the claim made in (6.3).

Let us consider the vector field

$$G(\xi) = (G_1(\xi), \dots, G_N(\xi)).$$

G is then continuous and, thanks to (6.3),

$$G(\xi) \cdot \xi < 0 \quad \text{for all } |\xi| = R$$

for any fixed small $R > 0$. Using this and degree theory we obtain the existence of ξ such that $c_i = 0$, $1 \le i \le N$. This concludes the proof. □

Remark 6.1. The proof of Theorem 1 in the case $p = \frac{N+1}{N-3}$ follows exactly the same lines with the following modified norms:

$$\|\phi\|_{*,\xi} = \sup_{|x-\xi|\le 1} |x-\xi|^\sigma |\phi(x)| + \sup_{|x-\xi|\ge 1} |x-\xi|^{\frac{2}{p-1}+\alpha} |\phi(x)|$$

$$\|h\|_{**,\xi} = \sup_{|x-\xi|\le 1} |x-\xi|^{2+\sigma} |h(x)| + \sup_{|x-\xi|\ge 1} |x-\xi|^{2+\frac{2}{p-1}+\alpha} |h(x)|$$

where $\alpha > 0$ is a small fixed number. With this slightly stronger norms Proposition 2.1 remains valid. Indeed, the stronger decay of h assures that the orthogonality condition (3.4) makes sense and one can verify that the estimates derived in §3 and the proof in §4 carry on. Moreover even with the modified norms the error $\|E_\lambda\|_{**,\xi}$ converges to zero. For this observe that following the calculation starting at (5.2):

$$\begin{aligned} \sup_{|x-\xi|\ge 1} |x-\xi|^{2+\frac{2}{p-1}+\alpha} \varphi_\lambda(x) w^{p-1}(x) &\le C\lambda^{N-2} \sup_{|x-\xi|\ge 1} |x-\xi|^{\frac{2}{p-1}-(N-2)+\alpha} \\ &\le C\lambda^{N-2} \end{aligned}$$

provided $0 < \alpha < N-2-\frac{2}{p-1}$.

7. The proof of Theorem 2

In this section we construct fast decay solutions to problem (1.1)-(1.2) when the exponent p is close to the Sobolev critical exponent $\frac{N+2}{N-2}$. In this case, we denote

$$p = q + \varepsilon, \quad q = \frac{N+2}{N-2}$$

where $\varepsilon > 0$ is small.

The proof of Theorem 2 is similar to that of Theorem 1, except that we need to adjust also the parameter λ.

The basic cell to construct a fast decay solution is the function w_{**} given by (1.15). For simplicity, but with slight abuse of notation, we will denote in what follows this function simply by w.

The main difference with the case treated in the previous sections, when p was a fixed exponent strictly above $\frac{N+2}{N-2}$, arises in the linearized problem. More precisely, in order to construct a proper inverse at mode 0 when the exponent is exactly the critical Sobolev exponent, an extra orthogonality condition is needed. The right hand side is now required to be orthogonal also to the generator of dilation, the function

$$z_0(r) = rw'(r) + \frac{N-2}{2} w.$$

We will denote now $Z_i = \eta w_{x_i}$, $i = 1, \dots, N$, for $\eta \in C_0^\infty(\mathbb{R}^N)$, $0 \le \eta \le 1$

$$\eta(x) = 1 \quad \text{for } |x| \le R_0, \quad \eta(x) = 0 \quad \text{for } |x| \ge R_0 + 1,$$

with $R_0 > 0$ fixed large enough. We define

$$Z_0 = \eta z_0.$$

For given $\xi \in \mathbb{R}$ and λ small, we first study existence and estimates for solutions $(\phi, c_0, c_1, \dots, c_N)$ to the problem

$$\begin{cases} \Delta\phi + qw^{q-1}\phi = N(\phi) + E + c_0 Z_0 + \sum_{i=1}^N c_i Z_i \text{ in } \mathbb{R}^N \setminus \bar{\mathcal{D}}_{\lambda,\xi} \\ \phi = 0 \text{ on } \partial\mathcal{D}_{\lambda,\xi}, \ \lim_{|x|\to+\infty} \phi(x) = 0. \end{cases} \tag{7.1}$$

Here

$$\begin{aligned} N(\phi) = &-|w - \varphi_\lambda + \phi|^{q+\varepsilon} + w^{q+\varepsilon} + (q+\varepsilon)w^{q+\varepsilon-1}(\phi - \varphi_\lambda) \\ &-[(q+\varepsilon)w^{q+\varepsilon-1} - qw^{q-1}]\phi - [qw^{q-1} - (q+\varepsilon)w^{q+\varepsilon-1}]\varphi_\lambda \end{aligned} \tag{7.2}$$

and

$$E = -w^{q+\varepsilon} + w^q + qw^{q-1}\varphi_\lambda. \tag{7.3}$$

Appropriate norms in this case are

$$\begin{aligned} \|\phi\|_{*,\xi} &= \sup_{|x-\xi|\le 1} |x-\xi|^\sigma |\phi(x)| + \sup_{|x-\xi|\ge 1} |x-\xi|^{N-2} |\phi(x)|, \\ \|h\|_{**,\xi} &= \sup_{|x-\xi|\le 1} |x-\xi|^{2+\sigma} |h(x)| + \sup_{|x-\xi|\ge 1} |x-\xi|^{N+1} |h(x)|. \end{aligned}$$

We will need to estimate the $\|\cdot\|_{**,\xi}$-norm of $N(\phi)$ and E.

Claim. If $0 < \sigma \leq \min(2, \frac{2}{p-1})$ there exists a positive constant C such that, if $\nu = \min(N-2, \sigma+2)$,

$$\|N(\phi)\|_{**,\xi} \leq C\left(\|\phi\|^2_{*,\xi} + \|\phi\|^q_{*,\xi} + \lambda^\nu + \varepsilon\|\phi\|_{*,\xi} + \varepsilon\lambda^\nu\right) \tag{7.4}$$

and

$$\|E\|_{**,\xi} \leq C\left(\lambda^\nu + \varepsilon\right). \tag{7.5}$$

Taking into account (5.5), in order to get (7.4) we are left to estimate the terms which appear in the second line of formula (7.2).

We write first

$$[(q+\varepsilon)w^{q+\varepsilon-1} - qw^{q-1}]\phi = A + B,$$

with

$$A = \varepsilon w^{q-1+\varepsilon}\phi \quad \text{and} \quad B = qw^{q-1}(w^\varepsilon - 1)\phi.$$

Then

$$\sup_{|x-\xi|\leq 1} |x-\xi|^{2+\sigma} A \leq C\varepsilon\|\phi\|_{*,\xi},$$

and

$$\sup_{|x-\xi|\geq 1} |x-\xi|^{N+1} A \leq C\varepsilon\|\phi\|_{*,\xi} \sup_{|x-\xi|\geq 1} |x-\xi|^{N+1-4-(N-2)} \leq C\varepsilon\|\phi\|_{*,\xi}.$$

Observe now that

$$\sup_{|x-\xi|\leq 1} |x-\xi|^{2+\sigma} B \leq C\varepsilon\|\phi\|_{*,\xi},$$

and

$$\sup_{|x-\xi|\geq 1} |x-\xi|^{N+1} B \leq C\varepsilon\|\phi\|_{*,\xi} \sup_{|x-\xi|\geq 1} |x-\xi|^{N+1-(N-2)} w^{q-1}|\log w| \leq C\varepsilon\|\phi\|_{*,\xi}.$$

These facts give the third term in the right hand side of (7.4).

The last term in (7.2) can be decomposed

$$[qw^{q-1} - (q+\varepsilon)w^{q+\varepsilon-1}]\varphi_\lambda = A + B$$

with

$$A = -\varepsilon w^{q-1}\varphi_\lambda \quad \text{and} \quad B = (q+\varepsilon)w^{q-1}(1-w^\varepsilon)\varphi_\lambda.$$

So we have

$$\sup_{|x-\xi|\leq 1} |x-\xi|^{2+\sigma} A \leq C\varepsilon\|w^{q-1}\|_\infty \lambda^{N-2} \sup_{\lambda\delta\leq|x-\xi|\leq 1} |x-\xi|^{2+\sigma-(N-2)} \leq C\varepsilon\lambda^\nu$$

and

$$\sup_{|x-\xi|\geq 1} |x-\xi|^{N+1} A \leq C\varepsilon\lambda^{N-2} \sup_{|x-\xi|\geq 1} |x-\xi|^{N+1-4-(N-2)} \leq C\varepsilon\lambda^{N-2}.$$

In a very analogous way, we obtain

$$\|B\|_{**,\xi} \leq C\varepsilon\lambda^\nu,$$

from which (7.4) follows.

We next show (7.5). We have

$$\sup_{|x-\xi|\leq 1} |x-\xi|^{2+\sigma} |w^{q+\varepsilon} - w^q| \leq C\varepsilon$$

and

$$\sup_{|x-\xi|\geq 1} |x-\xi|^{N+1} |w^{q+\varepsilon} - w^q| \leq C\varepsilon \sup_{|x-\xi|\geq 1} |x-\xi|^{N+1} w^q|\log w| \leq C\varepsilon.$$

Estimate (7.5) thus follows from an appropriate modification of the argument that leads to (5.2).

We are now ready to solve problem (7.1).

Lemma 7.1. *Let $\Lambda > 0$. Then there is $\varepsilon_0 >$ such that for $0 < \varepsilon < \varepsilon_0$, $|\xi| < \Lambda$ and $\lambda < \varepsilon_0$ there exist $\phi, c_0, \ldots, c_N$ solution to*

$$\begin{cases} \Delta\phi + qw^{q-1}\phi = N(\phi) + E + c_0 Z_0 + \sum_{i=1}^{N} c_i Z_i \text{ in } \mathbb{R}^N \setminus \bar{\mathcal{D}}_{\lambda,\xi} \\ \phi = 0 \text{ on } \partial\mathcal{D}_{\lambda,\xi}, \quad \lim_{|x|\to+\infty} \phi(x) = 0. \end{cases} \tag{7.6}$$

We have in addition

$$\|\phi\|_{*,\xi} + \max_{0\le i\le N} |c_i| \to 0 \quad \text{as } \lambda + \varepsilon \to 0,$$

and

$$\|\phi\|_{*,\xi} \le C(\lambda^\nu + \varepsilon), \quad \text{for all } 0 < \lambda < \lambda_0 \tag{7.7}$$

where

$$0 < \sigma < N-2, \quad \nu = \min(2+\sigma, N-2). \tag{7.8}$$

Proof. A first step is to solve the linear problem in $\mathbb{R}^N$. We have

Fact 1. *Let $|\xi| \le \Lambda$, $q = \frac{N+2}{N-2}$ and $0 < \sigma < N-2$. There is a linear map $(\phi, c_1, \ldots, c_N) = T(h)$ defined whenever $\|h\|_{**,\xi} < \infty$ such that*

$$\begin{cases} \Delta\phi + qw^{q-1}\phi = h + c_0 Z_0 + \sum_{i=1}^{N} c_i Z_i \quad \text{in } \mathbb{R}^N \\ \lim_{|x|\to+\infty} \phi(x) = 0 \end{cases} \tag{7.9}$$

and

$$\|\phi\|_{*,\xi} + \sum_{i=1}^{N} |c_i| \le C\|h\|_{**,\xi} \tag{7.10}$$

Moreover, $c_i = 0$ for all $0 \le i \le N$ if and only if h satisfies

$$\int_{\mathbb{R}^N} h z_0 = 0, \quad \int_{\mathbb{R}^N} h \frac{\partial w}{\partial x_i} = 0 \quad \forall 1 \le i \le N. \tag{7.11}$$

The proof of this fact follows exactly the steps to prove Proposition 3.1, except for the fact that the inverse in mode 0 exists under the extra orthogonality condition with respect to Z_0. Write ϕ and h in Fourier series as in Section 3, so that (7.9) yields

$$\phi_k'' + \frac{N-1}{r}\phi_k' + \left(pw^{q-1} - \frac{\lambda_k}{r^2}\right)\phi_k = h_k, \quad \text{for all } r > 0, \text{ for all } k \ge 0. \tag{7.12}$$

If $k = 0$, $\|h\|_{**} < +\infty$ and

$$\int_0^\infty h_0(r) z_0(r) r^{N-1}\, dr = 0 \tag{7.13}$$

then equation (7.12) has a solution ϕ_0 which depends linearly on h_0 and satisfies

$$\|\phi_0\|_* \le C\|h_0\|_{**}, \tag{7.14}$$

where in this case

$$\|\phi_0\|_* = \sup_{|x|\le 1} |x|^{\sigma}|\phi(x)| + \sup_{|x|\ge 1} |x|^{N-2}|\phi(x)|$$

and

$$\|h_0\|_{**} = \sup_{|x|\le 1} |x|^{2+\sigma}|h(x)| + \sup_{|x|\ge 1} |x|^{N+1}|h(x)|.$$

Indeed, since $|z_0(r)| \le Cr^{-(N-2)}$, we have that $\int_0^\infty h_0(r)z_0(r)r^{N-1} < \infty$, so that

$$\phi_0(r) = -z_0(r)\int_1^r z_0^{-2}(s)s^{1-N}\int_s^\infty z_0(\tau)h_0(\tau)\tau^{N-1}d\tau ds$$

solves (7.12) for $k = 0$ and satisfies (7.14).

Furthermore, observe that the construction of the inverse for mode 1 with the orthogonality condition with respect to Z_i, for $i = 1, \dots, N$, is still valid and that the corresponding estimate (3.9) holds true in the new norms. Indeed, taking into account that $z_1(r)h_1(r)r^{N-1}$ is integrable in $(0, \infty)$, where $z_1 = -w'$, and that the orthogonality condition holds true, we have that

$$\phi_1(r) = -w'(r)\int_1^r (w')^{-2}(s)s^{1-N}\int_s^\infty w'(\tau)h_1(\tau)\tau^{N-1}d\tau ds$$

solves (7.12) for $k = 1$ and satisfies, in the new norms (see (7.14))

$$\|\phi_1\|_* \le C\|h_1\|_{**}. \tag{7.15}$$

Fact 2. *Assume $q = \frac{N+2}{N-2}$, $0 < \sigma < N-2$ and let $|\xi| \le \Lambda$. Suppose $\|h\|_{**,\xi} < \infty$. Then for $\lambda > 0$ sufficiently small the problem*

$$\begin{cases} \Delta\phi + qw^{q-1}\phi = h + c_0 Z_0 + \sum_{i=1}^N c_i Z_i \quad \text{in } \mathbb{R}^N \setminus \bar{\mathcal{D}}_{\lambda,\xi} \\ \lim_{|x|\to+\infty} \phi(x) = 0, \quad \phi = 0 \text{ on } \partial\mathcal{D}_{\lambda,\xi} \end{cases} \tag{7.16}$$

has a solution $(\phi, c_0, c_1, \dots, c_N) = \mathcal{T}(h)$ that depends linearly on h and there is C such that

$$\|\phi\|_{*,\xi} + \max_{1\le i\le N} |c_i| \le C\|h\|_{**,\xi}.$$

The constant C is independent of λ.

To prove this fact, we argue as in the proof of Proposition 2.1.

Fact 3. Solving (7.1) reduces now to a fixed point problem. Namely, we need to find a fixed point for the map $A(\phi) = \mathcal{T}(N(\phi) + E)$. Define

$$F = \{\phi : \mathbb{R}^N \setminus \mathcal{D}_{\lambda,\xi} \to \mathbb{R} \,:\, \|\phi\|_{*,\xi} \le M(\lambda^\nu + \varepsilon)\}$$

for some $M > 0$ large and $\nu = \min(N-2, \sigma+2)$. Since

$$\|A(\phi)\|_{*,\xi} \le C\left(\|N(\phi)\|_{**,\xi} + \|E\|_{**,\xi}\right)$$

and taking into account (7.4)-(7.5), we easily get that $A(F) \subseteq F$ if $0 < \sigma \le \min(2+\sigma, N-2)$. To show that A is a contraction, we argue as in the proof of Proposition (5.1), taking into account that, in our case,

$$D_{\bar\phi}N(\bar\phi) = (q+\varepsilon)\left[(w - \varphi_\lambda + \bar\phi)^{q+\varepsilon-1} - w^{q+\varepsilon-1}\right] + \left[(q+\varepsilon)w^{q+\varepsilon-1} - qw^{q-1}\right]\bar\phi.$$

and that

$$\sup_{x\in\mathbb{R}^N\setminus\mathcal{D}_{\lambda,\xi}} |x|^2 |D_{\bar\phi} N(\bar\phi)|$$

is infinitesimal as $\lambda+\varepsilon\to 0$. In order to prove estimate (7.7) in the range $0<\sigma<N-2$ we proceed as in the proof of Lemma 5.1. □

Proof of Theorem 2. Let $\phi, c_0, c_1, \dots c_N$ be solution to problem (7.1). To prove the result contained in Theorem 2 it suffices to show that the parameter λ and the point ξ can be adjusted so that the constants $c_0,\dots,c_N$ are all contemporarily equal to zero. Under the assumption that the point ξ is bounded, fact that a posteriori will be true, it is just sufficient to show that

$$\int_{\mathbb{R}^N\setminus(\mathcal{D}_{\lambda,\xi})} (E+N(\phi))\,\frac{\partial w}{\partial x_j} + \int_{\partial\mathcal{D}_{\lambda,\xi}} \frac{\partial\phi}{\partial n}\frac{\partial w}{\partial x_j} = 0 \quad \forall 1\le j\le N$$

and

$$\int_{\mathbb{R}^N\setminus(\mathcal{D}_{\lambda,\xi})} (E+N(\phi))\,z_0 + \int_{\partial\mathcal{D}_{\lambda,\xi}} \frac{\partial\phi}{\partial n} z_0 = 0.$$

Define, for $1\le j\le N$,

$$G_j(\xi,\lambda) := \int_{\mathbb{R}^N\setminus(\mathcal{D}_{\lambda,\xi})} (E+N(\phi))\,\frac{\partial w}{\partial x_j} + \int_{\partial\mathcal{D}_{\lambda,\xi}} \frac{\partial\phi}{\partial n}\frac{\partial w}{\partial x_j} \tag{7.17}$$

and

$$G_0(\xi,\lambda) := \int_{\mathbb{R}^N\setminus(\mathcal{D}_{\lambda,\xi})} (E+N(\phi))\,z_0 + \int_{\partial\mathcal{D}_{\lambda,\xi}} \frac{\partial\phi}{\partial n} z_0. \tag{7.18}$$

Arguing as in (6.3) and taking into account that, by symmetry,

$$\int_{\mathbb{R}^N} w^{\frac{N+2}{N-2}} \log w \frac{\partial w}{\partial x_j} = 0 \quad \forall j=1,\dots,N,$$

we obtain

$$G_j(\xi,\lambda) = w(\xi) f_0 \frac{N+2}{N-2}\lambda^{N-2}\int_{\mathbb{R}^N} |x-\xi|^{-(N-2)} w(x)^{\frac{4}{N-2}} \frac{\partial w}{\partial x_j}(x) + o(\lambda^{N-2}+\varepsilon) \tag{7.19}$$

Observe that, again using symmetry, for $\xi=0$ the above integral is zero. Since the above integral depends smoothly on ξ, given $\delta>0$ small, for all λ and ε small we can find $\xi\in B(0,\delta)$, depending on λ and ε, so that all $c_j=0$, for $j=1,\dots,N$.

We are now left to show that also $c_0=0$. In order to get this fact, we need to adjust the parameter λ. Let us thus go to (7.18). Using the estimates obtained on ϕ, we first observe that

$$G_0(\xi,\lambda) = \int_{\mathbb{R}^N\setminus(\mathcal{D}_{\lambda,\xi})} E z_0 + o(\lambda^{N-2}+\varepsilon).$$

A direct computation now yields that

$$\int_{\mathbb{R}^N\setminus(\mathcal{D}_{\lambda,\xi})} E z_0 = -a\varepsilon + A(\xi)\lambda^{N-2} + o(\lambda^{N-2}+\varepsilon) \tag{7.20}$$

where

$$a = \int_{\mathbb{R}^N} w^{\frac{N+2}{N-2}} (\log w) z_0$$

and

$$A(\xi) = w(\xi)\frac{N+2}{N-2} f_0 \int_{\mathbb{R}^N} |x-\xi|^{-(N-2)} w^{\frac{4}{N-2}} z_0.$$

First we observe that the constant a is positive. Indeed, if we define

$$g(s) = \frac{1}{(p+1)^2}\int_{\mathbb{R}^N} w_s^{q+1} - \frac{1}{p+1}\int_{\mathbb{R}^N} w_s^{q+1}\log w_s,$$

where $w_s(x) = (\frac{s}{1+c_N s^2|x|^2})^{\frac{N-2}{2}}$, then a change of variables in the integrals gives that

$$g(s) = a_N - b_N \log s$$

for some constants a_N and $b_N > 0$, depending on N. Observing that $a = -g'(1)$, the conclusion thus follows.

We need now to prove that $A(\xi) > 0$, for ξ the point previously found. To do so, it is enough showing that

$$I = \int_{\mathbb{R}^N} |x|^{-(N-2)} \frac{1}{(1+|x|^2)^2} \frac{1-|x|^2}{(1+|x|^2)^{\frac{N}{2}}}\, dx > 0,$$

since ξ is close to 0. Now, writing ω_N for the volume of the $N-1$ dimensional unit sphere, we have

$$\begin{aligned} I &= \omega_N \Big(\int_0^\infty \frac{1}{(1+r^2)^{\frac{N}{2}+2}} r dr - \int_0^\infty \frac{1}{(1+r^2)^{\frac{N}{2}+2}} r^3 dr \Big) \\ &= \omega_N \frac{N-2}{N(N+2)} > 0 \end{aligned}$$

since $N > 2$.

From (7.20) we can find λ of order $\varepsilon^{\frac{1}{N-2}}$ so that $c_0 = 0$. This concludes the proof of the Theorem. □

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