Bubbling Solutions for an Anisotropic Emden-Fowler Equation

Juncheng Wei[1], Dong Ye[2], Feng Zhou[3]

[1]Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong.
E-mail address: wei@math.cuhk.edu.hk
[2]Département de Mathématiques, Université de Cergy-Pontoise, 95302 Cergy-Pontoise, France.
E-mail address: dong.ye@math.u-cergy.fr
[3]Department of Mathematics, East China Normal University, Shanghai 200062, China.
Email address: fzhou@math.ecnu.edu.cn

Abstract We consider the anisotropic Emden-Fowler equation : $\nabla(a(x)\nabla u) + \varepsilon^2 a(x)e^u = 0$ in Ω, $u = 0$ on $\partial\Omega$ where $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain and $a(x)$ is a positive, smooth function. We investigate the effect of anisotropic coefficient $a(x)$ on the existence of bubbling solutions. We show that at given strict local maximum points of a, there exist solutions with arbitrarily many bubbles. As a consequence, the quantity

$$\mathcal{T}_\varepsilon = \varepsilon^2 \int_\Omega a(x)e^u dx$$

can approach to $+\infty$ as $\varepsilon \to 0$. These results show a striking difference with the isotropic case ($a(x) \equiv$ *constant*).

Solutions à bulles pour une équation de Emden-Fowler anisotropique

Résumé : On considère l'équation de Emden-Fowler anisotropique : $\nabla(a(x)\nabla u) + \varepsilon^2 a(x)e^u = 0$ dans Ω, $u = 0$ sur $\partial\Omega$ où $\Omega \subset \mathbb{R}^2$ est un domaine régulier borné et a est une fonction régulière strictement positive. Nous étudions l'effet du coefficient anisotropique $a(x)$ sur l'existence des solutions à bulles. Nous montrons que pour un maximum local strict de la fonction a, il existe des solutions avec un nombre arbitraire de bulles. Par conséquent, la quantité

$$\mathcal{T}_\varepsilon = \varepsilon^2 \int_\Omega a(x)e^u dx$$

peut approcher $+\infty$ quand $\varepsilon \to 0$. Ces résultats montrent une différence frappante avec le cas isotropique ($a(x) \equiv$ *constante*).

1 Version française abrégée

Nous considérons l'équation de Emden-Fowler suivante:

$$\nabla(a(x)\nabla u) + \varepsilon^2 a(x)e^u = 0 \ \text{ dans } \Omega, \quad u = 0 \text{ sur } \partial\Omega \tag{1}$$

avec Ω un domaine régulier borné de $\mathbb{R}^2$, $\varepsilon > 0$ et a une fonction régulière, strictement positive sur $\overline{\Omega}$. La motivation vient de l'équation de Emden-Fowler classique

$$\Delta u + \varepsilon^2 k(x)e^u = 0 \ \text{ dans } \Omega, \quad u = 0 \text{ sur } \partial\Omega \tag{2}$$

avec Ω un domaine régulier et borné dans $\mathbb{R}^N$ et k une fonction régulière, strictement positive. Le cas $N = 2$ est lié au problème de l'existence de la surface de Riemann à courbure de Gauss prescrite, et quand $N \geq 3$, lié à des problèmes en physique comme l'émission thermique ou la combustion du gaz.

Quand $N = 2$, le comportement asymptotique des solutions de (2) quand ε tend vers 0 est caractérisé (voir [1, 10, 12, 8]) par celle de la quantité

$$\mathcal{T}_\varepsilon = \varepsilon^2 \int_\Omega k(x) e^{u_\varepsilon} dx.$$

Soit u_ε une famille de solutions telle que $\lim_{\varepsilon\to 0} \mathcal{T}_\varepsilon = l \in \mathbb{R}_+ \cup \{\infty\}$ existe, alors $l = 8\pi m$ avec $m \in \mathbb{N} \cup \{\infty\}$. Si $m = 0$, $\lim_{\varepsilon\to 0} \|u_\varepsilon\|_\infty = 0$; si $m = \infty$, alors $u_\varepsilon(x) \to +\infty$ pour tout $x \in \Omega$; et si $m \in \mathbb{N}^*$, à sous suite prés, u_ε explose sur m points $\{x_j\}$ telle que $\varepsilon^2 k(x) e^{u_\varepsilon} \to 8\pi \sum_j \delta_{x_j}$ et $(x_1, ..., x_m)$ est un point critique d'une fonctionnelle liée à la fonction de Green du domaine Ω. Réciproquement, des solutions explosives ont été construites (voir [2, 5, 7]). En particulier dans [5], ils montrent que si Ω est *non simplement connexe*, alors des solutions à m bulles existent pour n'importe quel $m \in \mathbb{N}^*$. Peu de résultats existent pour (2) si $N \geq 3$.

Notre motivation ici est double. Premièrement, l'équation (2) semble être une généralisation naturelle de (1), on pouvait espérer que des résultats similaires restent valables, nous allons voir pourtant que ce n'est pas le cas. Deuxièmement, l'équation (2) se transforme en (1) dans un cas particulier en dimension supérieure. En effet, si on cherche une solution à la symétrie rotationnelle sur un tore standard de $\mathbb{R}^N$, l'équation (2) se transforme en (1) avec $a(r, z) = r^{N-2}$. Dans [13], il est montré que si $\lim_{\varepsilon\to 0} \mathcal{T}_\varepsilon = l \in]0, \infty[$ existe, alors $l \in 8\pi\mathbb{N}^*$ et à sous suite prés, u_ε explose sur un ensemble fini $\mathcal{S} = \{x_1, \ldots, x_m\} \subset \Omega$. En plus, $u_\varepsilon \to u^*$ faiblement dans $W_0^{1,p}$ pour tout $p \in]1, 2[$ où u^* satisfait $\nabla(a\nabla u^*) + 8\pi \sum_i m_i a(x_i)\delta_{x_i} = 0$ dans Ω, $m_i \in \mathbb{N}^*$ et x_i sont des points critiques de la fonction a dans Ω. Ils ont construit aussi une famille de solutions avec un bulle dans le cas de symétrie radiale.

Plusieurs questions importantes restent ouvertes : peut on avoir $m_i > 1$ ou toutes les bulles sont simples? Comment peut on construire des solutions à bulles dans le cas général de a et de Ω? Les résultats suivants sont des réponses correspondantes à ces deux questions.

Théorème 1 *Soit $\bar{x} \in \Omega$ un maximum local strict de a, i.e. $\exists\ \delta > 0$ tel que $a(x) < a(\bar{x})$ pour tout $x \in B_\delta(\bar{x})\backslash\{\bar{x}\}$. Alors $\forall\ m \in \mathbb{N}^*$, (1) admet une famille de solutions u_ε telle que quand $\varepsilon \to 0^+$,*

$$\mathcal{T}_\varepsilon = \varepsilon^2 \int_\Omega a(x) e^{u_\varepsilon} \to 8\pi m a(\bar{x}), \quad u_\varepsilon \to u^* \quad \textit{dans} \quad C^2_{loc}(\overline{\Omega}\backslash\{\bar{x}\})$$

où $-\nabla(a(x)\nabla u^*) = 8\pi m a(\bar{x})\delta_{\bar{x}}$ dans Ω et $u^* = 0$ sur $\partial\Omega$. Plus précisément,

$$u_\varepsilon(x) = \sum_{j=1}^{m} \left[\log \frac{1}{(\varepsilon^2\mu_j^2 + |x - \xi_j^\varepsilon|^2)^2} + H(x, \xi_j^\varepsilon) \right] + o(1) \tag{3}$$

avec H la partie régulière de la fonction de Green G associée à l'opérateur $-\Delta_a = -a(x)^{-1}\nabla(a(x)\nabla\cdot)$ et à la condition de Dirichlet au bord. En plus,

$$\frac{1}{C} \leq \mu_j \leq |\log\varepsilon|^C, \quad \lim_{\varepsilon\to 0} \xi_j^\varepsilon = \bar{x} \quad \textit{et} \quad |\xi_i^\varepsilon - \xi_j^\varepsilon| > |\log\varepsilon|^{-\frac{m^2+1}{2}},\ \forall\ i \neq j.$$

Théorème 2 *Soit $\bar{x} \in \Omega$ un point critique topologiquement non trivial de a (voir la définition dans [6] ou dans la version en anglais), alors pour $\varepsilon > 0$ suffisamment petit, le problème (1) admet des solutions u_ε telle que*

$$\varepsilon^2 \int_\Omega a(x) e^{u_\varepsilon} \to 8\pi a(\bar{x}), \quad u_\varepsilon \to u^* \quad \textit{dans} \quad C^2_{loc}(\bar{\Omega}\backslash\{\bar{x}\}),$$

où u^* satisfait $-\nabla(a(x)\nabla u^*) = 8\pi a(\bar{x})\delta_{\bar{x}}$ dans Ω, $u^* = 0$ sur $\partial\Omega$.

Théorème 1 montre que l'accumulation des bulles peut apparaître dans le cas anisotropique même pour Ω *simplement connexe*, c'est assez surprenant car cela n'a pas lieu pour l'équation (2) (Un tel phénomène est possible seulement si la condition au bord est quelconque, voir [4]), et nous savions également par [13] que si $\bar{x}$ est un minimum local non dégénéré de a, alors $m = 1$. Dans le cadre du Théorème 1, par un processus de diagonale, on peut avoir une suite de solution u_ε telle que $\mathcal{T}_\varepsilon$ peut tend vers ∞, cela se contraste aussi avec l'équation (2), pour laquelle $\mathcal{T}_\varepsilon$ reste borné si Ω est *simplement connexe* (voir [9]).

Notre preuve utilise la méthode de l'énergie localisée, qui est une combinaison de la réduction de Liapunov-Schimidt et de la technique variationnelle. En fait, nous réduisons la résolution de (1) à un problème en dimension finie associée à une énergie réduite, et les solutions dans Théorèmes 1 et 2 sont générées comme des points critiques de l'énergie réduite. Ce genre de technique a été utilisé récemment dans beaucoup de travail. Ici nous allons suivre l'approche dans [5]. Notre situation est pourtant plus délicate, car les distances entre les différents bulles vont tendre vers zéro, ce qui nous oblige à bien étudier leurs intéractions et rend l'analyse plus difficile. Nous avons développé donc de nouveaux arguments pour établir un cadre fonctionnel convenable. Le détail de nos preuves se trouve dans [11].

2 Introduction

We consider the following generalized Emden-Fowler equation

$$\nabla(a(x)\nabla u) + \varepsilon^2 a(x)e^u = 0 \ \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{4}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain, $\varepsilon > 0$ and $a(x)$ is a smooth positive function over $\overline{\Omega}$. Equation (4) was motivated by the study of the following Emden-Fowler equation, or Gelfand's equation

$$\Delta u + \varepsilon^2 k(x)e^u = 0 \ \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{5}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded and smooth domain. When $N = 2$, (5) relates to the geometric problem of Riemannian surfaces with prescribed Gaussian curvature. When $N \geq 3$, it was also arised in the theory of thermionic emission, isothermal gas sphere, gas combustion and many other physical applications.

When $N = 2$, the asymptotic behavior of solutions to (5) has been studied in [1, 8, 10, 12]. Let

$$\mathcal{T}_\varepsilon = \varepsilon^2 \int_\Omega k(x)e^{u_\varepsilon} dx,$$

suppose $\max_{x\in\Omega} u_\varepsilon \to +\infty$. Then along a subsequence of solutions u_ε, $\mathcal{T}_\varepsilon$ tends to l as $\varepsilon \to 0$, there holds either $l = \infty$, $u_\varepsilon(x) \to +\infty$ for all $x \in \Omega$; or $l = 8\pi m$ with $m \in \mathbb{N}^*$ and u_ε makes m points (simple) blow-up on $\mathcal{S} = \{x_1, ..., x_m\} \subset \Omega$ such that $\varepsilon^2 k(x)e^{u_\varepsilon} \to 8\pi \sum_j \delta_{x_j}$. Conversely, many authors have tried to construct blow-up solutions, see [2, 5, 7].

Our motivation here are two-folds. First, equation (4) is a natural generalization of (5). One may expect similar results hold, but this is *not true* as we will show below. Secondly, in some special case, equation (5) in higher-dimension ($N \geq 3$) can be reduced to (4), for example, when we work with the cross-section of a N-dimensional torus having axial symmetry. In [3], Chanillo & Li studied (4) (and more general uniformly elliptic type problems) and generalized the Brezis-Merle results to (4). In [13], Ye & Zhou studied the asymptotic behavior of bubbling solutions to (4). They proved that if $\mathcal{T}_\varepsilon$ tends to $l \in (0, \infty)$, then there exists a finite set $\mathcal{S} = \{x_1, ..., x_m\} \subset \Omega$ such that $u_\varepsilon \to u^*$ weakly in $W_0^{1,p}$ for any $p \in (1, 2)$, where u^* satisfies $\nabla(a\nabla u^*) + 8\pi \sum_i m_i a(x_i)\delta_{x_i} = 0$ in Ω and $m_i \in \mathbb{N}^*$. Moreover, they proved that each x_i must be a critical point of a. In the case of $\Omega = B_1(0)$, $a = a(|x|)$, they also constructed a single blowing up family of radial solutions.

Several important questions have been left open: can we have $m_i > 1$, or are all the bubbles simple? How do we construct bubbling solution for the general (non-radial) a and Ω? In this note, we answer these two questions affirmatively. Our results are the following

Theorem 1 *Let $\bar{x} \in \Omega$ be a strict local maximum point of $a(x)$, i.e., there exists a neighborhood $B_\delta(\bar{x})$ such that $a(x) < a(\bar{x})$, $\forall\, x \in B_\delta(\bar{x})\backslash\{\bar{x}\}$. Then for any $m \in \mathbb{N}^*$, problem (4) has a family of solutions u_ε such that as $\varepsilon \to 0^+$,*

$$\mathcal{T}_\varepsilon = \varepsilon^2 \int_\Omega a(x)e^{u_\varepsilon} \to 8\pi m a(\bar{x}), \quad u_\varepsilon \to u^* \quad in \quad C^2_{loc}(\overline{\Omega}\backslash\{\bar{x}\})$$

where u^* satisfies $-\nabla(a(x)\nabla u^*) = 8\pi m a(\bar{x})\delta_{\bar{x}}$ in Ω, $u^* = 0$ on $\partial\Omega$. More precisely, we have

$$u_\varepsilon(x) = \sum_{j=1}^m \left[\log \frac{1}{(\varepsilon^2\mu_j^2 + |x - \xi_j^\varepsilon|^2)^2} + H(x, \xi_j^\varepsilon)\right] + o(1) \tag{6}$$

where H is the regular part of Green's function associated to the operator $-\Delta_a$ and the Dirichlet boundary condition (see Lemma 3.1). We have also

$$\frac{1}{C} \le \mu_j \le |\log\varepsilon|^C, \quad \xi_j^\varepsilon \to \bar{x} \quad \textit{and} \quad |\xi_i^\varepsilon - \xi_j^\varepsilon| > |\log\varepsilon|^{-\frac{m^2+1}{2}}, \; \forall\, i \neq j.$$

Let D be an open set in Ω. Following [6], we say $a(x)$ has a "topologically nontrivial critical point" at critical level c relative to B and B_0 if B and B_0 are closed subsets of $\overline{D}$ with B connected and $B_0 \subset B$ such that the following conditions hold: let Γ be the class of all maps $\Phi \in C(B, D)$ with the property that there exists a function $\Psi \in C([0,1] \times B, D)$ such that $\Psi(0,\cdot) = Id_B$, $\Psi(1,\cdot) = \Phi$, $\Psi(t,\cdot)|_{B_0} = Id_{B_0}$ for all $t \in [0,1]$. We assume that

$$\sup_{y \in B_0} a(y) < c = \inf_{\Phi \in \Gamma} \sup_{y \in B} a(\Phi(y))$$

and for all $y \in \partial D$ such that $a(y) = c$, there exists a vector τ_y tangent to ∂D at y verifying $\nabla a(y) \cdot \tau_y \neq 0$. Under these conditions, a critical point $\bar{y} \in D$ of $a(x)$ with $a(\bar{y}) = c$ exists. We call c a nontrivial critical level of a in D. It is easy to see that local maximum points, local minimum points, nondegenerate critical points of a are all "topologically nontrivial". We have then

Theorem 2 *Let $\bar{x} \in \Omega$ be a topologically nontrivial critical point of a. Then for $\varepsilon > 0$ sufficiently small, problem (4) has solutions u_ε such that*

$$\varepsilon^2 \int_\Omega a(x) e^{u_\varepsilon} \to 8\pi a(\bar{x}), \quad u_\varepsilon \to u^* \quad \textit{in} \quad C^2_{loc}(\bar{\Omega} \backslash \{\bar{x}\}),$$

where u^* satisfies $-\nabla(a(x)\nabla u^*) = 8\pi a(\bar{x})\delta_{\bar{x}}$ in Ω, $u^* = 0$ on $\partial\Omega$.

It is quite surprising that accumulation of bubbles can occur for anisotropic Emden-Fowler equation even for Ω *simply connected*. The only known result for such phenomena is due to [4]. In [13], it is shown that if $\bar{x}$ is a nondegenerate local minimum point of a, then $m = 1$. Here we show that if $\bar{x}$ is a strict local maximum point, then we can allow arbitrary $m > 1$. A consequence of Theorem 1 is that if a has a strict local maximum in Ω, then there exist solutions u_ε such that $\mathcal{T}_\varepsilon \to +\infty$. This is new and unexpected, since when Ω is *simply connected*, it was shown in [9] that $\mathcal{T}_\varepsilon$ is uniformly bounded for (5).

Theorems 1 and 2 are proved via the so-called "localized energy method"- a combination of Liapunov-Schmidt reduction method and variational techniques. Namely, we first use Liapunov-Schmidt reduction method to reduce the problem to a finite dimensional one, with some *reduced energy*. Then, the solutions in Theorems 1 and 2 turn out to be generated by critical points of the reduced energy functionals. Such an idea has been used in many other papers. Here we follow those of [5]. However, a new functional setting has to be introduced, since the distance between the bubbles is small, and an appropriate variational argument is developed in order to make the approach successful. In what follows, we shall sketch the proof for Theorem 1 since that of Theorem 2 is simpler and follows from the proof of Theorem 2 in [5], so we omit the details. We can also prove that the flatter the anisotropic coefficient is, the larger are the distances between the bubbles.

3 Sketch of the proof of Theorem 1

3.1 Ansatz for the solution

Given $\xi_j \in \Omega$, $\mu_j > 0$, we define

$$u_j(x) = \log \frac{8\mu_j^2}{(\varepsilon^2\mu_j^2 + |x - \xi_j|^2)^2}.$$

It is well known that $-\Delta u_j = \varepsilon^2 e^{u_j}$ in $\mathbb{R}^2$. We take the configuration space for $(\xi_1, ..., \xi_m)$ as follows

$$\Lambda := \left\{ \xi = (\xi_1, ..., \xi_m) \in B_\delta(\bar{x}) \times ... \times B_\delta(\bar{x}) \,\Big|\, \min_{i \neq j} |\xi_i - \xi_j| \ge \frac{1}{|\log\varepsilon|^M} \right\} \tag{7}$$

where $M = (m^2+1)/2$ and the choice of μ_j will be made later on. $U_\xi(x) = \sum_{1\le j\le m}\left[u_j(x) + H_j^\varepsilon(x)\right]$ is our ansatz where H_j^ε is a correction term defined as the solution of $\Delta_a H_j^\varepsilon + \nabla \log a(x)\nabla u_j = 0$ in Ω, $H_j^\varepsilon = -u_j$ on $\partial\Omega$. The behavior of H_j^ε is given by

Lemma 3.1 *For any $0<\alpha<1$, $H_j^\varepsilon(x) = H(x,\xi_j) - \log\left(8\mu_j^2\right) + O(\varepsilon^\alpha)$ uniformly in $\overline{\Omega}$, where $H(x,y) = G(x,y) + 4\log|x-y|$ with G the Green's function*

$$-\Delta_a G(x,y) = -\frac{1}{a(x)}\nabla\left[a(x)\nabla G\right] = 8\pi\delta(x-y) \text{ in } \Omega, \quad G(x,y) = 0 \text{ on } \partial\Omega.$$

Define $V(y) = U_\xi(\varepsilon y) + 4\log\varepsilon$, we will seek a solution v of the form $v = V + \phi$, the problem (4) can be stated as to find ϕ a solution to

$$\Delta_{a(\varepsilon y)}\phi + e^V\phi + N(\phi) + R = 0 \text{ in } \Omega_\varepsilon, \quad \phi = 0 \text{ on } \partial\Omega_\varepsilon,$$

where $\Omega_\varepsilon = \{y\in\mathbb{R}^2, \varepsilon y\in\Omega\}$, the nonlinear term is $N(\phi) = e^V(e^\phi - 1 - \phi)$ and the error term is given by $R = \Delta_{a(\varepsilon y)}V + e^V$. We choose μ_j by

$$\log\left(8\mu_j^2\right) = H(\xi_j,\xi_j) + \sum_{i\ne j} G(\xi_i,\xi_j), \quad \forall\, 1\le j\le m,$$

so that we make the error term R small and $W = e^V$ well controlled. Observe that μ_j is not $O(1)$ since $\xi_j\to\bar{x}$ implies that $\lim G(\xi_i,\xi_j) = \infty$. But we can derive that $\frac{1}{C}\le\mu_j\le|\log\varepsilon|^C$ for some fixed $C>0$.

3.2 Solvability of linear and nonlinear equations

A key step of our proof is to consider the following linear problem: Given $h\in L^\infty(\Omega_\varepsilon)$, find ϕ, $c_{11},...,c_{2m}$ such that

$$(8)\qquad \begin{cases} -\Delta_{a(\varepsilon y)}\phi = W\phi + h + \dfrac{1}{a(\varepsilon y)}\displaystyle\sum_{j=1}^m\sum_{i=1}^2 c_{ij}\chi_j Z_{ij} & \text{in } \Omega_\varepsilon \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon \\ \displaystyle\int_{\Omega_\varepsilon}\chi_j Z_{ij}\phi = 0, \quad \forall\, 1\le j\le m,\ i = 1,2 & \end{cases}$$

where $Z_{ij}(y) = z_{ij}(y-\xi_j')$ with $\xi_j' = \xi_j/\varepsilon$ and

$$z_{ij} = \frac{y_i}{\mu_j^2 + |y|^2}, \quad 1\le i\le 2, \quad 1\le j\le m.$$

Moreover χ_j are defined as $\chi_j(y) = \chi\left(\frac{|y-\xi_j'|}{\mu_j}\right)$ with χ a suitable cut-off function. The problem is solvable for all small $\varepsilon>0$. We have

Proposition 3.2 *Let m be a positive integer. Then there exist $\varepsilon_0>0$, $C>0$ such that for any $0<\varepsilon<\varepsilon_0$, any family of points $(\xi_1,\dots,\xi_m)\in\Lambda$ and any $h\in L^\infty(\Omega_\varepsilon)$, there is a unique solution $\phi\in L^\infty(\Omega_\varepsilon)$, $c_{ij}\in\mathbb{R}$ to (8). Moreover, $\|\phi\|_{L^\infty(\Omega_\varepsilon)}\le C|\log\varepsilon|\|h\|_*$.*

Here $\|h\|_*$ denotes the following norm

$$\|h\|_* = \sup_{y\in\Omega_\varepsilon}\frac{|h(y)|}{\varepsilon^2 + \sum_{j=1}^m \mu_j^{-2}\left(1+\frac{1}{\mu_j}|y-\xi_j'|\right)^{-3}}.$$

The proof is rather involved since the distance between different bubbles is small and the constants μ_j are no longer bounded when ε tends to zero, so we need to well control their interaction. The result of Proposition 3.2 implies that the unique solution $\phi = T(h)$ of (8) defines a continuous linear map from the Banach space $\mathcal{C}_*$ of $L^\infty(\Omega_\varepsilon)$ endowed with the norm $\|\cdot\|_*$ into L^∞. Using the differentiability of the operator T with respect to the variables ξ_j', we get then

Lemma 3.3 *Let $m \in \mathbb{N}^*$ and $\alpha \in (0,1)$. Consider the nonlinear equation with $h = R + N(\phi)$ in (8). Then there exist $\varepsilon_0 > 0$, $C > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and any $(\xi_1, \ldots, \xi_m) \in \Lambda$ the nonlinear problem $\phi = T(R+N(\phi))$ admits a unique solution ϕ, c_{ij} such that $\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C\varepsilon^\alpha |\log \varepsilon|$. Furthermore, the function $\xi' \mapsto \phi(\xi')$ is of class C^1 and $\|D_{\xi'}\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C\,\varepsilon^\alpha |\log \varepsilon|^3$.*

3.3 Variational reduction

Given $\xi = (\xi_1, \ldots, \xi_m) \in \Lambda$, we define $\phi(\xi)$ and $c_{ij}(\xi)$ to be the unique solution given by Lemma 3.3. Set $\mathcal{F}_\varepsilon(\xi) = J_\varepsilon(U_\xi + \tilde{\phi}(\xi))$ where J_ε is the functional defined by

$$J_\varepsilon(v) = \frac{1}{2}\int_\Omega a(x)|\nabla v|^2 dx - \varepsilon^2 \int_\Omega a(x)e^v dx$$

and $\tilde{\phi}(\xi)(x) = \phi\left(\varepsilon^{-1}x, \varepsilon^{-1}\xi\right)$ for $x \in \Omega$. With the estimate of Lemma 3.3, we can prove

Lemma 3.4 *If $\xi = (\xi_1, \ldots, \xi_m) \in \Lambda$ is a critical point of $\mathcal{F}_\varepsilon$ then $u = U(\xi) + \tilde{\phi}(\xi)$ is a critical point of J_ε, that is, a solution to* (4).

We get also the uniform closeness of $\mathcal{F}_\varepsilon(\xi)$ and $J_\varepsilon(U(\xi))$ as $\mathcal{F}_\varepsilon(\xi) = J_\varepsilon(U(\xi)) + \theta_\varepsilon(\xi)$ where $\|\theta_\varepsilon\|_{C^1(\Lambda)} \to 0$ when ε tends to 0. For proving Theorem 1, we use the uniform energy expansion over Λ

$$J_\varepsilon(U) = -16\pi \sum_{j=1}^m a(\xi_j)\log\varepsilon - 4\pi \sum_{i\neq j} a(\xi_j)G(\xi_i, \xi_j) - 4\pi \sum_j a(\xi_j)H(\xi_j, \xi_j) + O(1).$$

Finally, some suitable upper and lower bound estimates lead us to claim that the maximization problem $\max_{(\xi_1,\ldots,\xi_m)\in\bar{\Lambda}} \mathcal{F}_\varepsilon(\xi_1, ..., \xi_m)$ has a solution in the interior of Λ, hence guarantees the existence of a critical point of $\mathcal{F}_\varepsilon$, and then a solution u_ε for (4) by Lemma 3.4. Furthermore, from the ansatz and the decomposition of u_ε, we get easily the rest of the properties of u_ε.

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