

UNIQUENESS OF POSITIVE SOLUTIONS TO SOME COUPLED NONLINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT. We study the uniqueness of positive solutions of the following coupled nonlinear Schrödinger equations:

$$\begin{cases} \Delta u_1 - \lambda_1 u_1 + \mu_1 u_1^3 + \beta u_1 u_2^2 = 0 & \text{in } \mathbb{R}^N, \\ \Delta u_2 - \lambda_2 u_2 + \mu_2 u_2^3 + \beta u_1^2 u_2 = 0 & \text{in } \mathbb{R}^N, \\ u_1 > 0, u_2 > 0, u_1, u_2 \in H^1(\mathbb{R}^N) \end{cases}$$

where $N \leq 3$, $\lambda_1, \lambda_2, \mu_1, \mu_2$ are positive constants and $\beta \geq 0$ is a coupling constant. We prove first the uniqueness of positive solution for sufficiently small $\beta > 0$. Secondly, assuming that $\lambda_1 = \lambda_2$, we show that $u_1 = u_2 \sqrt{\beta - \mu_1} / \sqrt{\beta - \mu_2}$ when $\beta > \max\{\mu_1, \mu_2\}$ and thus obtain the uniqueness of positive solution using the corresponding result of scalar equation. Finally, for $N = 1$ and $\lambda_1 = \lambda_2$, we prove the uniqueness of positive solution when $0 \leq \beta \notin [\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}]$ and thus give a complete classification of positive solutions.

1. Introduction. We study the uniqueness of positive solutions in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ of the following coupled nonlinear Schrödinger equations:

$$\begin{cases} \Delta u - \lambda_1 u + \mu_1 u^3 + \beta uv^2 = 0 & \text{in } \mathbb{R}^N, \\ \Delta v - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0 & \text{in } \mathbb{R}^N, \end{cases} \quad (1.1)$$

where $N \leq 3$, $\lambda_1, \lambda_2, \mu_1, \mu_2$ are positive constants and β is a coupling constant. System (1.1) has been well studied in recent years, both theoretically and numerically, due to the fact that it gives solitary waves for Schrödinger systems that appears in a number of physical problems, for instance in nonlinear optics (see [1], [5], [9], [10], [12] and the references therein). In this paper, we concentrate on the *attractive* case, i.e. $\beta > 0$.

System (1.1) has unique semi-trivial solutions of the form $(U, 0)$ and $(0, V)$ where U and V are radially symmetric positive (nontrivial) solutions of

$$\Delta U - \lambda_1 U + \mu_1 U^3 = 0 \quad \text{in } \mathbb{R}^N$$

and

$$\Delta V - \lambda_2 V + \mu_2 V^3 = 0 \quad \text{in } \mathbb{R}^N.$$

(The uniqueness is proved in [8].) By a nontrivial solution of (1.1) we mean a pair (u, v) such that $u \neq 0 \neq v$.

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Suppose we have a nontrivial solution (u, v) of system (1.1), such that $u \geq 0$, $v \geq 0$ in \mathbb{R}^N . Note that u and v satisfy two linear equations

$$\Delta u - (\lambda_1 - \mu_1 u^2 - \beta v^2) u = 0$$

and

$$\Delta v - (\lambda_2 - \mu_2 v^2 - \beta u^2) v = 0.$$

So by the Strong Maximum Principle u and v are strictly positive in \mathbb{R}^N . By moving plane method as in [2] u and v are radial and decrease with respect to some point in \mathbb{R}^N . Note that it can be proved with the help of a classical "bootstrap" argument that solutions of (1.1) which are in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ are also in $C^2(\mathbb{R}^N) \times C^2(\mathbb{R}^N)$ and tend to zero as $x \rightarrow \infty$. In the following without loss of generality we assume that u and v are radial with respect to 0 and the system (1.1) becomes

$$\begin{cases} u''(r) + \frac{N-1}{r} u'(r) - \lambda_1 u + \mu_1 u^3 + \beta v^2 = 0, & \text{in } [0, \infty), \\ v''(r) + \frac{N-1}{r} v'(r) - \lambda_2 v + \mu_2 v^3 + \beta u^2 = 0, & \text{in } [0, \infty), \\ u(r), v(r) > 0 \text{ in } [0, \infty), \\ u'(0) = v'(0) = 0, \text{ and } u(r), v(r) \rightarrow 0 \text{ as } r \rightarrow \infty. \end{cases} \quad (1.2)$$

Nontrivial solutions does not always exist for all $\beta > 0$. (See [3] and [4].) In fact, we multiply the equation for u in (1.1) by v , the equation for v by u , and integrate resulting equations over \mathbb{R}^N . This yields

$$\begin{aligned} \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + \lambda_1 uv) &= \int_{\mathbb{R}^N} uv(\mu_1 u^2 + \beta v^2), \\ \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + \lambda_2 uv) &= \int_{\mathbb{R}^N} uv(\beta u^2 + \mu_2 v^2), \end{aligned}$$

from which we have

$$\int_{\mathbb{R}^N} uv [(\lambda_2 - \lambda_1) + (\mu_1 - \beta)u^2 + (\beta - \mu_2)v^2] = 0, \quad (1.3)$$

which is in a contradiction with the positivity of u and v as long as the three constants $(\lambda_2 - \lambda_1)$, $(\mu_1 - \beta)$, $(\beta - \mu_2)$ are of the same sign or zero, and one of them is not zero. This implies that the system (1.1) does not have a nontrivial solution with nonnegative components if $\lambda_1 = \lambda_2$, $\mu_1 \neq \mu_2$ and $\min\{\mu_1, \mu_2\} \leq \beta \leq \max\{\mu_1, \mu_2\}$.

When $\lambda_1 = \lambda_2 = \lambda$, system (1.1) admits a bound state of the form

$$(u_0, v_0) = \left(\sqrt{\frac{\lambda(\beta - \mu_2)}{\beta^2 - \mu_1\mu_2}} w(\sqrt{\lambda}x), \sqrt{\frac{\lambda(\beta - \mu_1)}{\beta^2 - \mu_1\mu_2}} w(\sqrt{\lambda}x) \right), \quad (1.4)$$

where w is the unique positive solution of

$$\Delta w - w + w^3 = 0 \text{ in } \mathbb{R}^N, \quad w(0) = \max_{x \in \mathbb{R}^N} w(x), \quad w(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty, \quad (1.5)$$

as long as $\beta \notin [\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}]$. An interesting question is whether the couple (u_0, v_0) is the unique positive solution to system (1.1) if $0 \leq \beta \notin [\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}]$. Note that when $\lambda = 1, \mu_1 = \mu_2 = \beta = 1$ system (1.1) has infinitely many positive solutions

$$(\cos \theta w, \sin \theta w), \quad \theta \in (0, \pi/2). \quad (1.6)$$

Another interesting question is whether they are all positive solutions to system (1.1) in this case.

In this paper we will give complete answers to the above questions in the case of $N = 1$ and some partial answers in the case of $N = 2, 3$. Our first result answers the uniqueness question.

Theorem 1.1. *Suppose that $N = 1$, $\lambda_1 = \lambda_2 = \lambda$ and $0 \leq \beta \notin [\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}]$. Then (u_0, v_0) is the unique positive solution to system (1.2).*

Our second result gives the classification of all the positive solutions of system (1.2) when $N = 1$, $\lambda_1 = \lambda_2 = \lambda$, $\mu_1 = \mu_2 = \beta$.

Theorem 1.2. *Suppose that $N = 1$, $\lambda_1 = \lambda_2 = \lambda > 0$ and $\mu_1 = \mu_2 = \beta > 0$. Then all the positive solutions of system (1.2) have the following form*

$$(u(x), v(x)) = \left(\sqrt{\frac{\lambda}{\beta}} w(\sqrt{\lambda}x) \cos \theta, \sqrt{\frac{\lambda}{\beta}} w(\sqrt{\lambda}x) \sin \theta \right), \quad \theta \in (0, \pi/2). \quad (1.7)$$

In the higher dimensional case, we have the following result.

Theorem 1.3. *Suppose that $N = 2, 3$.*

(a) *Solutions to system (1.2) are unique for sufficiently small $\beta > 0$.*

(b) *If $\beta > \max\{\mu_1, \mu_2\}$, then (u_0, v_0) (defined at (1.4)) is the unique positive solution to system (1.2).*

In Section 4, Theorem 1.3 can be extended to bounded or unbounded domains with or without trapping potentials. From Theorem 1.1-1.3, we can also deduce the *nondegeneracy* of positive solutions, an important property for constructing concentrating solutions. In Section 5, we use the nondegeneracy to construct single or multiple spike solutions to the following systems with trapping potential

$$\begin{cases} \epsilon^2 \Delta u - V_1(x)u + \mu_1 u^3 + \beta uv^2 = 0 & \text{in } \mathbb{R}^N, \\ \epsilon^2 \Delta v - V_2(x)v + \mu_2 v^3 + \beta u^2 v = 0 & \text{in } \mathbb{R}^N. \end{cases} \quad (1.8)$$

Before we end the introduction, let us compare our results with existing literature. In general, the question of uniqueness of positive solutions to nonlinear equations is difficult. For scalar equation, the shooting method and Pohozaev's identity can give uniqueness (a celebrated result is the uniqueness of solutions to (1.5) by Kwong [8]). However for systems, there are very few results on uniqueness.

Theorems 1.1-1.2 give a complete classification of positive solutions to (1.2). As far as we know, this seems to be the first such result in the literature. Part (a) of Theorem 1.3 is not new. In fact, in [6], the author proved part (a) of Theorem 1.3 and also extended to radially symmetric solutions to (1.2) with trapping potentials. The proof in [6] (and also our proof of (a)) is by perturbation argument. Part (b) of Theorem 1.3 seems to be **new**.

We remark that in [11], the author proved the uniqueness and nondegeneracy of symmetric positive solutions to a related ODE system

$$\begin{cases} -w'' + aw - wv = 0 \\ -v'' + bv - \frac{w^2}{2} = 0. \end{cases} \quad (1.9)$$

In this paper, we have classified the solutions in the case of $\lambda_1 = \lambda_2$. When $\lambda_1 \neq \lambda_2$, Dancer and Wei [5] proved that except for finite number of β 's, there exists a branch of nondegenerate solutions.

2. Proof of Theorem 1.1. Let (u, \bar{v}) is a positive solution of system (1.1). Under the hypotheses of Theorem 1.1 we define constant $a = \sqrt{\frac{\beta - \mu_1}{\beta - \mu_2}}$ and $v(r) = \bar{v}(r)/a$, then we only need to prove that $v(r) = u(r)$ for all $r \geq 0$ by the uniqueness result of the single scalar equation ([8]). Without loss of generality, we may assume that $\lambda_1 = \lambda_2 = 1$. Then (u, v) satisfies the following system

$$\begin{cases} u'' - u + \mu_1 u^3 + \beta a^2 u v^2 = 0 & \text{in } [0, \infty), \\ v'' - v + \mu_2 a^2 v^3 + \beta u^2 v = 0 & \text{in } [0, \infty), \\ u(r) > 0, v(r) > 0 & \text{in } [0, \infty), \\ u'(0) = v'(0) = 0, u(r), v(r) \rightarrow 0 & \text{as } r \rightarrow \infty. \end{cases} \quad (2.1)$$

Step 1: Multiplying the equation for u in system (2.1) by v we have

$$(u'v)' - u'v' - uv + \mu_1 u^3 v + \beta a^2 u v^3 = 0. \quad (2.2)$$

Similarly we get

$$(uv')' - u'v' - uv + \mu_2 a^2 u v^3 + \beta u^3 v = 0. \quad (2.3)$$

Subtracting (2.2) by (2.3) gives

$$(u'v - uv')' + (\mu_1 - \beta)uv(u^2 - v^2) = 0. \quad (2.4)$$

Integrating (2.4) over $(0, \infty)$ and using $u'(0) = v'(0) = 0 = u(\infty) = v(\infty)$, we have

$$(\mu_1 - \beta) \int_0^\infty uv(u^2 - v^2) = 0, \quad (2.5)$$

from which we know that $u = v$ if $u \geq v$ or $u \leq v$ when $\beta \neq \mu_1$.

Step 2: In this step we prove that either $u(r) \geq v(r)$ or $u(r) \leq v(r)$ for all r . Suppose not, then $(u - v)$ changes sign. Similar to equality (2.5) we can prove that $u - v$ can not equal 0 in any nonempty interval. Since $(u - v)$ satisfies

$$f'' - f + [\mu_1 u^2 + (\mu_1 - \beta)uv + \mu_2 a^2 v^2] f = 0 \text{ in } [0, \infty), \quad (2.6)$$

and $f(r) = u(r) - v(r) \rightarrow 0$ as $r \rightarrow \infty$, then by Maximum Principle $f(r) = u(r) - v(r)$ changes sign only finite time. Without loss of generality we may assume that $f(r) > 0$ for large r . Thus there exists $r_1 > 0$ such that

$$u(r_1) - v(r_1) = 0 \quad \text{and} \quad u(r) - v(r) > 0 \quad \text{for } r > r_1, \quad (2.7)$$

which implies

$$u'(r_1) - v'(r_1) \geq 0. \quad (2.8)$$

Integrating (2.4) over (r_1, ∞) we get

$$-(u'v - uv')(r_1) + (\mu_1 - \beta) \int_{r_1}^\infty uv(u^2 - v^2) = 0 \quad (2.9)$$

which then yields that

$$u'(r_1) - v'(r_1) > 0 \quad (2.10)$$

and

$$-(u'v - uv')(r_1) = -u(r_1) [u'(r_1) - v'(r_1)] > 0.$$

And from (2.7) we get

$$(\mu_1 - \beta) \int_{r_1}^\infty uv(u^2 - v^2) < 0 \quad \text{if } \beta > \max\{\mu_1, \mu_2\}.$$

Hence a contradiction follows for $\beta > \max\{\mu_1, \mu_2\}$.

Now we consider the case of $\beta < \max\{\mu_1, \mu_2\}$. we claim that there exists $r_2 > r_1$ such that

$$(u'v - uv')(r_2) = 0. \quad (2.11)$$

If this claim is true, then integrating (2.4) over (r_2, ∞) we get

$$0 = -(u'v - uv')(r_2) + (\mu_1 - \beta) \int_{r_2}^{\infty} uv(u^2 - v^2) = (\mu_1 - \beta) \int_{r_2}^{\infty} uv(u^2 - v^2) > 0,$$

a contradiction follows.

Step 3: In this step we will prove that our claim (2.11) is true. Suppose not, then

$$(u'v - uv')(r) > 0 \quad \text{for all } r > r_1 \quad (2.12)$$

since

$$(u'v - uv')(r_1) > 0 \quad (2.13)$$

and $(u'v - uv')$ is continuous in $[0, \infty)$. Multiplying the equation for u in (2.1) by u' we get

$$\frac{1}{2}[(u')^2]' - \frac{1}{2}(u^2)' + \frac{1}{4}\mu_1(u^4)' + \frac{1}{4}\beta a^2(u^2v^2)' + \frac{1}{2}\beta a^2uv(u'v - uv') = 0. \quad (2.14)$$

Similarly we get

$$\frac{1}{2}[(v')^2]' - \frac{1}{2}(v^2)' + \frac{1}{4}\mu_2a^2(v^4)' + \frac{1}{4}\beta(u^2v^2)' + \frac{1}{2}\beta uv(uv' - u'v) = 0. \quad (2.15)$$

Subtracting (2.14) by (2.15) and integrating the result equality over (r_1, ∞) gives

$$-\frac{1}{2}[(u')^2 - (v')^2](r_1) + \frac{a^2 + 1}{2}\beta \int_{r_1}^{\infty} uv(u'v - uv') = 0, \quad (2.16)$$

because $u(r_1) = v(r_1)$, $u'(\infty) = v'(\infty) = 0$ and $\mu_1 + \beta a^2 - \mu_2 a^2 - \beta = 0$. But using $0 > u'(r_1) > v'(r_1)$ and (2.12) we have

$$-\frac{1}{2}[(u')^2 - (v')^2](r_1) + \frac{a^2 + 1}{2}\beta \int_{r_1}^{\infty} uv(u'v - uv') > 0, \quad \text{since } \beta > 0, \quad (2.17)$$

a contradiction follows and we complete the proof.

3. Proof of Theorem 1.2. Without loss of generality we can assume that $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = \beta = 1$ via the transformation

$$(u(x), v(x)) \rightarrow \left(\sqrt{\frac{\beta}{\lambda}} u\left(\frac{x}{\sqrt{\lambda}}\right), \sqrt{\frac{\beta}{\lambda}} v\left(\frac{x}{\sqrt{\lambda}}\right) \right). \quad (3.1)$$

Define the energy functional $E(r)$ of the system (2.1) by

$$E(r) = \frac{1}{2}[(u')^2 + (v')^2] + \frac{1}{4}(u^4 + v^4) + \frac{\beta}{2}u^2v^2 - \frac{1}{2}(u^2 + v^2). \quad (3.2)$$

Thanks to $N = 1$ we can get $E'(r) = 0$. Since we assume that $u(r), v(r) \rightarrow 0$ as $r \rightarrow \infty$, then we have $E(r) = 0$ for all $r \geq 0$. In particular, letting $r = 0$ and using $u'(0) = v'(0) = 0$, we get

$$u^4(0) + v^4(0) + 2u^2(0)v^2(0) - 2u^2(0) - 2v^2(0) = 0, \quad (3.3)$$

which implies that

$$u^2(0) + v^2(0) = 2. \quad (3.4)$$

Thus $u(0) = \sqrt{2} \cos \theta$ and $v(0) = \sqrt{2} \sin \theta$ for some $\theta \in (0, \pi/2)$. By the uniqueness theorem of ODE system for initial value problem, we get $(u(r), v(r)) = (w(r) \cos \theta, w(r) \sin \theta)$.

4. Extensions and remarks: Proof of Theorem 1.3. In this section, we consider various extensions of Theorems 1.1.

4.1. Uniqueness for small β . We first consider the uniqueness of solution to system (1.2) when β is small. We work on the space $E = C_{r,0}(\mathbb{R}^N) \times C_{r,0}(\mathbb{R}^N)$, where $C_{r,0}(\mathbb{R}^N)$ denotes the space of continuous radial functions vanishing at ∞ . The following theorem is our first uniqueness result.

Theorem 4.1. *Suppose that $N = 2, 3$. Then solutions to system (1.2) are unique for sufficiently small $\beta > 0$.*

Proof. Let u_0 is the unique radial solution to system (1.2) when $\beta = 0$. Denote $u = (u_1, u_2) \in E$ and set $\Phi(\beta, u) = I'_\beta(u)$, where

$$\begin{aligned} I_\beta(u) &= \frac{1}{2} \int_{\mathbb{R}^N} [|\nabla u_1|^2 + |\nabla u_2|^2 + \lambda_1 u_1^2 + \lambda_2 u_2^2] dx \\ &\quad - \frac{1}{4} \int_{\mathbb{R}^N} [\mu_1 u_1^4 + \mu_2 u_2^4] dx - \frac{\beta}{2} \int_{\mathbb{R}^N} u_1^2 u_2^2 dx. \end{aligned} \quad (4.1)$$

Then it is clear that $\Phi(0, u_0) = 0$. Moreover, $\Phi_u(0, u_0) = I''_0(u_0)$ is invertible. By the implicit function theorem, there exist $\beta_0 > 0, r_0 > 0$ and $\phi : (-\beta_0, \beta_0) \rightarrow B_{r_0}(u_0)$ such that for any $\beta \in (-\beta_0, \beta_0)$, $\Phi(\beta, u) = 0$ has a unique solution $u = \phi(\beta)$ in $B_{r_0}(u_0)$. On the other hand, by Lemma 2.4 of [5], the set of radial solutions to system (1.2) is compact. Thus for β sufficiently small, the set of solutions to system (1.2) is contained in $B_{r_0}(u_0)$. Thus we complete the proof. \square

4.2. Uniqueness for large β . Next we consider generalizations of Theorem 1.1 to the following coupled nonlinear Schrödinger equations with trapping potentials:

$$\begin{cases} \Delta u_1 - V_1(x)u_1 + \mu_1 u_1^3 + \beta u_1 u_2^2 = 0 & \text{in } \Omega, \\ \Delta u_2 - V_2(x)u_2 + \mu_2 u_2^3 + \beta u_1^2 u_2 = 0 & \text{in } \Omega, \\ u_1, u_2 > 0 \text{ in } \Omega, u_1 = u_2 = 0 \text{ on } \partial\Omega, \end{cases} \quad (4.2)$$

where $\Omega \subset \mathbb{R}^N$, ($N \leq 3$) is a smooth (bounded or unbounded) domain, $V_1(x)$ and $V_2(x)$ are trapping potentials and μ_1, μ_2, β are positive constants.

Now we consider the case of large β and pose the following conditions on the trapping potentials and the coupling constant:

$$V_1(x) = V_2(x) = V(x) > 0 \quad \text{in } \Omega, \quad (4.3)$$

$$\beta > \max\{\mu_1, \mu_2\}. \quad (4.4)$$

Using only integration by part, we obtain the following result.

Theorem 4.2. *Let (u_1, u_2) is a solution of system (4.2), then under the conditions (4.3), (4.4), we obtain*

$$u_2(x) = a u_1(x), \quad \text{where } a = \sqrt{\frac{\beta - \mu_1}{\beta - \mu_2}} \text{ is a constant} \quad (4.5)$$

and u_1 satisfies the following scalar equation:

$$\Delta u - V(x)u + \frac{\beta^2 - \mu_1 \mu_2}{\beta - \mu_2} u^3 = 0. \quad (4.6)$$

By the above theorem and the uniqueness result for scalar equation in [8], we obtain the following uniqueness result.

Corollary 4.3. *Let $N = 2, 3$ and $\beta > \max\{\mu_1, \mu_2\}$, then the only solution to (1.2) is (u_0, v_0) (defined at (1.4)).*

Remark. (1) *The same conclusion is also true for the homogeneous Neumann boundary condition: $\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} = 0$ on $\partial\Omega$.*

(2) *The conclusion does not hold for all $\beta > 0$ as there are the examples in the case of $\Omega = \mathbb{R}^N$, $V_1 = V_2 \equiv 1$, $\mu_1 = \mu_2 = \beta = 1$.*

(3) *As conjectured in [12], we conjecture that Corollary 4.3 is also true when $0 < \beta < \min\{\mu_1, \mu_2\}$.*

(4) *It seems difficult to generalize the proof of Theorem 4.2 to nonlinear Schrodinger equations with more than two components.*

Proof. Define $\tilde{u}_2(x) = a^{-1}u_2(x)$. Then (u_1, \tilde{u}_2) satisfies

$$\begin{cases} \Delta u_1 - V_1(x)u_1 + \mu_1 u_1^3 + \beta a^2 u_1 \tilde{u}_2^2 = 0 & \text{in } \Omega, \\ \Delta \tilde{u}_2 - V_2(x)\tilde{u}_2 + \mu_2 a^2 \tilde{u}_2^3 + \beta u_1^2 \tilde{u}_2 = 0 & \text{in } \Omega, \\ u_1, \tilde{u}_2 > 0 \text{ in } \Omega, u_1 = \tilde{u}_2 = 0 \text{ on } \partial\Omega. \end{cases} \quad (4.7)$$

Let $\Omega_+ \equiv \{x \in \Omega \mid u_1(x) - \tilde{u}_2(x) > 0\}$. Then Ω_+ is a piecewise C^1 smooth domain. Multiplying the first equation in (4.7) by \tilde{u}_2 and the second equation in (4.7) by u_1 and then integrating by parts on Ω_+ and subtracting together, we obtain the following integral identity

$$\int_{\partial\Omega_+} (\tilde{u}_2 \frac{\partial u_1}{\partial n} - u_1 \frac{\partial \tilde{u}_2}{\partial n}) + \int_{\Omega_+} (\mu_1 - \beta) u_1 \tilde{u}_2 (u_1^2 - \tilde{u}_2^2) = 0, \quad (4.8)$$

where n denotes the unit outward normal to $\partial\Omega_+$.

On one hand, by the boundary condition and the definition of Ω_+ , we obtain

$$\int_{\partial\Omega_+ \cap \partial\Omega} (\tilde{u}_2 \frac{\partial u_1}{\partial n} - u_1 \frac{\partial \tilde{u}_2}{\partial n}) = 0, \quad (4.9)$$

since $u_1 = \tilde{u}_2 = 0$ on $\partial\Omega$, and

$$\int_{\partial\Omega_+ - \partial\Omega} (\tilde{u}_2 \frac{\partial u_1}{\partial n} - u_1 \frac{\partial \tilde{u}_2}{\partial n}) = \int_{\partial\Omega_+ - \partial\Omega} u_1 \frac{\partial(u_1 - \tilde{u}_2)}{\partial n} \leq 0, \quad (4.10)$$

since $u_1(x) - \tilde{u}_2(x) > 0$ in Ω_+ and $u_1(x) - \tilde{u}_2(x) = 0$ on $\partial\Omega_+ - \partial\Omega$.

On the other hand, because $\mu_1 - \beta < 0$ and $u_1(x) - \tilde{u}_2(x) > 0$ in Ω_+ , we have

$$\int_{\Omega_+} (\mu_1 - \beta) u_1 \tilde{u}_2 (u_1^2 - \tilde{u}_2^2) \leq 0. \quad (4.11)$$

Hence from the equalities (4.8)-(4.11), $\Omega_+ = \emptyset$. Similarly, we may prove that the set $\Omega_- \equiv \{x \in \Omega \mid u_1(x) - \tilde{u}_2(x) < 0\}$ is also an empty set. Therefore, $u_1(x) = \tilde{u}_2(x)$ in Ω and we complete the proof. \square

4.3. Nondegeneracy of positive solutions. Let (u_1, u_2) be a solution of (1.1). We say that (u_1, u_2) is *nondegenerate* if the solution set of the linearized equation

$$\begin{cases} \Delta\phi_1 - \lambda_1\phi_1 + 3\mu_1u_1^2\phi_1 + \beta u_2^2\phi_1 + 2\beta u_1u_2\phi_2 = 0, \\ \Delta\phi_2 - \lambda_2\phi_2 + 3\mu_2u_2^2\phi_2 + \beta u_1^2\phi_2 + 2\beta u_1u_2\phi_1 = 0, \\ |\phi_1| + |\phi_2| \leq 1 \end{cases} \quad (4.12)$$

is exactly N -dimensional, namely,

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \sum_{j=1}^N a_j \begin{pmatrix} \frac{\partial u_1}{\partial z_j} \\ \frac{\partial u_2}{\partial z_j} \end{pmatrix} \quad (4.13)$$

for some constants a_j .

Assume that $\lambda_1 = \lambda_2$. By Theorem 1.1 and Theorems 4.1-4.2, we conclude that $(u_1, u_2) = (u_0, v_0)$, where (u_0, v_0) is defined by (1.4), provided

$$\beta \in [\beta_0(N), \max\{\mu_1, \mu_2\}] \quad (4.14)$$

where $\beta_0(N) \leq \min\{\mu_1, \mu_2\}$ and $\beta_0(1) = \min\{\mu_1, \mu_2\}$. By Lemma 2.2 and Theorem 3.1 of [5], (u_0, v_0) is nondegenerate. We state it in the following corollary

Corollary 4.4. *Assume that $\lambda_1 = \lambda_2$ and (4.14) holds. Then the positive solution to (1.1) is nondegenerate.*

Finally, Theorem 1.3 is a consequence of Theorems 4.1-4.2.

5. Existence of bound states in systems with trapping potentials. In [10], the authors constructed ground states in two-component systems of nonlinear Schrödinger equations with trapping potentials. Using the nondegeneracy result, we can consider bound states for the following system

$$\begin{cases} \epsilon^2\Delta u - V_1(x)u + \mu_1u^3 + \beta uv^2 = 0 & \text{in } \mathbb{R}^1, \\ \epsilon^2\Delta v - V_2(x)v + \mu_2v^3 + \beta u^2v = 0 & \text{in } \mathbb{R}^1. \end{cases} \quad (5.1)$$

To simplify the technical difficulties, we assume that $C_1 \leq V_1, V_2 \leq C_2$. We have the following two results.

Theorem 5.1. *Assume that V_1 and V_2 have a local minimum at x_0 . That is, there exists $\delta > 0$ such that $V_1(x) > V_1(x_0), V_2(x) > V_2(x_0)$ for $x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$. Furthermore we assume that*

$$V_1(x_0) = V_2(x_0), \beta \notin [\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}] \quad (5.2)$$

Then for ϵ sufficiently small, problem (5.1) has a solution (u_ϵ, v_ϵ) with spikes near x_0 .

Theorem 5.2. *Assume that V_1 and V_2 have a local maximum at x_0 . That is, there exists $\delta > 0$ such that $V_1(x) < V_1(x_0), V_2(x) < V_2(x_0)$ for $x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$. Furthermore suppose (5.2) holds. Then for positive integer $K \geq 2$ and ϵ sufficiently small, problem (5.1) has a solution (u_ϵ, v_ϵ) with K spikes near x_0 .*

Theorem 5.2 seems to be the first result on the existence of bound states with multiple spikes.

Under the condition (5.2), we have uniqueness and nondegeneracy of the limiting equations. The proofs of both Theorem 5.1 and Theorem 5.2 follow from the same reduction procedure in [7] for single equations. We omit the details.

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REFERENCES

- [1] A. Ambrosetti and E. Colorado, *Bound and ground states of coupled nonlinear Schrödinger equations*, C.R. Math. Acad. Sci. Paris, **342** (2006), 453–458.
- [2] J. Busca and B. Sirakov, *Symmetry results for semilinear elliptic systems in the whole space*, J. Differential Equations, **163** (2000), 41–56.
- [3] T. Bartsch and Z. Q. Wang, *Note on ground states of nonlinear Schrödinger systems*, J. Part. Diff. Eqns., **19** (2006), 200–207.
- [4] T. Bartsch, Z. Q. Wang and J. Wei, *Bound states for a coupled Schrödinger system*, J. Fixed Point Theory Appl., **2** (2007), 353–367.
- [5] E. N. Dancer and J. Wei, *Spike solutions in coupled nonlinear Schrödinger equations with attractive interaction*, Trans. Amer. Math. Soc., **361** (2009), 1189–1208.
- [6] N. Ikoma, *Uniqueness of positive solutions for a nonlinear elliptic systems*, NoDEA, **16** (2009), 555–567.
- [7] X. Kang and J. Wei, *On interacting bumps of semiclassical states of nonlinear Schrödinger equations*, Adv. Diff. Eqns., **5** (2000), 899–928.
- [8] M. K. Kwong, *Uniqueness of positive solutions of $-\Delta u + u = u^p$ in \mathbb{R}^N* , Arch. Rat. Mech. Anal., **105** (1989), 243–266.
- [9] T. C. Lin and J. Wei, *Ground state of N coupled nonlinear Schrödinger equations in R^n , $n \leq 3$* , Communications in Mathematical Physics, **255** (2005), 629–653.
- [10] T. C. Lin and J. Wei, *Spikes in two-component systems of nonlinear Schrödinger equations with trapping potentials*, J. Diff. Eqns., **229** (2006), 538–569.
- [11] O. Lopes, *Uniqueness of a symmetric positive solutions to an ODE system*, Elect. J. Diff. Eqns., **162** (2009), 1–8.
- [12] B. Sirakov, *Least energy solitary waves for a system of nonlinear Schrödinger equations*, Comm. Math. Physics, **271** (2007), 199–221.

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