CHAPTER 2

GRAPH THEORY

1. Terminologies

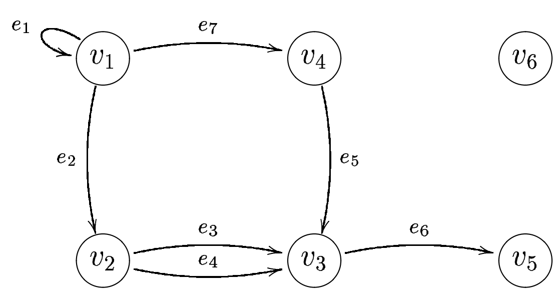
An *undirected graph* (or simply a *graph*) is an ordered pair $G = (V, E)$, where $V = \{v_1, v_2, \dots\}$, a non-empty set, is called the set of *vertices* and $E = \{e_1, e_2, \dots\}$ is called the set of *edges*, such that each edge e_k is identified with an unordered pair $\{v_i, v_j\}$ of vertices. The vertices v_i, v_j associated with edge e_k are called the end vertices of e_k. A graph with a finite number of vertices as well as a finite number of edges is called a *finite* graph, otherwise it is called an *infinite* graph.

An edge $\{v_i, v_j\}$ having the same vertex as both its end vertices is called a *self-loop*. Two edges with the same end vertices are referred to as *parallel edges*. A graph that has neither self-loops nor parallel edges is called a *simple graph*. In this section, we only consider finite undirected simple graphs.

If a vertex v_i is an end vertex of some edge e_j, v_i and e_j are said to be *incident* with each other. Two non-parallel edges are said to be *adjacent* if they are incident on a common vertex. Similarly, two vertices are said to be *adjacent* if they are the end vertices of the same edge.

The number of edges incident with a vertex v_i, with self-loops counted twice, is called the *degree*, $d(v_i)$, of vertex v_i. A vertex with odd (respectively even) degree is called an *odd* (respectively *even*) vertex. A vertex having no incident edge is called an *isolated vertex*. A vertex of degree one is called a *pendant vertex*. Two adjacent edges are said to be *in series* if their common vertex is of degree two.

EXAMPLE 2.1. Undirected graphs are usually represented by diagrams like the following one which shows a graph with six vertices and seven edges.



In this graph, we have edge e_1 is a self-loop and edges e_3 and e_4 are parallel edges. Vertex v_1 is incident with edge e_2. Vertices v_1 and v_2 are adjacent with degrees $d(v_1) = 4$ and $d(v_2) = 3$. Vertex v_5 is a pendant vertex and vertex v_6 is an isolated vertex. Edges e_5 and e_7 are adjacent and in series. Edges e_2 and e_3 are adjacent but not in series.

2. Some Theorems

THEOREM 2.1. *The number of vertices of odd degree in a graph is always even.*

PROOF. Let (V, E) be a graph with e edges and n vertices. Since each edge contributes two degrees, the sum of the degrees of all the vertices is twice the number of edges, i.e.

$$\sum_{i=1}^{n} d(v_i) = 2e \tag{1}$$

But the left side of (1) can only be an even number if there are an even number of vertices with odd degrees. □

A *walk* is defined as a sequence of edges $(e_{i_1}, e_{i_2}, \ldots, e_{i_n})$ such that $e_{i_j} = \{v_{i_j - 1}, v_{i_j}\}$ for $j = 1, 2, \ldots n$, (i.e. $e_{i_j - 1}$ and e_{i_j} are adjacent edges), and no edge is repeated in the sequence. Vertices with which a walk begins and ends are called its *terminal* vertices (i.e. vertices v_{i_0} and v_{i_n}). The two terminal vertices are also said to be *joined* by the walk. If a walk begins and ends at the same vertex, then it is called a *closed walk*. A walk that is not closed is called an *open walk*. We note that a walk may meet the same vertex more than once, i.e. $v_{i_j} = v_{i_k}$ for some $j \neq k$.

An open walk in which no vertex meets the walk more than once is called a *path* (or a *simple path* or an *elementary path*). The number of edges in a path is called the *length of a path*. A closed walk in which no vertex (except the two terminal vertices) appears more than once is called a *circuit* (or *cycle*), i.e. a circuit is a closed, non-intersecting walk.

In Example 1, the set of edges $\{e_5, e_7, e_2, e_3, e_6\}$ constitutes an open walk which is not a path for the vertex v_3 appears twice. However, the set of edges $\{e_7, e_2, e_3, e_6\}$ forms a path and the set $\{e_5, e_7, e_2, e_3\}$ is a circuit.

THEOREM 2.2. (i) *If a walk w joins two distinct vertices v_1 and v_2, then w contains a path joining v_1 and v_2.*

(ii) *If there are paths from a vertex v to w and from w to x where $v \neq x$, then there is a path from v to x.*

PROOF. (i) Let us traverse the walk from v_1 to v_2. If every vertex meets w at most once, then w is a path. If there exists a vertex v that meets w for a second time, then remove all the edges between the first time and the second time w meets v. The remaining edges still form a walk from v_1 to v_2. Repeat the process until a walk is obtained where every vertex meets it at most once. This is then a path from v_1 to v_2.

(ii) Combining the paths from v to w and from w to x will form a sequence of edges from v to x. By using the same process as in part (a) to remove edges, a path from v to x is obtained. □

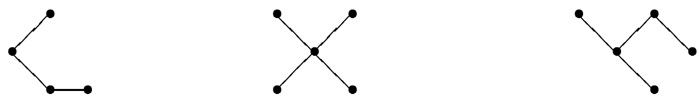
A graph $G = (V, E)$ is said to be *connected* if there is at least one path between every pair of distinct vertices in G. Otherwise, G is *disconnected*. A graph $H = (V', E')$ is said to be a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. A connected subgraph of a graph is called a *component* of the given graph if it is maximal (maximal in the set inclusion sense). Thus a component cannot be a proper subset of another connected subgraph of G. In fact, the sets of vertices in two distinct components of a graph must be disjoint. For if not, the union of two components will yield a larger connected subgraph of G. Components of a graph can easily be obtained by the following theorem. Its proof is left as an exercise.

THEOREM 2.3. *For any vertex v in a graph G, the component containing v is the subgraph consisting of v and all vertices of G that have paths to v, together with all the edges incident on them.* □

THEOREM 2.4. *If all the vertices of a graph G have degree at least 2, then G contains a circuit.*

PROOF. Take an edge, say $e_1 = \{v_{i_1},\ v_{i_2}\}$. Since $d(v_{i_2})$ must be at least 2, there exists an edge $e_2 = \{v_{i_2},\ v_{i_3}\}$. We can construct a walk with edges $\{e_1, e_2\}$ and extend this walk by repeating the above process. This process can be continued until the walk meets a vertex for a second time, i.e. $e_{m-1} = \{v_{i_{m-1}},\ v_{i_m}\}$ with $v_{i_m} = v_{i_k}$ for $k < m$. Then $\{e_k, e_{k+1}, \cdots, e_{m-1}\}$ is a circuit in G. ☐

A *tree* is a connected graph without any circuits. Thus a tree must be a simple graph. The following are examples of trees:



THEOREM 2.5. *A graph G without self-loop is a tree if and only if there is one and only one path between every pair of vertices.*

PROOF. (Necessity) Suppose that G is a tree. Since G is connected, there exists at least one path between every pair of vertices. On the other hand, if there exists a pair of vertices with two distinct paths between them, then these two paths together must contain a circuit and the graph cannot be a tree.

(Sufficiency) Suppose that there is exactly one path between every pair of vertices. Then G is connected. Now the existence of a circuit in G will imply that any pair of vertices in the circuit have at least two paths between them. Hence G has no circuits and is therefore a tree. ☐

THEOREM 2.6. *Let G be a connected graph and C be a circuit in G. The subgraph obtained by removing one of the edges on the circuit C is still connected.*

PROOF. Let $\{e_1, \cdots, e_k\}$ be the circuit and without loss of generality, let e_1 be the removed edge. Now given any two vertices and a path connecting them, if the path does not contain e_1, then we are done. If the path does contain e_1, then that removed segment of the path can be replaced be $\{e_k, e_{k-1}, \cdots, e_2\}$. Hence we see that the two vertices are still connected at least by a walk. Using Theorem 2 (i), the theoerm follows. ☐

THEOREM 2.7. *Suppose that a graph G has n vertices. Then whenever any two of the following conditions hold, the third condition will also hold.*

(i) *G has no circuits.*
(ii) *G is connected.*
(iii) *G has $n - 1$ edges.*

PROOF. ((i) and (ii) $\Rightarrow$ (iii)) We use induction on the number of vertices to prove that G has $n - 1$ edges. For $n = 1, 2$, the theorem clearly holds. Assume that the theorem holds for $n = k - 1$. Consider the case $n = k$. If all vertices of G have degree > 1, then by Theorem 4, G must contain a circuit and this contradicts (i). Hence there exists a vertex of degree zero or one. But the existence of a vertex of degree zero will imply that the graph is not connected. Hence there exists a vertex of degree one. Remove this vertex together with the edge incident with it to form a new graph H. Then H is a tree with $k - 1$ vertices. By induction assumption, H has $k - 2$ edges. Hence G has $k - 1$ edges.

((i) and (iii) $\Rightarrow$ (ii)) Assume that G consists of components $C_1, C_2, \ldots, C_k$. By the part we already proved, since each component is connected and has no circuits, each component has one less edge than the number of vertices. Hence G has $n-k$ edges. By (iii), we have $k=1$. Thus G has only one component and is therefore connected.

((ii) and (iii) $\Rightarrow$ (i)) Assume that G has a circuit C. By Theorem 6, we can remove one of the edges of C to obtain a connected subgraph with fewer edges. We can repeat this process of removing edges from circuits until we obtain a connected subgraph with no circuits. By the first part of the proof, this subgraph must have $n-1$ edges, the same number of edges as G. Hence this subgraph is G itself and in particular, G cannot have any circuit. □

It follows from the above theorem that a graph is a tree if it satisfies any two of the conditions stated in the theorem.

A subgraph T is said to be a *spanning tree* of a connected graph G if T is a tree and contains all vertices of G. The *distance*, denoted by $d(H,K)$, between two spanning trees H and K of a graph G is the number of edges of G present in one tree but not in the other.

THEOREM 2.8. *Any connected graph has at least one spanning tree.*

PROOF. If the graph has no circuits, then it is its own spanning tree. If the graph G has a circuit, then remove an edge from this circuit and by Theorem 6, the subgraph remaining is still connected. Repeat this process of removing circuits until a subgraph containing no circuits is obtained. Then this subgraph is a spanning tree. □

An edge in a spanning tree T is called a *branch* of T. An edge of G that is not in a given spanning tree is called a *chord*. The set of chords is called a *chord set*. A circuit formed by adding a chord to a spanning tree is called a *fundamental circuit* with respect to that tree. If after adding a chord to a spanning tree to form a fundamental circuit, a branch is removed from the circuit to generate another spanning tree, such a process is called a *cyclic exchange* or *elementary tree transformation*.

THEOREM 2.9. *Starting from any spanning tree of a graph G, we can obtain every spanning tree of G by successive cyclic exchanges.*

PROOF. Let H and K be two spanning trees of G. Let us prove that H can be transformed into K by cyclic exchanges by using induction on the distance between H and K. The case is trivially true if the distance is zero. Assume that the theorem is true if the distance between H and K is $\leq k-1$. Now if the distance between them is k, add an edge that is in K but not in H. This creates a circuit C in H by Theorem 7. The circuit cannot be contained in K. Thus we can find an edge in C that is not in K. Remove this edge from C to form a spanning tree H'. Clearly the distance between H' and K is $k-1$, and by induction assumption H' can be transformed to K by cyclic exchange. □

Recall that two subsets V_1 and V_2 of a set V are said to be a *partition* of V if $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \phi$. A *bipartite graph* G is a graph $G=(V,E)$ such that the vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins V_1 and V_2.

THEOREM 2.10. *A graph is bipartite if and only if all its circuits consists of even number of edges.*

PROOF. Suppose G is bipartite with vertex classes V_1 and V_2. Let $\{x_1, x_2, \cdots, x_j, x_1\}$ be a set of vertices that forms a circuit in G. We may assume that $x_1 \in V_1$. Then $x_2 \in V_2$,

$x_3 \in V_1$ etc, i.e. $x_i \in V_1$ if and only if i is odd. Since $x_j \in V_2$, we have j is even. Conversely, let us assume that all circuits of G have even number of edges. Since a graph is bipartite if and only if each component of it is, we may assume without loss of generality that G is connected. Pick a vertex $x \in V$ and let V_1 be the set of vertices y such that the length of the path joining x and y is odd. Let $V_2 = V \setminus V_1$. Then there is no edge joining two vertices of the same set since that will imply that G contains a circuit of odd number of edges. Thus G is bipartite. □

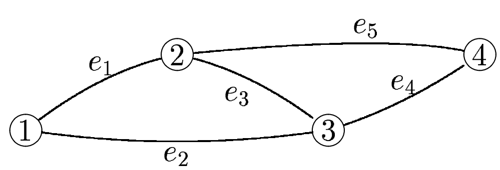
3. Incidence Matrices

Let $A = (a_{ij})$, $i = 1, \ldots, |V|$, $j = 1, \ldots, |E|$, be a matrix defined on an undirected graph as

$$a_{ij} = \begin{cases} 1 & \text{if edge } e_j \text{ and vertex } v_i \text{ are indicent} \\ 0 & \text{otherwise} \end{cases}$$

The matrix A is called the *incidence matrix* of the graph $G = (V, E)$. The ith row of A corresponds to vertex i and its ones indicate all of the edges incident with vertex i. Note that a graph is completely specified by its incidence matrix.

EXAMPLE 2.2. Consider the graph



Its incidence matirx is

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array} .$$

THEOREM 2.11. *Let A be the incidence matrix of an undirected graph G and let $\tilde{E} = \{e_{i_1}, \ldots, e_{i_n}\}$ be any set of edges of G. If $\tilde{E}$ contains no circuits, then the columns of A associated with the edges $e_{i_1}, \ldots, e_{i_n}$ are linearly independent.*

PROOF. Use induction on n. Clearly the theorem holds for $n = 1$. Assume that the theorem holds for $n = 1, \ldots, k-1$. Let $\{A_{*i_1}, \ldots, A_{*i_k}\}$ be the column vectors of A associated with the edges $\{e_{i_1}, \ldots, e_{i_k}\}$, and let $\tilde{V}$ be the set of vertices of G which are incident with some edge in $\tilde{E}$. Thus $\tilde{G} = (\tilde{V}, \tilde{E})$ is a subgraph of G. Since $\tilde{G}$ has no circuits, by Theorem 4, $\tilde{G}$ must contain a vertex x_s with degree one. Now suppose that for some scalars $\alpha_1, \ldots, \alpha_k$,

$$\sum_{j=1}^{k} \alpha_j A_{*i_j} = 0 \ .$$

Since x_s has degree one, it is incident with exactly one edge, say $e_{i_{j_0}}$. Hence the sth component of the sum on the L.H.S. is simply α_{j_0}. Thus $\alpha_{j_0} = 0$ and

$$\sum_{j=1 j\neq j_0}^{k} \alpha_j A_{*_{i_j}} = 0.$$

But this sum consists of $k-1$ column vectors corresponding to the edges $\{e_{i_j}\}$, $j = 1, \ldots, k$, $j \neq j_0$. By induction assumption, we must have $\alpha_j = 0$, for $j = 1, \ldots, k$, $j \neq j_0$. So we have shown that $\{A_{*_{i_j}}\}_{j=1,\ldots,k}$ are linearly independent. □

We remark that the converse of the above is not true in general unless the graph is bipartite.

THEOREM 2.12. *In a bipartite graph, if* $\{e_{i_1}, \ldots, e_{i_n}\}$ *forms a circuit, then the columns*

$$\{A_{*_{i_1}}, \ldots, A_{*_{i_n}}\}$$

of the incidence matrix of A *are linearly dependent.*

PROOF. Let $e_{i_j} = \{x_{i_{j-1}}, x_{i_j}\}$, $j = 1, \ldots, n$, where $x_{i_0} = x_{i_n}$. Inductively, we can easily prove that

$$A_{*_{i+1}} - A_{*_{i_2}} + A_{*_{i_3}} - \cdots, (-1)^{j+1} A_{*_{i_j}}$$

$$= \begin{cases} (\ldots\ 0 \ldots, \underset{i_0}{\overset{\uparrow}{\rightarrow}} 1, \ldots, \underset{i_j}{\overset{\uparrow}{\rightarrow}} (-1)^{j+1}, \ldots, 0\ \ldots) & \text{if } x_{i_0} \neq x_{i_j} \\ (\ldots\ 0 \ldots, \underset{i_0}{\overset{\uparrow}{\rightarrow}} 2, \ldots, 0\ \ldots,) & \text{if } x_{i_0} = x_{i_j} \text{ and } j \text{ is odd} \\ (\ldots\ 0\ \ldots) & \text{if } x_{i_0} = x_{i_j} \text{ and } j \text{ is even} \end{cases}$$

Since in a bipartite graph, all circuits consist of even number of edges. We must have n even. Hence

$$A_{*_{i_1}} - A_{*_{i_2}} + A_{*_{i_3}} - \cdots - A_{*_{i_n}} = 0$$

and $\{A_{*_{i_1}}, \ldots, A_{*_{i_n}}\}$ are linearly dependent. □

COROLLARY 2.13. *If a subgraph of a bipartite graph contains a circuit, then the column vectors, assoicated with the edges of this subgraph, in the incidence matrix are linearly dependent.*