Bubble Accumulations In An Elliptic Neumann Problem With Critical Sobolev Exponent

CHANGSHOU LIN*, LIPING WANG † and JUNCHENG WEI‡

October 2, 2006

Abstract

We consider the following critical elliptic Neumann problem $-\Delta u + \mu u = u^{\frac{N+2}{N-2}}$, $u > 0$ in Ω; $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$, Ω being a smooth bounded domain in $\mathbb{R}^N, N \geq 7, \mu > 0$ is a large number. We show that at a positive nondegenerate local minimum point Q_0 of the mean curvature, (we may assume that $Q_0 = 0$ and the unit normal at Q_0 is $-e_N$), for any fixed integer $K \geq 2$, there exists a $\mu_K > 0$ such that for $\mu > \mu_K$, the above problem has $K-$ *bubble* solution u_μ concentrating at the same point Q_0. More precisely, we show that u_μ has K local maximum points $Q_1^\mu, ..., Q_K^\mu \in \partial\Omega$ with the property that $u_\mu(Q_j^\mu) \sim \mu^{\frac{N-2}{2}}, Q_j^\mu \to Q_0, j = 1, ..., K$, and $\mu^{\frac{N-3}{N}}((Q_1^\mu)', ..., (Q_K^\mu)')$ approach an optimal configuration of the following functional

$(*)$ *Find out the optimal configuration that minimizes the following functional:*

$$R[Q_1', ..., Q_K'] = c_1 \sum_{j=1}^K \varphi(Q_j') + c_2 \sum_{i \neq j} \frac{1}{|Q_i' - Q_j'|^{N-2}}$$

where $Q_i^\mu = ((Q_i^\mu)', Q_{i,N}^\mu), c_1, c_2 > 0$ are two generic constants and $\varphi(Q) = Q^T \mathbf{G} Q$ with $\mathbf{G} = (\nabla_{ij} H(Q_0))$.

Subject class: Primary 35B40, 35J20; Secondary 35J55, 92C15, 92C40
Keywords: Critical Sobolev Exponent, Singularly Perturbations, Optimal Configuration

---

*Department of Mathematics, National Chung Cheng University, Minghsiung, Chia Yi, Taiwan. E-mail: cslin@math.ccu.edu.tw

†Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong. E-mail : lpwang@math.cuhk.edu.hk.

‡Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong. E-mail : wei@math.cuhk.edu.hk. Research supported in part by an Earmarked Grant from RGC of HK.

1 Introduction

In this paper we consider the following nonlinear elliptic Neumann problem

$$(P_{q,\mu}) \qquad \begin{cases} -\Delta u + \mu u &= u^q, \quad u > 0 \qquad \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= 0 \qquad\qquad\qquad \text{on } \partial\Omega \end{cases}$$

where $q = \frac{N+2}{N-2}$, $\mu > 0$ and Ω is a smooth and bounded domain in $\mathbb{R}^N$, $N \geq 7$.

Equation $(P_{q,\mu})$ arises in many branches of the applied sciences. For example, it can be viewed as a steady-state equation for the shadow system of the Gierer-Meinhardt system in biological pattern formation ([17], [26]) or of parabolic equations in chemotaxis, e.g. Keller-Segel model ([24]).

When q is subcritical, i.e. $q < \frac{N+2}{N-2}$, Lin, Ni and Takagi [24] proved that the only solution, for small μ, is the constant one, whereas nonconstant solutions appear for large μ, which concentrate, as μ goes to infinity, at one or several points. The least energy solution blows up at a boundary point which maximizes the mean curvature of the frontier [28][29]. Higher energy solutions exist which blow up at one or several points, located on the boundary [4][11][16][22][44][21], in the interior of the domain [5][10][12][14][15][19][41], or some of them on the boundary and others in the interior [20]. (A review up to 2004 can be found in [26].)

In particular, we mention the following result of [21]:

Theorem A ([21]). *Suppose that $1 < q < \frac{N+2}{N-2}$ and that $Q_0 \in \partial\Omega$ is a strictly local minimum point of the mean curvature function $H(P)$. Then given any positive integer K, there exists a $\mu_K > 0$ such that for $\mu > \mu_K$, problem $(P_{q,\mu})$ has a solution u_μ with K spikes $Q_j^\mu, j = 1, ..., K$ such that $Q_j^\mu \to Q_0$ and $|Q_i^\mu - Q_j^\mu| \geq C\frac{1}{\sqrt{\mu}} \log \mu$.*

In the critical case, i.e. $q = \frac{N+2}{N-2}$, there also have been many works on $P_{q,\mu}$. For large μ, nonconstant solutions exist [1][36]. As in the subcritical case the least energy solution blows up, as μ goes to infinity, at a unique point which maximizes the mean curvature of the boundary [3][27]. Higher energy solutions have also been exhibited, blowing up at one [2][37][31][18] or several (separated) boundary points [25][38][39]. The question of interior blow-up is still open. However, in contrast with the subcritical situation, at least one blow-up point has to lie on the boundary [32]. In the case of μ small, Zhu [45] proved that, for convex three dimensional domains, the only solution is the constant one . A different proof is given in [42]. For μ small , Rey and Wei [35] have proved there exists arbitrarily many bubble solutions if $N = 5$ and Wang, Wei [43] also proved there exists arbitrarily many bubble solutions if $N = 4, 6$.

In the slightly supercritical case, i.e., $q = \frac{N+2}{N-2} + \delta$ where $\delta > 0, \delta \to 0$, Rey and Wei [33]-[34], del Pino, Musso and Pistoia [9], proved the existence of boundary bubble solutions for fixed $\mu > 0$. Furthermore, a new type of solutions, i.e., bubble-towers, has been constructed in [9].

Our aim, in this paper, is to prove a version of Theorem A, in the critical exponent case. Furthermore, we can identify the optimal configurations inside the clustered bubbles.

Let $H(Q)$ denote the (outward) boundary mean curvature function at $Q \in \partial\Omega$. Our basic assumption is the following:

$(H1)$ $N \geq 7$,

$(H2)$ $Q_0 = 0$ is a nondegenerate *local minimum* point of $H(Q)$ and $H(Q_0) > 0$.

From now on, we assume that (H1) and (H2) hold. Without loss of generality, we may assume that the unit outward normal at Q_0 is $-e_N = (0, ..., 0, -1)$. By (H2), the eigenvalues of the matrix $\mathbf{G} = (\nabla_{ij} H(Q_0))_{(N-1)\times(N-1)}$ are all positive. For $Q' \in \mathbb{R}^{N-1}$, set

$$\varphi(Q') = (Q')^T \mathbf{G} Q'. \tag{1.1}$$

For $(Q_1', ..., Q_K') \in \mathbb{R}^{(N-1)K}, Q_i' \neq Q_j'$, we define

$$R[Q_1', ..., Q_K'] := c_1 \sum_{j=1}^{K} \varphi(Q_j') + c_2 \sum_{i \neq j} \frac{1}{|Q_i' - Q_j'|^{N-2}} \tag{1.2}$$

where c_1 and c_2 are two generic constants to be defined later (see (2.6)).

It turns out that the following optimal configuration problem plays an important role in our studies:

$(*)$ *Find out the optimal configuration* $(Q_1', ..., Q_K')$ *that minimizes the functional* $R[Q_1', ..., Q_K']$.

For normalization reasons, we consider throughout the paper the following equation

$$-\Delta u + \mu u = \alpha_N u^{\frac{N+2}{N-2}}, \quad u > 0 \text{ in } \Omega, \ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \tag{1.3}$$

instead of the original one, where $\alpha_N = N(N-2)$. The solutions are identical, up to the multiplicative constant $(\alpha_N)^{-\frac{N-2}{4}}$. We recall that, according to [6], the functions

$$U_{\varepsilon,Q}(x) = \frac{\varepsilon^{\frac{N-2}{2}}}{(\varepsilon^2 + |x-Q|^2)^{\frac{N-2}{2}}} \qquad \varepsilon > 0\,,\ Q \in \mathbb{R}^N \tag{1.4}$$

are the only solutions to the problem

$$-\Delta u = \alpha_N u^{\frac{N+2}{N-2}}, \quad u > 0 \qquad \text{in} \quad \mathbb{R}^N.$$

The following is the main result of this paper.

Theorem 1.1 *Suppose that (H1) and (H2) hold. Let $K \geq 2$ be a fixed integer. Then there exists a $\mu_K > 0$ such that for $\mu > \mu_K$, $(P_{\frac{N+2}{N-2},\mu})$ has a nontrivial solution u_μ with the following properties*

(1) $u(x) = \sum_{j=1}^{K} U_{\frac{1}{\mu}\Lambda_j, Q_0+\mu^{\frac{3-N}{N}}\hat{Q}_j^\mu} + O(\mu^{\frac{N-4}{2}})$, where $Q_0 + \mu^{\frac{3-N}{N}}\hat{Q}_j^\mu \in \partial\Omega$, $\Lambda_j \to \Lambda_0 := A_0 H(Q_0) > 0, j = 1, ..., K$, and

(2) $((\hat{Q}_1^\mu)', ..., (\hat{Q}_K^\mu)')$ approach an optimal configuration in the problem (*), as $\mu \to +\infty$, where $(\hat{Q}_i^\mu)' = (\hat{Q}_{i,1}^\mu, ..., \hat{Q}_{i,N-1}^\mu), i = 1, ..., K$. Here A_0 is a generic positive constant (see (2.1)).

(3) u_ε has exactly K local maximum points $q_j^\varepsilon \in \partial\Omega, j = 1, ..., K$. Moreover, after a permutation, $q_j^\varepsilon = Q_0 + \mu^{\frac{3-N}{N}}\hat{Q}_j^\mu + o(\mu^{\frac{3-N}{N}}), j = 1, ..., K$.

Remarks.

1. The construction of one bubble with the assumption (H1) and (H2) has been done in [3]. So we focus on the case of $K \geq 2$.

2. The fact that $H(Q_0) > 0$ seems to be necessary. In [18], it is proved that at least when $K = 1$, the blow-up point Q_0 must have nonnegative mean curvature. Another notable geometric effect is that there are no solutions with its peaks staying in a portion of the boundary which is totally flat. See [23].

3. We need the dimension $N \geq 7$. We do not know what happens when $4 \leq N \leq 6$. At least when $N = 4, 5$, our computations show that bubble accumulations can not occur. (See the remark below.) As far as the authors know, it seems to be the first such result for the critical exponent case. (For slightly supercritical case, multiple bubbles (bubble-towers) can exist [9].)

Let $Q_0 \in \partial\Omega$ satisfy the assumption (H2). Without loss of generality, we may assume that $Q_0 = 0$. Set

$$\varepsilon = \frac{1}{\mu}, \ \Omega_\varepsilon := \{z | \varepsilon z + Q_0 \in \Omega\}. \tag{1.5}$$

By suitably scaling, equation $P_{q,\mu}$ becomes the following rescaled problem which we work with

$$\Delta u - \varepsilon u + \alpha_N u^q = 0, u > 0 \ \text{ in } \Omega_\varepsilon, \qquad \frac{\partial u}{\partial n} = 0 \ \text{ on } \ \partial\Omega_\varepsilon, \tag{1.6}$$

where $q = \frac{N+2}{N-2}$.

We need to define another small constant

$$d = \varepsilon^{\frac{3}{N}}. \tag{1.7}$$

$\frac{1}{d}$ measures the distance between the bubbles in the rescaled domain Ω_ε. In the original domain Ω, the distance between bubbles is $O(\frac{\varepsilon}{d}) = O(\varepsilon^{\frac{N-3}{N}})$.

Let us now explain the differences between $N \geq 7$ and $N = 4, 5$. Consider the fundamental solution of $-\Delta + \varepsilon$. We denote it by $K_\varepsilon(|y|)$. It is easy to see that

$$K_\varepsilon(y) = \varepsilon^{\frac{N-2}{2}} K_1(\sqrt{\varepsilon} y) \tag{1.8}$$

Thus when $N \geq 7$, $K_\varepsilon(y)$ behaves like $\frac{1}{|y|^{N-2}}$ for $|y| << \frac{1}{\sqrt{\varepsilon}}$. For $N = 4, 5$, $\frac{1}{d} >> \frac{1}{\sqrt{\varepsilon}}$ and K_ε behaves like $\varepsilon^{\frac{N-2}{2}} e^{-\frac{\sqrt{\varepsilon}}{d}}$.

The dimension $N = 6$ seems to be a critical case.

We set

$$S_\varepsilon[u] := \Delta u - \varepsilon u + \alpha_N u_+^q, u_+ = \max(u, 0), \tag{1.9}$$

We introduce the following functional defined in $H^1(\Omega_\varepsilon)$

$$J_\varepsilon[v] = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla v|^2 + \frac{\varepsilon}{2}\int_{\Omega_\varepsilon} v^2 - \frac{\alpha_N}{q+1}\int_{\Omega_\varepsilon} v_+^{q+1} \tag{1.10}$$

whose nontrivial critical points are solutions to (1.6).

The main idea for proving Theorem 1.1 is the so-called "localized energy method", which reduces the problem to a finite dimensional problem, for which we can use min-max theorems. This kind of argument has been used in many papers [10], [7], [9], [19], [20], [21], [33] and [34] and the references therein. We follow closely to that of [9], [33] and [34], where slightly supercritical elliptic Neumann problems are studied. More precisely, by choosing the height for each bubble to be $\Lambda_i = \Lambda_0(1 + \frac{\varepsilon}{d}\hat{\Lambda}_i)$ and the location of bubble to be $Q_i = Q_0 + \frac{\varepsilon}{d}\hat{Q}_i$, and by using Liapunov-Schmidt reduction ([33], [34]), we reduce the problem to finding a critical point for the following reduced problem:

$$\hat{I}_\varepsilon(\hat{\mathbf{\Lambda}}, \hat{\mathbf{Q}}) = C_N \Lambda_0^2 \sum_{j=1}^{K} \hat{\Lambda}_j^2 - R[\hat{Q}_1', ..., \hat{Q}_K'] + o(1). \tag{1.11}$$

Problem (1.11) has a saddle point: minimizing in $\hat{\mathbf{\Lambda}} = (\hat{\Lambda}_1, ..., \hat{\Lambda}_K)$ and maximizing in $\hat{\mathbf{Q}}' = (\hat{Q}_1', ..., \hat{Q}_K') \in \mathbb{R}^{(N-1)K}$. We have to use a max-min argument to conclude.

The organization of the papers is as follows: In Section 2, we have various preliminaries including the construction of $K-$bubble approximate solutions. Section 3 is devoted to the finite-dimensional reduction process: we first study a linear problem then we study the nonlinear reduction. In Section 4, we define and expand a reduced energy functional. Afterwards, we use a max-min argument to conclude the existence of a critical point to the reduced energy. This finishes the proof of Theorem 1.1. Several estimates are included in the Appendix.

Throughout the paper, the constant C represents constants independent of $\varepsilon > 0$.

2 Some Preliminaries

This section contains various preliminaries.

2.1 Several Generic Constants

Let us first define several generic constants to be used throughout this paper.

Let

$$A_0 = (\int_{\mathbb{R}^N_+} (U_{1,0}^2))^{-1} \int_{\mathbb{R}^N_+} \left[2y_N \sum_{j=1}^{N-1} \frac{\partial^2 U_{1,0}}{\partial y_j^2} - (N-1) \frac{\partial U_{1,0}}{\partial y_N} \right] \left[\left. \frac{\partial U_{\Lambda,0}}{\partial \Lambda} \right|_{\Lambda=1} \right], \tag{2.1}$$

$$B_N = \frac{N-1}{4} \int_{\mathbb{R}^N_+} y_N (\frac{\partial U_{1,0}}{\partial y_N})^2, \tag{2.2}$$

$$C_N = \frac{1}{2} \int_{\mathbb{R}^N_+} U_{1,0}^2, \tag{2.3}$$

and

$$E_N = \frac{\alpha_N}{2} \int_{\mathbb{R}^N_+} U_{1,0}^{\frac{N+2}{N-2}}. \tag{2.4}$$

The constants A_0, B_N, C_N can be calculated straightforwardly by the following formula:

$$\int_0^\infty \frac{r^m}{(1+r^2)^l} dr = \frac{\Gamma(\frac{1+m}{2})\Gamma(\frac{2l-m-1}{2})}{2\Gamma(l)}$$

Then we obtain the following relation between A_0, B_N and C_N:

$$A_0 = \frac{B_N}{2C_N}. \tag{2.5}$$

Set

$$c_1 = \frac{B_N}{2} \Lambda_0, \qquad c_2 = E_N \Lambda_0^{N-2} \tag{2.6}$$

where $\Lambda_0 = A_0 H(Q_0)$.

2.2 A Simple Lemma on Problem (*)

Let R be defined as in (1.2). We state a simple fact on Problem (*).

Lemma 2.1 *Assume that the matrix* $\mathbf{G}$ *is positive definite. Then the following minimization problem*

$$R_0 := \inf_{\hat{\mathbf{Q}}' \in \mathbb{R}^{(N-1)K}} R[\hat{Q}_1', ..., \hat{Q}_K'] \tag{2.7}$$

is attained.

Proof: This follows from simple calculus. □

We remark that even in the case of $\mathbf{G}$ being the identity matrix, it is unclear what the optimal configuration is.

2.3 Boundary Deformations

Fix $Q \in \partial\Omega$. We introduce a boundary deformation which strengthens the boundary near Q. Without loss of generality, after rotation and translation of the coordinate system we may assume that the inward normal to $\partial\Omega$ at Q is the direction of the positive x_N-axis. Denote $x' = (x_1, \ldots, x_{N-1})$, $B'(\delta) = \{x' \in \mathbb{R}^{N-1} : |x' - Q'| < \delta\}$, and $\Omega_1 = \Omega \cap B(Q, 4\delta)$, where $B(Q, 4\delta) = \{x \in \mathbb{R}^N : \quad |x - Q| < 4\delta\}$.

Then, since $\partial\Omega$ is smooth, we can find a constant $\delta > 0$ such that $\partial\Omega \cap B(Q, 4\delta)$ can be represented by the graph of a smooth function $\rho_Q : B'(4\delta) \to R$, where $\rho_Q(Q) = 0$, $\nabla\rho_Q(Q) = 0$, and

$$\Omega \cap B(Q, 4\delta) = \{(x', x_N) \in B(Q, 4\delta) : x_N > \rho_Q(x')\}. \tag{2.8}$$

Moreover, we may write

$$\rho_Q(x') = \frac{1}{2}\sum_{i=1}^{N-1} k_i(x_i - Q_i)^2 + O(|x - Q|^3) \tag{2.9}$$

where $k_i, i = 1, ..., N-1$, are the principal curvatures at Q. Furthermore, the average of the principal curvatures of $\partial\Omega$ at Q is the mean curvature $H(Q) = \frac{1}{N-1}\sum\limits_{i=1}^{N-1} k_i$.

On $\partial\Omega \cap B(Q, \delta)$, the normal derivative $n(x)$ writes as

$$n(x) = \frac{1}{\sqrt{1 + |\nabla' \rho_Q|^2}}(\nabla' \rho_Q, -1) \tag{2.10}$$

and the tangential derivatives are given by

$$\frac{\partial}{\partial\tau_i} = \frac{1}{\sqrt{1 + |\frac{\partial\rho_Q}{\partial x_i}|^2}}(0, ..., 1, ..., \frac{\partial\rho_Q}{\partial x_i}) \qquad i = 1, ..., N-1. \tag{2.11}$$

As in [29], for $\bar{x} \in \mathbb{R}^N$ and $|\bar{x}|$ sufficiently small, we define a mapping $x - Q = \Phi_Q(\bar{x})$ with $\Phi_Q(\bar{x}) = (\Phi_{Q,1}(\bar{x}), ..., \Phi_{Q,N}(\bar{x}))$ by

$$\Phi_{Q,j}(\bar{x}) = \begin{cases} \bar{x}_j - \bar{x}_N \frac{\partial\rho_Q}{\partial x_j}(\bar{x}'), \text{ for } j = 1, ..., N-1, \\ \bar{x}_N + \rho_Q(\bar{x}') \quad \text{for } j = N. \end{cases} \tag{2.12}$$

Since $\nabla\rho_Q(Q) = 0$, the differential map $D\Phi_Q$ of Φ_Q satisfies $D\Phi_Q(0) = I$, the identity map. This Φ_Q has the inverse mapping $\bar{x} = \Phi_Q^{-1}(x - Q)$ for $|x - Q| < \delta'$. We write as $\Psi_Q(x - Q) = (\Psi_{Q,1}(x - Q), ..., \Psi_{Q,N}(x - Q))$ instead of $\Phi_Q^{-1}(x - Q)$.

2.4 An Auxiliary Linear Problem

Let Λ be a fixed positive constant. Recall $U_{\varepsilon,Q}$ is defined in (1.4). For the convenience of notations, we also denote $U_{\Lambda,0}$ as U_Λ. Set

$$\beta(\Lambda, Q) = \frac{A_0 H(Q)}{\Lambda}. \tag{2.13}$$

Recall that $\Lambda_0 = A_0 H(Q_0)$. Thus

$$\beta(\Lambda, Q) - 1 = O(|\Lambda - \Lambda_0|) + O(|Q - Q_0|). \tag{2.14}$$

In this subsection, we study an auxiliary linear problem

Lemma 2.2 *The following linear problem*

$$\begin{cases} \Delta u + p\alpha_N U_\Lambda^{p-1} u - \beta(\Lambda, Q) U_\Lambda + 2y_N \sum_{j=1}^{N-1} k_j(Q) \frac{\partial^2 U_\Lambda}{\partial y_j^2} - (N-1) H(Q) \frac{\partial U_\Lambda}{\partial y_N} = 0, y \in \mathbb{R}^N_+ \\ \frac{\partial u}{\partial y_N} = 0 \text{ on } \partial \mathbb{R}^N_+ \end{cases} \tag{2.15}$$

has a unique solution $\hat{U}_\Lambda$ with the following properties:
(1) $\hat{U}_\Lambda(y', y_N)$ is even in $y_1, ..., y_{N-1}$,
(2) $\hat{U}_\Lambda$ has the following decay

$$\hat{U}_\Lambda(y) = -\frac{B}{|y|^{N-4}}(1 + O(\frac{1}{|y|})) \quad \text{for } |y| > 1 \tag{2.16}$$

for some constant $B > 0$ (depending on Λ and N).

Proof: Let

$$L_0[\phi] = \Delta \phi + p\alpha_N U_\Lambda^{p-1} \phi. \tag{2.17}$$

We need two weighted Sobolev spaces

$$W^{2,t_0}_{\beta_0}(\mathbb{R}^N) = \{u : \quad |< y >^{\beta_0} u| \in W^{2,t_0}(\mathbb{R}^N)\}, \;\; L^{t_0}_{\beta_0+2}(\mathbb{R}^N) = \{u : \quad |< y >^{\beta_0+2} u| \in L^{t_0}(\mathbb{R}^N)\} \tag{2.18}$$

where we choose β_0, t_0 such that

$$t_0 > N, \beta_0 = N - 4. \tag{2.19}$$

By Proposition 2.3 of [44], the operator L_0 is an invertible operator from

$$X_0 = W^{2,t_0}_{\beta_0}(\mathbb{R}^N) \cap \{u(y) \text{ is even in } y_1, ..., y_N, \int_{\mathbb{R}^N} \frac{\partial U_\Lambda}{\partial \Lambda} u = 0\} \tag{2.20}$$

to

$$Y_0 = L^{t_0}_{\beta_0+2}(\mathbb{R}^N) \cap \{u(y) \text{ is even in } y_1, ..., y_N, \int_{\mathbb{R}^N} \frac{\partial U_\Lambda}{\partial \Lambda} u = 0\}. \tag{2.21}$$

Let

$$f(y) := -\beta(\Lambda, Q)U_\Lambda + 2y_N \sum_{j=1}^{N-1} k_j(Q) \frac{\partial^2 U_\Lambda}{\partial y_j^2} - (N-1)H(Q)\frac{\partial U_\Lambda}{\partial y_N}$$

We extend $f(y)$ evenly to $\mathbb{R}^N$. Observe that by our choice of $\beta(Q)$, (2.13),

$$\int_{\mathbb{R}^N_+} \left[-\beta(\Lambda, Q)U_\Lambda + 2y_N \sum_{j=1}^{N-1} k_j(Q) \frac{\partial^2 U_\Lambda}{\partial y_j^2} - (N-1)H(Q)\frac{\partial U_\Lambda}{\partial y_N}\right] \frac{\partial U_\Lambda}{\partial \Lambda} = 0. \tag{2.22}$$

This implies $f(y) \in Y_0$. Thus there exists a unique solution $\hat{U}_\Lambda$ satisfying:

$$L_0 \hat{U}_\Lambda + f(y) = 0, \hat{U}_\Lambda \in X_0.$$

By restricting $\hat{U}_\Lambda$ to $\mathbb{R}^N_+$, we obtain a solution to (2.15). The decay follows from the fact that the decay of $(-\Delta)^{-1}(U_\Lambda)$ is $|y|^{4-N}$.

□

We denote the unique solution in Lemma 2.2 as $\hat{U}_\Lambda$. (Note that $\hat{U}_\Lambda$ also depends on Q.) This function will be used to improve our approximate functions.

2.5 Construction of One-Bubble Approximate Solution

We first construct one bubble approximate solution.

Let $Q \in \partial\Omega$ and Λ be such that $|Q - Q_0| \le C\frac{\varepsilon}{d}$, $|\Lambda - \Lambda_0| \le C\frac{\varepsilon}{d}$. We write

$$Q = Q_0 + \frac{\varepsilon}{d}\hat{Q}, \quad \Lambda = \Lambda_0(1 + \frac{\varepsilon}{d}\hat{\Lambda}). \tag{2.23}$$

Define

$$z - \frac{1}{d}\hat{Q} = \frac{1}{\varepsilon}\Phi_Q(\varepsilon y), \; y = \frac{1}{\varepsilon}\Psi_Q(\varepsilon z - Q), \; u(z) = v(y) \tag{2.24}$$

where Φ_Q and Ψ_Q are the two functions defined at Q at Section 2.1.

Then it is easy to see that in the y-coordinate, the equation for v becomes

$$\sum_{i,j=1}^{N} a_{ij}(y)\frac{\partial^2 v}{\partial y_i \partial y_j} + \varepsilon \sum_{j=1}^{N} b_j(y)\frac{\partial v}{\partial y_j} - \varepsilon v + \alpha_N v^{\frac{N+2}{N-2}} = 0 \tag{2.25}$$

By expanding further and using (2.9), we obtain

$$S_\varepsilon[u] = \Delta v(y) + \varepsilon \left\{ 2y_N \sum_{j=1}^{N-1} k_j(Q)\frac{\partial^2 v}{\partial y_j^2} - (N-1)H(Q)\frac{\partial v}{\partial y_N}\right\} - \varepsilon v + \alpha_N v^{\frac{N+2}{N-2}} \tag{2.26}$$

$$+\varepsilon^2 \left\{ \sum_{i,j=1}^{N} c_{ij}(y)\frac{\partial^2 v}{\partial y_i \partial y_j} + \sum_{j=1}^{N} c_j(y)\frac{\partial v}{\partial y_j}\right\} = 0$$

where $|c_{ij}(y)| = O(|y|^2)$ and $|c_j(y)| = O(|y|)$ for $0 \le |y| \le \frac{4\delta}{\varepsilon}$. Moreover, the boundary condition becomes

$$\frac{\partial v}{\partial y_N} = 0 \qquad \text{for} \;\; y_N = 0, \;\; |y| \le \frac{4\delta}{\varepsilon}. \tag{2.27}$$

Let $\chi(y)$ be a cut-off function such that $\chi(y) = 1$ for $|y| < \frac{r_0}{\sqrt{\varepsilon}}$ and $\chi(y) = 0$ for $|y| > \frac{2r_0}{\sqrt{\varepsilon}}$, where r_0 is a small but fixed number.

Let $\eta(x)$ be another cut-off function such that $\eta(x) = 0$ for $|x - Q_0| > 3\delta$ and $\eta(x) = 1$ for $|x - Q_0| < 2\delta$.

Now we choose one bubble approximate solution: for $z \in \Omega_\varepsilon$,

$$U_Q(z) := \eta(\varepsilon z) \left\{ U_\Lambda\left(\frac{\Psi_Q(\varepsilon z - Q)}{\varepsilon}\right) + \varepsilon \hat{U}_\Lambda\left(\frac{\Psi_Q(\varepsilon z - Q)}{\varepsilon}\right)\chi\left(\frac{\Psi_Q(\varepsilon z - Q)}{\varepsilon}\right) \right\} + (1-\eta(\varepsilon z))\varepsilon^{N-2}. \tag{2.28}$$

Note that by our choice of U_Q, $U_Q = \varepsilon^{N-2}$ for $|z - \frac{1}{d}\hat{Q}| \ge \frac{2\delta}{\varepsilon}$ and $U_Q \sim (1 + |z - \frac{1}{d}\hat{Q}|)^{2-N}$ for all $z \in \Omega_\varepsilon$.

The following lemma gives error and energy estimates. The proof of it is delayed to Appendix.

Lemma 2.3 *Let $z \in \Omega_\varepsilon$. We have*

$$|S_\varepsilon[U_Q](z)| \le C\varepsilon^{1.5 - \frac{1}{N}} (1 + |z - \frac{1}{d}\hat{Q}|)^{-\beta_N}, \tag{2.29}$$

$$\left| \frac{\partial S_\varepsilon[U_Q]}{\partial \Lambda}(z) \right| \le C\varepsilon (1 + |z - \frac{1}{d}\hat{Q}|)^{-\beta_N}, \tag{2.30}$$

where $\beta_N = N - 3 + \frac{1}{N}$ if $N = 7, 8, 9, 10$ and $\beta_N = \frac{N}{2} + \frac{7}{3}$ if $N \ge 11$, and

$$J_\varepsilon[U_Q] = A_N - B_N \varepsilon \Lambda H(Q) + C_N \varepsilon \Lambda^2 + D_N \varepsilon^2 + O(\frac{\varepsilon^3}{d}), \tag{2.31}$$

$$\frac{\partial J_\varepsilon[U_Q]}{\partial \Lambda} = -B_N \varepsilon H(Q) + 2C_N \varepsilon \Lambda + O(\varepsilon^2) \tag{2.32}$$

where A_N, B_N, C_N, D_N are generic constants with $A_N > 0, B_N > 0, C_N > 0$.

2.6 Construction of K-Bubble Approximate Solution

We now construct multiple bubble solutions.

Let $\mathbf{Q} = (Q_1, ..., Q_K) = \frac{\varepsilon}{d}(\hat{Q}_1, ..., \hat{Q}_K)$. Set

$$\bar{Q}_j = \frac{1}{d}\hat{Q}_j, j = 1, ..., K, \quad \bar{\mathbf{Q}} = (\bar{Q}_1, ..., \bar{Q}_K).$$

Note that $\hat{Q}_i = (\hat{Q}_i', \hat{Q}_{i,N})$ where $\hat{Q}_{i,N} = \frac{d}{\epsilon}\rho(\frac{\epsilon}{d}\hat{Q}_i') = O(\frac{\epsilon}{d})$.

We also choose $\mathbf{\Lambda} = (\Lambda_1, ..., \Lambda_K)$ with

$$\Lambda_j = \Lambda_0(1 + \frac{\varepsilon}{d}\hat{\Lambda}_j), j = 1, ..., K, \quad \hat{\mathbf{\Lambda}} = (\hat{\Lambda}_1, ..., \hat{\Lambda}_K). \tag{2.33}$$

We assume that $\hat{\mathbf{\Lambda}}, \hat{\mathbf{Q}}$ satisfy the following assumption

$$(\hat{\Lambda}_1, ..., \hat{\Lambda}_K) \in \Gamma^1_{C_1}, \quad (\hat{Q}_1, ..., \hat{Q}_K) \in \Gamma^2_{C_2} \tag{2.34}$$

where

$$\Gamma^1_{C_1} = \{(\hat{\Lambda}_1, ..., \hat{\Lambda}_K) \in \mathbb{R}^K, |\hat{\mathbf{\Lambda}}| < C_1\}, \quad \Gamma^2_{C_2} = \{R[\hat{Q}'_1, ..., \hat{Q}'_K] < C_2\}, \tag{2.35}$$

and $C_1 > C_2 > R_0$ are two positive numbers to be chosen later and $R[\hat{Q}'_1, ..., \hat{Q}'_K]$ is defined at (1.2).

Let U_{Q_i} be defined in Section 2.5. To simplify our notations, from now on, we denote U_{Q_i} as U_i. We now define the K-bubble approximate solution

$$W_{\mathbf{\Lambda}, \bar{\mathbf{Q}}} := U_1 + ... + U_K. \tag{2.36}$$

Note that $W_{\mathbf{\Lambda}, \bar{\mathbf{Q}}}$ depends smoothly on $\mathbf{\Lambda}, \bar{\mathbf{Q}}$. To avoid clumsy notations, we omit the dependence of W on $\mathbf{\Lambda}, \bar{\mathbf{Q}}$. Furthermore, we have

$$C^{-1}\sum_{j=1}^{K} \frac{1}{|z - \bar{Q}_i|^{N-2}} \le W(z) \le C\sum_{j=1}^{K} \frac{1}{|z - \bar{Q}_i|^{N-2}}. \tag{2.37}$$

Let us define

$$< z - \bar{\mathbf{Q}} >= \min_{j=1}^{K}(1 + |z - \frac{1}{d}\hat{Q}_i|^2)^{\frac{1}{2}}. \tag{2.38}$$

By our choice of U_i, we have the following error and energy estimates. The proof of it is delayed to Appendix.

Lemma 2.4 *We have*

$$|S_\varepsilon[W](z)| \le C\varepsilon^{1.5 - \frac{1}{N}}(1 + |z - \bar{\mathbf{Q}}|)^{-\beta_N}, \tag{2.39}$$

$$|\frac{\partial S_\varepsilon[W]}{\partial \Lambda_j}(z)| \le C\varepsilon(1 + |z - \bar{\mathbf{Q}}|)^{-\beta_N}, j = 1, ..., K, \tag{2.40}$$

where β_N is defined as before and

$$J_\varepsilon[W] = KA_N - B_N\sum_{j=1}^{K}\varepsilon\Lambda_j H(Q_j) + C_N\varepsilon\sum_{j=1}^{K}\Lambda_j^2 + KD_N\varepsilon^2 - E_N d^{N-2}\sum_{i\neq j}^{K}\frac{\Lambda_i^{\frac{N-2}{2}}\Lambda_j^{\frac{N-2}{2}}}{|\hat{Q}'_i - \hat{Q}'_j|^{N-2}} + O(\frac{\varepsilon^3}{d}) \tag{2.41}$$

where A_N, B_N, C_N, D_N are given in Lemma (2.3) and $E_N > 0$ is another generic constant, given at (2.4), and

$$\frac{\partial J_\varepsilon[W]}{\partial \Lambda_j} = -B_N\varepsilon H(Q_j) + 2C_N\varepsilon\Lambda_j + O(\varepsilon^2) \tag{2.42}$$

3 The finite dimensional reduction

We perform a finite dimensional reduction which is similar to those of [7] and [33]-[35].

3.1 Inversion of the linearized problem

We first consider the linearized problem at the function W.

Equipping $H^1(\Omega_\varepsilon)$ with the scalar product

$$(u,v)_\varepsilon = \int_{\Omega_\varepsilon} (\nabla u \cdot \nabla v + \varepsilon uv)$$

orthogonality to the functions

$$Y_{j,0} = \frac{\partial W}{\partial \Lambda_j}, j = 1, ..., K, \qquad Y_{j,i} = \frac{\partial W}{\partial \bar{Q}_{j,i}} \qquad 1 \le i \le N-1, j = 1, ..., K \tag{3.1}$$

in that space is equivalent, setting

$$Z_{j,0} = -\Delta \frac{\partial W}{\partial \Lambda_j} + \varepsilon \frac{\partial W}{\partial \Lambda_j} \qquad Z_{j,i} = -\Delta \frac{\partial W}{\partial \bar{Q}_{j,i}} + \varepsilon \frac{\partial W}{\partial \bar{Q}_{j,i}} \qquad 1 \le i \le N-1, j = 1, ..., K \tag{3.2}$$

to the orthogonality in $L^2(\Omega_\varepsilon)$, equipped with the usual scalar product $\langle \cdot, \cdot \rangle$, to the functions $Z_{j,i}$, $1 \le j \le K, 0 \le i \le N-1$. Then, we consider the following problem : h being given, find a function ϕ which satisfies

$$\begin{cases} -\Delta\phi + \varepsilon\phi - \alpha_N q W^{q-1}\phi & = h + \sum_{j,i} c_{j,i} Z_{j,i} & \text{in} \quad \Omega_\varepsilon \\ \frac{\partial \phi}{\partial n} & = 0 & \text{on} \quad \partial\Omega_\varepsilon \\ \langle Z_{j,i}, \phi \rangle & = 0 & 0 \le i \le N-1, 1 \le j \le K \end{cases} \tag{3.3}$$

for some numbers $c_{j,i}$.

Existence and uniqueness of ϕ will follow from an inversion procedure in suitable functional spaces. To this end, we define two norms:

$$\|\phi\|_* = \max_{z \in \Omega_\varepsilon} (< z - \bar{\mathbf{Q}} >^{\beta_N - 2} |\phi(z)|), \quad \|f\|_{**} = \max_{z \in \Omega_\varepsilon} (< z - \bar{\mathbf{Q}} >^{\beta_N} |\phi(z)|). \tag{3.4}$$

We are in need of the following lemma, whose proof is given in the Appendix :

Lemma 3.1 *Let u and f satisfy*

$$-\Delta u + \varepsilon u = f \quad in\ \Omega_\varepsilon, \ \frac{\partial u}{\partial n} = 0 \ on\ \partial\Omega_\varepsilon.$$

Then we have

$$|u(x)| \le C \int_{\Omega_\varepsilon} \frac{|f(y)|}{|x-y|^{N-2}} dy \tag{3.5}$$

and as a consequence,

$$\|u\|_* \le C \|f\|_{**}. \tag{3.6}$$

The main result of this subsection is :

Proposition 3.1 *There exists $\varepsilon_0 > 0$ and a constant $C > 0$, independent of $\varepsilon, \mathbf{\Lambda}$ and $\bar{\mathbf{Q}}$ satisfying (2.34), such that for all $0 < \varepsilon < \varepsilon_0$ and all $h \in L^\infty(\Omega_\varepsilon)$, problem* (3.3) *has a unique solution $\phi \equiv L_\varepsilon(h)$. Besides,*

$$\|L_\varepsilon(h)\|_* \leq C\|h\|_{**} \qquad\qquad |c_{j,i}| \leq C\|h\|_{**}. \tag{3.7}$$

Moreover, the map $L_\varepsilon(h)$ is C^1 with respect to $\mathbf{\Lambda}, \hat{\mathbf{Q}}$ and the $**$-norm, and

$$\|D_{\mathbf{\Lambda}}\, L_\varepsilon(h)\|_* \leq C\|h\|_{**}. \tag{3.8}$$

PROOF. The argument follows closely the ideas in [7], [33] and [34]. We repeat it since we use different norms. The proof relies on the following result:

Lemma 3.2 *Assume that ϕ_ε solves (3.3) for $h = h_\varepsilon$. If $\|h_\varepsilon\|_{**}$ goes to zero as ε goes to zero, so does $\|\phi_\varepsilon\|_*$.*

PROOF Arguing by contradiction, we may assume that $\|\phi_\varepsilon\|_* = 1$. Multiplying the first equation in (3.3) by $Y_{k,l}$ and integrating in Ω_ε we find

$$\sum_{j,i} c_{j,i}\langle Z_{j,i}, Y_{k,l}\rangle = \Big\langle -\Delta Y_{k,l} + \varepsilon Y_{k,l} - q\alpha_N W^{q-1} Y_{k,l}, \phi_\varepsilon \Big\rangle - \langle h_\varepsilon, Y_{k,l}\rangle.$$

On one hand we check, in view of the definition of $Z_{j,i}$, $Y_{k,l}$

$$\langle Z_{j,0}, Y_{j,0}\rangle = \|Y_{j,0}\|_\varepsilon^2 = \gamma_0 + o(1) \qquad \langle Z_{j,i}, Y_{j,i}\rangle = \|Y_{j,i}\|_\varepsilon^2 = \gamma_1 + o(1) \quad 1 \leq i \leq N-1,\ 1 \leq j \leq K \tag{3.9}$$

where γ_0, γ_1 are strictly positive constants, and

$$\langle Z_{j,i}, Y_{k,l}\rangle = o(1) \qquad j \neq k, i \neq l. \tag{3.10}$$

On the other hand, in view of the definition of $Y_{k,l}$ and W, straightforward computations yield

$$\Big\langle -\Delta Y_{k,l} + \varepsilon Y_{k,l} - q\alpha_N W^{q-1} Y_{k,l}, \phi_\varepsilon \Big\rangle = o(\|\phi_\varepsilon\|_*)$$

(since $N \geq 7$) and

$$\langle h_\varepsilon, Y_{k,l}\rangle = O(\|h_\varepsilon\|_{**}).$$

Consequently, inverting the quasi diagonal linear system solved by the $c_{j,i}$'s, we find

$$c_{j,i} = O(\|h_\varepsilon\|_{**}) + o(\|\phi_\varepsilon\|_*). \tag{3.11}$$

In particular, $c_{j,i} = o(1)$ as ε goes to zero.

Since $\|\phi_\varepsilon\|_*=1$, elliptic theory shows that along some subsequence, $\phi_{\varepsilon,j}(y)=\phi_\varepsilon(y-\bar{Q}_j)$ converges uniformly in any compact subset of $\mathbb{R}^N_+$ to a nontrivial solution of

$$-\Delta\phi_j=\alpha_N\frac{N+2}{N-2}U_{\Lambda_0,0}^{\frac{4}{N-2}}\phi_j$$

since $\Lambda_j\to\Lambda_0$. Moreover, $|\phi_j(y)|\in C(1+|y|)^{-\beta_N+2}$. A bootstrap argument (see e.g. Proposition 2.2 of [40]) implies $|\phi_j(y)|\le C(1+|y|)^{-\beta_N-2}$. As a consequence, ϕ_j writes as

$$\phi_j=\alpha_0\frac{\partial U_{\Lambda_0,0}}{\partial\Lambda_0}+\sum_{i=1}^{N-1}\alpha_i\frac{\partial U_{\Lambda_0,0}}{\partial y_i}$$

(see [30]). On the other hand, equalities $\langle Z_{j,i},\phi_\varepsilon\rangle=0$ provide us with the equalities

$$\begin{aligned}&\int_{\mathbb{R}^N_+}-\Delta\frac{\partial U_{\Lambda_0,0}}{\partial\Lambda_0}\phi_j=\int_{\mathbb{R}^N_+}U_{\Lambda_0,0}^{\frac{4}{N-2}}\frac{\partial U_{\Lambda_0,0}}{\partial\Lambda_0}\phi_j=0\\&\int_{\mathbb{R}^N_+}-\Delta\frac{\partial U_{\Lambda_0,0}}{\partial y_i}\phi_j=\int_{\mathbb{R}^N_+}U_{\Lambda_0,0}^{\frac{4}{N-2}}\frac{\partial U_{\Lambda_0,0}}{\partial y_i}\phi_j=0\qquad 1\le i\le N-1.\end{aligned}$$

As we have also

$$\int_{\mathbb{R}^N_+}|\nabla\frac{\partial U_{\Lambda_0,0}}{\partial\Lambda_0}|^2=\gamma_0>0\qquad\qquad\int_{\mathbb{R}^N_+}|\nabla\frac{\partial U_{\Lambda_0,0}}{\partial y_i}|^2=\gamma_1>0\quad 1\le i\le N-1$$

and

$$\int_{\mathbb{R}^N_+}\nabla\frac{\partial U_{\Lambda_0,0}}{\partial\Lambda_0}.\nabla\frac{\partial U_{\Lambda_0,0}}{\partial y_i}=\int_{\mathbb{R}^N_+}\nabla\frac{\partial U_{\Lambda_0,0}}{\partial y_{i'}}\cdot\nabla\frac{\partial U_{\Lambda_0,0}}{\partial y_i}=0\quad i\neq i'$$

the α_i's solve a homogeneous quasi diagonal linear system, yielding $\alpha_i=0$, $0\le i\le N-1$, and $\phi_j=0$. So $\phi_\varepsilon(z-\bar{Q}_j)\to 0$ in $C^1_{loc}(\Omega_\varepsilon)$. Now since

$$|\langle z-\bar{\mathbf{Q}}\rangle^{\beta_N}W^{\frac{4}{N-2}}\phi_\varepsilon|\le C\|\phi_\varepsilon\|_*\langle z-\hat{\mathbf{Q}}\rangle^{-2}.$$

So we obtain

$$\|W^{\frac{4}{N-2}}\phi_\varepsilon\|_{**}=o(1).$$

On the other hand,

$$\langle z-\bar{\mathbf{Q}}\rangle^{\beta_N}|Z_{j,i}|\le C\langle z-\bar{\mathbf{Q}}\rangle^{-1}$$

applying Lemma 3.2 we obtain

$$\|\phi_\varepsilon\|_*\le C\|W^{\frac{4}{N-2}}\phi_\varepsilon\|_{**}+C\|h_\varepsilon\|_{**}+C\sum_{j,i}|c_{j,i}|\|Z_{j,i}\|_{**}=o(1)$$

that is, a contradiction.

Proof of Proposition 3.1 completed. We set

$$H = \left\{ \phi \in H^1(\Omega_\varepsilon),\ \langle Z_{j,i}, \phi \rangle = 0 \quad 0 \le i \le N-1, 1 \le j \le K \right\}$$

equipped with the scalar product $(\cdot,\cdot)_\varepsilon$. Problem (3.3) is equivalent to finding $\phi \in H$ such that

$$(\phi, \theta)_\varepsilon = \left\langle q\alpha_N W^{q-1}\phi + h\,,\, \theta \right\rangle \qquad \forall \theta \in H$$

that is

$$\phi = T_\varepsilon(\phi) + \tilde{h} \tag{3.12}$$

$\tilde{h}$ depending linearly on h, and T_ε being a compact operator in H. Fredholm's alternative ensures the existence of a unique solution, provided that the kernel of $Id - T_\varepsilon$ is reduced to 0. We notice that any $\phi_\varepsilon \in Ker(Id - T_\varepsilon)$ solves (3.3) with $h = 0$. Thus, we deduce from Lemma 3.2 that $\|\phi_\varepsilon\|_* = o(1)$ as ε goes to zero. As $Ker(Id - T_\varepsilon)$ is a vector space, $Ker(Id - T_\varepsilon) = \{0\}$. The inequalities (3.7) follow from Lemma 3.2 and (3.11). This completes the proof of the first part of Proposition 3.1.

The smoothness of L_ε with respect to $\mathbf{\Lambda}$ and $\bar{\mathbf{Q}}$ is a consequence of the smoothness of T_ε and $\tilde{h}$, which occur in the implicit definition (3.12) of $\phi \equiv L_\varepsilon(h)$, with respect to these variables. Inequalities (3.8) are obtained differentiating (3.3), writing the derivatives of ϕ with respect to $\mathbf{\Lambda}$ as a linear combination of the $Z_{j,i}$' and an orthogonal part, and estimating each term using the first part of the proposition - see [7] for detailed computations. □

3.2 The reduction

Let $S_\varepsilon[u]$ be defined at (1.9). Then (1.6) is equivalent to

$$S_\varepsilon[u] = 0 \text{ in } \partial\Omega_\varepsilon,\ u_+ \not\equiv 0,\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega_\varepsilon \tag{3.13}$$

for if u satisfies (3.13), the Maximum Principle ensures that $u > 0$ in Ω_ε and (1.6) is satisfied. Observe that

$$S_\varepsilon[W + \phi] = -\Delta(W+\phi) + \varepsilon(W+\phi) - \alpha_N (W+\phi)_+^q$$

may be written as

$$S_\varepsilon[W+\phi] = -\Delta\phi + \varepsilon\phi - q\alpha_N W^{q-1}\phi - R^\varepsilon - \alpha_N N_\varepsilon(\phi) \tag{3.14}$$

with

$$N_\varepsilon[\phi] = (W+\phi)_+^q - W^q - qW^{q-1}\phi \tag{3.15}$$

$$R^\varepsilon = S_\varepsilon[W] = \Delta W - \varepsilon W + \alpha_N W^q \tag{3.16}$$

From Lemma 2.4, we derive the following estimates:

$$\|R^\varepsilon\|_{**} \le C\varepsilon^{1.5 - \frac{1}{N}}, \quad \|D_{\mathbf{\Lambda}} R^\varepsilon\|_{**} \le C\varepsilon. \tag{3.17}$$

(Note that we do not need the estimate for $D_{\bar{\mathbf{Q}}} R^\varepsilon$.)

We consider now the following nonlinear problem : finding ϕ such that, for some numbers $c_{j,i}$

$$\begin{cases} -\Delta(W+\phi)+\varepsilon(W+\phi)-\alpha_N(W+\phi)_+^q & = \sum_{j,i} c_{j,i} Z_{j,i} \quad \text{in} \quad \Omega_\varepsilon \\ \frac{\partial\phi}{\partial n} & = 0 \quad \text{on} \quad \partial\Omega_\varepsilon \\ \langle Z_{j,i},\phi\rangle & = 0. \quad 1\le j\le K,\ 0\le i\le N-1 \end{cases} \tag{3.18}$$

The first equation in (3.18) writes as

$$-\Delta\phi+\varepsilon\phi-q\alpha_N W^{q-1}\phi = \alpha_N N_\varepsilon(\phi)+R^\varepsilon+\sum_{j,i} c_{j,i} Z_{j,i} \tag{3.19}$$

for some numbers $c_{j,i}$. We now obtain some estimates on N_ε.

Lemma 3.3 *There exist $\varepsilon_1>0$, independent of $\mathbf{\Lambda},\bar{\mathbf{Q}}$, and C, independent of ε, $\mathbf{\Lambda},\bar{\mathbf{Q}}$, such that for $|\varepsilon|\le\varepsilon_1$, and $\|\phi\|_*\le 1$*

$$\|N_\varepsilon(\phi)\|_{**}\le C\varepsilon^{\gamma_N}\|\phi\|_*^q \tag{3.20}$$

and, for $\|\phi_i\|_*\le 1$

$$\|N_\varepsilon(\phi_1)-N_\varepsilon(\phi_2)\|_{**}\le C\varepsilon^{\gamma_N}\big(\max(\|\phi_1\|_*,\|\phi_2\|_*)\big)^{q-1}\|\phi_1-\phi_2\|_*, \tag{3.21}$$

where $\gamma_N=-\frac{2}{7}$ if $N=7$ and $\gamma_N=0$ if $N\ge 8$.

Proof. The proof is similar to Lemma 3.1 and Proposition 3.5 of [40]. For the convenience of the reader, we include a proof here. We deduce from (3.15) that

$$|N_\varepsilon(\phi)|\le C|\phi|^q \quad \text{since } q<2. \tag{3.22}$$

By definition, we have

$$\begin{aligned} \||\phi|^q\|_{**} &= \max_{z\in\Omega_\varepsilon}\langle z-\bar{\mathbf{Q}}\rangle^{\beta_N}|\phi|^q \\ &\le C\|\phi\|_*^q\max_{z\in\Omega_\varepsilon}\langle z-\bar{\mathbf{Q}}\rangle^{\max\{\frac{-2N+16-\frac{4}{N}}{N-2},-\frac{16}{3(N-2)}\}} \\ &\le C\varepsilon^{\gamma_N}\|\phi\|_*^q \end{aligned}$$

Concerning (3.21), we write

$$N_\varepsilon(\phi_1)-N_\varepsilon(\phi_2)=\partial_\eta N_\varepsilon(\eta)(\phi_1-\phi_2)$$

for some $\eta=t\phi_1+(1-t)\phi_2$, $t\in[0,1]$. From

$$\partial_\eta N_\varepsilon(\eta)=q\big((W+\eta)_+^{q-1}-W^{q-1}\big)$$

we deduce

$$|\partial_\eta N_\varepsilon(\eta)| \le C|\eta|^{q-1} \quad \text{since } N \ge 7. \tag{3.23}$$

The proof of (3.21) is similar. $\square$

We state now the following result :

Proposition 3.2 *There exists C, independent of ε and $\bar{\mathbf{Q}}$, $\mathbf{\Lambda}$ satisfying (2.34), such that for small ε problem (3.18) has a unique solution $\phi = \phi(\mathbf{\Lambda}, \bar{\mathbf{Q}}, \varepsilon)$ with*

$$\|\phi\|_* \le C\varepsilon^{1.5-\frac{1}{N}}, \tag{3.24}$$

Moreover, $(\mathbf{\Lambda}, \bar{\mathbf{Q}}) \mapsto \phi(\mathbf{\Lambda}, \bar{\mathbf{Q}}, \varepsilon)$ is C^1 with respect to the $*$-norm and $**$-norm, and

$$\|D_{\mathbf{\Lambda}}\phi\|_* \le C\varepsilon. \tag{3.25}$$

Proof. Following [7], we consider the map A_ε from $\mathcal{F} = \{\phi \in H^1(\Omega_\varepsilon) \, : \, \|\phi\|_* \le C_0\varepsilon^{1.5-\frac{1}{N}}\}$ to $H^1(\Omega_\varepsilon)$ defined as

$$A_\varepsilon(\phi) = L_\varepsilon(\alpha_N N_\varepsilon(\phi) + R^\varepsilon).$$

Here C_1 is a large number, to be determined later, and L_ε is give by Proposition 3.1. We remark that finding a solution ϕ to problem (3.18) is equivalent to finding a fixed point of A_ε. On one hand we have, for $\phi \in \mathcal{F}$

$$\|A_\varepsilon(\phi)\|_* \le \|L_\varepsilon(N_\varepsilon(\phi))\|_* + \|L_\varepsilon(R^\varepsilon)\|_* \le \|N_\varepsilon(\phi)\|_{**} + C\varepsilon^{1.5-\frac{1}{N}} \le 2C\varepsilon^{1.5-\frac{1}{N}}.$$

for ε small enough, implying that A_ε sends $\mathcal{F}$ into itself, if we choose $C_0 = 2C$. On the other hand A_ε is a contraction. Indeed, for ϕ_1 and ϕ_2 in $\mathcal{F}$, we write

$$\begin{aligned}\|A_\varepsilon(\phi_1) - A_\varepsilon(\phi_2)\|_* &\le C\|N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2)\|_{**} \\ &\le C\varepsilon^{(1.5-\frac{1}{N})(q-1)+\gamma_N}\|\phi_1 - \phi_2\|_* \\ &\le \frac{1}{2}\|\phi_1 - \phi_2\|_*\end{aligned}$$

by Lemma 3.3. Contraction Mapping Theorem implies that A_ε has a unique fixed point in $\mathcal{F}$, that is problem (3.18) has a unique solution ϕ such that $\|\phi\|_* \le C_0\varepsilon^{1.5-\frac{1}{N}}$.

In order to prove that $(\mathbf{\Lambda}, \bar{\mathbf{Q}}) \mapsto \phi(\mathbf{\Lambda}, \bar{\mathbf{Q}})$ is C^1, we remark that setting for $\eta \in \mathcal{F}$

$$B(\mathbf{\Lambda}, \bar{\mathbf{Q}}, \eta) \equiv \eta - L_\varepsilon(\alpha_N N_\varepsilon(\eta) + R^\varepsilon)$$

ϕ is defined as

$$B(\mathbf{\Lambda}, \bar{\mathbf{Q}}, \phi) = 0. \tag{3.26}$$

We have

$$\partial_\eta B(\mathbf{\Lambda}, \bar{\mathbf{Q}}, \eta)[\theta] = \theta - \alpha_N L_\varepsilon\big(\theta\,(\partial_\eta N_\varepsilon)(\eta)\big).$$

Using Proposition 3.1, (3.23) and (2.34) we obtain for $N \geq 7$

$$
\begin{aligned}
\|L_\varepsilon\big(\theta\,(\partial_\eta N_\varepsilon)(\eta)\big)\|_* &\leq C\|\theta\,(\partial_\eta N_\varepsilon)(\eta)\|_{**} \\
&\leq C\|\langle z-\bar{\mathbf{Q}}\rangle^{-\beta_N+2}(\partial_\eta N_\varepsilon)(\eta)\|_{**}\|\theta\|_* \\
&\leq C\|\langle z-\bar{\mathbf{Q}}\rangle^{-\beta_N+2}|\eta|^{q-1}\|_{**}\|\theta\|_* \\
&\leq C\varepsilon^{\gamma_N}\|\eta\|_*^{q-1}\|\theta\|_* \\
&\leq C\varepsilon^{(1.5-\frac{1}{N})(q-1)+\gamma_N}\|\theta\|_*.
\end{aligned}
$$

Therefore we can write, for any $N \geq 7$

$$
\|L_\varepsilon\big(\theta\,(\partial_\eta N_\varepsilon)(\eta)\big)\|_* \leq C\varepsilon^{(1.5-\frac{1}{N})(q-1)+\gamma_N}\|\theta\|_*.
$$

Consequently, $\partial_\eta B(\mathbf{\Lambda}, \bar{\mathbf{Q}}, \phi)$ is invertible in with uniformly bounded inverse. Then, the fact that $(\mathbf{\Lambda}, \bar{\mathbf{Q}}) \mapsto \phi(\mathbf{\Lambda}, \bar{\mathbf{Q}})$ is C^1 follows from the fact that $(\mathbf{\Lambda}, \bar{\mathbf{Q}}, \eta) \mapsto L_\varepsilon(N_\varepsilon(\eta))$ is C^1 and the implicit functions theorem.

Finally, let us show how estimates (3.25) may be obtained. Derivating (3.26) with respect to Λ_i, we have

$$
\partial_{\Lambda_i}\phi = (\partial_\eta B(\mathbf{\Lambda}, \bar{\mathbf{Q}}, \phi))^{-1}\Big(\alpha_N(\partial_{\Lambda_i}L_\varepsilon)(N_\varepsilon(\phi)) + \alpha_N L_\varepsilon((\partial_{\Lambda_i}N_\varepsilon)(\phi)) + \partial_{\Lambda_i}(L_\varepsilon(R^\varepsilon))\Big)
$$

whence, according to Proposition 3.1

$$
\begin{aligned}
\|\partial_{\Lambda_i}\phi\|_* &\leq C\Big(\|(\partial_{\Lambda_i}L_\varepsilon)(N_\varepsilon(\phi))\|_* + \|(L_\varepsilon(\partial_{\Lambda_i}N_\varepsilon)(\phi))\|_* + \|(\partial_{\Lambda_i}(L_\varepsilon(R^\varepsilon))\|_*\Big) \\
&\leq C\Big(\|N_\varepsilon(\phi)\|_{**} + \|(\partial_{\Lambda_i}N_\varepsilon)(\phi)\|_{**} + \|(\partial_{\Lambda_i}(L_\varepsilon(R^\varepsilon))\|_*\Big).
\end{aligned}
$$

From (3.20) and (3.24) we know that

$$
\|N_\varepsilon(\phi)\|_{**} \leq C\varepsilon^{(1.5-\frac{1}{N})q+\gamma_N}.
$$

Concerning the next term, we notice that according to the definition (3.15) of N_ε

$$
\begin{aligned}
&|(\partial_{\Lambda_i}N_\varepsilon)(\phi)| \\
&\quad = q\Big|(W+\phi)_+^{q-1} - W^{q-1} - (q-1)W^{\frac{6-N}{N-2}}\phi\Big||\partial_{\Lambda_i}W| \\
&\quad \leq CW^{q-1}|\phi| \\
&\quad \leq C\langle z-\bar{\mathbf{Q}}\rangle^{-2-\beta_N}\|\phi\|_* \\
&\quad \leq C\varepsilon^{1.5-\frac{1}{N}}\langle z-\bar{\mathbf{Q}}\rangle^{-2-\beta_N}
\end{aligned}
$$

where we used successively the fact that $W > 0$ (see (2.37)) and $|\partial_{\Lambda_i}W| \leq CW$.

We obtain

$$\|(\partial_{\Lambda_i}N_\varepsilon)(\phi)\|_{**} \le C\varepsilon^{1.5-\frac{1}{N}}.$$

From Proposition 3.1 we deduce the estimate for the last term

$$\|\partial_{\Lambda_i}(L_\varepsilon(R^\varepsilon))\|_* \le C\|\partial_{\Lambda_i}R^\varepsilon\|_{**} \le C\varepsilon$$

and finally

$$\|\partial_{\Lambda_i}\phi\|_* \le C\varepsilon.$$

This concludes the proof of Proposition 3.2. (The first derivatives of ϕ with respect to $\bar{\mathbf{Q}}$ may be estimated in the same way, but this is not needed here.) $\square$

3.3 Coming back to the original problem

We define a reduced energy functional in a finite dimension:

$$I_\varepsilon(\mathbf{\Lambda}, \bar{\mathbf{Q}}) \equiv J_\varepsilon\left[W_{\mathbf{\Lambda},\bar{\mathbf{Q}}} + \phi_{\varepsilon,\mathbf{\Lambda},\bar{\mathbf{Q}}}\right] \tag{3.27}$$

We then have :

Proposition 3.3 *The function $u = W + \phi$ is a solution to problem $(P'_{\frac{N+2}{N-2},\mu})$ if and only if $(\mathbf{\Lambda}, \bar{\mathbf{Q}})$ is a critical point of I_ε.*

Proof. We notice that $u = W + \phi$ being a solution to $(P'_{\frac{N+2}{N-2},\mu})$ is equivalent to being a critical point of J_ε. It is also equivalent to the cancellation of the $c_{j,i}$'s in (3.18) or, in view of (3.9) (3.10)

$$J'_\varepsilon[W+\phi][Y_{j,i}] = 0 \qquad 1 \le j \le K, \quad 0 \le i \le N-1. \tag{3.28}$$

On the other hand, we deduce from (3.27) that $I'_\varepsilon(\mathbf{\Lambda}, \bar{\mathbf{Q}}) = 0$ is equivalent to the cancellation of $J'_\varepsilon(W+\phi)$ applied to the derivatives of $W+\phi$ with respect to $\mathbf{\Lambda}$ and $\bar{\mathbf{Q}}$. According to the definition (3.1) of the $Y_{j,i}$'s, (3.17) and Proposition 3.2 we have

$$\frac{\partial(W+\phi)}{\partial\Lambda_j} = Y_{j,0} + y_{j,0}, \qquad \frac{\partial(W+\phi)}{\partial\bar{Q}_{j,i}} = Y_{j,i} + y_{j,i}, \qquad 1 \le i \le N-1, 1 \le j \le K$$

with $\|y_{j,i}\|_* = o(1)$, $1 \le j \le K, 0 \le i \le N-1$. Writing

$$y_{j,i} = y'_{j,i} + \sum_{k,l} a_{ji,kl}Y_{k,l}, \qquad \langle y'_{j,i}, Z_{k,l}\rangle = (y'_{j,i}, Y_{j,i})_\varepsilon = 0, \qquad 0 \le i \le N-1, 1 \le j \le K$$

and

$$J'_\varepsilon[W+\phi][Y_{j,i}] = \alpha_{j,i}$$

it turns out that $I'_\varepsilon(\mathbf{\Lambda}, \bar{\mathbf{Q}}) = 0$ is equivalent, since $J'_\varepsilon[W+\phi][\theta] = 0$ for $\langle\theta, Z_{j,i}\rangle = (\theta, Y_{j,i})_\varepsilon = 0$, $1 \le j \le K$, $0 \le i \le N-1$, to

$$(Id + [a_{ji,kl}])[\alpha_{ji}] = 0.$$

As $a_{ji,kl} = O(\|y_{k,l}\|_*) = o(1)$, we see that $I'_\varepsilon(\mathbf{\Lambda}, \bar{\mathbf{Q}}) = 0$ means exactly that (3.28) is satisfied. $\square$

4 Proofs of Theorems 1.1

In view of Proposition 3.3 we have, for proving the theorem, to find critical points of I_ε. We establish first a C^1-expansion of I_ε.

4.1 Expansion of I_ε

Proposition 4.1 *There exist A_N, B_N, C_N, E_N, strictly positive constants such that*

$$I_\varepsilon(\mathbf{\Lambda},\bar{\mathbf{Q}}) = KA_N + KD_N\varepsilon^2 + C_N\Lambda_0^2\varepsilon^{3-\frac{6}{N}}(\hat{\Lambda}_1^2+...+\hat{\Lambda}_K^2) - \varepsilon^{3-\frac{6}{N}}R[\hat{Q}_1',...,\hat{Q}_K'] + \varepsilon^{3-\frac{6}{N}}\hat{\sigma}_\varepsilon(\hat{\mathbf{\Lambda}},\bar{\mathbf{Q}})$$

where $R[\hat{Q}_1',...,\hat{Q}_K']$ is defined at (1.2) with $c_1 = \frac{B_N}{2}\Lambda_0, c_2 = E_N\Lambda_0^{N-2}$, and $\hat{\sigma}_\varepsilon$ and $\partial_{\hat{\Lambda}_i}\sigma_\varepsilon$ go to zero as ε goes to zero, uniformly with respect to $\hat{\mathbf{\Lambda}},\bar{\mathbf{Q}}$ satisfying (2.34).

Proof. By Lemma (2.4), we proved

$$J_\varepsilon[W] = KA_N - B_N\sum_{i=1}^K \varepsilon\Lambda_i H(Q_i) + C_N\sum_{i=1}^K\varepsilon\Lambda_i^2 + KD_N\varepsilon^2 - d^{N-2}\sum_{i\neq j}\frac{E_N\Lambda_i^{\frac{N-2}{2}}\Lambda_j^{\frac{N-2}{2}}}{|\hat{Q}_i'-\hat{Q}_j'|^{N-2}} + o(\varepsilon^{3-\frac{6}{N}}). \tag{4.1}$$

By expanding

$$\Lambda_i = \Lambda_0(1+\frac{\varepsilon}{d}\hat{\Lambda}_i), H(Q_i) = H(Q_0) + \frac{1}{2}(\frac{\varepsilon}{d})^2\sum_{j=1}^K\varphi(\hat{Q}_j') + o((\frac{\varepsilon}{d})^2),$$

and using the fact that

$$\Lambda_0 = A_0H(Q_0) = \frac{B_N}{2C_N}H(Q_0), \tag{4.2}$$

we obtain

$$J_\varepsilon[W] = KA_N + KD_N\varepsilon^2 + C_N\Lambda_0^2\varepsilon(\frac{\varepsilon}{d})^2(\hat{\Lambda}_1^2+...+\hat{\Lambda}_K^2) - \varepsilon(\frac{\varepsilon}{d})^2R[\hat{Q}_1',...,\hat{Q}_K'] + \varepsilon(\frac{\varepsilon}{d})^2\sigma_\varepsilon(\Lambda,a) \tag{4.3}$$

where

$$R[\hat{Q}_1',...,\hat{Q}_K'] = \frac{B_N}{2}\Lambda_0\varphi(\hat{Q}_j') + \sum_{i\neq j}\frac{E_N\Lambda_0^{N-2}}{|\hat{Q}_i'-\hat{Q}_j'|^{N-2}}. \tag{4.4}$$

Using $d=\varepsilon^{\frac{3}{N}}$ we get the desired result.

Next we show that

$$I_\varepsilon(\mathbf{\Lambda},\bar{\mathbf{Q}}) - J_\varepsilon[W] = o(\varepsilon^{3-\frac{6}{N}}). \tag{4.5}$$

Actually, in view of (3.27), a Taylor expansion and the fact that $J'_\varepsilon[W+\phi][\phi]=0$ yield

$$\begin{aligned}
I(\mathbf{\Lambda},\bar{\mathbf{Q}})-J_\varepsilon[W] &= J_\varepsilon[W+\phi]-J_\varepsilon[W]\\
&= \int_0^1 J''_\varepsilon(W+t\phi)[\phi,\phi]tdt\\
&= \int_0^1\Big(\int_{\Omega_\varepsilon}\big(|\nabla\phi|^2+\varepsilon\phi^2-q\alpha_N(W+t\phi)_+^{q-1}\phi^2+R^\varepsilon\phi\big)\Big)tdt\\
&= \int_0^1\Big(\alpha_N\int_{\Omega_\varepsilon}\big(N_\varepsilon(\phi)\phi+q\big[W^{q-1}-(W+t\phi)_+^{q-1}\big]\phi^2\big)\Big)tdt\\
&\quad+\frac{1}{2}\int_{\Omega_\varepsilon}R^\varepsilon\phi.
\end{aligned}$$

The first term can be estimated as follows. Using (3.22) and Proposition 3.2, we have, for $N\geq 7$

$$\Big|\int_{\Omega_\varepsilon}N_\varepsilon(\phi)\phi\Big|\leq C\|\phi\|_*^{\frac{2N}{N-2}}\int_{\Omega_\varepsilon}\langle z-\bar{\mathbf{Q}}\rangle^{-(\beta_N-2)(\frac{2N}{N-2})}\leq C\varepsilon^{3-\frac{4}{N}}. \tag{4.6}$$

For the second term, the same arguments as previously yield

$$\begin{aligned}
\int_{\Omega_\varepsilon}\Big|W^{q-1}-(W+t\phi)_+^{q-1}\Big|\phi^2 &\leq C\int_{\Omega_\varepsilon}\big(W^{q-1}|\phi|^2+|\phi|^{2+q-1}\big)\\
&\leq C\Big(\|\phi\|_*^2\int_{\Omega_\varepsilon}\langle z-\bar{\mathbf{Q}}\rangle^{-2(\beta_N-2)-4}\\
&\quad+\|\phi\|_*^{2+q-1}\int_{\Omega_\varepsilon}\langle z-\bar{\mathbf{Q}}\rangle^{-(\beta_N-2)(2+q-1)}\Big)
\end{aligned}$$

whence using Proposition 3.2 again

$$\int_{\Omega_\varepsilon}\big|W^{q-1}-(W+t\phi)_+^{q-1}\big|\phi^2\leq C\varepsilon^{3-\frac{4}{N}}. \tag{4.7}$$

Concerning the last term, we remark that according to (3.17)

$$|R^\varepsilon|\leq C\varepsilon^{1.5-\frac{1}{N}}\langle z-\bar{\mathbf{Q}}\rangle^{-\beta_N}$$

uniformly in Ω_ε. Therefore

$$\int_{\Omega_\varepsilon}|R^\varepsilon\phi|\leq C\varepsilon^{1.5-\frac{1}{N}}\|\phi\|_*\int_{\Omega_\varepsilon}\langle z-\bar{\mathbf{Q}}\rangle^{-2\beta_N+2}$$

yielding, through Proposition 3.2

$$\int_{\Omega_\varepsilon}|R^\varepsilon\phi|\leq C\varepsilon^{3-\frac{4}{N}} \tag{4.8}$$

for $N \geq 8$. For $N = 7$, we have to use the refined estimate (5.4)

$$|R^\varepsilon| \leq C\varepsilon\langle z - \bar{\mathbf{Q}}\rangle^{2-N}$$

then we can get

$$\int_{\Omega_\varepsilon} |R^\varepsilon \phi| \leq C\varepsilon^{2+\frac{5}{14}} = o(\varepsilon^{3-\frac{5}{7}}).$$

The desired result follows from (4.6), (4.7) and (4.8).

To estimate the derivative with respect to Λ_j, we note that

$$\begin{aligned}
\frac{\partial}{\partial\hat{\Lambda}_j} J_\varepsilon[W+\phi] &= \int_{\Omega_\varepsilon} S_\varepsilon[W+\phi]\frac{\partial(W+\phi)}{\partial\hat{\Lambda}_j} \\
&= \frac{\varepsilon}{d}\int_{\Omega_\varepsilon} S_\varepsilon[W+\phi]\frac{\partial(W+\phi)}{\partial\Lambda_j} \\
&= \frac{\varepsilon}{d}\int_{\Omega_\varepsilon} S_\varepsilon[W]\frac{\partial W}{\partial\Lambda_j} + \frac{\varepsilon}{d}\int_{\Omega_\varepsilon}(\Delta\phi - \varepsilon\phi + q\alpha_N W^{q-1}\phi)\frac{\partial W}{\partial\Lambda_j} + o(d^{N-2}) \\
&= \frac{\varepsilon}{d}\frac{\partial J_\varepsilon[W]}{\partial\Lambda_j} + \frac{\varepsilon}{d}\int_{\Omega_\varepsilon}(\Delta\frac{\partial W}{\partial\Lambda_j} - \varepsilon\frac{\partial W}{\partial\Lambda_j} + q\alpha_N W^{q-1}\frac{\partial W}{\partial\Lambda_j})\phi + o(d^{N-2}) \\
&= \frac{\varepsilon}{d}\frac{\partial}{\partial\Lambda_j} J_\varepsilon[W] + o(d^{N-2})
\end{aligned}$$

The rest follows from Lemma 2.4 and Proposition 3.2. □

4.2 Proofs of Theorem 1.1 completed

We prove Theorem 1.1 in this section. We have to use a min-max argument.

Set

$$\hat{I}_\varepsilon(\hat{\mathbf{\Lambda}}, \hat{\mathbf{Q}}) = \frac{I_\varepsilon(\mathbf{\Lambda}, \bar{\mathbf{Q}}) - A_N - D_N\varepsilon^2}{\varepsilon^{3-\frac{6}{N}}} \tag{4.9}$$

By Proposition 4.1, we have the following asymptotic expansion for $\hat{I}_\epsilon(\hat{\mathbf{\Lambda}}, \hat{\mathbf{Q}})$:

$$\hat{I}_\varepsilon(\hat{\mathbf{\Lambda}}, \hat{\mathbf{Q}}) = C_N\Lambda_0^2\sum_{j=1}^{K}\hat{\Lambda}_j^2 - R[\hat{Q}_1', ..., \hat{Q}_K'] + \tilde{\sigma}_\varepsilon(\hat{\mathbf{\Lambda}}, \hat{\mathbf{Q}}) \tag{4.10}$$

with

$$\tilde{\sigma}_\varepsilon(\hat{\mathbf{\Lambda}}, \hat{\mathbf{Q}}) = o(1) \quad \partial_{\hat{\Lambda}_j}\tilde{\sigma}_\varepsilon(\hat{\mathbf{\Lambda}}, \hat{\mathbf{Q}}) = \frac{\varepsilon}{d}\partial_{\Lambda_j}\sigma_\varepsilon(\mathbf{\Lambda}, \hat{\mathbf{Q}}) = o(1) \text{ as } \varepsilon \to 0. \tag{4.11}$$

We set

$$\Sigma_0 = \{(\hat{\mathbf{\Lambda}}, \hat{\mathbf{Q}}) | \hat{\mathbf{\Lambda}} \in \Gamma^1_{2C_1}, \quad \hat{\mathbf{Q}} \in \Gamma^2_{2C_2}\} \tag{4.12}$$

where $C_1 > C_2 > R_0$ are two numbers such that

$$C_N\Lambda_0^2 C_1^2 > 2C_2 = 4R_0. \tag{4.13}$$

We define also

$$B = \{(\hat{\mathbf{\Lambda}}, \hat{\mathbf{Q}}) | \hat{\mathbf{\Lambda}} \in \Gamma_{C_1}, \hat{\mathbf{Q}} \in \Gamma^2_{C_2}\}, \qquad B_0 = (\partial \Gamma_{C_1}) \times \Gamma^2_{C_2}$$

where C_1, C_2 are chosen as in (4.13).

It is trivial to see that $B_0 \subset B \subset \Sigma_0$, B_0, B are compact and B is connected. Let Γ be the class of continuous functions $\varphi : B \to \Sigma_0$ with the property that $\varphi(y) = y$ for all $y \in B_0$. Define the max-min value c as

$$c = \max_{\varphi \in \Gamma} \min_{y \in B} \hat{I}_\varepsilon(\varphi(y)). \tag{4.14}$$

We now show that c defines a critical value. To this end, we just have to verify the following two conditions

(T1) $\min_{y \in B_0} \hat{I}_\varepsilon(\varphi(y)) > c, \forall \varphi \in \Gamma$,

(T2) For all $y \in \partial \Sigma_0$ such that $\hat{I}_\epsilon(y) = c$, there exists a vector τ_y tangent to $\partial \Sigma_0$ at y such that

$$\partial_{\tau_y} \hat{I}_\epsilon(y) \neq 0.$$

Suppose (T1) and (T2) hold. Then standard deformation argument ensures that the max-min value c is a (topologically nontrivial) critical value for $\hat{I}_\varepsilon(\hat{\mathbf{\Lambda}}, \hat{\mathbf{Q}})$ in Σ_0. (Similar notion has been introduced in [8] for degenerate critical points of mean curvature.)

To check (T1) and (T2), we define $y = (y_1, y_2) \in B, y_1 \in \Gamma^1_{C_1}, y_2 \in \Gamma^2_{C_2}, \varphi(y) = (\varphi_1(y), \varphi_2(y))$ where $\varphi_1(y) \in \Gamma^1_{C_1}$ and $\varphi_2(y) \in \Gamma^2_{C_2}$.

By taking φ to be identity map, we obtain

$$c \geq \max_{\hat{\mathbf{Q}} \in \Gamma^2_{C_2}} \min_{\hat{\mathbf{\Lambda}} \in \Gamma^1_{C_1}} \hat{I}_\varepsilon(\hat{\mathbf{\Lambda}}, \hat{\mathbf{Q}}) = -R_0 + o(1). \tag{4.15}$$

On the other hand, for any $\varphi \in \Gamma$ and $y_2 \in \Gamma^2_{C_2}$, the map $y_1 \to \varphi_1(y_1, y_2)$ is a continuous function from $\Gamma^1_{C_1}$ to $\Gamma^1_{2C_1}$ such that $\varphi_1(y_1, y_2) = y_1$ for $y_1 \in \partial \Gamma^1_{C_1}$. By Brouwer's fixed point theorem, there exists $y_1^{'} \in \Gamma^1_{C_1}$ such that $\varphi_1(y_1^{'}, y_2) = 0$, whence

$$\min_{y \in B} \hat{I}_\varepsilon(\varphi(y)) \leq \hat{I}_\varepsilon(0, \varphi_2(y_1^{'}, y_2)) = 0 - R[\varphi_2(y_1^{'}, y_2)] + o(1) \leq -R_0 + o(1). \tag{4.16}$$

As a consequence

$$c \leq -R_0 + o(1), \quad c = -R_0 + o(1). \tag{4.17}$$

For $y \in B_0$, we have $|\varphi_1(y)| = C_1$. So we have $\hat{I}_\varepsilon(y) = C_N \Lambda_0^2 C_1^2 - R[\varphi_2(y)] + o(1) > C_N \Lambda_0^2 C_1^2 - C_2 + o(1) > 0$, by our choice of C_1 (4.13). So (T1) is verified.

To verify (T2), we observe that $\partial(\Sigma_0) = (\partial \Gamma_{2C_1}) \times \Gamma^2_{2C_2} \cup \Gamma_{2C_1} \times \partial \Gamma^2_{2C_2}$.

Let $y = (y_1, y_2) \in \partial \Sigma_0$ be such that $\hat{I}_\varepsilon(y) = c$.

On $\partial\Gamma^1_{2C_1} \times \Gamma^2_{2C_2}$, previous arguments show that $\hat{I}_\varepsilon(y) > 0 > c$. On $\Gamma^1_{2C_1} \times (\partial\Gamma^2_{2C_2})$, we claim that there exists j such that $\tau_y = \frac{\partial}{\partial \Lambda_j}$ to obtain that

$$\partial_{\tau_y} \hat{I}_\epsilon(y) = 2\Lambda_j + o(1) \neq 0$$

since otherwise $\partial_{\tau_y} \hat{I}_\epsilon(y) = 0$ yields $\Lambda_j = o(1)$ for all j, and

$$\hat{I}_\varepsilon(y) = o(1) - R[\varphi_2(y)] \leq -C_2 + o(1) = -2R_0 + o(1) < c,$$

again by the choice of C_2 (4.13). A contradiction to the assumption. So (T2) is also verified.

In conclusion, we proved that for ε small enough, c is a critical value, i.e. a critical point $(\hat{\mathbf{\Lambda}}^\varepsilon, \hat{\mathbf{Q}}^\varepsilon) \in \Sigma_0$ of $\hat{I}_\varepsilon$ exists. Let $u_\varepsilon = W_{\mathbf{\Lambda}^\varepsilon, \bar{\mathbf{Q}}^\varepsilon, \mu, \varepsilon} + \phi_{\mathbf{\Lambda}^\varepsilon, \bar{\mathbf{Q}}^\varepsilon, \mu, \varepsilon}$. u_ε is a nontrivial solution to the problem (1.6). Then, the strong maximum principle shows that $u_\varepsilon > 0$ in Ω_ε. Let $u_\mu = \varepsilon^{-\frac{N-2}{2}} u_\varepsilon(\frac{x-Q_0}{\varepsilon})$. The same argument in [27] shows that u_ε has exactly K local maximum points $q_j^\varepsilon \in \partial\Omega, j = 1, ..., K$. Moreover, after a permutation, $q_j^\varepsilon = Q_0 + \frac{\varepsilon}{d}\hat{Q}_j^\varepsilon + o(\frac{\varepsilon}{d}), j = 1, ..., K$. By our construction, u_μ satisfies all the properties of Theorem 1.1.

This concludes the proof of Theorem 1.1.

5 Appendix

5.1 Proof of Lemma 2.3

We first prove (2.29). We divide the domain into three regions: $|z - \frac{1}{d}\hat{Q}| > \frac{\delta}{\varepsilon}$, $\frac{r_0}{\sqrt{\varepsilon}} \leq |z - \frac{1}{d}\hat{Q}| \leq \frac{\delta}{\varepsilon}$, $|z - \frac{1}{d}\hat{Q}| < \frac{r_0}{\sqrt{\varepsilon}}$.

In the first region, $|z - \frac{1}{d}\hat{Q}| > \frac{\delta}{\varepsilon}$, we have

$$|U_Q| \leq C\varepsilon^{N-2}, \quad |\Delta U_Q| \leq C\varepsilon^N, \quad |U_Q|^p \leq C\varepsilon^{2+N}$$

which gives

$$|S_\varepsilon[U_Q](z)| \leq C\varepsilon^{1.5-\frac{1}{N}}(1 + |z - \frac{1}{d}\hat{Q}|)^{3-N-\frac{1}{N}}. \tag{5.1}$$

In the second region, $\frac{r_0}{\sqrt{\varepsilon}} \leq |z - \frac{1}{d}\hat{Q}| \leq \frac{\delta}{\varepsilon}$, we then have

$$|U_Q| \leq \frac{C}{(1 + |z - \frac{1}{d}\hat{Q}|)^{N-2}}, \quad |\nabla U_Q| \leq \frac{C}{(1 + |z - \frac{1}{d}\hat{Q}|)^{N-1}},$$

and hence

$$|S_\varepsilon[U_Q](z)| \leq C\varepsilon|U_\Lambda(z)| \leq C\varepsilon^{1.5-\frac{1}{N}}(1 + |z - \frac{1}{d}\hat{Q}|)^{3-N-\frac{1}{N}}. \tag{5.2}$$

In the last region, $|z-\frac{1}{d}\hat{Q}|<\frac{r_0}{\sqrt{\varepsilon}}$, we introduce the y-coordinate as in (2.24)

$$z-\frac{1}{d}\hat{Q}=\frac{1}{\varepsilon}\Phi_Q(\varepsilon y)$$

and then have (see (2.26))

$$S_\varepsilon[U_Q]=\Delta U_Q-\varepsilon U_Q+\varepsilon\left\{2y_N\sum_{j=1}^{N-1}k_j(Q)\frac{\partial^2 U_\Lambda}{\partial y_j^2}-(N-1)H(Q)\frac{\partial U_\Lambda}{\partial y_N}\right\}+\alpha_N U_Q^q+O(\varepsilon^2(1+|y|)^{2-N})$$

Using the equations for U_Λ and $\hat{U}_\Lambda$, we obtain

$$S_\varepsilon[U_Q]=-\varepsilon^2\hat{U}_\Lambda+\varepsilon(\beta(\Lambda,Q)-1)U_\Lambda+O(\varepsilon^2(1+|y|)^{2-N} \tag{5.3}$$

and hence

$$\begin{aligned}|S_\varepsilon[U_Q]|&\le\varepsilon^2|\hat{U}_\Lambda|+\varepsilon|(\beta(\Lambda,Q)-1)|U_\Lambda+O(\varepsilon^2(1+|y|)^{2-N})\\&\le C\varepsilon^{1.5-\frac{1}{N}}(1+|y|)^{3-N-\frac{1}{N}}\le C\varepsilon^{1.5-\frac{1}{N}}(1+|z-\frac{1}{d}\hat{Q}|)^{3-N-\frac{1}{N}}\end{aligned} \tag{5.4}$$

since by (2.14)

$$\varepsilon|\beta(\Lambda,Q)-1|\le C\varepsilon\frac{\varepsilon}{d}=C\varepsilon^{1+\frac{N-3}{N}} \tag{5.5}$$

because of $N\ge 7$. Combining (5.1), (5.2) and (5.4), (2.29) is proved.

To show (2.30), we note that for $|z-\frac{1}{d}\hat{Q}|<\frac{r_0}{\sqrt{\varepsilon}}$,

$$\frac{\partial S_\varepsilon[U_Q]}{\partial\Lambda}=-\varepsilon^2\frac{\partial\hat{U}_\Lambda}{\partial\Lambda}+\varepsilon\frac{\partial\beta(\Lambda,Q)}{\partial\Lambda}U_\Lambda+O(\frac{\varepsilon^2}{d}(1+|y|)^{2-N}) \tag{5.6}$$

Since

$$|\frac{\partial\hat{U}_\Lambda}{\partial\Lambda}|\le C|\hat{U}_\Lambda|,\quad\frac{\partial\beta(\Lambda,Q)}{\partial\Lambda}=-A_0\frac{H(Q_0)}{\Lambda_0^2}+o(1), \tag{5.7}$$

we obtain (2.30) from (5.6).

Next we compute the energy. Observe that

$$J_\varepsilon[U_Q]=\frac{1}{2}\int_{\Omega_\varepsilon}|\nabla U_Q|^2+\frac{\varepsilon}{2}\int_{\Omega_\varepsilon}|U_Q|^2-\frac{\alpha_N}{q+1}\int_{\Omega_\varepsilon}U^{q+1}. \tag{5.8}$$

To prove (2.31), we note that for $|z-\frac{1}{d}\hat{Q}|\ge\frac{r_0}{\sqrt{\varepsilon}}$, we have

$$\begin{aligned}\varepsilon\int_{|z-\frac{1}{d}\hat{Q}|\ge\frac{r_0}{\sqrt{\varepsilon}}}|U_Q|^2&\le C\varepsilon\int_{|z-\frac{1}{d}\hat{Q}|\ge\frac{r_0}{\sqrt{\varepsilon}}}\frac{C}{|z-\frac{1}{d}\hat{Q}|^{2(N-2)}}\\&\le C\varepsilon\int_{\frac{r_0}{\sqrt{\varepsilon}}}^{+\infty}r^{4-N}dr\le\varepsilon^{\frac{N-1}{2}}=O(\varepsilon^3).\end{aligned} \tag{5.9}$$

Similarly we have

$$\int_{|z-\frac{1}{d}\hat{Q}|\ge\frac{r_0}{\sqrt{\varepsilon}}}(|\nabla U_Q|^2+U_Q^{q+1})\le C\varepsilon^3. \tag{5.10}$$

So we just need to compute $J_\varepsilon[U_Q]$ in the region $I := |z - \frac{1}{d}\hat{Q}| < \frac{r_0}{\sqrt{\varepsilon}}$. Let

$$J_{\varepsilon,I}[v] = \frac{1}{2}\int_I |\nabla v|^2 + \frac{\varepsilon}{2}\int_I v^2 - \frac{\alpha_N}{q+1}\int_I v^{q+1}. \tag{5.11}$$

Previous arguments show that

$$\begin{aligned} J_\varepsilon[U_Q] = J_{\varepsilon,I}[U_Q] + o(d^{N-2}) = J_{\varepsilon,I}[U_\Lambda] + \varepsilon \int_I S_\varepsilon[U_\Lambda]\hat{U}_\Lambda \\ +\varepsilon^2 \left[\frac{1}{2}\int_I |\nabla \hat{U}_\Lambda|^2 + \frac{\varepsilon}{2}\int_I \hat{U}_\Lambda^2 - \frac{q\alpha_N}{2}\int_I U_\Lambda^{q-1}\hat{U}_\Lambda^2\right] \\ -\frac{\alpha_N}{q+1}\int_I \left[(U_\Lambda + \varepsilon\hat{U}_\Lambda)^{q+1} - U_\Lambda^{q+1} - (q+1)U_\Lambda^q \hat{U}_\Lambda - \frac{q(q+1)}{2}U_\Lambda^{q-1}\hat{U}_\Lambda^2\right] + O(d^{N-2}) \end{aligned} \tag{5.12}$$

Let us compute each term in (5.12).

To compute $J_{\varepsilon,I}[U_\Lambda]$, we may use the y-coordinate introduced in (2.24) to expand

$$\det(D\Phi_Q(y)) = 1 - (N-1)\varepsilon H(Q) y_N + \varepsilon^2 \sum_{k,l} R_{kl}(Q) y_k y_l + O(\varepsilon^3 |y|^3) \tag{5.13}$$

where $R_{kl}(Q)$ is a smooth function of Q. (We do not need to know the exact form of $R_{kl}(Q)$.) To prove this, we refer to Lemma A.1 of Appendix of [28].

Since $Q = Q_0 + O(\frac{\varepsilon}{d})$,

$$\det(D\Phi_Q(y)) = 1 - (N-1)\varepsilon H(Q) y_N + \varepsilon^2 \sum_{k,l} R_{k,l}(Q_0) + O(\frac{\varepsilon^3}{d}|y|^3). \tag{5.14}$$

So we have

$$\begin{aligned} &\int_{|y|\le \frac{\delta}{\sqrt{\varepsilon}}} |\nabla_z U_\Lambda(\frac{\Psi_Q(\varepsilon z - Q)}{\varepsilon})|^2 \det D\Phi_Q(y) dy \\ &= A_{N,1} - B_{N,1}\varepsilon \Lambda H(Q) + D_{N,1}\varepsilon^2 + O(\frac{\varepsilon^3}{d}) \end{aligned} \tag{5.15}$$

since we have

$$\int_{\mathbb{R}^N} |\nabla U_\Lambda|^2 |y|^3 < +\infty.$$

Similarly, we have

$$\begin{aligned} &\int_{|y|\le \frac{\delta}{\sqrt{\varepsilon}}} |U_\Lambda(\frac{\Psi_Q(\varepsilon z - Q)}{\varepsilon})|^{q+1} \det D\Phi_Q(y) dy \\ &= A_{N,2} - B_{N,2}\varepsilon \Lambda H(Q) + D_{N,2}\varepsilon^2 + O(\frac{\varepsilon^3}{d}) \end{aligned} \tag{5.16}$$

and

$$\varepsilon \int_{|y|\le \frac{\delta}{\sqrt{\varepsilon}}} |U_\Lambda(\frac{\Psi_Q(\varepsilon z - Q)}{\varepsilon})|^2 \det D\Phi_Q(y) dy$$

$$= 2\varepsilon C_N \Lambda^2 + D_{N,3}\varepsilon^2 + O(\frac{\varepsilon^3}{d}) \tag{5.17}$$

where C_N is given at (2.3).

Combining (5.15), (5.16) and (5.17), we obtain

$$J_{\varepsilon,I}[U_Q] = A_N - B_N \varepsilon \Lambda H(Q_i) + C_N \varepsilon \Lambda^2 + D_N \varepsilon^2 + o(d^{N-2}) \tag{5.18}$$

where $B_N = \frac{1}{2}B_{N,1} - \frac{\alpha_N}{q+1}B_{N,2}$ and C_N are given by (2.2) and (2.3) respectively.

To compute the second term in (5.12), we note that

$$\begin{array}{rcl} \varepsilon \int_I S_\varepsilon[U_\Lambda]\hat{U}_\Lambda & = & \varepsilon^2 \int_{|y|\le \frac{r_0}{\sqrt{\varepsilon}}} [U_\Lambda - 2y_N \sum_{j=1}^{N-1} k_j(Q) \frac{\partial^2 U_\Lambda}{\partial y_j^2} + (N-1)H(Q)\frac{\partial U_\Lambda}{\partial y_N}]\hat{U}_\Lambda + o(d^{N-2}) \\ & = & D_{N,4}\varepsilon^2 + o(d^{N-2}) \end{array} \tag{5.19}$$

Similarly, we have

$$\varepsilon^2 \left[\int_I |\nabla \hat{U}_\Lambda|^2 + \varepsilon \int_I \hat{U}_\Lambda^2 - q \int_I U_\Lambda^{q-1}\hat{U}_\Lambda^2 \right] = D_{N,5}\varepsilon^2 + o(d^{N-2}) \tag{5.20}$$

The last term in (5.12) can be estimated as follows:

$$\int_I \left| (U_\Lambda + \varepsilon \hat{U}_\Lambda)^{q+1} - U_\Lambda^{q+1} - (q+1)U_\Lambda^q \hat{U}_\Lambda - \frac{q(q+1)}{2} U_\Lambda^{q-1}\hat{U}_\Lambda^2 \right|$$

$$\le C\varepsilon^3 \int_{|y|\le \frac{r_0}{\sqrt{\varepsilon}}} U_\Lambda^{q-2}\hat{U}_\Lambda^3 \le C\varepsilon^3 \tag{5.21}$$

since $N \ge 7$.

Combining (5.18), (5.19), (5.20) and (5.21), we obtain (2.31).

To prove (2.32), we note that using (5.3)

$$\frac{\partial}{\partial \Lambda} J_\varepsilon[U_Q] = \int_{\Omega_\varepsilon} S_\varepsilon[U_Q] \frac{\partial U_Q}{\partial \Lambda}$$

$$= \int_{|z - \frac{1}{d}\hat{Q}| < \frac{r_0}{\sqrt{\varepsilon}}} \left[-\varepsilon^2 \hat{U}_Q + \varepsilon(\beta(\Lambda, Q) - 1)U_\Lambda + O(\varepsilon^2(1+|y|)^{2-N}) \right] \frac{\partial U_Q}{\partial \Lambda} + O(\varepsilon^2)$$

Now using the fact that $|\frac{\partial U_Q}{\partial \Lambda_j}| \le C U_Q$, we can perform similar estimates to obtain (2.32). $\square$

5.2 Proof of Lemma 2.4

The proof is similar as in Lemma 2.3, except now that we have to compute the interaction terms.

For $S_\varepsilon[W]$, we have

$$S_\varepsilon[W] = \sum_{j=1}^{K} S_\varepsilon[U_j] + I[W]$$

where $I[W] = (\sum_{j=1}^{K} U_j)^q - \sum_{j=1}^{K} U_j^q$ denotes the interaction term.

For $I[W]$, we have for $|z - \frac{1}{d}\hat{Q}_i| \leq \frac{C}{d}$

$$I[W] = O(\sum_{j\neq i} |U_i|^{q-1}|U_j|) = O(d^{N-2} < z - \frac{1}{d}\hat{\mathbf{Q}} >^{-4}) = O(\varepsilon^{1.5-\frac{1}{N}} < z - \frac{1}{d}\hat{\mathbf{Q}} >^{-\frac{7}{3}-\frac{N}{2}}) \tag{5.22}$$

For $|z - \frac{1}{d}\hat{Q}_i| \geq \frac{C}{d}$ where C is large, we have

$$I[W] = O(d^{-(N-2)q}) = O(\varepsilon^{\frac{3(N+2)}{N}}) = O(\varepsilon^{1.5-\frac{1}{N}} < z - \frac{1}{d}Q_i >^{-\frac{7}{3}-\frac{N}{2}}) \tag{5.23}$$

Combining with Lemma 2.3, we obtain (2.39) and (2.30).

For the energy expansion, we have

$$J_\varepsilon[W] = \sum_{j=1}^{K} J_\varepsilon[U_j] + \sum_{i<j}[\int_{\Omega_\varepsilon} \nabla U_i \nabla U_j + \varepsilon \int_{\Omega_\varepsilon} U_i U_j] - \frac{\alpha_N}{q+1}\int_{\Omega_\varepsilon}[\sum_{j=1}^{K} U_j]^{q+1} - \sum_{j=1}^{K} U_j^{q+1}]$$

For the interaction terms in the energy expansion, we have

$$\int_{\Omega_\varepsilon} \nabla U_i \nabla U_j + \varepsilon \int_{\Omega_\varepsilon} U_i U_j - \alpha_N \int_{\Omega_\varepsilon} U_i^q U_j = -\int_{\Omega_\varepsilon} S_\varepsilon[U_i]U_j + o(d^{N-2}). \tag{5.24}$$

By Lemma 2.2, we have

$$\int_{\Omega_\varepsilon} S_\varepsilon[U_i]U_j = O(\varepsilon^2 \int_{|z-\frac{1}{d}\hat{Q}_i|\leq \frac{r_0}{\sqrt{\varepsilon}}} \hat{U}_i U_j) = O(\varepsilon^2 d^{N-6}) = o(d^{N-2}) \tag{5.25}$$

since $N \geq 7$.

Similar arguments give

$$\int_{\Omega_\varepsilon} [\sum_{j=1}^{K} U_j]^{q+1} - \sum_{j=1}^{K} U_j^{q+1} - \sum_{i\neq j}(q+1)U_i^q U_j] = o(d^{N-2}) \tag{5.26}$$

Thus,

$$
\begin{aligned}
J_\varepsilon[\sum_{j=1}^K U_j] &= \sum_{j=1}^K J_\varepsilon[U_j] - \alpha_N \sum_{i>j} \int_{\Omega_\varepsilon} U_i^q U_j + o(d^{N-2}) \\
&= \sum_{j=1}^K J_\varepsilon[U_j] - \frac{\alpha_N}{2} \sum_{i \neq j} \int_{|z-\frac{1}{d}Q_i| \leq \frac{\delta}{\varepsilon}} U_i^q U_j + o(d^{N-2}) \\
&= \sum_{j=1}^K J_\varepsilon[U_j] - \frac{\alpha_N}{2} \sum_{i \neq j} \int_{|z-\frac{1}{d}\hat{Q}_i| \leq \frac{r_0}{\sqrt{\varepsilon}}} U_i^q U_j + O(\varepsilon^{\frac{N-1}{2}}) + o(d^{N-2}).
\end{aligned}
$$

It is easy to see

$$
\begin{aligned}
\frac{\alpha_N}{2} \int_{|z-\frac{1}{d}\hat{Q}_i| \leq \frac{r_0}{\sqrt{\varepsilon}}} U_i^q U_j &= \frac{\alpha_N}{2} \int_{\mathbb{R}^N_+} U_{\Lambda_i}^q \frac{\Lambda_j^{\frac{N-2}{2}}}{|\bar{Q}_i - \bar{Q}_j|^{N-2}} + o(d^{N-2}) \\
&= E_N d^{N-2} \frac{\Lambda_i^{\frac{N-2}{2}} \Lambda_j^{\frac{N-2}{2}}}{|\hat{Q}_i' - \hat{Q}_j'|^{N-2}} + o(d^{N-2})
\end{aligned}
$$

where E_N is given by (2.4).

Thus

$$
J_\varepsilon[W] = \sum_{j=1}^K J_\varepsilon[U_j] - E_N d^{N-2} \sum_{i \neq j} \frac{\Lambda_i^{\frac{N-2}{2}} \Lambda_j^{\frac{N-2}{2}}}{|\hat{Q}_i' - \hat{Q}_j'|^{N-2}} + o(d^{N-2}).
$$

Combining with (2.31), we obtain (2.41).

The proof (2.42) is similar by noting that $|\frac{\partial W}{\partial \Lambda_j}| \leq CW$.

$\square$

5.3 Proof of Lemma 3.1

We prove (3.5) first. Through scaling, we may assume that $\varepsilon = 1$. Let $G_\mu(x, y)$ be the Green's function satisfying

$$
-\Delta G_\mu(x,y) + \mu G_\mu(x,y) = \delta_y \text{ in } \Omega, \quad \frac{\partial G_\mu(x,y)}{\partial n} = 0 \text{ on } \partial\Omega.
$$

Note that for $\mu \geq 1$

$$
G_\mu(x,y) \leq G_1(x,y). \tag{5.27}
$$

So we may assume also that $\mu = 1$.

Then we have for $x \in \Omega$,

$$
u(x) = \int_\Omega G(x,y) f(y) dy.
$$

So it is enough to show that there exists a constant C, independent of x and y, such that

$$|G(x,y)| \le \frac{C}{|x-y|^{N-2}}.$$

To this end, we decompose G into two parts:

$$G(x,y) = K(|x-y|) + H(x,y)$$

where $K(|x-y|)$ is the singular part of G and $H(x,y)$ is the regular part of G. Certainly we have $|K(|x-y|)| \le \frac{C}{|x-y|^{N-2}}$. It remains to show that

$$|H(x,y)| \le \frac{C}{|x-y|^{N-2}}. \tag{5.28}$$

Note that, if $d(x,\partial\Omega) > d_0 > 0$ or $d(y,\partial\Omega) > d_0 > 0$, then $|H(x,y)| \le C$ and hence (5.28) also holds. So we just need to estimate $H(x,y)$ for $d(x,\partial\Omega)$ and $d(y,\partial\Omega)$ small. Let $y \in \Omega$ be such that $d = d(y,\partial\Omega)$ is small. So there exists a unique point $\bar{y} \in \partial\Omega$ such that $d = |y-\bar{y}|$. Without loss of generality, we may assume $\bar{y} = 0$ and the inner normal at $\bar{y}$ is pointing toward x_N-direction. Let y^* be the reflection point $y^* = (0,...,0,-d)$ and consider the following auxiliary function

$$H^*(x,y) = K(|x-y^*|)$$

Then H^* satisfies $\Delta H^* - \mu H^* = 0$ in Ω and on $\partial\Omega$

$$\frac{\partial}{\partial n}(H^*(x,y)) = -\frac{\partial}{\partial n}(K(|x-y|)) + O(\frac{1}{d^{N-3}}).$$

Hence we derive that

$$H(x,y) = -H^*(x,y) + O(\frac{1}{d^{N-3}})$$

which proves (5.28) for $x, y \in \Omega$. This implies that for $x \in \Omega$

$$|u(x)| \le C\int_\Omega \frac{|f(y)|}{|x-y|^{N-2}}dy. \tag{5.29}$$

If $x \in \partial\Omega$, we consider a sequence of points $x_i \in \Omega, x_i \to x \in \partial\Omega$ and take the limit in (5.29). Lebesgue's Dominated Convergence Theorem applies and (3.5) is proved.

Since

$$\langle z - \bar{\mathbf{Q}}\rangle^{\beta_N - 2}\int_{R^N} \frac{1}{|x-y|^{N-2}}\langle z-\bar{\mathbf{Q}}\rangle^{-\beta_N}dx \le C < \infty$$

(3.6) is proved.

$\square$

References

[1] Adimurthi, G. Mancini, *The Neumann problem for elliptic equations with critical nonlinearity, "A tribute in honour of G. Prodi"*, Scuola Norm. Sup. Pisa (1991) 9-25.

[2] Adimurthi, G. Mancini, *Geometry and topology of the boundary in the critical Neumann problem*, J. Reine Angew. Math. **456** (1994) 1-18.

[3] Adimurthi, F. Pacella, S.L. Yadava, *Interaction between the geometry of the boundary and positive solutions of a semilinear Neumann problem with critical nonlinearity*, J. Funct. Anal. **113** (1993) 318-350.

[4] P. Bates, E.N. Dancer, J. Shi, *Multi-spike stationary solutions of the Cahn-Hilliard equation in higher-dimension and instability*, Adv. Differential Equations **4** (1999), 1-69.

[5] P. Bates, G. Fusco, *Equilibria with many nuclei for the Cahn-Hilliard equation*, J. Diff. Equ. **160** (2000) 283-356.

[6] L. Caffarelli, B. Gidas, J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. **42** (1989) 271-297.

[7] M. del Pino, P. Felmer, M. Musso, *Two-bubble solutions in the super-critical Bahri-Coron's problem*, Calc. Var. and Part. Diff. Eq. **16** (2003) 113-145.

[8] M. del Pino, P. Felmer, J. Wei, *On the role of mean curvature in some singularly perturbed Neumann problems*, SIAM J. Math. ANal. **31**(2000), 63-79.

[9] M. del Pino, M. Musso, A. Pistoia, *Supercritical boundary bubbling in a semilinear Neumann problem*, Ann. Inst. H. Poincare Anal. Non Linearie 22(2005), no. 1, 45-82.

[10] G. Cerami, J. Wei, *Multiplicity of multiple interior peaks solutions for some singularly perturbed Neumann problems*, Intern. Math. Res. Notes **12** (1998) 601-626.

[11] E.N. Dancer, S. Yan, *Multipeak solutions for a singular perturbed Neumann problem*, Pacific J. Math. **189** (1999), 241-262.

[12] E.N. Dancer, S. Yan, *Interior and boundary peak solutions for a mixed boundary value problem*, Indiana Univ. Math. J. **48** (1999), 1177-1212.

[13] N. Ghoussoub, C. Gui, M. Zhu, *On a singularly perturbed Neumann problem with the critical exponent.* Comm. Partial Differential Equations **26** (2001) 1929-1946.

[14] M. Grossi, A. Pistoia, *On the effect of critical points of distance function in superlinear elliptic problems*, Adv. Diff. Equ. **5** (2000) 1397-1420.

[15] M. Grossi, A. Pistoia, J. Wei, *Existence of multipeak solutions for a semilinear elliptic problem via nonsmooth critical point theory*, Calc. Var. PDE **11** (2000) 143-175.

[16] C. Gui, *Multi-peak solutions for a semilinear Neumann problem*, Duke Math. J. **84** (1996) 739-769.

[17] A. Gierer, H. Meinhardt, *A theory of biological pattern formation,* Kybernetik (Berlin) **12** (1972) 30-39.

[18] C. Gui, C.S. Lin, *Estimates for boundary-bubbling solutions to an elliptic Neumann problem*, J. Reine Angew. Math. **546** (2002) 201–235.

[19] C. Gui, J. Wei, *Multiple interior peak solutions for some singularly perturbed Neumann problems*, J. Diff. Equ. **158** (1999) 1-27.

[20] C. Gui, J. Wei, *On multiple mixed interior and boundary peak solutions for some singularly perturbed Neumann problems*, Canad. J. Math. **52** (2000) 522-538.

[21] C. Gui, J. Wei, M. Winter, *Multiple boundary peak solutions for some singularly perturbed Neumann problems*, Ann. Inst. H. Poincare, Anal. Non-lineaire **17** (2000) 47-82.

[22] Y.Y. Li, *On a singularly perturbed equation with Neumann boundary condition*, Comm. Part. Diff. Equ. **23** (1998) 487-545.

[23] C.S.Lin, *Locating the peaks of solutions via the maximum principle, I. Neumann problem*, Comm. Pure Appl. Math., to appear.

[24] C.S. Lin, W.N. Ni, I. Takagi, *Large amplitude stationary solutions to a chemotaxis system*, J. Diff Equ. **72** (1988) 1-27.

[25] S. Maier-Paape, K. Schmitt, Z.Q. Wang, *On Neumann problems for semilinear elliptic equations with critical nonlinearity : existence and symmetry of multi-peaked solutions*, Comm. Part. Diff. Equ. **22** (1997), 1493-1527.

[26] W.-M. Ni, *Qualitative properties of solutions to elliptic problems.* Stationary partial differential equations. Vol. I, 157-233, Handb. Differ. Equ., North-Holland, Amsterdam, 2004.

[27] W.N. Ni, X.B. Pan, I. Takagi, *Singular behavior of least-energy solutions of a semi-linear Neumann problem involving critical Sobolev exponents*, Duke Math. J. **67** (1992) 1-20.

[28] W.N. Ni, I. Takagi, *On the shape of least-energy solutions to a semi-linear problem Neumann problem*, Comm. Pure Appl. Math. **44** (1991) 819-851.

[29] W.M. Ni, I. Takagi, *Locating the peaks of least-energy solutions to a semi-linear Neumann problem*, Duke Math. J. **70** (1993) 247-281.

[30] O. Rey, *The role of the Green's function in a nonlinear elliptic problem involving the critical Sobolev exponent*, J. Funct. Anal. **89** (1990) 1-52.

[31] O. Rey, *An elliptic Neumann problem with critical nonlinearity in three dimensional domains*, Comm. Contemp. Math. **1** (1999) 405-449.

[32] O. Rey, *The question of interior blow-up points for an elliptic Neumann problem : the critical case*, J. Math. Pures Appl. **81** (2002) 655-696.

[33] O. Rey, J. Wei, *Blow-uo solutions for an elliptic neumann problem with sub-or-supcritical nonlinearity, I: $N = 3$*, J. Funct. Anal. **212** (2004), 472-499.

[34] O. Rey, J. Wei, *Blow-up solutions for an elliptic neumann problem with sub-or-supcritical nonlinearity, II: $N \geq 4$*, Ann. Inst. H. Poincare, Anal. Non-lineaire, to appear.

[35] O. Rey, J. Wei, *Arbitrary number of positive solutions for an elliptic problem with critical nonlinearity*, J. Euro. Math. Soc., **7** (2005), 449-476.

[36] X.J. Wang, *Neumann problem of semilinear elliptic equations involving critical Sobolev exponents*, J. Diff. Equ. **93** (1991) 283-310.

[37] Z.Q. Wang, *The effect of domain geometry on the number of positive solutions of Neumann problems with critical exponents*, Diff. and Integ. Equ. **8** (1995) 1533-1554.

[38] Z.Q. Wang, *High energy and multi-peaked solutions for a nonlinear Neumann problem with critical exponent*, Proc. Roy. Soc. Edimburgh **125 A** (1995) 1003-1029.

[39] Z.Q. Wang, *Construction of multi-peaked solution for a nonlinear Neumann problem with critical exponent*, J. Nonlinear Anal. TMA **27** (1996), 1281-1306.

[40] X. Wang, J. Wei, *On the equation $\Delta u + K(x)u^{\frac{n+2}{n-2}\pm\epsilon^2} = 0$ in R^n*, Rend. Circolo Matematico di Palermo **II** (1995), 365-400.

[41] J. Wei, *On the interior spike layer solutions of singularly perturbed semilinear Neumann problems*, Tohoku Math. J. **50** (1998) 159-178.

[42] J. Wei, X. Xu, *Uniqueness and a priori estimates for some nonlinear elliptic Neumann equations in $\mathbb{R}^3$*. Pacific J. Math. **221** (2005), 159-165.

[43] L. Wang, J. Wei, *Arbitrary number of positive solutions for an elliptic problem with critical nonlinearity: $N = 4$ and $N = 6$*, preprint.

[44] J. Wei, M. Winter, *Stationary solutions for the Cahn-Hilliard equation*, Ann. Inst. H. Poincaré, Anal. Non linearie. **15** (1998) 459-482.

[45] M. Zhu, *Uniqueness results through a priori estimates, I. A three dimensional Neumann problem*, J. Diff. Equ. **154** (1999) 284-317.