NONRADIAL CLUSTERED SPIKE SOLUTIONS FOR SEMILINEAR ELLIPTIC PROBLEMS ON $\mathbf{S}^n$

CATHERINE BANDLE AND JUNCHENG WEI

Abstract. We consider the following superlinear elliptic equation on $\mathbf{S}^n$

$$\varepsilon^2 \Delta_{\mathbf{S}^n} u - u + u^p = 0 \text{ in } \mathbf{S}^n, \ u > 0 \text{ in } \mathbf{S}^n$$

where $\Delta_{\mathbf{S}^n}$ is the Laplace-Beltrami operator on $\mathbf{S}^n$. We prove that for any $k = 1, ..., n-1$, there exists $p_k > 1$ such that for $1 < p < p_k$ and ε sufficiently small, there exist at least $n-k$ positive solutions concentrating on $k-$dimensional subset of the equator. We also discuss the problem on geodesic balls of $\mathbf{S}^n$ and establish the existence of positive **nonradial** solutions. The method extends to Dirichlet problems with more general nonlinearities. The proofs are based on the finite-dimensional reduction procedure which was successfully used by the second author in singular perturbation problems.

1. Introduction

Let D be a geodesic ball in the n-dimensional sphere $\mathbf{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$, centered at the North pole with geodesic radius θ_n^*. We consider singularly perturbed elliptic problems of the following type

$$\varepsilon^2 \Delta_{\mathbf{S}^n} u - u + u^p = 0, u > 0 \text{ in } D, \ u = 0 \text{ on } \partial D. \tag{1.1}$$

where $\Delta_{\mathbf{S}^n}$ is the Laplace-Beltrami operator on $\mathbf{S}^n$ and $p > 1$. It is well-known that the moving plane method applies to the balls contained in the hemisphere, see [34] and [21], and implies that all positive solutions are radial in the sense that they depend only on the geodesic distance θ_n from the North Pole. For large balls containing the hemisphere, the moving plane device fails in general. However in some special cases it is still true that all positive solutions are radial as was observed by Brock and Prajapat [10]. For instance if $p \leq \frac{n+2}{n-2}$ and $\varepsilon^2 > \frac{4}{(n-2)n}$ then (1.1) admits only radial solutions.

The main goal of this paper is to construct non radial solutions for small ε and for balls with radius $\theta_n^* > \pi/2$.

1991 *Mathematics Subject Classification.* Primary 35B40, 35B45; Secondary 35J40.

Key words and phrases. Multiple Clustered Spikes, Elliptic Equation on $\mathbf{S}^n$.

Throughout this paper we shall assume that $\theta_n^* > \pi/2$

We shall also be interested in non radial solutions of the corresponding problem in $\mathbf{S}^n$

$$\varepsilon^2 \Delta_{\mathbf{S}^n} u - u + u^p = 0, \ \ u > 0 \ \text{ in } \mathbf{S}^n. \tag{1.2}$$

In [9] Brezis and Peletier conjectured that nonradial solutions will bifurcate as $\varepsilon \to 0$. It is one of the purposes of this paper to answer this question affirmatively. More precisely we have, setting for fixed $k = 1, ..., n-1$,

$$p_k = \begin{cases} \frac{n-k+2}{n-k-2}, & \text{if } k < n-2, \\ +\infty, & \text{if } k \geq n-2, \end{cases} \tag{1.3}$$

Theorem 1.1. *For each integer $k = 1, ..., n-1$, there exists $\varepsilon_k > 0$ such that for $1 < p < p_k$, $0 < \varepsilon < \varepsilon_k$ problem (1.2) has at least $n-k$ solutions concentrating on a k-dimensional surface of the equator.*

The analogous problem for an arbitrary domain in $\mathbb{R}^n$ and for a power nonlinearity

$$\begin{cases} \varepsilon^2 \Delta u - u + u^p = 0, u > 0 \ \text{ in } \Omega; \\ u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.4}$$

has attracted a lot of attention in recent years. For $p < \frac{n+2}{n-2}$, and ε small, problem (1.4) admits solutions with spike layers concentrating at (local or global) maximum points of the distance function. See [13], [15], [24], [30], [38], and the references therein. We expect according to numerical computations carried out in [36], that such a result remains true also for (1.1). For the critical case similar results have been established in[7].

In [6] the authors studied (1.1) for ε small and general p and proved for balls of geodesic radius $\theta_n^* > \pi/2$ the existence of radially symmetric clustered layer solutions (i.e., solutions depending on θ_n only) for (1.1) as $\varepsilon \to 0$. The same result was obtained independently by Brezis and Peletier [9] for the special case $n = 3$ and $p = 5$ by means of a completely different technique.

Our results extend to problems with more general nonlinearities

$$\varepsilon^2 \Delta_{\mathbf{S}^n} u - u + f(u) = 0 \quad u > 0 \quad \text{in} \quad D \subset \mathbf{S}^n, \quad u = 0 \quad \text{on} \quad \partial D, \tag{1.5}$$

where f is subject to the following conditions:

(f1) $f(t) \equiv 0$ for $t < 0, f(0) = f'(0) = 0$ and $f \in C^{1+\sigma}[0, \infty) \cap C^2(0, \infty)$,

(f2) the following problem has a unique solution

$$\Delta w - w + f(w) = 0 \ \text{ in } \mathbb{R}^{n-k}, w(0) = \max_{y \in \mathbb{R}^{n-k}} w(y), w(y) \to 0 \text{ as } |y| \to +\infty. \tag{1.6}$$

Assumption (f2) implies that $w(y) = w(|y|)$ is radial and that the only solution of the linearization of (1.6) at w

$$\Delta v - v + f'(w)v = 0 \ \text{ in } \mathbb{R}^{n-k}, v(y) \to 0 \text{ for } |y| \to +\infty \tag{1.7}$$

is a linear combination of $\frac{\partial w}{\partial y_j}, j = 1, ..., n-k$. The proof of this fact can be found in Appendix C of [33].

We will show that problem (1.5) possesses three types of solutions:

(1) Type (I) solutions with a spike near $k-$dimensional subset of the boundary ∂D
(2) Type (II) solutions with clustered spikes on k-dimensional subset of spheres near the equator
(3) Type (III) solutions with clustered spikes both on $k-$dimensional subset of spheres near the equator and a spike near $k-$dimensional subset of the boundary ∂D.

Remark: The symmetry results in [31] and [10] imply that in balls contained of radius $\theta_n^* < \pi/2$ problem (1.5) has only radial solutions. As for problem (1.1) nonradial solutions can therefore only be expected for $\theta_n^* > \pi/2$.

To state our results, we introduce the polar coordinates in $\mathbb{R}^{n+1}$:

$$\begin{cases} x_1 = r \sin\theta_n \sin\theta_{n-1} ... \sin\theta_2 \sin\varphi, \\ x_2 = r \sin\theta_n \sin\theta_{n-1} ... \sin\theta_2 \cos\varphi, \\ x_3 = r \sin\theta_n \sin\theta_{n-1} ... \cos\theta_2, \\ \vdots \\ x_{n+1} = r\cos\theta_n \end{cases} \tag{1.8}$$

where $r = \sqrt{x_1^2 + ... + x_{n+1}^2}, \quad 0 \le \varphi < 2\pi, \quad 0 \le \theta_j \le \pi, \quad j = 2, ..., n$. So a parametrization of $\mathbf{S}^n$ is $r = 1$, $\{0 \le \varphi < 2\pi,\ 0 \le \theta_j \le \pi, \quad j = 2, ..., n\}$. We also define

$$\xi_j = \cos\theta_j, j = 2, ..., n. \tag{1.9}$$

We look for **nonradial** solutions of (1.1) of the form

$$u = u(\xi_n, ..., \xi_{k+1}), k = 1, ..., n-2.$$

Define a $k-$dimensional spherical cap on $\mathbf{S}^n$

$$C_r = \{\theta_{k+1} = ... = \theta_{n-1} = \frac{\pi}{2}, \theta_n = \arccos r\}. \tag{1.10}$$

We also need a quantity: let ρ_ε satisfy

$$w(\rho_\varepsilon) = A_0 \varepsilon^2 \rho_\varepsilon, \rho_\varepsilon >> 1 \tag{1.11}$$

where A_0 is some generic constant to be given in (3.10). It is not difficult to see that there exists a unique solution ρ_ε which satisfies $(2-\delta)\log\frac{1}{\varepsilon} \le \rho_\varepsilon \le 2\log\frac{1}{\varepsilon}$ for all $\varepsilon \le \varepsilon_0(\delta)$. The location of the boundary layer is given by the unique solution of

$$\frac{\tilde{r}^\varepsilon + R}{\sqrt{1-R^2}} - \frac{n-k-1}{4}\varepsilon \ln \frac{\varepsilon\sqrt{1-R^2}}{\tilde{r}^\varepsilon + R} = \frac{1}{2}\varepsilon \log\frac{1}{\varepsilon} + O(\varepsilon). \tag{1.12}$$

Our main result in this paper is the following

Theorem 1.2. *Let $k = 1, ..., n-1$ be a fixed integer. Let $N > 0$ be another fixed positive integer. Set $-R = \arccos \theta_n^*$. Then there exists $\varepsilon_{N,k} > 0$ such that for all $\varepsilon < \varepsilon_{N,k}$, problem (1.1) admits three types of solutions $u_\varepsilon^1(\xi_n, ..., \xi_{k+1}), u_\varepsilon^2(\xi_n, ..., \xi_{k+1}), u_\varepsilon^3(\xi_n, ..., \xi_{k+1})$, with the following properties*

(1) (Type I) u_ε^1 concentrates at $\Sigma_1 = C_{r_1^\varepsilon}$ where r_1^ε satisfies (1.12)
More precisely, we have $u_\varepsilon^1(0, ..., 0, r_1^\varepsilon) \to w(0)$, where $w(y)$ is the unique solution of (1.6), and there exist two constants a and b such that

$$u_\varepsilon^1(\xi_n, ..., \xi_{k+1}) \le a e^{-b\varepsilon^{-1} dist((\xi_n,...,\xi_{k+1}),\Sigma_1)}. \tag{1.13}$$

(2) (Type II) $u_\varepsilon^2(\xi_n, ..., \xi_{k+1})$ concentrates at $\Sigma_2 = \cup_{l=1}^N C_{r_2^{\varepsilon,l}}$ with

$$r_2^{\varepsilon,l} = (l - \frac{N+1}{2})\varepsilon\rho_\varepsilon + O(\varepsilon), l = 1, ..., N \tag{1.14}$$

where ρ_ε is defined by (1.11).

(3) (Type III) $u_\varepsilon^3(\xi_n, ..., \xi_{k+1})$ concentrates at k-dimensional subset $\Sigma_3 = \cup_{l=1}^N C_{r_3^{\varepsilon,l}} \cup C_{\tilde{r}_0^\varepsilon}$ where r_0^ε satisfies (1.12) *and*

$$r_3^{\varepsilon,l} = (j - \frac{N+1}{2})\varepsilon\rho_\varepsilon + O(\varepsilon), l = 1, ..., N. \tag{1.15}$$

As a consequence, for each $N \ge 1$, there exists at least $(2N+1)$ solutions for ε sufficiently small.

Remarks:

(1) When $k = n-1$, this corresponds to result in [6] and [9]. In this case the solutions are radial. When $k = 1, ..., n-2$, all solutions in Theorem 1.2 are nonradial.

(2) It was proved in [16] that all solutions of (1.6) are radial. In general there are not necessarily unique. Examples of functions for which there is a unique solution are found for instance in [28].

Going back to equation (1.2), we have the following

Theorem 1.3. *Let $k = 1, ..., n-1$ and $k+1 \leq j \leq n$ and N be a fixed positive integer. Then there exists $\varepsilon_{N,k} > 0$ such that for all $\varepsilon < \varepsilon_{N,k}$, problem (1.2) admits n- k solutions $u_{\varepsilon,j}(\xi_n, ..., \xi_{k+1})$ such that $u_{\varepsilon,j}$ concentrates on a $k-$dimensional subset $\{\theta_i = \frac{\pi}{2}, i = k+1, ..., n, i \neq j, \theta_j = \arccos r^{\varepsilon}_{l,j}\}$ where*

$$r^{\varepsilon}_{l,j} = (l - \frac{N+1}{2})\varepsilon\rho_{\varepsilon} + O(\varepsilon), l = 1, ..., N \tag{1.16}$$

Theorem 1.1 then follows from Theorem 1.3.

Radially symmetric Type II solutions are studied for the following singularly perturbed problem

$$\begin{cases} \varepsilon^2\Delta u - u + f(u) = 0, u > 0 \text{ in } \Omega; \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \end{cases} \tag{1.17}$$

where Ω is the unit ball in $\mathbb{R}^n$. See [1], [2], [11], and [26]. In particular, we mention the results of [26] which state that for any positive integer $N \geq 1$, there exists a layered solution u_ε to (1.17) with the property that u_ε concentrates on N spheres $r^{\varepsilon}_1 > ... > r^{\varepsilon}_N$ satisfying $1 - r^{\varepsilon}_1 = \varepsilon \log\frac{1}{\varepsilon} + O(\varepsilon), r^{\varepsilon}_{j-1} - r^{\varepsilon}_j = \varepsilon\log\frac{1}{\varepsilon} + O(\varepsilon), j = 2, ..., N$. This is in contrast to the Dirichlet problem where near the boundary the solution concentrates at most on one sphere(cf. solutions of Type I of [6]).

Our approach mainly relies upon a finite dimensional reduction procedure. Such a method has been used successfully in many papers, see e.g. [1], [2], [14], [18], [19], [26]. In particular, we shall follow the one used in [26].

This method consists of three main steps:

Step 1: Choose good approximate solutions which concentrate on some circles $C_{r_1}, ..., C_{r_N}$.

This is done in Section 3.

Step 2: Solve the nonlinear PDE modulo the projections of finite dimensions corresponding to translation modes. This reduces the problem to a finite dimensional problem.

This is done in Section 4.

Step 3: Use degree theory (depending on nondegeneracy of some reduced functional) to solve the reduced problem.

This is done in Section 5.

The interested reader may consult the survey article of Ni [29] (pages 170-173) for more details.

Throughout this paper, unless otherwise stated, the letter C will always denote various generic constants which are independent of ε, for ε sufficiently small. The notation $A_\varepsilon = 0(B_\varepsilon)$ means that $|\frac{A_\varepsilon}{B_\varepsilon}| \le C$, while $A_\varepsilon = o(B_\varepsilon)$ means that $\lim_{\varepsilon\to 0} \frac{|A_\varepsilon|}{|B_\varepsilon|} = 0$.

Acknowledgment. The research was completed while the first author (C.B.) was visiting the Institute of Mathematical Sciences of The Chinese University of Hong Kong. She would like to express her gratitude for the hospitality and the stimulating atmosphere. The research of the second author (J.W.) is supported by an Earmarked Grant from RGC of Hong Kong.

2. Coordinates on $\mathbf{S}^n$

In this section, we introduce the spherical coordinates on $\mathbf{S}^n$ and derive the equation for u. For this purpose we find it convenient to map $\mathbf{S}^n$ stereographically onto $\mathbb{R}^n$. The equator is mapped into $\mathbf{S}^{n-1}$ and we shall assume that the upper hemisphere is mapped into the unit ball of $\mathbb{R}^n$.

Let $x \in \mathbb{R}^n$ and let $(r, \theta_{n-1}, \theta_{n-2}, \dots, \theta_2, \varphi)$ be its spherical coordinates given at (1.8), such that $\theta_k \in [0, \pi)$ for $k = 2, \dots, n-1$ and $\varphi \in [0, 2\pi)$.

Then

$$\begin{aligned}|dx|^2 = dr^2 + r^2 d\theta_{n-1}^2 + r^2 \sin\theta_{n-1}^2 d\theta_{n-2}^2 + \dots \\ + r^2 \sin\theta_{n-1}^2 \sin\theta_{n-2}^2 \sin\theta_{n-3}^2 \dots \sin\theta_2^2 d\varphi^2.\end{aligned}$$

For our purposes it will be convenient to use the new variables

$$\xi_k = \cos\theta_k \text{ for } k = 2, \dots n-1, \;\; \xi_1 = \varphi.$$

Then

$$|dx|^2 = dr^2 + \frac{r^2}{1-\xi_{n-1}^2} d\xi_{n-1}^2 + \frac{r^2(1-\xi_{n-1}^2)}{1-\xi_{n-2}^2} d\xi_{n-2}^2 + \dots + r^2(1-\xi_{n-1}^2)\dots(1-\xi_2^2) d\xi_1^2.$$

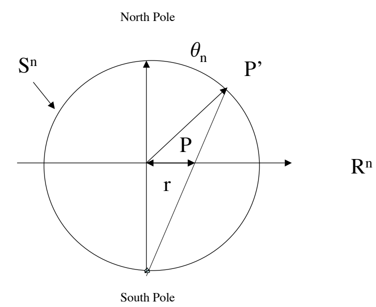

FIGURE 1. Stereographic projection

After a stereographic projection of $\mathbf{S}^n \subset \mathbb{R}^{n+1}$ onto $\mathbb{R}^n$ we have for the line element on $\mathbf{S}^n$

$$ds^2 = \left(\frac{2}{1+r^2}\right)^2 |dx|^2.$$

Let θ_n be the geodesic distance from a point on $\mathbf{S}^n$ to the North pole and set

$$\xi_n = \cos\theta_n.$$

Then

$$r = \tan\frac{\theta_n}{2} = \sqrt{\frac{1-\xi_n}{1+\xi_n}}, \quad \frac{2}{1+r^2} = 1+\xi_n.$$

We have

$$\left(\frac{2}{1+r^2}\right)^2 r^2 = 1-\xi_n^2 \text{ and } \left(\frac{dr}{d\xi_n}\right)^2 = (1-\xi_n)^{-1}(1+\xi_n)^{-3}.$$

Hence

$$\begin{aligned} ds^2 = \frac{1}{1-\xi_n^2} d\xi_n^2 + \frac{1-\xi_n^2}{1-\xi_{n-1}^2} d\xi_{n-1}^2 + \frac{(1-\xi_n^2)(1-\xi_{n-1}^2)}{1-\xi_{n-2}^2} d\xi_{n-2}^2 + \\ \cdots + (1-\xi_n^2)(1-\xi_{n-1}^2)\dots(1-\xi_2^2) d\xi_1^2 \\ = \sum_{i=1}^n g_{ii} d\xi_i^2. \end{aligned}$$

The Laplace-Beltrami operator on $\mathbf{S}^n$ takes the form

$$\Delta_{\mathbf{S}^n} = \frac{1}{\sqrt{g}} \sum_{i=1}^{n} \frac{\partial}{\partial \xi_i} (\sqrt{g} g_{ii}^{-1} \frac{\partial}{\partial \xi_i}),$$

where

$$\sqrt{g} = \prod_{k=3}^{n} (1-\xi_k^2)^{\frac{k-2}{2}}.$$

For $m = 2, \ldots n$ denote

$$\Delta_m := \frac{1}{(1-\xi_m^2)^{\frac{m-2}{2}}} \frac{\partial}{\partial \xi_m} \left((1-\xi_m^2)^{\frac{m}{2}} \frac{\partial}{\partial \xi_m} \right).$$

Then

$$\Delta_{\mathbf{S}^n} = \Delta_n + \frac{\Delta_{n-1}}{1-\xi_n^2} + \cdots + \frac{\Delta_m}{\prod_{i=m+1}^n (1-\xi_i^2)} + \cdots + \frac{1}{\prod_{i=2}^n (1-\xi_i^2)} \frac{\partial^2}{\partial \xi_1^2}. \tag{2.1}$$

Remark Observe that Δ_m is the Laplace- Beltrami operator on $\mathbf{S}^m$ depending only on the geodesic distance from the North pole of $\mathbf{S}^m$, that is $(0, \cdots, 0, \underbrace{1}_{m+1}, 0, \cdots, 0)$.

Finally for $u = u(\xi_n, ..., \xi_{k+1})$ satisfying (1.1), we have

$$\varepsilon^2 \left[\Delta_n u + \frac{\Delta_{n-1} u}{1-\xi_n^2} + \cdots + \frac{\Delta_{k+1} u}{\prod_{i=k+2}^n (1-\xi_i^2)} \right] - u + f(u) = 0 \text{ in } D, \quad u = 0 \text{ on } \partial D \tag{2.2}$$

Let

$$z_j = \frac{\xi_j}{\varepsilon}, \quad j = n, ..., k+1, \quad z = (z_n, ..., z_{k+1}). \tag{2.3}$$

Then (2.2) becomes

$$\Delta' u - u + f(u) = 0 \text{ in } I_\varepsilon; u = 0 \text{ on } I_\varepsilon \tag{2.4}$$

where

$$\Delta' u = \Delta'_n u + + \frac{\Delta'_{n-1} u}{1-\varepsilon^2 z_n^2} + \cdots + \frac{\Delta'_{k+1} u}{\prod_{i=k+2}^n (1-\varepsilon^2 z_i^2)}, \tag{2.5}$$

$$\Delta'_m u = \frac{1}{(1-\varepsilon^2 z_m^2)^{\frac{m-2}{2}}} \frac{\partial}{\partial z_m} \left((1-\varepsilon^2 z_m^2)^{\frac{m}{2}} \frac{\partial}{\partial z_m} \right) u$$

and

$$I_\varepsilon = \left[-\frac{R}{\varepsilon}, \frac{1}{\varepsilon} \right) \times \left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon} \right)^{n-k-1}$$

We also write (2.4) in the following form:

$$\sum_{i=k+1}^{n} \frac{1}{\sqrt{g(\varepsilon z)}} \frac{\partial}{\partial z_i} \left(\sqrt{g(\varepsilon z)} (g_{ii}(\varepsilon z))^{-1} \frac{\partial u}{\partial z_i} \right) - u + f(u) = 0 \text{ in } I_\varepsilon; u = 0 \text{ on } I_\varepsilon \tag{2.6}$$

where

$$\sqrt{g} = \prod_{i=k+1}^{n} (1 - \varepsilon^2 z_i^2)^{\frac{i-2}{2}}, g_{ii}^{-1} = \frac{1 - \varepsilon^2 z_i^2}{\prod_{l=i+1}^{n} (1 - \varepsilon^2 z_l^2)} \tag{2.7}$$

Let us illustrate the solutions described in Theorem (1.2) in the three-dimensional case. If we project the ball $D \subset \mathbf{S}^3$ stereographically into $\mathbb{R}^3$ we obtain again a ball of Euclidean radius $R_0 = \tan \frac{\theta_3^*}{2}$ centered at the origin. The equator corresponds to the concentric ball of Euclidean radius 1. The figures indicate in the meridian plane the maxima of the solutions. We show type III solutions.

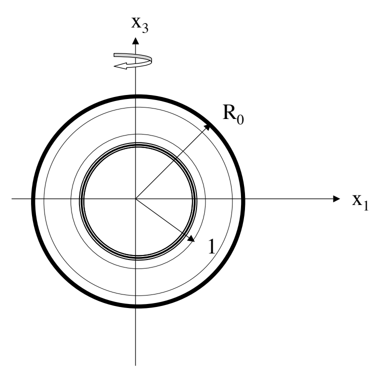

FIGURE 2. radial solutions with boundary layer and clustered spikes near the equator

Near the boundary the solutions have only one maximum whereas near the equator the number of spikes is any number $N \leq N_0(\varepsilon)$. This number increases as ε decreases. This is the same for nonradial solutions. The spikes here lie on circles near the boundary and near the equator.

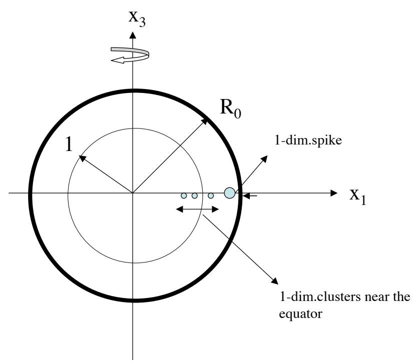

FIGURE 3. nonradial solutions with boundary layer and clustered spikes near the equator

3. Approximate Solutions

In this section we introduce a family of approximate solutions to (2.6) and derive some useful estimates. Since the construction of type III of Theorem 1.2 is the most complicated, we shall focus on the existence of u_ε^3 in Theorem 1.2. The proof of existence of $u_\varepsilon^1, u_\varepsilon^2$ can be modified accordingly. We shall use a unified approach to prove both Theorem 1.2 and Theorem 1.3.

Let us now fix $j_0 \in \{k+1, ..., n\}$. (To prove Theorem 1.2, we take $j_0 = n$.)

Let w be the unique solution of (1.6), (see assumption (f2)). Using ODE analysis, it is standard [17] to see that

$$w(y) = A_n r^{-\frac{n-k-1}{2}} e^{-r} + O(r^{-\frac{n-k+1}{2}} e^{-r}),\ y \in \mathbb{R}^{n-k},\ r = |y|, \tag{3.1}$$
$$w'(r) = -A_n r^{-\frac{n-k-1}{2}} e^{-r} + O(r^{-\frac{n-k+1}{2}} e^{-r}) \quad \text{for} \quad r \geq 1,$$

where $A_n > 0$ is a generic constant depending on n and f only. We state the following lemma, which measure the interactions of spikes. The proof of it follows from (3.1) and Lebesgue's Dominated Convergence Theorem. See Lemma 2.3 of [25].

Lemma 3.1. *Let $k_1 \geq k_2 > 0$ and let w be a function satisfying* (3.1). *Then for* $|P_1 - P_2| >> 1$ *we have*

$$w^{k_1}(\mathbf{z} - P_1)w^{k_2}(\mathbf{z} - P_2) = O\left(w^{k_2}(|P_1 - P_2|)\right) \tag{3.2}$$

and for $k_1 > k_2$

$$\int_{\mathbb{R}^{n-k}} w^{k_1}(\mathbf{z}-P_1)w^{k_2}(\mathbf{z}-P_2) = w^{k_2}(|P_1-P_2|)(1+O(\frac{1}{|P_1 - P_2|}))\int_{\mathbb{R}^{n-k}} w^{k_1}(\mathbf{z})e^{-k_2<\frac{P_1-P_2}{|P_1-P_2|},\mathbf{z}>} \tag{3.3}$$

For $\mathbf{t} = (t_n, ..., t_{k+1})$ with $t_n \in (-\frac{R}{4}, \frac{1}{4})$, we start with the approximation of solutions concentrating at $\mathbf{z} = \mathbf{t}/\varepsilon$. Let

$$w_{\mathbf{t}}(\mathbf{z}) := w\left((z_n - \frac{t_n}{\varepsilon})\sqrt{g_{nn}(\mathbf{t})}, \dots, (z_{k+1} - \frac{t_{k+1}}{\varepsilon})\sqrt{g_{(k+1)(k+1)}(\mathbf{t})}\right), \ \mathbf{z} \in I_\varepsilon \tag{3.4}$$

and

$$w_{\varepsilon,\mathbf{t}}(\mathbf{z}) = w_{\mathbf{t}}(\mathbf{z})\eta(\varepsilon\mathbf{z} - \mathbf{t}), \quad \mathbf{z} \in I_\varepsilon, \tag{3.5}$$

where

$$\eta(x) = \begin{cases} 1 \text{ for } |x| < \frac{1-R}{100}; \\ 0 \text{ for } |x| > \frac{3(1-R)}{100} \end{cases} \tag{3.6}$$

The approximation of the type I solution with a boundary layer is more complicated.

For $\mathbf{t}^0 = (t_n^0, 0, ..., 0)$ where $t_n^0 \in (-R, -\frac{R}{4})$, we have to define $w_{\varepsilon,\mathbf{t}^0}$ differently. First we set

$$\alpha_\varepsilon(\mathbf{t}^0) = w\left(\frac{(-R - t_n^0)\sqrt{g_{nn}(\mathbf{t}^0)}}{\varepsilon}, 0, ..., 0\right), \ \beta_\varepsilon(\mathbf{z}) = e^{-\frac{(\varepsilon z_n + R)\sqrt{g_{nn}(\mathbf{t}^0)}}{\varepsilon}}, \quad \mathbf{z} \in I_\varepsilon. \tag{3.7}$$

(3.1) implies that for $\frac{R+t_n^0}{\varepsilon} >> 1$

$$\alpha_\varepsilon(\mathbf{t}^0) = (A_n + O(\varepsilon))\left(\frac{\varepsilon\sqrt{1 - (t_n^0)^2}}{R + t_n^0}\right)^{\frac{n-k-1}{2}} e^{-\frac{(R+t_n^0)}{\varepsilon\sqrt{1-(t_n^0)^2}}} \tag{3.8}$$

A first ansatz for type I solutions of Theorem 1.2 is

$$\tilde{w}(\mathbf{z}) = \left(w_{\mathbf{t}^0}(\mathbf{z}) - \alpha_\varepsilon(\mathbf{t}^0)\beta_\varepsilon(\mathbf{z})\right)\eta(\varepsilon\mathbf{z} - \mathbf{t}^0).$$

If we compute $S_\varepsilon[\tilde{w}]$ first order terms in ε remain (cf. Section 6.2, in particular the computation following (6.24)). The correction which takes care of these terms is described next.

Let $\Psi_0(y)$ be the unique solution of the problem

(3.9)
$$\begin{cases} \Delta\Psi_0 - \Psi_0 + f'(w)\Psi_0 = (-\frac{R}{\sqrt{1-R^2}})(2y_1\frac{\partial^2 w}{\partial y_1^2} + n\frac{\partial w}{\partial y_1}) + \frac{2R}{\sqrt{1-R^2}}y_1\sum_{j=2}^{n-k}\frac{\partial^2 w}{\partial y_j^2} - A_0 e^{-2c_1}f'(w)e^{-y_1}, \text{in } \mathbb{R}^{n-k} \\ \int_{\mathbb{R}^{n-k}}\Psi_0\frac{\partial w}{\partial y_j} = 0, j = 1, ..., n-k \end{cases}$$

where A_0 and c_1 are defined by

$$A_0 = \frac{k\int_{\mathbb{R}^{n-k}}|\nabla w|^2 dy}{A_n\int_{\mathbb{R}^{n-k}}f(w)e^{-y_1}dy} \text{ and } e^{-2c_1} = \frac{R}{\sqrt{1-R^2}}A_0. \tag{3.10}$$

(Observe that by (3.10), the right hand of (3.9) is perpendicular to $\frac{\partial w}{\partial y_j}, j = 1, ..., n-k$. Hence by the assumption (**f2**) concerning (1.7), there exists a unique solution to (3.9).)

Since Ψ_0 does not satisfy the Dirichlet boundary conditions we have to modify it as follows: let

(3.11)
$$\hat\Psi_0(\mathbf{z}) = \Psi_0\left((z_n - \frac{t_n^0}{\varepsilon})\sqrt{g_{nn}(\mathbf{t}^0)}, \dots, z_{k+1}\sqrt{g_{(k+1)(k+1)}(\mathbf{t}^0)}\right) - \Psi_0\left((-R - \frac{t_n^0}{\varepsilon})\sqrt{g_{nn}(\mathbf{t}^0)}, 0, ..., 0\right)\beta_\varepsilon(\mathbf{z}).$$

The approximate solution of type I assumes now the form:

$$w_{\varepsilon,\mathbf{t}^0}(\mathbf{z}) = \left(w_{\mathbf{t}^0}(\mathbf{z}) - \alpha_\varepsilon(\mathbf{t}^0)\beta_\varepsilon(\mathbf{z}) - \varepsilon\hat\Psi_0(\mathbf{z})\right)\eta(\varepsilon\mathbf{z} - \mathbf{t}^0), \tag{3.12}$$

where $t_n^0 \in (-R, -\frac{R}{4})$.

Note that for $z_n \geq \frac{1}{4\varepsilon}$, we have

$$|w_{\varepsilon,\mathbf{t}}(\mathbf{z})| + |\nabla_{\mathbf{z}} w_{\varepsilon,\mathbf{t}}(\mathbf{z})| + |\nabla_{\mathbf{z}}^2 w_{\varepsilon,\mathbf{t}}(\mathbf{z})| \leq e^{-\frac{1}{C\varepsilon}}. \tag{3.13}$$

Observe also that, by construction, $w_{\varepsilon,\mathbf{t}}$ satisfies the Dirichlet boundary condition, i.e., $w_{\varepsilon,\mathbf{t}}(-\frac{R}{\varepsilon}, z_{n-1}, ..., z_{k+1}) = 0$. Furthermore, $w_{\varepsilon,\mathbf{t}}$ depends smoothly on $\mathbf{t}$ as a map with values in $C^2(I_\varepsilon)$.

Next we describe the approximate location of the concentration points $\mathbf{t}$. We first describe the boundary concentration point. Let $\bar t_\varepsilon^0$ be the unique solution such that

$$\frac{\bar t_\varepsilon^0 + R}{\sqrt{1-R^2}} - \frac{n-k-1}{4}\varepsilon\ln\frac{\varepsilon\sqrt{1-R^2}}{\bar t_\varepsilon^0 + R} = \frac{1}{2}\varepsilon\log\frac{1}{\varepsilon} + c_1\varepsilon \tag{3.14}$$

where c_1 is defined at (3.10).

To describe the concentration points near the equator, we have to consider the auxiliary functional

$$E_\varepsilon(t^1,...,t^N)=\frac{A_0}{2}\sum_{i=1}^{N}(t^i)^2+\sum_{i=2}^{N}G(\frac{t^i-t^{i-1}}{\varepsilon}) \tag{3.15}$$

where A_0 is defined in (3.10) and the function G satisfies

$$G(T)=\frac{1}{\int_{\mathbb{R}^{n-k}}f(w)e^{-y_1}}\int_{\mathbb{R}^{n-k}}f(w(y_1,...,y_{n-k}))w(T+y_1,y_2,...,y_{n-k}) \tag{3.16}$$

The properties of G are given in the following lemma whose proof is given in Section 6.

Lemma 3.2. *The function G is radially symmetric and for T large, we have*

$$G(T)=(1+O(\frac{1}{T}))w(T),G^{'}(T)=(1+O(\frac{1}{T}))w^{'}(T),G^{''}(T)=(1+O(\frac{1}{T}))w^{''}(T) \tag{3.17}$$

Using Lemma 3.2, we have the following result, the proof of which is carried out in Section 6.

Lemma 3.3. *The functional $E_\varepsilon(t^1,...,t^N)$ has a unique minimizer $(\bar{t}_\varepsilon^1,...,\bar{t}_\varepsilon^N)$ in the set*

$$\{(t^1,...,t^N)|t^j-t^{j-1}>\varepsilon, j=2,...,N\}.$$

Moreover, we have

$$\bar{t}_\varepsilon^j=(j-\frac{N+1}{2})\varepsilon\rho_\varepsilon+O(\varepsilon) \tag{3.18}$$

where ρ_ε is the unique solution of

$$G^{'}(\rho_\varepsilon)=A_0\varepsilon^2\rho_\varepsilon,\rho_\varepsilon>>1. \tag{3.19}$$

Furthermore, the smallest eigenvalue of the matrix

$$\mathcal{M}=(\frac{\partial^2E_\varepsilon}{\partial t^i\partial t^j})$$

is greater than or equal to A_0. As a consequence, we have that

$$|\mathcal{M}^{-1}\mathbf{x}|\leq C|\mathbf{x}|. \tag{3.20}$$

Remark: Note that for any $0<\delta<1$ there exists $\varepsilon_0>0$ such that.

$$(2-\delta)\log\frac{1}{\varepsilon}\leq\rho_\varepsilon\leq 2\log\frac{1}{\varepsilon}\text{ for all }\varepsilon\leq\varepsilon_0. \tag{3.21}$$

We introduce the following set

$$\Lambda=\left\{(\mathbf{t}^0,\mathbf{t}^1,...,\mathbf{t}^N)\left|\begin{array}{l}\mathbf{t}^0=(t_n^0,0,...,0),\mathbf{t}^i=(0,...,t_{j_0}^i,0,...,0),\text{where}\\|t_n^0-\bar{t}_\varepsilon^0|\leq\varepsilon^{1+\tau_0},\\|t_{j_0}^i-\bar{t}_\varepsilon^i|\leq\varepsilon,i=1,...,N\end{array}\right.\right\}, \tag{3.22}$$

where $0 < \tau_0 < \frac{\sigma}{4}$, σ being defined in (**f1**).

For $(\mathbf{t}^0, ..., \mathbf{t}^N) \in \Lambda$, we define

(3.23) $t_n^0 = \bar{t}_0^\varepsilon + \varepsilon^{1+\tau_0}\hat{t}^0, t_{j_0}^i = \bar{t}_\varepsilon^i + \varepsilon\hat{t}^i, i = 1, ..., N, \mathbf{t}^0 = (t_n^0, 0, ..., 0), \mathbf{t}^i = (0, ..., t_{j_0}^i, 0, ..., 0),$

$$W_i = w_{\varepsilon,\mathbf{t}^i}(z), i = 0, 1, ..., N, W(z) := \sum_{i=0}^N W_i, \tag{3.24}$$

and

(3.25)

$$I_{\varepsilon,i} = \left[\frac{(-R + t_n^i)(g_{nn}(\mathbf{t}^i))^{1/2}}{\varepsilon}, \frac{(1 - t_n^i)(g_{nn}(\mathbf{t}^i))^{1/2}}{\varepsilon}\right) \times \prod_{l=n-1}^{k+1} \left(\frac{(-1 + t_l^i)(g_{ll}(\mathbf{t}^i))^{1/2}}{\varepsilon}, \frac{(1 - t_l^i)(g_{ll}(\mathbf{t}^i))^{1/2}}{\varepsilon}\right).$$

Then we have

$$(\mathbf{t}^0, \mathbf{t}^1, ..., \mathbf{t}^N) \in \Lambda \ \text{ iff } \ |\hat{t}^j| < 1, j = 0, 1, ..., N$$

and

$$\alpha_\varepsilon(\mathbf{t}^0) = O(\sqrt{\varepsilon}), \ t_{j_0}^j = O(\varepsilon|\ln\varepsilon|), j = 1, ..., N, \ |\mathbf{t^i} - \mathbf{t}^j| \geq 2|i - j|\varepsilon\log\frac{1}{\varepsilon}. \tag{3.26}$$

The choice of the approximated location of the concentration points comes from the computations carried out in the proof of formula (5.1).

Let

$$\mathcal{S}_\varepsilon(u) = \Delta' u - u + f(u). \tag{3.27}$$

Finally we state the following important lemma on the error estimates. The proof of them will be delayed to Section 6.2.

Lemma 3.4. *Let $(\mathbf{t}^0, ..., \mathbf{t}^N) \in \Lambda$ and let ε be sufficiently small. Then*

(i) if $\varepsilon z_j = t_j^0 + \varepsilon(g_{nn}(\mathbf{t}^0))^{-1/2} y_{n+1-j}, j = n, ..., k+1$, we have

$$\mathcal{S}_\varepsilon[W_0](z) = -\frac{2A_n + O(\frac{1}{|\ln\varepsilon|})}{\sqrt{1 - R^2}}\varepsilon^{1+\tau_0}\hat{t}^0 e^{-2c_1} f'(w) e^{-y_1} + O(\varepsilon^{1+\frac{\sigma}{2}}), \tag{3.28}$$

where σ is defined in the condition (**f1**).

(ii) if $\varepsilon z_j = t_j^i + \varepsilon(g_{nn}(\mathbf{t}^i))^{-1/2} y_{n+1-j}, j = k+1, ..., n, i = 1, ..., N$, we have

$$\mathcal{S}_\varepsilon[W_i] = -2\varepsilon t_{j_0}^i \frac{\partial^2 w}{\partial y_{n+1-j_0}^2} + 2\varepsilon t_{j_0}^i \sum_{m<j_0} \frac{\partial^2 w}{\partial y_{n+1-m}^2} \tag{3.29}$$

$$-\varepsilon j_0 t_{j_0}^i \frac{\partial w}{\partial y_{n+1-j_0}} + \varepsilon^2 E^i(y) + O(\varepsilon^3(1 + |y|^3)e^{-|y|})$$

where $E^i, i = 1, ..., N$ are bounded and integrable functions which are even in $y_j, j = 1, ..., n-k$. As a consequence, we obtain

$$\|\mathcal{S}_\varepsilon[W]\|_{L^\infty(I_\varepsilon)} \leq C\varepsilon^{1+\tau_0}. \tag{3.30}$$

4. Finite-Dimensional Reduction

In this section we perform a finite dimensional reduction process. Since most of the analysis here is similar to that of [6], we just point out the main differences.

Fix $(\mathbf{t}^0, ..., \mathbf{t}^N) \in \Lambda$. We define two norms:

$$(u,v)_\varepsilon = \int_{I_\varepsilon} \sqrt{g}\left[\sum_{i=k+1}^{n} g_{ii}^{-1}|\frac{\partial u}{\partial z_i}|^2 + uv\right], \qquad < u, v >_\varepsilon = \int_{I_\varepsilon} \sqrt{g} uv. \tag{4.1}$$

Here $\sqrt{g}$ corresponds to volume element related to $\mathbb{S}^{n-k}$ cf. (2.7) Integration by parts implies

$$(u,v)_\varepsilon = - < u, \Delta' v - v >_\varepsilon .$$

Define

$$z_{0,\varepsilon} = \frac{\partial W}{\partial t_n^0}, z_{i,\varepsilon} = \frac{\partial W}{\partial t_{j_0}^i}, i = 1, ..., N, \qquad Z_i = \Delta^{'} z_{i,\varepsilon} - z_{i,\varepsilon}, i = 0, ..., N \tag{4.2}$$

and

$$\mathcal{H} = \left\{ \begin{array}{c} (u,u)_\varepsilon < +\infty, u(-\frac{R}{\varepsilon}, z_{n-1}, ..., z_{k+1}) = 0, \\ u \text{ even in } \ \xi_j, j = k+1, ..., n, j \neq j_0 \\ (u, z_{i,\varepsilon})_\varepsilon = 0, \ \ j = 0, \dots, N \end{array} \right\}.$$

Note that, integrating by parts, one has

$$u \in \mathcal{H} \qquad \text{if and only if} \qquad < u, Z_i >_\varepsilon = -(u, z_{i,\varepsilon})_\varepsilon = 0, \quad i = 0, 1, ..., N.$$

Let us consider first the following linear problem: for given $h \in L^\infty(I_\varepsilon)$ find $\phi \in \mathcal{H}$ such that

$$(\phi, \psi)_\varepsilon - < f'(W)\phi, \psi >_\varepsilon = < h, \psi >_\varepsilon, \ \forall \psi \in \mathcal{H}.$$

This equation can be rewritten as a differential equation

$$\left\{ \begin{array}{l} L_\varepsilon[\phi] := \Delta^{'}\phi - \phi + f'(W)\phi = h + \sum_{i=0}^N c_i Z_i; \\ \phi \in \mathcal{H}, \quad < \phi, Z_i >_\varepsilon = 0, \quad i = 0, 1, ..., N, \end{array} \right. \tag{4.3}$$

for some constants $c_i, i = 0, 1, ..., N$ or in an abstract form as

$$\phi + \mathcal{S}(\phi) = \bar{h} \qquad \text{in } \mathcal{H}, \tag{4.4}$$

where $\overline{h}$ is defined by duality and $\mathcal{S} : \mathcal{H} \to \mathcal{H}$ is a linear compact operator. Using Fredholm's alternative, showing that equation (4.4) has a unique solution for each $\bar{h}$, is equivalent to showing that the equation has a unique solution for $\bar{h} = 0$. Next it will be shown that this is the case when ε is sufficiently small.

In order to derive an a priori bound for ϕ in terms of h we need the asymptotic behaviour of $z_{i,\varepsilon}$ and Z_i in ε. By elementary computations we obtain, setting $y_{n+1-j} = \frac{(\varepsilon z_j - t_j^i)(g_{jj}(\mathbf{t}^i))^{1/2}}{\varepsilon}$

$$z_{i,\varepsilon} = -\frac{(g_{j_0 j_0}(\mathbf{t}^i))^{1/2}}{\varepsilon} \frac{\partial w}{\partial y_{n+1-j_0}} + R_1(y) \tag{4.5}$$

$$Z_i = \frac{(g_{j_0 j_0}(\mathbf{t}^i))^{1/2}}{\varepsilon} f'(w) \frac{\partial w}{\partial y_{n+1-j_0}} + R_2(y) \tag{4.6}$$

where $R_i(y)$ $i = 1, 2$ are bounded and integrable over $\mathbb{R}^{n-k}$.

Let us define the norm

$$\|\phi\|_* = \sup_{\mathbf{z} \in I_\varepsilon} |\phi(\mathbf{z})|. \tag{4.7}$$

We have the following result.

Proposition 4.1. *Let ϕ satisfy (4.3). Then for ε sufficiently small, we have*

$$\|\phi\|_* \leq C \|h\|_* \tag{4.8}$$

where C is a positive constant independent of ε and $(\mathbf{t}^0, ..., \mathbf{t}^N) \in \Lambda$.

Proof. The proof of this proposition is similar to that of Proposition 4.1 of [6]. We just point out the differences here. the key point here is to use the symmetry assumption (i.e., ϕ is even in all variables except xi_{j_0}) to exlcude many degeneracies. It is important to note that $W \in \mathcal{H}$ and the the equation is invariant under reflections in $\xi_j, j = n, ..., k+1$.

Arguing by contradiction, assume that

$$\|\phi\|_* = 1; \qquad \|h\|_* = o(1). \tag{4.9}$$

Similar to the proof of (4.12) in [6], we obtain

$$c_i = O(\varepsilon \|h\|_*) + o(\varepsilon \|\phi\|_*) = o(\varepsilon), \qquad i = 0, 1, ..., N. \tag{4.10}$$

Also, since we are assuming that $\|h\|_* = o(1)$ and since $\|Z_i\|_* = O\left(\frac{1}{\varepsilon}\right)$, there holds

$$\|h + \sum_{i=0}^{N} c_i Z_i\|_* = o(1). \tag{4.11}$$

Thus (4.3) yields

$$\begin{cases} \Delta' \phi - \phi + f'(W)\phi = o(1); \\ \phi \in \mathcal{H} \quad < \phi, Z_i >_\varepsilon = 0, \quad i = 0, 1, ..., N, \end{cases} \tag{4.12}$$

We show that (4.12) is incompatible with our assumption $\|\phi\|_* = 1$. First we claim that, for arbitrary, fixed $R_0 > 0$, there holds

$$|\phi(\mathbf{z})| \to 0 \quad \text{on} \quad \bigcup_{i=0}^{N} \{|\mathbf{z} - \frac{\mathbf{t}^i}{\varepsilon}| < R_0\} \qquad \text{as } \varepsilon \to 0 \tag{4.13}$$

Indeed, assuming the contrary, there exist $\delta_0 > 0$, $i \in \{0, 1, \dots, N\}$ and sequences $\varepsilon_k, \phi_k, \mathbf{z}_k \in B_{R_0}(\mathbf{t}^i)$ such that $\varepsilon_k \to 0$ as $k \to \infty$ and ϕ_k satisfies (4.3) and

$$|\phi_k(\mathbf{z}_k)| \geq \delta_0 \text{ for all } k. \tag{4.14}$$

Without loss of generality, we may assume that $i = 1$. The proof of the other cases is similar. We also omit the index k for simplicity. Let $\tilde{\phi}(\mathbf{y}) = \phi(\mathbf{z})$ where $z_j = \frac{t_j^1}{\varepsilon} + (g_{jj}(\mathbf{t}^i))^{-1/2} y_{n+1-j}$. Then using (4.12) and $\|\phi\|_* = 1$, as $\varepsilon_k \to 0$ $\tilde{\phi}_k$ converges weakly in $H^2_{loc}(\mathbb{R}^{n-k})$ and strongly in $C^1_{loc}(\mathbb{R}^{n-k})$ to a bounded function ϕ_0 which satisfies

$$\Delta \phi_0 - \phi_0 + f'(w)\phi_0 = 0 \qquad \text{in } \mathbb{R}^{n-k}.$$

Hence ϕ_0 must tend to zero at infinity and so, by (1.7), $\phi_0 = \sum_{j=1}^{n-k} c_j \frac{\partial w}{\partial y_j}$ for some c_j. Since ϕ is even in $y_j, j = n, ..., k+1, j \neq n+1-j_0$, we see that

$$\phi_0 = c_{j_0} \frac{\partial w}{\partial y_{n+1-j_0}}$$

On the other hand,$\tilde{\phi}_k \perp Z_i$ in $\mathcal{H}$, we conclude that $\int_{\mathbb{R}^{n-k}} \phi_0 f'(w) \frac{\partial w}{\partial y_{n+1-j_0}} = 0$, which yields $c_{j_0} = 0$. Hence $\phi_0 = 0$ and $\tilde{\phi}_k \to 0$ in $B_{2R}(0)$. This contradicts (4.14), so (4.13) holds true.

Given $\delta > 0$, the decay of w and (4.13) (with R_0 sufficiently large) imply

$$\|f'(W)\phi\|_* \leq \delta + \frac{1}{2}\|\phi\|_*. \tag{4.15}$$

Using (4.12) and the Maximum Principle one finds

$$\begin{aligned}\|\phi\|_* &\leq \|f'(W)\phi\|_* + \sum_{i=0}^{N} |c_i|\|Z_i\|_* + \|h\|_* \\ &\leq 2\delta + \frac{1}{2}\|\phi\|_*,\end{aligned}$$

and hence

$$\|\phi\|_* \leq 4\delta < 1$$

if we choose $\delta < \frac{1}{4}$. This contradicts (4.9). □

Using Proposition 4.1 and standard contraction mapping principle, we have the following finite dimensional reduction theorem

Theorem 4.2. *For $(\mathbf{t}^0, ..., \mathbf{t}^N) \in \Lambda$ and ε sufficiently small, there exists a unique $(\Phi, c) = (\Phi_{\varepsilon,\mathbf{t}^0,...,\mathbf{t}^N}, c_\varepsilon(\mathbf{t}^0, ..., \mathbf{t}^N))$ such that the following holds*

$$\left\{\begin{array}{l}\Delta'(W+\Phi) - (W+\Phi) + f(W+\Phi) = \sum_{i=0}^{N} c_i Z_i \ \text{in } I_\varepsilon, \\ \Phi \in \mathcal{H}\end{array}\right. \tag{4.16}$$

Moreover, the map $(\mathbf{t}^0, ..., \mathbf{t}^N) \mapsto (\Phi_{\varepsilon,\mathbf{t}^0,...,\mathbf{t}^N}, c_\varepsilon(\mathbf{t}^0, ..., \mathbf{t}^N))$ is of class C^0, and we have

$$\|\Phi_{\varepsilon,\mathbf{t}^0,...,\mathbf{t}^N}\|_* \leq C\varepsilon^{1+\tau_0}. \tag{4.17}$$

Proof. The proof is exactly the same as Proposition 4.2 of [6]. we omit the details. □

5. Proof of Theorems 1.2 and 1.3

From (4.16), we see that, to prove the existence of Type III solutions of Theorem 1.2 and Theorem 1.3, it is enough to find a zero of the vector $\mathbf{c}_\varepsilon(\mathbf{t}^0, ..., \mathbf{t}^N) = (c_{0,\varepsilon}, c_{1,\varepsilon}, ..., c_{N,\varepsilon})^T$.

The next Proposition computes the asymptotic formula for $\mathbf{c}_\varepsilon(t_0, \mathbf{t})$: (Recalling (3.23))

Proposition 5.1. *For ε sufficiently small, we have the following asymptotic expansion*

$$\frac{1}{\varepsilon^{2+\tau_0}\int_{\mathbb{R}^{n-k}}(f'(w)(\frac{\partial w}{\partial y_1})^2)} c_{0,\varepsilon}(\mathbf{t}^0, ..., \mathbf{t}^N) = d_0\hat{t}^0 + \beta_{0,\varepsilon}(\hat{t}^0, ..., \hat{t}^N), \tag{5.1}$$

(5.2)

$$\frac{1}{\varepsilon^{3}\int_{\mathbb{R}^{n-k}}(f'(w)(\frac{\partial w}{\partial y_1})^2)} c_{i,\varepsilon}(\mathbf{t}^0, ..., \mathbf{t}^N) = d_i\frac{\partial(E_\varepsilon(\hat{t}^1, ..., \hat{t}^N))}{\partial \hat{t}^i} + \beta_{i,\varepsilon}(\hat{t}^0, ..., \hat{t}^N), i = 1, ..., N$$

where $d_i \neq 0, i = 0,1,...,N$ *are positive constants, and* $\beta_{i,\varepsilon}(\hat{t}^0,...,\hat{t}^N), i = 0,1,...,N$ *are continuous functions in* $(\hat{t}^0,...,\hat{t}^N)$ *with*

$$\beta_{i,\varepsilon}(\hat{t}^0,...,\hat{t}^N) = O(\varepsilon^{\tau_0} + \sum_{i=0}^{N} |\hat{t}^i|^2), i = 0,...,N. \tag{5.3}$$

We delay the proof of the proposition at the end of the section. Let us now use it to prove Theorem 1.2 and Theorem 1.3

Proof of Theorem 1.2:

To prove Theorem 1.2, we set $j_0 = n$.

To find a zero of $\mathbf{c}_\varepsilon(\mathbf{t}^0,...,\mathbf{t}^N)$, it is enough to solve the following systems of equations

$$d_0\hat{t}^0 + \beta_{0,\varepsilon}(\hat{t}^0,...,\hat{t}^N) = 0,\ d_i\nabla_{\hat{t}^i}(E_\varepsilon(\hat{t}^1,...,\hat{t}^N) + \beta_{i,\varepsilon}(\hat{t}^0,...,\hat{t}^N) = 0, i = 1,...,N. \tag{5.4}$$

By Lemma 3.3, the matrix $\mathcal{M}$ is invertible with uniform bound, (5.4) is equivalent to

$$(\hat{t}^0,...,\hat{t}^N) = \hat{\beta}_\varepsilon(\hat{t}^0,...,\hat{t}^N) \tag{5.5}$$

where $\hat{\beta}_\varepsilon(\hat{t}^0,...,\hat{t}^N)$ is a continuous function in $(\hat{t}^0,...,\hat{t}^N)$ satisfying

$$\hat{\beta}_\varepsilon(\hat{t}^0,...,\hat{t}^N) = O(\varepsilon^{\tau_0} + \sum_{i=0}^{N} \hat{t}_i^2). \tag{5.6}$$

Let $\mathbf{B} = \{(\hat{t}^0,...,\hat{t}^N) |\ |(\hat{t}^0,...,\hat{t}^N)| < \varepsilon^{\frac{\tau_0}{2}}\}$. Then Brouwer's fixed point theorem gives a solution in $\mathbf{B}$, called $(\hat{t}^0_\varepsilon,...,\hat{t}^0_\varepsilon)$, to (5.5), which in turn, gives a solution

$$u^3_\varepsilon = w_{\varepsilon,\mathbf{t}^0_\varepsilon,...,\mathbf{t}^N_\varepsilon} + \Phi_{\varepsilon,\mathbf{t}^0_\varepsilon,...,\mathbf{t}^N_\varepsilon}$$

to equation (2.6), where

$$\mathbf{t}^0_\varepsilon = (\bar{t}^0_\varepsilon + \hat{t}^0_\varepsilon, 0,...,0), \mathbf{t}^i_\varepsilon = (0,...,\bar{t}^i_\varepsilon + \hat{t}^0_\varepsilon, 0,...,0). \tag{5.7}$$

It is easy to see that u^3_ε satisfies all the properties listed in Theorem 1.2.

□

Proof of Theorem 1.3:

To prove Theorem 1.3, we let $j_0 \in \{k+1,...,n\}$. Then for each fixed N, there are at least $n-k$ solutions $u_{\varepsilon,j_0}, j_0 = k+1,...,n$.

□

Now we are ready to prove (5.1).

Proof of (5.1):

Similar to the proof of (5.8) of [6], we multiply equation (4.16) by $\sqrt{g} z_{i,\varepsilon}$, integrate by parts and obtain, using Lemma 3.4 and Theorem 4.2,

$$\sum_{l=0}^{N} c_{l,\varepsilon} < Z_l, z_{i,\varepsilon} >_\varepsilon = \int_{I_\varepsilon} \sqrt{g(\varepsilon z)} \mathcal{S}_\varepsilon[W + \Phi_{\varepsilon,\mathbf{t}^0,...,\mathbf{t}^N}] z_{i,\varepsilon} = \int_{I_\varepsilon} ????\mathcal{S}_\varepsilon[W] z_{i,\varepsilon} + O(\varepsilon^{1+\tau_0}) \tag{5.8}$$

$$= \int_{I_\varepsilon} (\sum_{l=0}^{N} \mathcal{S}_\varepsilon[W_l] + f(\sum_{l=1}^{N} W_l) - \sum_{l=1}^{N} f(W_l)) z_{i,\varepsilon} + O(\varepsilon^{1+\tau_0}).$$

For $i = 0, 1, ..., N$, we make use of (4.5) and deduce that

$$\int_{I_\varepsilon} \mathcal{S}_\varepsilon[W_l] z_i = -\frac{1}{\varepsilon} (g_{j_0 j_0}(\mathbf{t}^i))^{1/2} \int_{I_{\varepsilon,i}} \mathcal{S}_\varepsilon[W_l] \frac{\partial w}{\partial y_{n+1-j_0}} + O(\varepsilon^{1+\tau_0}). \tag{5.9}$$

For $i = 0, j_0 = n$, we have, using (3.28),

$$\int_{I_{\varepsilon,0}} \mathcal{S}_\varepsilon[W_0] z_{i,\varepsilon} = \frac{2A_n + O(\frac{1}{|\ln \varepsilon|})}{1 - R^2} \varepsilon^{\tau_0} \hat{t}_0 e^{-2c_1} \int_{\mathbb{R}^{n-k}} f'(w) \frac{\partial w}{\partial y_1} e^{-y_1} dy + O(\varepsilon^{\frac{\sigma}{2}})$$

$$= d_0 \varepsilon^{\tau_0} \hat{t}_0 + O(\varepsilon^{2\tau_0}) \tag{5.10}$$

where

$$d_0 = \frac{2A_n + O(\frac{1}{|\ln \varepsilon|})}{1 - R^2} e^{-2c_1} \int_{\mathbb{R}^{n-k}} f'(w) \frac{\partial w}{\partial y_1} e^{-y_1} = \frac{2A_n + O(\frac{1}{|\ln \varepsilon|})}{1 - R^2} e^{-2c_1} \int_{\mathbb{R}^{n-k}} f(w) e^{-y_1} \neq 0.$$

For $i = 1, 2, ..., N$, we have, using (3.29),

$$\int_{I_{\varepsilon,t_j}} \mathcal{S}_\varepsilon[W_l] z_i dz = O(\varepsilon^{1+\tau_0}) \text{ if } l \neq i,$$

$$\int_{I_{\varepsilon,t_j}} \mathcal{S}_\varepsilon[W_i] z_i dz = -\frac{1}{\varepsilon} \int_{I_{\varepsilon,i}} \sum_{i=1}^{N} \mathcal{S}_\varepsilon[W_i] \frac{\partial w}{\partial y_{n+1-j_0}} + O(\varepsilon^3)$$

$$= 2t^i_{j_0} \int_{\mathbb{R}^{n-k}} y_{n+1-j_0} \frac{\partial^2 w}{\partial y^2_{n+1-j_0}} \frac{\partial w}{\partial y_{n+1-j_0}} - 2\varepsilon t^i_{j_0} \sum_{m<j_0} \int_{\mathbb{R}^{n-k}} y_{n+1-j_0} \frac{\partial^2 w}{\partial y^2_{n+1-m}} \frac{\partial w}{\partial y_{n+1-j_0}}$$

$$+ j_0 t^i_{j_0} \int_{\mathbb{R}^{n-k}} (\frac{\partial w}{\partial y_{n+1-j_0}})^2 + O(\varepsilon^2)$$

$$= -k t^i_{j_0} \int_{\mathbb{R}^{n-k}} (\frac{\partial w}{\partial y_1})^2 + O(\varepsilon^2 |\ln \varepsilon|)$$

for

$$\int_{\mathbb{R}^{n-k}} y_{j_0} \frac{\partial^2 w}{\partial y^2_{j_0}} \frac{\partial w}{\partial y_{j_0}} = -\frac{1}{2} \int_{\mathbb{R}^{n-k}} (\frac{\partial w}{\partial y_{j_0}})^2 \tag{5.11}$$

and

$$\int_{\mathbb{R}^{n-k}} y_{j_0} \frac{\partial^2 w}{\partial y_m^2} \frac{\partial w}{\partial y_{j_0}} = \frac{1}{2} \int_{\mathbb{R}^{n-k}} (\frac{\partial w}{\partial y_{j_0}})^2 \tag{5.12}$$

where $m \neq j_0$.

On the other hand, we have

$$\int_{I_\varepsilon} (f(\sum_{l=1}^N W_l) - \sum_{l=1}^N f(W_l)) z_i = -\frac{1}{\varepsilon} \int_{\mathbb{R}^{n-k}} (f'(w)(w_{\mathbf{t}^{i-1}} + w_{\mathbf{t}^{i+1}}) \frac{\partial w}{\partial y_{n+1-j_0}} + O(\varepsilon^2)$$

$$= \sum_{l=i-1,i+1} \int_{\mathbb{R}^{n-k}} f(w) \frac{\partial (w_{\mathbf{t}^{i-1}} + w_{\mathbf{t}^{i+1}})}{\varepsilon \partial y_{n+1-j_0}} + O(\varepsilon^2)$$

$$= B_0 \sum_{l=i-1,i+1} \frac{\partial G(\frac{t^i_{j_0} - t^l_{j_0}}{\varepsilon})}{\partial t^i_{j_0}} + O(\varepsilon^2)$$

where

$$B_0 = \int_{\mathbb{R}^{n-k}} f(w) e^{-y_{n+1-j_0}} \neq 0$$

Thus we have for $i = 1, ..., N$,

$$\int_{I_\varepsilon} (\sum_{l=0}^N \mathcal{S}_\varepsilon[W_l] + f(\sum_{l=1}^N W_l) - \sum_{l=1}^N f(W_l)) z_i$$

$$= -kt^i_{j_0} \int_{\mathbb{R}^{n-k}} |\frac{\partial w}{\partial y_1}|^2 + B_0 \sum_{l=i-1,i+1} \frac{\partial G(\frac{t^i_{j_0} - t^l_{j_0}}{\varepsilon})}{\partial t^i_{j_0}} + O(\varepsilon^{1+\tau_0})$$

$$= -B_0 [A_0 t^i_{j_0} - \sum_{i=2}^N \frac{\partial G(\frac{t^i_{j_0} - t^{i-1}_{j_0}}{\varepsilon})}{\partial t^i_{j_0}} + O(\varepsilon^{1+\tau_0})]$$

So we have for $i = 1, ..., N$,

$$\int_{I_\varepsilon} \mathcal{S}_\varepsilon[W] z_i dz = -B_0 \frac{\partial E_\varepsilon(\mathbf{t})}{\partial t^i_{j_0}} + O(\varepsilon^{1+\tau_0})$$

$$= -B_0 \frac{\partial E_\varepsilon}{\partial t_j}|_{\mathbf{t}=\mathbf{t}^\varepsilon} + B_0 \Big(\mathcal{M}(\mathbf{t} - \mathbf{t}^\varepsilon) \Big)_j + O(\varepsilon^{1+\tau_0})$$

$$= d_j \varepsilon \Big(\mathcal{M} \hat{t} \Big)_j + O(\varepsilon^{1+\tau_0}) \tag{5.13}$$

where $d_j = -B_0, j = 1, ..., N$.

Since

$$< Z_l, z_{i,\varepsilon} >_\varepsilon = \frac{1}{\varepsilon^2(1-(t^i_{j_0})^2)}(\delta_{li}\int_{\mathbb{R}^{n-k}} f'(w)(\frac{\partial w}{\partial y_1})^2 + O(\varepsilon)) \tag{5.14}$$

we derive Proposition 5.1 from (5.8), (5.10) and (5.13). (The fact that all the error terms are continuous in $(\hat{t}^0, \hat{t}^1, ..., \hat{t}^N)$ follows from the continuity of $\Phi_{\varepsilon,\mathbf{t}^0,...,\mathbf{t}^N}$ in $(\mathbf{t}^0, ..., \mathbf{t}^N)$.)

□

6. Proof of three lemmas

6.1. **Proof of Lemma 3.2.** The fact that G is radially symmetric follows from the radial symmetry of w.

To study the properties of G when T is large, we note that by (3.1) for T large

$$w(T+y_1, y_2, ..., y_{n-k}) \sim A_n((T+y_1)^2 + (\sum_{j=2}^{n-k} y_j^2)^{\frac{k+1-n}{4}} e^{-\sqrt{(T+y_1)^2+\sum_{j=2}^{n-k} y_j^2}}$$

$$\sim A_n T^{\frac{k+1-n}{2}} e^{-T} e^{-y_1+O(\frac{1}{T})}$$

$$\sim w(T) e^{-y_1+O(\frac{1}{T})}$$

Then by Lebesgue's Dominated Convergence Theorem, we derive that

$$\int_{\mathbb{R}^{n-k}} f(w) w(T+y_1, y_2, ..., y_{n-k}) = (1+O(\frac{1}{T}))w(T)\int_{\mathbb{R}^{n-k}} f(w)e^{-y_1} \tag{6.15}$$

and so

$$G(T) = (1+O(\frac{1}{T}))w(T)$$

The other estimates can be proved in a similar way.

□

6.2. **Proof of Lemma 3.3.** Note that for T large, $w''(T) > 0$ and hence $G(T) > 0$. It is easy to see that a global minimizer of $E_\varepsilon(\mathbf{t})$ exists since $E_\varepsilon(\mathbf{t})$ is convex. Let us denote it by $(\bar{t}^1_\varepsilon, ..., \bar{t}^N_\varepsilon)$ which is unique. Setting $\bar{t}^i_\varepsilon = \varepsilon(i - \frac{N+1}{2})\rho_\varepsilon + \varepsilon s^i_\varepsilon$, where ρ_ε satisfies

$$A_0 \varepsilon^2 \rho_\varepsilon = -G'(\rho_\varepsilon),$$

then s_i^ε satisfies

$$A_0(i - \frac{N+1}{2}) + A_0 \varepsilon s_j^\varepsilon + e^{-(s_j^\varepsilon - s_{j-1}^\varepsilon)} - e^{-(s_{j+1}^\varepsilon - s_j^\varepsilon)} + O(\sum_{j=1}^N \frac{|s_j^\varepsilon|}{\rho_\varepsilon}) = 0, j = 1, ..., N \tag{6.16}$$

which admits a unique solution $\mathbf{s}^\varepsilon = (s_1^\varepsilon, ..., s_N^\varepsilon) = O(1)$.

Let $\mathcal{M} = (\frac{\partial^2 E_\varepsilon(\mathbf{t})}{\partial t_i \partial t_j})$. We show that the smallest eigenvalue of $\mathcal{M}$ is uniformly bounded from below. In fact, let $\eta = (\eta_1, ..., \eta_N)^T$ and we compute

$$\sum_{i,j} \mathcal{M}_{ij}\eta_i\eta_j = A_0 \sum_{j=1}^{N} \eta_j^2 + \frac{1}{\varepsilon^2}\sum_{j=2}^{N} G''(\frac{|t_j^\varepsilon - t_{j-1}^\varepsilon|}{\varepsilon})(\eta_j - \eta_{j-1})^2 \geq A_0|\eta|^2 \tag{6.17}$$

which implies that the smallest eigenvalue of $\mathcal{M}$, denoted by λ_1, satisfies

$$\lambda_1 \geq A_0 > 0. \tag{6.18}$$

Now we consider

$$|\mathcal{M}^{-1}\eta|^2 = \eta^t \mathcal{M}^{-2}\eta \leq \lambda_1^{-2}|\eta|^2$$

which proves (3.20).

□

6.3. **Proof of Lemma 3.4.** Using (1.6) it is easy to see that

$$\mathcal{S}_\varepsilon[W] = \mathcal{S}_\varepsilon[W_0] + \mathcal{S}_\varepsilon[\sum_{l=1}^{N} W_l] + O(e^{-\frac{1}{C\varepsilon}})$$

$$= \mathcal{S}_\varepsilon[W_0] + \sum_{l=1}^{N}\mathcal{S}_\varepsilon[W_l] + f(\sum_{l=1}^{N} W_l) - \sum_{l=1}^{N} f(W_l) + O(e^{-\frac{1}{C\varepsilon}}). \tag{6.19}$$

Let us compute each term in the right hand side of (6.19): to this end, we first compute

$$\Delta' w_{\mathbf{t}} = \sum_{l=k+1}^{n} \frac{1}{\sqrt{g(\varepsilon\mathbf{z})}}(\frac{\partial}{\partial z_l}\sqrt{g(\varepsilon\mathbf{z})}(g_{ll}(\varepsilon\mathbf{z}))^{-1}\frac{\partial w_{\mathbf{t}}}{\partial z_l}) \tag{6.20}$$

$$= \sum_{l=k+1}^{n} \frac{1}{g_{ll}(\varepsilon\mathbf{z})}\frac{\partial^2 w_{\mathbf{t}}}{\partial z_l^2} + \sum_{l=k+1}^{n} \frac{1}{g_{ll}(\varepsilon\mathbf{z})}(\frac{\partial}{\partial z_l}\ln(\sqrt{g(\varepsilon\mathbf{z})}))\frac{\partial w_{\mathbf{t}}}{\partial z_l}$$

$$+ \sum_{l=k+1}^{n} (\frac{\partial}{\partial z_l}(g_{ll}(\varepsilon\mathbf{z}))^{-1})\frac{\partial w_{\mathbf{t}}}{\partial z_l}$$

Let $\varepsilon z_j = t_j^i + \varepsilon(g_{jj}(\mathbf{t}))^{-1/2}y_{n+1-j}, j = n, ..., k+1$ or

$$\mathbf{z} = \frac{\mathbf{t}}{\varepsilon} + \mathbf{D}\mathbf{y}$$

where $\mathbf{D}$ is a diagonal matrix.

For $i = 0, j_0 = n, n+1-j_0 = 1$, we have

$$\sqrt{g} = (1 - \varepsilon^2 z_n^2)^{\frac{n-2}{2}}(1 - \sum_{l=k+1}^{n-1} \frac{l-2}{2}\varepsilon^2 z_l^2 + O(\varepsilon^4)), \tag{6.21}$$

$$g_{mm}^{-1} = \begin{cases} 1-\varepsilon^2 z_n^2 \text{ if } m=n \\ \frac{1}{1-\varepsilon^2 z_n^2}(1-\varepsilon^2(z_m^2 - \sum_{j=m+1}^{n-1} z_j^2) + O(\varepsilon^4)) \text{ if } m<n \end{cases} \tag{6.22}$$

and

$$\frac{\partial \ln \sqrt{g}}{\partial z_m} = -\frac{(m-2)\varepsilon^2 z_m}{1-\varepsilon^2 z_m^2}, \quad \frac{\partial g_{mm}^{-1}}{\partial z_m} = \begin{cases} -2\varepsilon^2 z_n \text{ if } m=n \\ -\frac{2\varepsilon^2 z_m}{1-\varepsilon^2 z_n^2} + O(\varepsilon^4) \text{ if } m<n \end{cases} \tag{6.23}$$

Substituting (6.23) into (6.20) and after a lengthy computation, we have

$$\begin{aligned} &\mathcal{S}_\varepsilon[w_{\mathbf{t}^0}] \\ &= -2\varepsilon \frac{t_n^0}{\sqrt{1-(t_n^0)^2}} y_1 \frac{\partial^2 w}{\partial y_1^2} + 2\varepsilon \frac{t_n^0}{\sqrt{1-(t_n^0)^2}} \sum_{m<n} y_1 \frac{\partial^2 w}{\partial y_{n+1-m}^2} \\ &\quad -\varepsilon n \frac{t_n^0}{\sqrt{1-(t_n^0)^2}} \frac{\partial w}{\partial y_1} + O(\varepsilon^2 |\ln \varepsilon| (1+|y|^2) e^{-|y|}) \end{aligned} \tag{6.24}$$

On the other hand since β_ε depends on z_n only,

$$\begin{aligned} &\Delta' \beta_\varepsilon - \beta_\varepsilon \\ &= (1-\varepsilon^2 z_n^2)\beta_\varepsilon'' - n\varepsilon^2 z_n \beta_\varepsilon' - \beta_\varepsilon \\ &= \left[\frac{1-\varepsilon^2 z_n^2}{1-(t_n^0)^2} + n\varepsilon^2 z_n \frac{1}{\sqrt{1-(t_n^0)^2}} - 1 \right] \beta_\varepsilon = O(\varepsilon)\beta_\varepsilon \end{aligned}$$

and hence

$$\begin{aligned} &\mathcal{S}_\varepsilon[w_{\mathbf{t}_0} - \alpha_\varepsilon(\mathbf{t}_0)\beta_\varepsilon] = f(w - \alpha_\varepsilon(\mathbf{t}_0)\beta_\varepsilon) - f(w) \\ &-2\varepsilon \frac{t_n^0}{\sqrt{1-(t_n^0)^2}} y_1 \frac{\partial^2 w}{\partial y_1^2} + 2\varepsilon \frac{t_n^0}{\sqrt{1-(t_n^0)^2}} \sum_{m<n} y_1 \frac{\partial^2 w}{\partial y_{n+1-m}^2} \\ &-\varepsilon \frac{n t_n^0}{\sqrt{1-(t_n^0)^2}} \frac{\partial w}{\partial y_1} + O(\varepsilon^{3/2}(1+|y|^2)e^{-|y|}) \\ &= -\alpha_\varepsilon(\mathbf{t}^0) e^{-\frac{R+t_n^0}{\varepsilon\sqrt{1-(t_n^0)^2}}} f'(w) e^{-y_1} - 2\varepsilon \frac{R}{\sqrt{1-R^2}} \frac{\partial^2 w}{\partial y_1^2} + 2\varepsilon \frac{R}{\sqrt{1-R^2}} \sum_{m<n} \frac{\partial^2 w}{\partial y_{n+1-m}^2} \\ &-\varepsilon \frac{n t_n^0}{\sqrt{1-R^2}} \frac{\partial w}{\partial y_1} + O(\varepsilon^{1+\frac{\sigma}{2}}) \\ &= -A_n (1-R^2)^{\frac{k+3-n}{4}} \left(\frac{\varepsilon}{R+t_n^0}\right)^{\frac{n-k-1}{2}} e^{-\frac{2(R+t_n^0)}{\sqrt{1-R^2}}} f'(w) e^{-y_1} - 2\varepsilon \frac{R}{\sqrt{1-R^2}} \frac{\partial^2 w}{\partial y_1^2} + 2\varepsilon \frac{R}{\sqrt{1-R^2}} \sum_{m<n} \frac{\partial^2 w}{\partial y_{n+1-m}^2} \\ &-\varepsilon \frac{nR}{\sqrt{1-R^2}} \frac{\partial w}{\partial y_1} + O(\varepsilon^{1+\frac{\sigma}{2}}) \end{aligned}$$

Since

$$(\frac{\varepsilon}{R+t_n^0})^{\frac{n-k-1}{2}}e^{-\frac{2(R+t_n^0)}{\sqrt{1-R^2}}}$$

$$=(\frac{\varepsilon}{R+\bar{t}_\varepsilon^0})^{\frac{n-k-1}{2}}(1+O(\frac{\varepsilon^{\tau_0}}{|\ln\varepsilon|}\hat{t}_0)e^{-\frac{2(R+\bar{t}_\varepsilon^0)}{\sqrt{1-R^2}}}e^{-\frac{2\varepsilon^{\tau_0}\hat{t}_0}{\sqrt{1-R^2}}}=\varepsilon e^{-2c_1}(1+O(\frac{\varepsilon^{\tau_0}}{|\ln\varepsilon|}\hat{t}_0)e^{-\frac{2\varepsilon^{\tau_0}\hat{t}_0}{\sqrt{1-R^2}}},$$

we deduce that (using the equation (3.9))

(6.25)
$$\mathcal{S}_\varepsilon[W_0]=\mathcal{S}_\varepsilon[w_{\mathbf{t}_0}-\alpha_\varepsilon(\mathbf{t}^0)\beta_\varepsilon-\varepsilon\hat{\Psi}_0]=-\frac{2A_n+O(\frac{1}{|\ln\varepsilon|})}{\sqrt{1-R^2}}\varepsilon^{1+\tau_0}\hat{t}_0e^{-2c_1}f^{'}(w)e^{-y_1}+O(\varepsilon^{1+\frac{\sigma}{2}})$$

which is just (3.28).

Next we consider the case $i=1,...,n$. In this case, we have that

$$g_{mm}^{-1}=1-\varepsilon^2(z_m^2-\sum_{j=m+1}^n z_j^2)+O(\varepsilon^4|\mathbf{z}|^4),\ \sqrt{g}=1-\sum_{j=k+1}^n\frac{j-2}{2}\varepsilon^2z_j^2+O(\varepsilon^4|\mathbf{z}|^4) \tag{6.26}$$

Hence

$$\Delta^{'}w_{\mathbf{t}^i}=\Delta w+\sum_{m=k+1}^n[\frac{g_{mm}(\mathbf{t})}{g_{mm}(\mathbf{t}^i+\varepsilon\mathbf{D}\mathbf{y})}-1]\frac{\partial^2 w}{\partial y_{n+1-m}^2}$$

$$-2\varepsilon^2\sum_{m=k+1}^n z_m(g_{mm}(\mathbf{t}^i))^{1/2}\frac{\partial w}{\partial y_{n+1-m}}$$

$$+\sum_{m=k+1}^n\frac{1}{\sqrt{g(\mathbf{t}^i+\varepsilon\mathbf{D}\mathbf{y})}}(-(m-2)\varepsilon^2z_m)\frac{\partial w}{\partial y_{n+1-m}}+O(\varepsilon^3(1+|y|^3)e^{-|y|})$$

$$=\Delta w-2\varepsilon t_{j_0}^i y_{n+1-j_0}\frac{\partial^2 w}{\partial y_{n+1-j_0}^2}+2\varepsilon t_{j_0}^i\sum_{m<j_0}y_{n+1-j_0}\frac{\partial^2 w}{\partial y_{n+1-m}^2} \tag{6.27}$$

$$-\varepsilon j_0 t_{j_0}^i\frac{\partial w}{\partial y_{n+1-j_0}}+\varepsilon^2E^i(y)+O(\varepsilon^3(1+|y|^3)e^{-|y|})$$

where $E^i(y)$ is a bounded and integrable function which is even in $y_j, j=1,...,n-k$. (The last term is harmless since it is even in y_j.)

Hence from (6.27) we have

$$\mathcal{S}_\varepsilon[W_i]=-2\varepsilon t_{j_0}^i\frac{\partial^2 w}{\partial y_{n+1-j_0}^2}+2\varepsilon t_{j_0}^i\sum_{m<j_0}\frac{\partial^2 w}{\partial y_{n+1-m}^2} \tag{6.28}$$

$$-\varepsilon j_0 t_{j_0}^i(g_{j_0j_0}(\mathbf{t}^i))^{-1/2}\frac{\partial w}{\partial y_{n+1-j_0}}+\varepsilon^2E^i(y)+O(\varepsilon^3|\ln\varepsilon|(1+|y|^3)e^{-|y|})=O((\varepsilon^2|\ln\varepsilon|e^{-|y|}).$$

On the other hand, the interaction terms can be estimated as follows: for $\varepsilon z_j = t^i_j + \varepsilon(g_{jj}(\mathbf{t}^i))^{-1/2} y_{n+1-j}, |l-i| \geq 2, g_{jj}(\mathbf{t}^i) = 1 + O(\varepsilon^2 |\ln \varepsilon|)$

$$w_{\mathbf{t}^l}(z) = O(e^{-\frac{|\mathbf{t}^i - \mathbf{t}^l|}{\varepsilon}}) = O(\varepsilon^4 |\ln \varepsilon|^4)$$

Therefore

$$f(\sum_{i=1}^N w_{\mathbf{t}^i}) - \sum_{i=1}^N f(w_{t_i}) = f'(w(y))(w_{\mathbf{t}^{i-1}} + w_{\mathbf{t}^{i+1}}) + O(\varepsilon^{2+\sigma}). \tag{6.29}$$

Combining (6.28) and (6.29), we obtain (3.29). (3.30) follows from (3.28) and (3.29). □

7. Emden equations of $\mathbf{S}^n$

Our interest in the existence of positive nonradial solutions grew out from the study of the problem

$$\Delta_{\mathbf{S}^n} v - \lambda v + v^p = 0, \quad v > 0 \text{ in } D \subset \mathbf{S}^n, v = 0 \text{ on } \partial D, \ \lambda > 0. \tag{7.1}$$

in particular when $p = \frac{n+2}{n-2}$ is the critical exponent. As in the previous sections we shall assume that D is a geodesic ball, centered at the North pole with geodesic radius θ_n^*. As already mentioned in the Introduction if $\theta_n^* < \pi/2$ all positive solutions are radial. We therefore assume that $\theta_n^* > \pi/2$. If we set $u = \lambda^{-\frac{1}{p-1}} v$ and $\varepsilon^{-2} = \lambda$ then u satisfies problem (1.1).

In [6] it was shown that in contrast to the corresponding problem in the Euclidean space (7.1) possesses for arbitrary $p > 1$ and large λ radial solutions. There are three types of solutions as in Theorem (1.2) with k replaced by $n-1$. As λ increases there are more and more solutions with an increasing number of spikes on $n-1$- dimensional spheres near the equator. These solutions have been observed numerically in [36]. From Theorem (1.2) we obtain also the existence of nonradial solutions for (7.1) or equivalently to (1.1). yields

Theorem 7.1. *Let $1 < p < p_k$. Then the statements of Theorem (1.2) remain valid for $u = \lambda^{\frac{1}{p-1}} v$, v being a positive solution of (7.1). In particular we have $\lambda^{-\frac{1}{p-1}} v(\theta) \to c_0 > 0$ as $\theta \to \Sigma_i$, $i = 1, 2, 3$ where $\theta = (\theta_n, \cdots, \theta_2, \varphi)$.*

Proof. We only have to show that the assumptions **(f1), (f2)** are satisfied for

$$\Delta w - w + w^p = 0 \text{ in } \mathbb{R}^{n-k}.$$

The existence of a ground state was proved in [37]. The radial symmetry follows from [16], see also the remark in [35]. The uniqueness has been proved in [22]. □

Theorem (1.1) follows from the above result.

References

[1] A.Ambrosetti, A. Malchiodi and W.-M. Ni, Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres, Part I, *Comm. Math. Phys.* 235 (2003), 427-466.

[2] A.Ambrosetti, A. Malchiodi and W.-M. Ni, Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres, Part II, *Indiana Univ. Math. J.* , 53 (2004), no.2, 297-329.

[3] C. Bandle and R. Benguria, The Brezis-Nirenberg problem on S^3, *J. Diff. Eqns.* 178 (2002), 264-279.

[4] C. Bandle and L.A. Peletier, Best constants and Emden equations for the critical exponent in S^3, *Math. Ann.* 313 (1999), 83-93.

[5] C. Bandle, A. Brillard, M. Flucher, Green's function, harmonic transplantation and best Sobolev constant in spaces of constant curvature,*Trans. Amer. Math. Soc.* 350 (1998), 1103-1128

[6] C. Bandle and J. Wei, Multiple Clustered Layer Solutions for Semilinear Elliptic Problems on $\mathbf{S}^n$, submitted.

[7] C. Bandle and J. Wei, Solutions with an interior bubble and clustered layers for elliptic equations with critical exponents on spherical caps of $\mathcal{S}^3$, in preparation.

[8] H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.* 36(1983), 437-477.

[9] H. Brezis and L. A. Peletier, Elliptic equations with critical exponent on spherical caps of $\mathbf{S^3}$, *J'Analyse Math.* 2006, to appear.

[10] F. Brock and J. Prajapat, Some new symmetry results for elliptic problems on the sphere and the Euclidean space, *Rend. Circ. Mat. Palermo* (2), 49 (2000), 445-462.

[11] C.J. Budd, M.C. Knaap and L.A. Peletier, Asymptotic behavior of solutions of elliptic equations with critical exponents and Neumann boundary data, *Proc. Roy. Soc. Edinb.* 117**A**(1991), 225-250.

[12] W. Chen and J. Wei, On the Brezis-Nirenber problem on $\mathbf{S}^3$ and a conjecture of Bandle-Benguria, *C.R. Math.Acad. Sci. Paris* 341(2005), no. 3, 153-156.

[13] M. del Pino, P. Felmer and J. Wei, On the role of distance function in some singularly perturbed problems, *Comm. PDE* 25(2000), 155-177.

[14] E.N. Dancer and S. Yan, Multipeak solutions for a singular perturbed Neumann problem, *Pacific J. Math.* 189(1999), 241-262.

[15] M. del Pino, P. Felmer and J. Wei, Mutiple peak solutions for some singular perturbation problems, *Cal. Var. PDE* 10(2000), 119-134.

[16] B. Gidas, W,-M. Ni and L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in* $\mathbb{R}^n$ Math. Anal. and Appl., part A, Advances in Math. Suppl. Studies 7 A, Academic Press (1981), 369-402.

[17] R. Gardner and L. A. Peletier, The set of positive solutions of semilinear equations in large balls, *Proc. R. Soc. Edinb., Sect. A*, 104 (1986), 53-72.

[18] C. Gui and J. Wei, Multiple interior spike solutions for some singular perturbed Neumann problems, *J. Diff. Eqns.* 158(1999), 1-27.

[19] C. Gui and J. Wei, On multiple mixed interior and boundary peak solutions for some singularly perturbed Neumann problems, *Can. J. Math.* 52(2000), 522-538.

[20] C. Gui, J. Wei and M. Winter, Multiple boundary peak solutions for some singularly perturbed Neumann problems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 17 (2000), 249-289.

[21] S. Kumaresan and J. Prajapat, Serrin's result for hyperbolic space and sphere, *Duke Math. J.* 91 (1998), 17-28.
[22] M. K. Kwong Uniqueness of positive radial solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$, *Arch. Rat. Mech. Anal.*105(1989), 243-266.
[23] Y.-Y. Li, On a singularly perturbed equation with Neumann boundary condition, *Comm. P.D.E.* 23(1998), 487-545.
[24] Y.-Y. Li and L. Nirenberg, The Dirichlet problem for singularly perturbed elliptic equations, *Comm. Pure Appl. Math.* 51 (1998), 1445-1490.
[25] F.-H. Lin, W.-M. Ni and J.-C. Wei, On the number of interior peak solutions for a singularly perturbed Neumann problem, *Comm. Pure Appl. Math.*, to appear.
[26] A.Malchiodi, W.-M. Ni and J.-C. Wei, Multiple clustered layer solutions for semilinear Neumann problems on a ball, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 22(2005),no. 2, 143-163.
[27] A.Malchiodi, W.-M. Ni and J.-C. Wei, Boundary clustered interfaces for the Allen-Cahn equation, submitted.
[28] K. McLeod, Uniqueness of positive radial solutions of $\Delta u + f(u) = 0$ in $\mathbb{R}^n$.II, *Trans. Amer. Math. Soc.* 339 (1993), 495-505.
[29] W.-M. Ni, *Qualitative properties of solutions to elliptic problems.* Stationary partial differential equations. Vol. I, 157–233, Handb. Differ. Equ., North-Holland, Amsterdam, 2004.
[30] W.-M. Ni and J. Wei, On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems, *Comm. Pure Appl. Math.* 48 (1995), 731-768.
[31] P. Padilla, Symmetry properties of positive solutions of elliptic equations on symmetric domains, *J. Appl. Anal.* 64 (1997), 153-169.
[32] M. Struwe, "Variational Methods and Applications to Nonlinear Partial Differential Eqtaions and Hamiltonian Systems", Springer-Verlag, Berlin, 1990.
[33] Ni W.-M., Takagi I., Locating the peaks of least energy solutions to a semilinear Neumann problem, *Duke Math. J.* 70 (1993), 247-281.
[34] P. Padilla, Symmetry properties of positive solutions of elliptic equations on symmetric domains, *Appl. Anal.* 64(1997), 153-169.
[35] , A. Simon and P. Volkmann, Remarks on a theorem of Gidas, Ni and Nirenberg, *Ren. Circ. Mat. Palermo* (2) 45 (1996), 116-118.
[36] S.I. Stingelin, Das Brezis-Nirenberg auf der Sphare $\mathbf{S}^N$, Inauguraldissertation, Univeritat Basel, 2004.
[37] W. A. Strauss, Existence and solitary waves in higher dimensions, *Comm. Math. Physics* 55 (1977), 149-162.
[38] J. Wei, On the construction of single-peaked solutions to a singularly perturbed semilinear Dirichlet problem, *J. Diff. Eqns.* 129 (1996), 315-333.

Mathematisches Institut, Universitat Basel, Rheinsprung 21, CH-4051 Basel, Switzerland
E-mail address: catherine.bandle@math.unibas.ch

Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong
E-mail address: wei@math.cuhk.edu.hk