Blowing up Solutions for an Elliptic Neumann Problem with Sub- or Supercritical Nonlinearity. Part II: $N \geq 4$

Olivier REY[*] and Juncheng WEI[†]

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Abstract

We consider the sub- or supercritical Neumann elliptic problem $-\Delta u + \mu u = u^{\frac{N+2}{N-2}+\varepsilon}$, $u > 0$ in Ω; $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$, Ω being a smooth bounded domain in $\mathbb{R}^N$, $N \geq 4, \mu > 0$ and $\varepsilon \neq 0$ a small number. We show that for $\varepsilon > 0, N \geq 4$, there always exists a solution to the slightly supercritical problem, which blows up at the most curved part of the boundary as ε goes to zero. On the other hand, for $\varepsilon < 0, N \geq 4$, assuming that the domain is not convex, there also exists a solution to the slightly subcritical problem, which blows up at the least curved part of the domain.

1 Introduction

In this paper we consider the nonlinear Neumann elliptic problem

$$(P_{q,\mu}) \qquad \begin{cases} -\Delta u + \mu u = u^q \quad u > 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

[*]Centre de Mathématiques de l'Ecole Polytechnique, 91128 Palaiseau Cedex, France. E-mail: rey@math.polytechnique.fr

[†]Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong. E-mail : wei@math.cuhk.edu.hk. Research supported in part by an Earmarked Grant from RGC of HK.

where $1 < q < +\infty$, $\mu > 0$ and Ω is a smooth and bounded domain in $\mathbb{R}^N, N \geq 4$.

Equation $(P_{q,\mu})$ arises in many branches of the applied sciences. For example, it can be viewed as a steady-state equation for the shadow system of the Gierer-Meinhardt system in biological pattern formation ([15], [29]) or of parabolic equations in chemotaxis, e.g. Keller-Segel model ([26]).

When q is subcritical, i.e. $q < \frac{N+2}{N-2}$, Lin, Ni and Takagi proved that the only solution, for small μ, is the constant one, whereas nonconstant solutions appear for large μ [26] which blow up, as μ goes to infinity, at one or several points. The least energy solution blows up at a boundary point which maximizes the mean curvature of the frontier [31][32]. Higher energy solutions exist which blow up at one or several points, located on the boundary [8][14][24][37][20], in the interior of the domain [5][7][12][13][17][22][43][45],

or some of them on the boundary and others in the interior [19]. (A good review can be found in [29].) In the critical case, i.e. $q = 5$, Zhu [46] proved that, for convex domains, the only solution is the constant one for small μ (see also [44]). For large μ, nonconstant solutions exist [1][38]. As in the subcritical case the least energy solution blows up, as μ goes to infinity, at a unique point which maximizes the mean curvature of the boundary [3][30]. Higher energy solutions have also been exhibited, blowing up at one [2][39][34][16] or several boundary points [28][40][41][18]. The question of interior blow-up is still open. However, in contrast with the subcritical situation, at least one blow-up point has to lie on the boundary [35].

Very few is known about the supercritical case, save the uniqueness of the radial solution on a ball for small μ [25]. In [29], Ni raised the following conjecture.

Conjecture: For any exponent $q > 1$, and μ large, there always exists a *nonconstant* solution to $(P_{q,\mu})$.

Our aim, in this paper, is to continue our study ([36]) on the problem for fixed μ, when the exponent q is close to the critical one, i.e. $q = \frac{N+2}{N-2} + \varepsilon$ and ε is a small nonzero number. Whereas the previous results, concerned with peaked solutions, always assume that μ goes to infinity, we are going to prove that a single interior or boundary peak solution may exist for finite μ, provided that q is close enough to the critical exponent. In [36], we showed that for $N = 3$, a single interior bubble solution exists for finite μ, as $\varepsilon \to 0$. In this paper, we establish the existence of a single boundary bubble for *any* finite μ and for any smooth bounded domain $\Omega \subset \mathbb{R}^N, N \geq 4$, provided that $\varepsilon > 0$ is sufficiently small.

Let $H(a)$ denote the boundary mean curvature function at $a \in \partial\Omega$. The following result partially answers Ni's conjecture:

Theorem 1.1 *Suppose that $N \geq 4$. Then $(P_{\frac{N+2}{N-2}+\varepsilon,\mu})$ has a nontrivial solution, for $\varepsilon > 0$ close enough to zero, which blows up as ε goes to zero at a point $a \in \partial\Omega$, such that $H(a) = \max_{P \in \partial\Omega} H(P)$.*

In the case of $\varepsilon < 0$, i.e. slightly subcritical case, we then have the following theorem.

Theorem 1.2 *Assume that $N \geq 4$ and Ω is not convex. Then $(P_{\frac{N+2}{N-2}+\varepsilon,\mu})$ has a nontrivial solution, for $\varepsilon < 0$ close enough to zero, which blows up as ε goes to zero at a point $a \in \partial\Omega$, such that $H(a) = \min_{P \in \partial\Omega} H(P)$.*

Remark. Theorem 1.2 agrees with the following result of Gui and Lin: in [16], it is proved that if there exists a sequence of single boundary blowing up solutions u_{ε_i} to $P_{\frac{N+2}{N-2}+\varepsilon_i,\mu}$ with $\varepsilon_i \leq 0$, then necessarily, u_{ε_i} blows up at a boundary point $a \in \partial\Omega$ such that $H(a) \leq 0$ and a is a critical point of H. Here we have established a partial converse to [16].

A similar slightly supercritical Dirichlet problem

$$(Q_\varepsilon) \qquad \begin{cases} -\Delta u &= u^{\frac{N+2}{N-2}+\varepsilon^2} \quad u > 0 \qquad \text{in } \Omega \\ \quad u &= 0 \qquad\qquad\qquad\quad \text{on } \partial\Omega \end{cases}$$

has been studied in [9], where the existence of solutions with two bubbles in domains with a small hole is established, provided that ε is small. It is interesting to note that, here, and also in [36], we have no condition on the domain, in the slightly supercritical Neumann case.

The scheme of the proof is similar to [36] (see also [9]). However, we use a different framework - i.e. weighted Sobolev spaces - to treat the case of $N \geq 5$. In the next section, we define a two-parameters set of approximate solutions to the problem, and we look for a true solution in a neighborhood of this set. Considering in Section 3 the linearized problem at an approximate solution, and inverting it in suitable functional spaces, the problem reduces to a finite dimensional one, which is solved in Section 4. Some useful facts and computations are collected in Appendix.

2 Some Preliminaries

2.1 Approximate solutions and rescaling

For sake of simplicity, we consider in the following the supercritical case, i.e. we assume that $\varepsilon > 0$. The subcritical case may be treated exactly in the same way. For normalization reasons, we consider throughout the paper the equation

$$-\Delta u + \mu u = \alpha_N u^{\frac{N+2}{N-2}+\varepsilon}, \quad u > 0 \tag{2.1}$$

instead of the original one, where $\alpha_N = N(N-2)$. The solutions are identical, up to the multiplicative constant $(\alpha_N)^{-\frac{N-2}{4+(N-2)\varepsilon}}$. We recall that, according to [6], the functions

$$U_{\lambda,a}(x) = \frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda^2|x-a|^2)^{\frac{N-2}{2}}} \qquad \lambda > 0\,,\ a \in \mathbb{R}^N \tag{2.2}$$

are the only solutions to the problem

$$-\Delta u = \alpha_N u^{\frac{N+2}{N-2}}, \quad u > 0 \qquad \text{in} \quad \mathbb{R}^N.$$

As $a \in \partial\Omega$ and λ goes to infinity, these functions provide us with approximate solutions to the problem that we are interested in. However, in view of the additional linear term μu which occurs in $(P_{\frac{N+2}{N-2}+\varepsilon,\mu})$, the approximation needs to be improved.

Integral estimates (see Appendix) suggest to make the additional *a priori* assumption that λ behaves as $1/\varepsilon$ as ε goes to zero. Namely, we set

$$\lambda = \frac{1}{\Lambda\varepsilon} \qquad\qquad \frac{1}{\delta'} < \Lambda < \delta' \tag{2.3}$$

with δ' some strictly positive number. Now, fix $a \in \partial\Omega$. We define $V_{\Lambda,a,\mu,\varepsilon} = V$ satisfying

$$\begin{cases} -\Delta V + \mu V &= \alpha_N U_{\frac{1}{\Lambda\varepsilon},a}^{\frac{N+2}{N-2}} \quad \text{in} \ \Omega \\ \frac{\partial V}{\partial n} &= 0 \qquad\qquad\qquad \text{on} \ \ \partial\Omega. \end{cases} \tag{2.4}$$

The $V_{\Lambda,a,\mu,\varepsilon}$'s are the suitable approximate solutions in the neighborhood of which we shall find a true solution to the problem. In order to make further computations easier, we proceed to a rescaling. We set

$$\Omega_\varepsilon = \frac{\Omega}{\varepsilon}$$

and define in Ω_ε the functions

$$W_{\Lambda,\xi,\mu,\varepsilon}(x)=\varepsilon^{\frac{N-2}{2}}V_{\Lambda,a,\mu,\varepsilon}(\varepsilon x) \qquad\qquad \xi=\frac{a}{\varepsilon}. \tag{2.5}$$

Then, $W_{\Lambda,\xi,\mu,\varepsilon}=W$ satisfies the uniform estimate (see Lemma 5.1)

$$|W_{\Lambda,\xi,\mu,\varepsilon}|\le CU_{\frac{1}{\Lambda},\xi} \quad \text{in} \ \ \Omega_\varepsilon. \tag{2.6}$$

Moreover, we have

$$\begin{cases} -\Delta W+\mu\varepsilon^2 W & =\alpha_N U_{\frac{1}{\Lambda},\xi}^{\frac{N+2}{N-2}} \quad \text{in} \ \ \Omega_\varepsilon \\ \qquad\quad \frac{\partial W}{\partial n} & =0 \qquad\qquad\quad \text{on} \ \ \partial\Omega_\varepsilon \end{cases} \tag{2.7}$$

and, since $U_{\frac{1}{\Lambda},\xi}\ge C\epsilon^{N-2}$ and $\Delta W\ge 0$ at a minimum point of W in the closure of Ω

$$W\ge C\epsilon^N \quad \text{in} \ \ \bar{\Omega}. \tag{2.8}$$

Another fact that we shall use later is the following: observe that $\partial_\Lambda W$ satisfies

$$\begin{cases} -\Delta(\partial_\Lambda W)+\mu\varepsilon^2\partial_\Lambda W & =\alpha_N\partial_\Lambda(U_{\frac{1}{\Lambda},\xi}^{\frac{N+2}{N-2}}) \quad \text{in} \ \ \Omega_\varepsilon \\ \qquad\qquad \frac{\partial(\partial_\Lambda W)}{\partial n} & =0 \qquad\qquad\qquad\quad \text{on} \ \ \partial\Omega_\varepsilon. \end{cases}$$

Since $|\partial_\Lambda(U_{\frac{1}{\Lambda},\xi}^{\frac{N+2}{N-2}})|\le CU_{\frac{1}{\Lambda},\xi}^{\frac{N+2}{N-2}}$, by comparison principle we obtain

$$|\partial_\Lambda W|\le CW. \tag{2.9}$$

Finding a solution to $(P_{\frac{N+2}{N-2}+\varepsilon,\mu})$ in a neighbourhood of the functions $V_{\Lambda,a,\mu,\varepsilon}$ is equivalent, through the following rescaling

$$u(x)\to\varepsilon^{-\frac{2(N-2)}{4+(N-2)\varepsilon}}u(\frac{x}{\varepsilon})$$

to solving the problem

$$(P'_{\frac{N+2}{N-2}+\varepsilon,\mu}) \qquad \begin{cases} -\Delta u+\mu\varepsilon^2 u & =\alpha_N u^{\frac{N+2}{N-2}+\varepsilon} \quad u>0 \qquad \text{in} \ \ \Omega_\varepsilon \\ \qquad\quad \frac{\partial u}{\partial n} & =0 \qquad\qquad\qquad\qquad\quad \text{on} \ \ \partial\Omega_\varepsilon \end{cases} \tag{2.10}$$

in a neighbourhood of the functions $W_{\Lambda,\xi,\mu,\varepsilon}$. (From now on, we shall work with $(P'_{\frac{N+2}{N-2}+\varepsilon,\mu})$.) For that purpose, we have to use some local inversion procedure. Namely, we are going to look for a solution to $(P'_{\varepsilon,\mu})$ writing as

$$w=W_{\Lambda,\xi,\mu,\varepsilon}+\omega$$

with ω small and orthogonal at $W_{\Lambda,\xi,\mu,\varepsilon}$, in a suitable sense, to the manifold

$$M = \left\{ W_{\Lambda,\xi,\mu,\varepsilon},\ \Lambda \text{ satisfying (2.3) }, \xi \in \partial\Omega_\varepsilon \right\}.$$

The general strategy consists in finding first, using an inversion procedure, a smooth map $(\Lambda, \xi) \mapsto \omega(\Lambda, \xi)$ such that $W_{\Lambda,\xi,\mu,\varepsilon} + \omega(\Lambda, \xi, \mu, \varepsilon)$ solves the problem in an orthogonal space to M. Then, we are left with a finite dimensional problem, for which a solution may be found using the assumptions of the theorems. In the subcritical or critical case, the first step may be performed in H^1 (see e.g. [4][33][34]). However, this approach is not valid any more in the supercritical case, for H^1 does not inject into L^q as $q > \frac{2N}{N-2}$. In [9], a weighted Hölder space approach was used. In the present paper, we use weighted Sobolev spaces to reduce the problem to a finite dimensional one.

2.2 Boundary Deformations

Fix $a \in \partial\Omega$. We introduce a boundary deformation which strengthens the boundary near a. Without loss of generality, we may assume that $a = 0$ and after rotation and translation of the coordinate system we may assume that the inward normal to $\partial\Omega$ at a is the direction of the positive x_N-axis. Denote $x' = (x_1, \dots, x_{N-1})$, $B'(\delta) = \{x' \in \mathbb{R}^{N-1} : |x'| < \delta\}$, and $\Omega_1 = \Omega \cap B(a, \delta)$, where $B(a, \delta) = \{x \in \mathbb{R}^N : \quad |x - a| < \delta\}$.

Then, since $\partial\Omega$ is smooth, we can find a constant $\delta > 0$ such that $\partial\Omega \cap B(a, \delta)$ can be represented by the graph of a smooth function $\rho_a : B'(\delta) \to R$, where $\rho_a(0) = 0$, $\nabla\rho_a(0) = 0$, and

$$\Omega \cap B(a, \delta) = \{(x', x_N) \in B(a, \delta) \ : \ x_N > \rho_a(x')\}. \tag{2.11}$$

Moreover, we may write

$$\rho_a(x') = \frac{1}{2}\sum_{i=1}^{N-1} k_i x_i^2 + O(|x|^3) \tag{2.12}$$

Here $k_i, i = 1, ..., N-1$, are the principal curvatures at a. Furthermore, the average of the principal curvatures of $\partial\Omega$ at a is the mean curvature $H(a) = \frac{1}{N-1}\sum_{i=1}^{N-1} k_i$. To avoid clumsy notations, we drop the index a in ρ.

On $\partial\Omega \cap B(a, \delta)$, the normal derivative $n(x)$ writes as

$$n(x) = \frac{1}{\sqrt{1 + |\nabla'\rho|^2}}(\nabla'\rho, -1) \tag{2.13}$$

and the tangential derivatives are given by

$$\frac{\partial}{\partial \tau_{i,x}} = \frac{1}{\sqrt{1+|\frac{\partial \rho}{\partial x_i}|^2}}(0,...,1,....,\frac{\partial \rho}{\partial x_i}) \qquad i=1,...,N-1. \tag{2.14}$$

When there is no confusion, we also drop the dependence of $\frac{\partial}{\partial \tau_{i,x}}$ on x.

2.3 Expansion of V and W

In appendix (Lemma 5.1), we derive the following asymptotic expansion of V: For $N \geq 4$, we have the expansion

$$V = U_{\frac{1}{\Lambda\varepsilon},a} - (\Lambda\varepsilon)^{\frac{4-N}{2}}\varphi_0(\frac{x-a}{\Lambda\varepsilon}) + O(\varepsilon^{\frac{6-N}{2}|\ln\varepsilon|^m}) \tag{2.15}$$

where φ_0 solves some linear problem and $m=1$ for $N=4$ and $m=0$ for $N \geq 5$. Let τ be a small fixed number. (We may choose $\tau = \frac{1}{100}$.) This then implies that

$$W = U_{\frac{1}{\Lambda},\xi}(x) - \hat{\varphi}(x) \tag{2.16}$$

where

$$\hat{\varphi}(x) = \varepsilon(\Lambda)^{\frac{4-N}{2}}\varphi_0(\frac{x-\xi}{\Lambda}) + O(\varepsilon^{2-\tau}). \tag{2.17}$$

Furthermore, we have the following upper bound

$$|\hat{\varphi}(x)| \leq \frac{C\varepsilon|\ln\varepsilon|^n}{(1+|x-\xi|)^{N-3}},\ x \in \Omega_\varepsilon, \tag{2.18}$$

where $n=1$ for $N=4,5$ and $n=0$ for $N \neq 6$.

3 The finite dimensional reduction

3.1 Inversion of the linearized problem

We first consider the linearized problem at a function $W_{\Lambda,\xi,\mu,\varepsilon}$, and we invert it in an orthogonal space to M. From now on, we omit for sake of simplicity the indices in the writing of $W_{\Lambda,\xi,\mu,\varepsilon}$. Equipping $H^1(\Omega_\varepsilon)$ with the scalar product

$$(u,v)_\varepsilon = \int_{\Omega_\varepsilon}(\nabla u.\nabla v + \mu\varepsilon^2 uv)$$

orthogonality to the functions

$$Y_0 = \frac{\partial W}{\partial \Lambda} \qquad Y_i = \frac{\partial W}{\partial \tau_i} \qquad 1 \leq i \leq N-1 \tag{3.1}$$

in that space is equivalent, setting

$$Z_0 = -\Delta \frac{\partial W}{\partial \Lambda} + \mu\varepsilon^2 \frac{\partial W}{\partial \Lambda} \qquad Z_i = -\Delta \frac{\partial W}{\partial \tau_i} + \mu\varepsilon^2 \frac{\partial W}{\partial \tau_i} \qquad 1 \leq i \leq N-1 \tag{3.2}$$

to the orthogonality in $L^2(\Omega_\varepsilon)$, equipped with the usual scalar product $\langle \cdot, \cdot \rangle$, to the functions Z_i, $0 \leq i \leq N-1$. Then, we consider the following problem : h being given, find a function ϕ which satisfies

$$\begin{cases} -\Delta\phi + \mu\varepsilon^2\phi - \alpha_N(\frac{N+2}{N-2} + \varepsilon) W^{\frac{4}{N-2}+\varepsilon}\phi = h + \sum_i c_i Z_i & \text{in} \quad \Omega_\varepsilon \\ \frac{\partial\phi}{\partial n} = 0 & \text{on} \quad \partial\Omega_\varepsilon \\ \langle Z_i, \phi \rangle = 0 & 0 \leq i \leq N-1 \end{cases} \tag{3.3}$$

for some numbers c_i.

Existence and uniqueness of ϕ will follow from an inversion procedure in suitable functional spaces. For $N = 3$, the weighted Hölder spaces in [9] or [36] work well. For $N \geq 4$, we use a weighted Sobolev approach which seems more suitable in treating the large dimensions case. (Special attention is needed for the case of $N = 4$.) Similar approach has been used in [42] in dealing with slightly supercritical exponent problem.

Let $\mathcal{U}$ be an open set in $\mathbb{R}^N$ and $\xi \in \mathcal{U}$. For $1 < t < +\infty$, a nonnegative integer l, and a real number β, we define a weighted Sobolev norm

$$\|\phi\|_{W^{l,t}_\beta(\mathcal{U})} = \sum_{|\alpha|=0}^{l} \|\langle x - \xi \rangle^{\beta + |\alpha|} \partial^\alpha \phi\|_{L^t(\mathcal{U})}$$

where $\langle x - \xi \rangle = (1 + |x - \xi|^2)^{\frac{1}{2}}$. When $l = 0$, we denote $W^{0,t}_\beta(\mathcal{U})$ as $L^t_\beta(\mathcal{U})$.

Let f be a function in Ω_ε. We define the following two weighted Sobolev norms

$$\|f\|_* = \|f\|_{W^{2,t}_\beta(\Omega_\varepsilon)}$$

and

$$\|f\|_{**} = \|f\|_{L^t_{\beta+2}(\Omega_\varepsilon)}.$$

We choose t and β such that

$$N < t < +\infty \qquad \frac{N-2}{2} + \frac{N(N-2)}{4t} < \beta < \frac{N}{t'} - 2 \tag{3.4}$$

where $t^{'}$ is the conjugate exponent of t, i.e., $\frac{1}{t} + \frac{1}{t'} = 1$. (It is easily checked that such a choice of t and β is always possible.) Since $t > N$, by Sobolev embedding theorem, we have

$$|\nabla \phi(x)| + |\phi(x)| \leq C \langle x - \xi \rangle^{-\beta} \|\phi\|_*, \ \forall x \in \Omega_\varepsilon. \tag{3.5}$$

We recall the following result :

Lemma 3.1 *(Corollary 1 of [27].) The integral operator*

$$Tu(x) = \int_{\mathbb{R}^N} \frac{u(y)}{|x-y|^{N-2}} dy$$

is a bounded operator from $L^t_{\beta+2}(\mathbb{R}^N)$ to $L^t_\beta(\mathbb{R}^N)$, provided that $-\frac{N}{t} < \beta < \frac{N}{t'} - 2$.

We are also in need of the following lemma, whose proof is given in the appendix :

Lemma 3.2 *Let $f \in L^t_{\beta+2}(\Omega_\varepsilon)$ and u satisfy*

$$-\Delta u + \mu \varepsilon^2 u = f \quad in \ \Omega_\varepsilon, \ \frac{\partial u}{\partial n} = 0 \ on \ \partial\Omega_\varepsilon.$$

Then we have

$$|u(x)| \leq \int_{\Omega_\varepsilon} \frac{C}{|x-y|^{N-2}} |f(y)| dy \tag{3.6}$$

and

$$\|u\|_* \leq C \|f\|_{**}. \tag{3.7}$$

The main result of this subsection is :

Proposition 3.1 *There exists $\varepsilon_0 > 0$ and a constant $C > 0$, independent of ε and ξ, Λ satisfying (2.3), such that for all $0 < \varepsilon < \varepsilon_0$ and all $h \in L^t_{\beta+2}(\Omega_\varepsilon)$, problem* (3.3) *has a unique solution $\phi \equiv L_\varepsilon(h)$. Besides,*

$$\|L_\varepsilon(h)\|_* \leq C \|h\|_{**} \qquad |c_i| \leq C \|h\|_{**}. \tag{3.8}$$

Moreover, the map $L_\varepsilon(h)$ is C^1 with respect to Λ, ξ and the $W^{2,t}_\beta(\Omega_\varepsilon)$-norm, and

$$\|D_{(\Lambda,\xi)} L_\varepsilon(h)\|_* \leq C \|h\|_{**}. \tag{3.9}$$

PROOF. The argument follows closely the ideas in [9] and [36]. We repeat it since we use a different norm. The proof relies on the following result:

Lemma 3.3 *Assume that ϕ_ε solves (3.3) for $h = h_\varepsilon$. If $\|h_\varepsilon\|_{**}$ goes to zero as ε goes to zero, so does $\|\phi_\varepsilon\|_*$.*

PROOF OF LEMMA 3.3. Arguing by contradiction, we may assume that $\|\phi_\varepsilon\|_* = 1$. Multiplying the first equation in (3.3) by Y_j and integrating in Ω_ε we find

$$\sum_i c_i \langle Z_i, Y_j\rangle = \Big\langle -\Delta Y_j + \mu\varepsilon^2 Y_j - \alpha_N\Big(\frac{N+2}{N-2} + \varepsilon\Big) W^{\frac{4}{N-2}+\varepsilon} Y_j, \phi_\varepsilon \Big\rangle - \langle h_\varepsilon, Y_j\rangle.$$

On one hand we check, in view of the definition of Z_i, Y_j

$$\langle Z_0, Y_0\rangle = \|Y_0\|_\varepsilon^2 = c_0 + o(1) \qquad \langle Z_i, Y_i\rangle = \|Y_i\|_\varepsilon^2 = c_1 + o(1) \quad 1 \le i \le N-1 \tag{3.10}$$

where c_0, c_1 are strictly positive constants, and

$$\langle Z_i, Y_j\rangle = o(1) \qquad i \ne j. \tag{3.11}$$

On the other hand, in view of the definition of Y_j and W, straightforward computations yield

$$\Big\langle -\Delta Y_j + \mu\varepsilon^2 Y_j - \alpha_N\Big(\frac{N+2}{N-2} + \varepsilon\Big) W^{\frac{4}{N-2}+\varepsilon} Y_j, \phi_\varepsilon \Big\rangle = o(\|\phi_\varepsilon\|_*)$$

and

$$\langle h_\varepsilon, Y_j\rangle = O(\|h_\varepsilon\|_{**}).$$

Consequently, inverting the quasi diagonal linear system solved by the c_i's, we find

$$c_i = O(\|h_\varepsilon\|_{**}) + o(\|\phi_\varepsilon\|_*). \tag{3.12}$$

In particular, $c_i = o(1)$ as ε goes to zero.

Since $\|\phi_\varepsilon\|_* = 1$, elliptic theory shows that along some subsequence, $\tilde{\phi}_\varepsilon(x) = \phi_\varepsilon(x - \xi)$ converges uniformly in any compact subset of $\mathbb{R}^N_+$ to a nontrivial solution of

$$-\Delta\tilde{\phi} = \alpha_N \frac{N+2}{N-2} U_{\tilde{\Lambda},0}^{\frac{4}{N-2}} \tilde{\phi}$$

for some $\tilde{\Lambda} > 0$. Moreover, $|\tilde{\phi}(x)| \in L^t_\beta(\mathbb{R}^N)$. A bootstrap argument (see e.g. Proposition 2.2 of [42]) implies $|\tilde{\phi}(x)| \le C/|x|^{N-2}$. As a consequence, $\tilde{\phi}$ writes as

$$\tilde{\phi} = \alpha_0 \frac{\partial U_{\tilde{\Lambda},0}}{\partial \tilde{\Lambda}} + \sum_{i=1}^{N-1} \alpha_i \frac{\partial U_{\tilde{\Lambda},0}}{\partial a_i}$$

(see [33]). On the other hand, equalities $\langle Z_i, \phi_\varepsilon\rangle = 0$ provide us with the equalities

$$\begin{aligned}
&\int_{\mathbb{R}^N_+} -\Delta \frac{\partial U_{\tilde{\Lambda},0}}{\partial \tilde{\Lambda}} \tilde{\phi} = \int_{\mathbb{R}^N_+} U_{\tilde{\Lambda},0}^{\frac{4}{N-2}} \frac{\partial U_{\tilde{\Lambda},0}}{\partial \tilde{\Lambda}} \tilde{\phi} = 0 \\
&\int_{\mathbb{R}^N_+} -\Delta \frac{\partial U_{\tilde{\Lambda},0}}{\partial a_i} \tilde{\phi} = \int_{\mathbb{R}^N_+} U_{\tilde{\Lambda},0}^{\frac{4}{N-2}} \frac{\partial U_{\tilde{\Lambda},0}}{\partial a_i} \tilde{\phi} = 0 \qquad 1 \le i \le N-1.
\end{aligned}$$

As we have also

$$\int_{\mathbb{R}^N_+} |\nabla \frac{\partial U_{\tilde{\Lambda},0}}{\partial \tilde{\Lambda}}|^2 = c_0 > 0 \qquad\qquad \int_{\mathbb{R}^N_+} |\nabla \frac{\partial U_{\tilde{\Lambda},0}}{\partial a_i}|^2 = c_1 > 0 \quad 1 \le i \le N-1$$

and

$$\int_{\mathbb{R}^N_+} \nabla \frac{\partial U_{\tilde{\Lambda},0}}{\partial \tilde{\Lambda}} . \nabla \frac{\partial U_{\tilde{\Lambda},0}}{\partial a_i} = \int_{\mathbb{R}^N_+} \nabla \frac{\partial U_{\tilde{\Lambda},0}}{\partial a_j} . \nabla \frac{\partial U_{\tilde{\Lambda},0}}{\partial a_i} = 0 \quad i \ne j$$

the α_j's solve a homogeneous quasi diagonal linear system, yielding $\alpha_j = 0$, $0 \le \alpha_j \le N-1$, and $\tilde{\phi} = 0$. So $\phi_\varepsilon(x-\xi) \to 0$ in $C^1_{loc}(\Omega_\varepsilon)$. Now since

$$|\langle x-\xi\rangle^{\beta+2} W^{\frac{4}{N-2}+\varepsilon} \phi_\varepsilon|^t \le C\|\phi_\varepsilon\|_* \langle x-\xi\rangle^{-(2+(N-2)\varepsilon)t} \in L^1(\mathbb{R}^N),$$

by the Dominated Convergence Theorem we obtain

$$\int_{\Omega_\varepsilon} |\langle x-\xi\rangle^{\beta+2} W^{\frac{4}{N-2}+\varepsilon} \phi_\varepsilon|^t = o(1) \qquad \text{i.e.} \qquad \|W^{\frac{4}{N-2}+\varepsilon}\phi_\varepsilon\|_{**} = o(1).$$

On the other hand,

$$\langle x-\xi\rangle^{\beta+2} |Z_i| \le C\langle x-\xi\rangle^{\beta-N} \in L^t(\mathbb{R}^N),$$

applying Lemma 3.2 we obtain

$$\|\phi_\varepsilon\|_* \le C\|W^{\frac{4}{N-2}+\varepsilon}\phi_\varepsilon\|_{**} + C\|h_\varepsilon\|_{**} + C\sum_i |c_i| \|Z_i\|_{**} = o(1)$$

that is, a contradiction.

Proof of Proposition 3.1 completed. We set

$$H = \left\{ \phi \in H^1(\Omega_\varepsilon),\, \langle Z_i, \phi \rangle = 0 \quad 0 \le i \le N-1 \right\}$$

equipped with the scalar product $(\cdot,\cdot)_\varepsilon$. Problem (3.3) is equivalent to finding $\phi \in H$ such that

$$(\phi, \theta)_\varepsilon = \left\langle \alpha_N (\frac{N+2}{N-2} + \varepsilon) W^{\frac{4}{N-2}+\varepsilon} \phi + h \,,\, \theta \right\rangle \qquad \forall \theta \in H$$

that is

$$\phi = T_\varepsilon(\phi) + \tilde{h} \tag{3.13}$$

$\tilde{h}$ depending linearly on h, and T_ε being a compact operator in H. Fredholm's alternative ensures the existence of a unique solution, provided that the kernel of $Id - T_\varepsilon$ is reduced to 0. We notice that any $\phi_\varepsilon \in Ker(Id - T_\varepsilon)$ solves (3.3) with $h = 0$. Thus, we deduce from Lemma 3.3 that $\|\phi_\varepsilon\|_* = o(1)$ as ε goes to zero. As $Ker(Id - T_\varepsilon)$ is a vector space, $Ker(Id - T_\varepsilon) = \{0\}$. The inequalities (3.8) follow from Lemma 3.3 and (3.12). This completes the proof of the first part of Proposition 3.1.

The smoothness of L_ε with respect to Λ and ξ is a consequence of the smoothness of T_ε and $\tilde{h}$, which occur in the implicit definition (3.13) of $\phi \equiv L_\varepsilon(h)$, with respect to these variables. Inequalities (3.9) are obtained differentiating (3.3), writing the derivatives of ϕ with respect to Λ and ξ as a linear combination of the Z_i' and an orthogonal part, and estimating each term using the first part of the proposition - see [9] [21] for detailed computations. □

3.2 The reduction

Let

$$S_\varepsilon(u) = -\Delta u + \mu \varepsilon^2 u - \alpha_N u_+^{\frac{N+2}{N-2}+\varepsilon}$$

where $u_+ = \max(0, u)$. Then (2.10) is equivalent to

$$S_\varepsilon(u) = 0 \text{ in } \partial\Omega_\varepsilon,\ u_+ \not\equiv 0,\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega_\varepsilon \tag{3.14}$$

for if u satisfies (3.14), the Maximum Principle ensures that $u > 0$ in Ω_ε and (2.10) is satisfied. Observe that

$$S_\varepsilon(W + \phi) = -\Delta(W + \phi) + \mu\varepsilon^2(W + \phi) - \alpha_N (W + \phi)_+^{\frac{N+2}{N-2}+\varepsilon}$$

may be written as

$$S_\varepsilon(W+\phi) = -\Delta\phi + \mu\varepsilon^2\phi - (\frac{N+2}{N-2}+\varepsilon)\alpha_N W^{\frac{4}{N-2}+\varepsilon}\phi - R^\varepsilon - \alpha_N N_\varepsilon(\phi) \tag{3.15}$$

with

$$N_\varepsilon(\phi) = (W+\phi)_+^{\frac{N+2}{N-2}+\varepsilon} - W^{\frac{N+2}{N-2}+\varepsilon} - (\frac{N+2}{N-2}+\varepsilon)W^{\frac{4}{N-2}+\varepsilon}\phi \tag{3.16}$$

$$R^\varepsilon = \Delta W - \mu\varepsilon^2 W + \alpha_N W^{\frac{N+2}{N-2}+\varepsilon} = \alpha_N\big(W^{\frac{N+2}{N-2}+\varepsilon} - U_{\frac{1}{\Lambda},\xi}^{\frac{N+2}{N-2}}\big). \tag{3.17}$$

We first have :

Lemma 3.4 *There exists C, independent of ξ, Λ satisfying (2.3), such that*

$$\|R^\varepsilon\|_{**} \le C\varepsilon \qquad \|D_{(\Lambda,\xi)}R^\varepsilon\|_{**} \le C\varepsilon.$$

PROOF. According to (2.16) and (2.18), $W = U + O(\varepsilon U^{\frac{N-3}{N-2}})$ uniformly in Ω_ε. Consequently, noticing that $U \ge C\varepsilon^{N-2}$ in Ω_ε, C independent of ε, easy computations yield

$$R^\varepsilon = O\big(\varepsilon U^{\frac{N+2}{N-2}}|\ln U| + \varepsilon U^{\frac{N+1}{N-2}}\big) \tag{3.18}$$

uniformly in Ω_ε whence, using (3.4)

$$\begin{aligned}\|R^\varepsilon\|_{**} &= \|\langle x-\xi\rangle^{\beta+2}(U^{\frac{N+2}{N-2}} - W^{\frac{N+2}{N-2}+\varepsilon})\|_{L^t(\Omega_\varepsilon)} \\ &\le C\varepsilon\|\langle x-\xi\rangle^{\beta+2}(U^{\frac{N+2}{N-2}}|\ln U| + U^{\frac{N+1}{N-2}})\|_{L^t(\Omega_\varepsilon)} \le C\varepsilon.\end{aligned}$$

The first estimate of the lemma follows. The others are obtained in the same way, differentiating (3.17) and estimating each term as previously. □

We consider now the following nonlinear problem : finding ϕ such that, for some numbers c_i

$$\left\{\begin{array}{rlll} -\Delta(W+\phi) + \mu\varepsilon^2(W+\phi) - \alpha_N(W+\phi)_+^{\frac{N+2}{N-2}+\varepsilon} & = \sum_i c_i Z_i & \text{in} & \Omega_\varepsilon \\ \frac{\partial\phi}{\partial n} & = 0 & \text{on} & \partial\Omega_\varepsilon \\ 0 \le i \le N-1 \qquad \langle Z_i,\phi\rangle & = 0. & & \end{array}\right. \tag{3.19}$$

The first equation in (3.19) writes as

$$-\Delta\phi + \mu\varepsilon^2\phi - (\frac{N+2}{N-2}+\varepsilon)\alpha_N W^{\frac{4}{N-2}+\varepsilon}\phi = \alpha_N N_\varepsilon(\phi) + R^\varepsilon + \sum_i c_i Z_i \tag{3.20}$$

for some numbers c_i. We now obtain some estimates on N_ε.

Lemma 3.5 *Assume that $N \geq 4$ and (3.4) holds. There exist $\varepsilon_1 > 0$, independent of Λ, ξ, and C, independent of ε, Λ, ξ, such that for $|\varepsilon| \leq \varepsilon_1$, and $\|\phi\|_* \leq 1$*

$$\|N_\varepsilon(\phi)\|_{**} \leq C\|\phi\|_*^{\min(2,\frac{N+2}{N-2}+\varepsilon)} \tag{3.21}$$

and, for $\|\phi_i\|_* \leq 1$

$$\|N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2)\|_{**} \leq C\big(\max(\|\phi_1\|_*, \|\phi_2\|_*)\big)^{\min(1,\frac{4}{N-2}+\varepsilon)}\|\phi_1 - \phi_2\|_*. \tag{3.22}$$

Proof. The proof is similar to Lemma 3.1 and Proposition 3.5 of [42]. For the convenience of the reader, we include a proof here. We deduce from (3.16) that

$$\begin{cases} |N_\varepsilon(\phi)| \leq C(W^{\frac{6-N}{N-2}+\varepsilon}|\phi|^2 + |\phi|^{\frac{N+2}{N-2}+\varepsilon}) & \text{if } N \leq 6 \\ |N_\varepsilon(\phi)| \leq C|\phi|^{\frac{N+2}{N-2}+\varepsilon} & \text{if } N \geq 7 \end{cases} \tag{3.23}$$

Using (3.4) and (3.5) we have

$$\begin{aligned} \||\phi|^{\frac{N+2}{N-2}+\varepsilon}\|_{**} &= \Big(\int_{\Omega_\varepsilon} (\langle x-\xi\rangle^{\beta+2}|\phi|^{\frac{N+2}{N-2}+\varepsilon})^t\Big)^{\frac{1}{t}} \\ &\leq C\|\phi\|_*^{\frac{N+2}{N-2}+\varepsilon}\Big(\int_{\Omega_\varepsilon}\langle x-\xi\rangle^{t(\beta+2-(\frac{N+2}{N-2}+\varepsilon)\beta)}\Big)^{\frac{1}{t}} \\ &\leq C\|\phi\|_*^{\frac{N+2}{N-2}+\varepsilon} \end{aligned}$$

and, for $N = 4, 5, 6$

$$\begin{aligned} \|W^{\frac{6-N}{N-2}+\varepsilon}|\phi|^2\|_{**} &= \Big(\int_{\Omega_\varepsilon} (\langle x-\xi\rangle^{\beta+2}W^{\frac{6-N}{N-2}+\varepsilon}|\phi|^2)^t\Big)^{\frac{1}{t}} \\ &\leq C\|\phi\|_*^2\Big(\int_{\Omega_\varepsilon}\langle x-\xi\rangle^{\big(N-4-(N-2)\varepsilon-\beta\big)t}\Big)^{\frac{1}{t}} \\ &\leq C\|\phi\|_*^2 \end{aligned}$$

whence (3.21). Concerning (3.22), we write

$$N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2) = \partial_\eta N_\varepsilon(\eta)(\phi_1 - \phi_2)$$

for some $\eta = x\phi_1 + (1-x)\phi_2$, $x \in [0,1]$. From

$$\partial_\eta N_\varepsilon(\eta) = \big(\frac{N+2}{N-2} + \varepsilon\big)\big((W+\eta)_+^{\frac{4}{N-2}+\varepsilon} - W^{\frac{4}{N-2}+\varepsilon}\big)$$

we deduce

$$\begin{cases} |\partial_\eta N_\varepsilon(\eta)| \le C(W^{\frac{6-N}{N-2}+\varepsilon}|\eta| + |\eta|^{\frac{4}{N-2}+\varepsilon}) & \text{if } N \le 6 \\ |\partial_\eta N_\varepsilon(\eta)| \le C|\eta|^{\frac{4}{N-2}+\varepsilon} & \text{if } N \ge 6 \end{cases} \tag{3.24}$$

whence (3.22), using as previously (3.4) and (3.5). □

We state now the following result :

Proposition 3.2 *There exists C, independent of ε and ξ, Λ satisfying (2.3), such that for small ε problem (3.19) has a unique solution $\phi = \phi(\Lambda, \xi, \mu, \varepsilon)$ with*

$$\|\phi\|_* \le C\varepsilon. \tag{3.25}$$

Moreover, $(\Lambda, \xi) \mapsto \phi(\Lambda, \xi, \mu, \varepsilon)$ is C^1 with respect to the $W^{2,t}_\beta(\Omega_\varepsilon)$-norm, and

$$\|D_\Lambda \phi\|_* \le C\varepsilon. \tag{3.26}$$

Proof. Following [9], we consider the map A_ε from $\mathcal{F} = \{\phi \in H^1 \cap W^{2,t}_\beta(\Omega_\varepsilon) : \|\phi\|_* \le C_0\varepsilon\}$ to $H^1 \cap W^{2,t}_\beta(\Omega_\varepsilon)$ defined as

$$A_\varepsilon(\phi) = L_\varepsilon(\alpha_N N_\varepsilon(\phi) + R^\varepsilon).$$

Here C_1 is a large number, to be determined later, and L_ε is give by Proposition 3.1. We remark that finding a solution ϕ to problem (3.19) is equivalent to finding a fixed point of A_ε. One the one hand we have, for $\phi \in \mathcal{F}$

$$\|A_\varepsilon(\phi)\|_* \le \|L_\varepsilon(N_\varepsilon(\phi))\|_* + \|L_\varepsilon(R^\varepsilon)\|_* \le \|N_\varepsilon(\phi)\|_{**} + C\varepsilon \le 2C\varepsilon$$

for ε small enough, implying that A_ε sends $\mathcal{F}$ into itself, if we choose $C_0 = 2C$. On the other hand A_ε is a contraction. Indeed, for ϕ_1 and ϕ_2 in $\mathcal{F}$, we write

$$\begin{aligned} \|A_\varepsilon(\phi_1) - A_\varepsilon(\phi_2)\|_* &\le C\|N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2)\|_{**} \\ &\le C\varepsilon^{\min(1,\frac{4}{N-2})}\|\phi_1 - \phi_2\|_* \\ &\le \frac{1}{2}\|\phi_1 - \phi_2\|_* \end{aligned}$$

by Lemma (3.5). Contraction Mapping Theorem implies that A_ε has a unique fixed point in $\mathcal{F}$, that is problem (3.19) has a unique solution ϕ such that $\|\phi\|_* \le C_0\varepsilon$.

In order to prove that $(\Lambda, \xi) \mapsto \phi(\Lambda, \xi)$ is C^1, we remark that setting for $\eta \in \mathcal{F}$

$$B(\Lambda, \xi, \eta) \equiv \eta - L_\varepsilon(\alpha_N N_\varepsilon(\eta) + R^\varepsilon)$$

ϕ is defined as

$$B(\Lambda, \xi, \phi) = 0. \tag{3.27}$$

We have

$$\partial_\eta B(\Lambda, \xi, \eta)[\theta] = \theta - \alpha_N L_\varepsilon\big(\theta\,(\partial_\eta N_\varepsilon)(\eta)\big).$$

Using Proposition 3.1, (3.5), (3.24) and (3.4) we obtain for $N \geq 7$

$$\begin{aligned} \|L_\varepsilon\big(\theta\,(\partial_\eta N_\varepsilon)(\eta)\big)\|_* &\leq C\|\theta\,(\partial_\eta N_\varepsilon)(\eta)\|_{**} \\ &\leq C\|\langle x-\xi\rangle^{-\beta}(\partial_\eta N_\varepsilon)(\eta)\|_{**}\|\theta\|_* \\ &\leq C\|\langle x-\xi\rangle^2|\eta|^{\frac{4}{N-2}+\varepsilon}\|_{L^t(\Omega_\varepsilon)}\|\theta\|_* \\ &\leq C\|\eta\|_*^{\frac{4}{N-2}+\varepsilon}\|\theta\|_* \\ &\leq C\varepsilon^{\frac{4}{N-2}}\|\theta\|_* \end{aligned}$$

and, proceeding in the same way, we find as $N = 4, 5, 6$

$$\|L_\varepsilon\big(\theta\,(\partial_\eta N_\varepsilon)(\eta)\big)\|_* \leq C\varepsilon\|\theta\|_*.$$

Therefore we can write, for any $N \geq 4$

$$\|L_\varepsilon\big(\theta\,(\partial_\eta N_\varepsilon)(\eta)\big)\|_* \leq C\varepsilon^{\min(1,\frac{4}{N-2})}\|\theta\|_*.$$

Consequently, $\partial_\eta B(\Lambda, \xi, \phi)$ is invertible in $W^{2,t}_\beta(\Omega_\varepsilon)$ with uniformly bounded inverse. Then, the fact that $(\Lambda, \xi) \mapsto \phi(\Lambda, \xi)$ is C^1 follows from the fact that $(\Lambda, \xi, \eta) \mapsto L_\varepsilon(N_\varepsilon(\eta))$ is C^1 and the implicit functions theorem.

Finally, let us show how estimates (3.26) may be obtained. Derivating (3.27) with respect to Λ, we have

$$\partial_\Lambda\phi = (\partial_\eta B(\Lambda, \xi, \phi))^{-1}\Big(\alpha_N(\partial_\Lambda L_\varepsilon)(N_\varepsilon(\phi)) + \alpha_N L_\varepsilon((\partial_\Lambda N_\varepsilon)(\phi)) + \partial_\Lambda(L_\varepsilon(R^\varepsilon))\Big)$$

whence, according to Proposition 3.1

$$\begin{aligned} \|\partial_\Lambda\phi\|_* &\leq C\Big(\|(\partial_\Lambda L_\varepsilon)(N_\varepsilon(\phi))\|_* + \|(L_\varepsilon(\partial_\Lambda N_\varepsilon)(\phi))\|_* + \|(\partial_\Lambda(L_\varepsilon(R^\varepsilon))\|_*\Big) \\ &\leq C\Big(\|N_\varepsilon(\phi)\|_{**} + \|(\partial_\Lambda N_\varepsilon)(\phi)\|_{**} + \|(\partial_\Lambda(L_\varepsilon(R^\varepsilon))\|_*\Big). \end{aligned}$$

From (3.21) and (3.25) we know that

$$\|N_\varepsilon(\phi)\|_{**} \leq C\varepsilon^{\min(2,\frac{N+2}{N-2})}.$$

Concerning the next term, we notice that according to the definition (3.16) of N_ε

$$
\begin{aligned}
&|(\partial_\Lambda N_\varepsilon)(\phi)| \\
&\quad = (\frac{N+2}{N-2}+\varepsilon)\Big|(W+\phi)_+^{\frac{4}{N-2}+\varepsilon} - W^{\frac{4}{N-2}+\varepsilon} - (\frac{4}{N-2}+\varepsilon)W^{\frac{6-N}{N-2}+\varepsilon}\phi\Big| |\partial_\Lambda W| \\
&\quad \le C\Big[W^{\frac{4}{N-2}+\varepsilon}|\phi| \quad \text{if } N\ge 7\ ;\ W^{\frac{4}{N-2}+\varepsilon}|\phi| + W|\phi|^{\frac{4}{N-2}+\varepsilon} \quad \text{if } N\le 6\Big] \\
&\quad \le C\Big[\langle x-\xi\rangle^{-4-(N-2)\varepsilon-\beta}\|\phi\|_* \quad \text{if } N\ge 7\ ; \\
&\qquad \langle x-\xi\rangle^{-4-(N-2)\varepsilon-\beta}\|\phi\|_* + \langle x-\xi\rangle^{-(N-2)-(\frac{4}{N-2}+\varepsilon)\beta}\|\phi\|_*^{\frac{4}{N-2}+\varepsilon} \quad \text{if } N\le 6\Big] \\
&\quad \le C\Big[\varepsilon\langle x-\xi\rangle^{-4-\beta} \quad \text{if } N\ge 7\ ; \\
&\qquad \varepsilon\langle x-\xi\rangle^{-4-\beta} + \varepsilon^{\frac{4}{N-2}}\langle x-\xi\rangle^{-(N-2)-\frac{4}{N-2}\beta} \quad \text{if } N\le 6\Big]
\end{aligned}
$$

where we used successively the fact that $W>0$ (see (2.8)) and $|\partial_\Lambda W|\le CW$ (see (2.9)), inequality (3.5) and the fact that $W\le CU\le C\langle x-\xi\rangle^{-(N-2)}$, (3.21) and (3.25).

As (3.4) ensures that $\langle x-\xi\rangle^{-4-\beta}$, and $\langle x-\xi\rangle^{-(N-2)-(\frac{4}{N-2}+\varepsilon)}$ for $N\le 6$, are in $L^t_{\beta+2}(\mathbb{R}^N)$, we obtain

$$\|(\partial_\Lambda N_\varepsilon)(\phi)\|_{**}\le C\varepsilon.$$

From Proposition 3.1 we deduce the estimate for the last term

$$\|\partial_\Lambda(L_\varepsilon(R^\varepsilon))\|_*\le C\|R^\varepsilon\|_{**}\le C\varepsilon$$

and finally

$$\|\partial_\Lambda\phi\|_*\le C\varepsilon.$$

This concludes the proof of Proposition 3.2. (The first derivatives of ϕ with respect to ξ may be estimated in the same way, but this is not needed here.) $\square$

3.3 Coming back to the original problem

We introduce the following functional defined in $H^1(\Omega_\varepsilon)\cap W^{2,t}_\beta(\Omega_\varepsilon)$

$$J_\varepsilon(u) = \frac{1}{2}\int_{\Omega_\varepsilon}(|\nabla u|^2+\mu\varepsilon^2u^2) - \frac{\alpha_N}{\frac{2N}{N-2}+\varepsilon}\int_{\Omega_\varepsilon}u_+^{\frac{2N}{N-2}+\varepsilon} \tag{3.28}$$

whose nontrivial critical points are solutions to $(P'_{\frac{N+2}{N-2}+\varepsilon,\mu})$. Setting

$$I_\varepsilon(\Lambda,a)\equiv J_\varepsilon\big(W_{\Lambda,a}+\phi_{\varepsilon,\Lambda,a}\big) \tag{3.29}$$

we have :

Proposition 3.3 *The function $u = W + \phi$ is a solution to problem $(P'_{\frac{N+2}{N-2}+\varepsilon,\mu})$ if and only if (Λ, a) is a critical point of I_ε.*

PROOF. We notice that $u = W + \phi$ being a solution to $(P'_{\frac{N+2}{N-2}+\varepsilon,\mu})$ is equivalent to being a critical point of J_ε. It is also equivalent to the cancellation of the c_i's in (3.19) or, in view of (3.10) (3.11)

$$J'_\varepsilon(W + \phi)[Y_i] = 0 \qquad 0 \le i \le N - 1. \tag{3.30}$$

On the other hand, we deduce from (3.29) that $I'_\varepsilon(\Lambda, a) = 0$ is equivalent to the cancellation of $J'_\varepsilon(W + \phi)$ applied to the derivatives of $W + \phi$ with respect to Λ and ξ. According to the definition (3.1) of the Y_i's, Lemma 3.4 and Proposition 3.2 we have

$$\frac{\partial(W + \phi)}{\partial\Lambda} = Y_0 + y_0 \qquad\qquad \frac{\partial(W + \phi)}{\partial\xi_j} = Y_j + y_j \qquad 1 \le j \le N - 1$$

with $\|y_i\|_* = o(1)$, $0 \le i \le N - 1$. Writing

$$y_i = y'_i + \sum_{j=0}^{N-1} a_{ij} Y_j \qquad\qquad \langle y'_i, Z_j\rangle = (y'_i, Y_j)_\varepsilon = 0 \qquad 0 \le i, j \le N - 1$$

and

$$J'_\varepsilon(W + \phi)[Y_i] = \alpha_i$$

it turns out that $I'_\varepsilon(\Lambda, a) = 0$ is equivalent, since $J'_\varepsilon(W + \phi)[\theta] = 0$ for $\langle\theta, Z_j\rangle = (\theta, Y_j)_\varepsilon = 0$, $0 \le j \le N - 1$, to

$$(Id + [a_{ij}])[\alpha_i] = 0.$$

As $a_{ij} = O(\|y_i\|_*) = o(1)$, we see that $I'_\varepsilon(\Lambda, a) = 0$ means exactly that (3.30) is satisfied. □

4 Proofs of Theorems 1.1 and 1.2

In view of Proposition 3.3 we have, for proving the theorem, to find critical points of I_ε. We establish first a C^1-expansion of I_ε.

4.1 Expansion of I_ε

Proposition 4.1 *There exist A, B, C, strictly positive constants such that*

$$I_\varepsilon(\Lambda,a)=A-B\Lambda\varepsilon H(a)+\frac{(N-2)^2}{4}A\varepsilon\ln(\Lambda)+\varepsilon(C+\frac{(N-2)^2}{4N}A)+\varepsilon\sigma_\varepsilon(\Lambda,a)$$

with σ_ε and $\partial_\Lambda\sigma_\varepsilon$ going to zero as ε goes to zero, uniformly with respect to Λ satisfying (2.3).

Proof. In Appendix, we shall prove

$$J_\varepsilon(W)=A-B\Lambda\varepsilon H(a)+\frac{(N-2)^2}{4}A\varepsilon\ln(\Lambda)+\varepsilon(C+\frac{(N-2)^2}{4N}A)+o(\varepsilon). \tag{4.1}$$

Then it remains to show that

$$I_\varepsilon(\Lambda,a)-J_\varepsilon(W+\phi)=o(\varepsilon). \tag{4.2}$$

Actually, in view of (3.29), a Taylor expansion and the fact that $J'_\varepsilon(W+\phi)[\phi]=0$ yield

$$\begin{aligned}
I(\Lambda,a)-J_\varepsilon(W)&=J_\varepsilon(W+\phi)-J_\varepsilon(W)\\
&=\int_0^1 J''_\varepsilon(W+t\phi)[\phi,\phi]tdt\\
&=\int_0^1\Big(\int_{\Omega_\varepsilon}\big(|\nabla\phi|^2+\mu\varepsilon^2\phi^2-\alpha_N(\frac{N+2}{N-2}+\varepsilon)(W+t\phi)_+^{\frac{4}{N-2}+\varepsilon}\phi^2+R^\varepsilon\phi\big)\Big)tdt\\
&=\int_0^1\Big(\alpha_N\int_{\Omega_\varepsilon}\big(N_\varepsilon(\phi)\phi+(\frac{N+2}{N-2}+\varepsilon)\big[W^{\frac{4}{N-2}+\varepsilon}-(W+t\phi)_+^{\frac{4}{N-2}+\varepsilon}\big]\phi^2\big)\Big)tdt\\
&\quad+\frac12\int_{\Omega_\varepsilon}R^\varepsilon\phi.
\end{aligned}$$

The first term can be estimated as follows. Using (3.23), (3.5), (3.4) and Proposition 3.2, we have, for $N\geq 7$

$$\Big|\int_{\Omega_\varepsilon}N_\varepsilon(\phi)\phi\Big|\leq C\|\phi\|_*^{\frac{2N}{N-2}+\varepsilon}\int_{\Omega_\varepsilon}\langle x-\xi\rangle^{-\beta(\frac{2N}{N-2}+\varepsilon)}\leq C\varepsilon^{\frac{2N}{N-2}}.$$

In the same way we obtain for $N=4,5,6$, in view of (3.23) and $W\leq C\langle x-\xi\rangle^{-(N-2)}$

$$\Big|\int_{\Omega_\varepsilon}N_\varepsilon(\phi)\phi\Big|\leq C\varepsilon^{\frac{2N}{N-2}}+C\|\phi\|_*^3\int_{\Omega_\varepsilon}\langle x-\xi\rangle^{-3\beta-(6-N+(N-2)\varepsilon)}\leq C\varepsilon^3$$

whence finally, for any $N \geq 4$

$$|\int_{\Omega_\varepsilon} N_\varepsilon(\phi)\phi| \leq C\varepsilon^{\min(3,\frac{2N}{N-2})}. \tag{4.3}$$

For the second term, the same arguments as previously yield

$$\int_{\Omega_\varepsilon} \Big|W^{\frac{4}{N-2}+\varepsilon} - (W+t\phi)_+^{\frac{4}{N-2}+\varepsilon}\Big|\phi^2 \leq C\int_{\Omega_\varepsilon}\big(W^{\frac{4}{N-2}+\varepsilon}|\phi|^2 + |\phi|^{2+\frac{4}{N-2}+\varepsilon}\big)$$
$$\leq C\Big(\|\phi\|_*^2\int_{\Omega_\varepsilon}\langle x-\xi\rangle^{-2\beta-(4+(N-2)\varepsilon)} + \|\phi\|_*^{2+\frac{4}{N-2}+\varepsilon}\int_{\Omega_\varepsilon}\langle x-\xi\rangle^{-\beta(2+\frac{4}{N-2}+\varepsilon)}\Big)$$

whence, using again (3.4)

$$\int_{\Omega_\varepsilon}\big|W^{\frac{4}{N-2}+\varepsilon} - (W+t\phi)_+^{\frac{4}{N-2}+\varepsilon}\big|\phi^2 \leq C\varepsilon^2. \tag{4.4}$$

Concerning the last term, we remark that according to (3.18)

$$R^\varepsilon \leq C\big(\varepsilon U^{\frac{N+2}{N-2}}|\ln U| + \varepsilon U^{\frac{N+1}{N-2}}\big) \leq C\varepsilon\langle x-\xi\rangle^{-(N+1)}$$

uniformly in Ω_ε. Therefore

$$\int_{\Omega_\varepsilon}|R^\varepsilon\phi| \leq C\varepsilon\|\phi\|_*\int_{\Omega_\varepsilon}\langle x-\xi\rangle^{-(N+1)-\beta}$$

yielding, through Proposition 3.2

$$\int_{\Omega_\varepsilon}|R^\varepsilon\phi| \leq C\varepsilon^2. \tag{4.5}$$

The desired result follows from (4.3), (4.4) and (4.5). The same estimate holds for the first derivative with respect to Λ, obtained similarly with more delicate computations - see Proposition 3.4 of [21]. □

4.2 Proofs of Theorem 1.1 and Theorem 1.2 completed

We first prove Theorem 1.1. We have to use a max-min argument. Since Ω is smooth and bounded, $\max_{P\in\partial\Omega} H(P) = \gamma > 0$. For $\delta < \gamma$, we define

$$(\partial\Omega)_\delta = \Big\{a \in \partial\Omega \text{ s.t. } H(a) > \delta\Big\},$$

and

$$\hat{I}_\varepsilon(\Lambda, a) = \frac{A - I_\varepsilon(\Lambda, a)}{B\varepsilon} + B^{-1}(C + \frac{(N-2)^2}{4N}A). \tag{4.6}$$

By Proposition 4.1, we have the following asymptotic expansion for $\hat{I}_\epsilon(\Lambda, a)$:

$$\hat{I}_\varepsilon(\Lambda, a) = \Lambda H(a) - \alpha \ln(\Lambda) - \tilde{\sigma}_\varepsilon(\Lambda, a). \tag{4.7}$$

with

$$\alpha = \frac{(N-2)^2}{4B}A > 0 \qquad \text{and} \qquad \tilde{\sigma}_\varepsilon(\Lambda, a) = o(1) \quad \partial_\Lambda \tilde{\sigma}_\varepsilon(\Lambda, a) = o(1) \text{ as } \varepsilon \to O.$$

We set

$$\Sigma_0 = \{(\Lambda, a) | \frac{c_1}{2} < \Lambda < \frac{2}{c_1}, a \in (\partial\Omega)_{\gamma_0}\} \tag{4.8}$$

where c_1 is a small number, to be chosen later, and $0 < \gamma_0 < \gamma$. We define also

$$B = \{(\Lambda, a) | c_1 \leq \Lambda \leq \frac{1}{c_1}, a \in (\partial\Omega)_{\gamma_1}\} \qquad B_0 = \{c_1\} \times (\partial\Omega)_{\gamma_1} \cup \{\frac{1}{c_1}\} \times (\partial\Omega)_{\gamma_1}$$

where $\gamma_0 < \gamma_1 < \gamma$. (Here we choose a connected component of $(\partial\Omega)_{\gamma_1}$ so that B becomes connected.)

It is trivial to see that $B_0 \subset B \subset \Sigma_0$, B_0, B are closed and B is connected. Let Γ be the class of continuous functions $\varphi : B \to \Sigma_0$ with the property that $\varphi(y) = y$ for all $y \in B_0$. Define the max-min value c as

$$c = \max_{\varphi \in \Gamma} \min_{y \in B} \hat{I}_\varepsilon(\varphi(y)). \tag{4.9}$$

We now show that c defines a critical value. To this end, we just have to verify the following two conditions

(H1) $\min_{y \in B_0} \hat{I}_\varepsilon(\varphi(y)) > c, \forall \varphi \in \Gamma$,

(H2) For all $y \in \partial\Sigma_0$ such that $\hat{I}_\epsilon(y) = c$, there exists a vector τ_y tangent to $\partial\Sigma_0$ at y such that

$$\partial_{\tau_y} \hat{I}_\epsilon(y) \neq 0.$$

Suppose (H1) and (H2) hold. Then standard deformation argument ensures that the max-min value c is a (topologically nontrivial) critical value for $\hat{I}_\varepsilon(\Lambda, a)$

in Σ_0. (Similar notion has been introduced in [10] for degenerate critical points of mean curvature.)

To check (H1) and (H2), we define $\varphi(y) = (\varphi_1(y), \varphi_2(y))$ where $\varphi_1(y) \in [c_1, \frac{1}{c_1}]$ and $\varphi_2(y) \in (\partial\Omega)_{\gamma_1}$.

By taking φ to be identity map, we obtain

$$
\begin{aligned}
c \geq d_0 := & \max_{a\in(\partial\Omega)_{\gamma_1}} \min_{\Lambda\in[c_1,\frac{1}{c_1}]} \hat{I}_\varepsilon(\Lambda, a) \\
&= \alpha - \alpha\ln\alpha + \alpha\ln\gamma + o(1).
\end{aligned}
$$

Let $(\Lambda_0, a_0) \in B$ be such that $H(a_0) = \gamma, \Lambda_0 = \frac{\alpha}{\gamma}$. Thus $\hat{I}_\varepsilon(\Lambda_0, a_0) = d_0 + o(1)$.

On the other hand, for any $\varphi \in \Gamma$, φ_1 is a continuous function from B to $[\frac{c_1}{2}, \frac{2}{c_1}]$ such that $[c_1, \frac{1}{c_1}] \subset \varphi_1(B)$. Thus, assuming that c_1 is chosen samll enough, there exists $y_0 \in B$ such that $\varphi_1(y_0) = \Lambda_0$, whence

$$
\begin{aligned}
\min_{y\in B} \hat{I}_\varepsilon(\varphi(y)) &\leq \hat{I}_\varepsilon(\Lambda_0, \varphi_2(y_0)) \\
&\leq \frac{\alpha}{\gamma} H(\varphi_2(y_0)) - \alpha\ln\alpha + \alpha\ln\gamma + o(1) \\
&\leq d_0 = o(1).
\end{aligned}
$$

As a consequence

$$
c = d_0 + o(1) = \alpha - \alpha\ln\alpha + \alpha\ln\gamma + o(1). \tag{4.10}
$$

For $y \in B_0$, we have $\varphi_1(y) = c_1$ or $\varphi_1(y) = \frac{1}{c_1}$. In the first case, we have $\hat{I}_\varepsilon(y) = c_1 H(\varphi_2(y)) - \alpha\ln c_1 + o(1) > \alpha\ln\frac{1}{c_1} + o(1) > 2d_0 > c$, provided c_1 is small enough. In the latter case, we have $\hat{I}_\varepsilon(y) = \frac{1}{c_1} H(\varphi_2(y)) + \alpha\ln c_1 + o(1) > \frac{\gamma_1}{c_1} + \alpha\ln c_1 + o(1) > 2d_0 > c$, provided again c_1 is small enough. So (H1) is verified.

To verify (H2), we observe that $\partial(\Sigma_0) = (\{\frac{c_1}{2}\} \times (\partial\Omega)_{\gamma_0}) \cup (\{\frac{2}{c_1}\} \times (\partial\Omega)_{\gamma_0}) \cup ([c_1, \frac{1}{c_1}] \times (\partial(\partial\Omega)_{\gamma_0}))$. Let $y = (y_1, y_2) \in \partial\Sigma_0$ be such that $\hat{I}_\varepsilon(y) = c$.

On $(\{\frac{c_1}{2}\} \times (\partial\Omega)_{\gamma_0}) \cup (\{\frac{2}{c_1}\} \times (\partial\Omega)_{\gamma_0})$, previous arguments show that $\hat{I}_\varepsilon(y) > c$ as c_1 is chosen sufficiently small. On $([c_1, \frac{1}{c_1}] \times (\partial((\partial\Omega)_{\gamma_0}))$, we take $\tau_y = \frac{\partial}{\partial\Lambda}$ to obtain that

$$
\partial_{\tau_y} \hat{I}_\epsilon(y) = H(y_2) - \frac{\alpha}{\Lambda} + o(1) \neq 0
$$

since $\partial_{\tau_y} \hat{I}_\epsilon(y) = 0$ yields $\Lambda H(y_2) = \alpha + o(1)$, and

$$
\hat{I}_\varepsilon(y) = \alpha - \alpha\ln\alpha + \alpha\ln H(\varphi_2(y)) + o(1) = \alpha - \alpha\ln\alpha + \alpha\ln\gamma_0 + o(1).
$$

Then, (4.10) shows that $\hat{I}_\varepsilon(y) < c$, a contradiction to the assumption. So (H2) is also verified.

In conclusion, we proved that for ε small enough, c is a critical value, i.e. a critical point $(\Lambda_\varepsilon, a_\varepsilon) \in \Sigma_0$ of $\hat{I}_\varepsilon$ exists. Let $u_\varepsilon = W_{\Lambda_\varepsilon,\xi_\varepsilon,\mu,\varepsilon} + \phi_{\Lambda_\varepsilon,\xi_\varepsilon,\mu,\varepsilon}$. u_ε is a nontrivial solution to the problem

$$-\Delta u + \mu\varepsilon^2 u = u_+^{\frac{N+2}{N-2}+\varepsilon} \text{ in } \Omega_\varepsilon\,; \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega_\varepsilon.$$

Then, the strong maximum principle shows that $u_\varepsilon > 0$ in Ω_ε. The fact that u_ε blows up, as ε goes to zero, at a point a such that $H(a) = \max_{P\in\partial\Omega} H(P)$, follows from the construction of u_ε. This concludes the proof of Theorem 1.1.

In the case of $\varepsilon < 0$, we have

$$\hat{I}_\varepsilon(\Lambda, a) = \Lambda H(a) + \alpha \ln(\Lambda) - \tilde{\sigma}_\varepsilon(\Lambda, a).$$

We assume that Ω is nonconvex. Similarly as before, we define

$$(\partial\Omega)_\delta = \{a \in \partial\Omega | H(a) < -\delta\}$$

where $0 < \delta < \gamma = -\min_{a\in\partial\Omega} H(a) > 0$, and

$$\Sigma_0 = \{(\Lambda, a) | \frac{c_1}{2} \le \Lambda \le \frac{2}{c_1}, a \in (\partial\Omega)_{\gamma_0}\}$$

$$B = \{(\Lambda, a) | c_1 \le \Lambda \le \frac{1}{c_1}, a \in (\partial\Omega)_{\gamma_1}\} \qquad B_0 = \{c_1\} \times (\partial\Omega)_\gamma \cup \{\frac{1}{c_1}\} \times (\partial\Omega)_{\gamma_1}$$

with $\gamma_0 < \gamma_1 < \gamma$.

Let Γ be the class of continuous functions $\varphi : B \to \Sigma_0$ with the property that $\varphi(y) = y$ for all $y \in B_0$. Define the min-max value c as

$$c = \min_{\varphi\in\Gamma} \max_{y\in B} \hat{I}_\varepsilon(\varphi(y)).$$

Arguing as previously, we find that c is a critical point of $\hat{I}_\varepsilon$. This proves Theorem 1.2.

5 Appendix

5.1 Error estimates

We recall that, according to the definition of $V_{\Lambda,a,\mu,\varepsilon}$ in Section 2

$$V_{\Lambda,a,\mu,\varepsilon}(x) = U_{\frac{1}{\Lambda\varepsilon},a}(x) - \varphi_{\Lambda,a,\mu,\varepsilon} \tag{5.1}$$

with $\varphi_{\Lambda,a,\mu,\varepsilon}$ satisfying

$$\begin{cases} -\Delta\varphi_{\Lambda,a,\mu,\varepsilon} + \mu\varphi_{\Lambda,a,\mu,\varepsilon} &= \mu U_{\frac{1}{\Lambda\varepsilon},a} & \text{in } \Omega \\ \frac{\partial\varphi_{\Lambda,a,\mu,\varepsilon}}{\partial n} &= \frac{\partial U_{\frac{1}{\Lambda\varepsilon},a}}{\partial n} & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

This subsection is devoted to an expansion of $\varphi_{\Lambda,a,\mu,\varepsilon}$.

We recall that, through space translation and rotation, we assume that $a = 0$ and Ω is given, in a neighbourhood of a, by (2.11) and (2.12). We introduce an auxiliary function φ_0: let φ_0 be such that

$$\begin{cases} \Delta\varphi_0 &= 0 & \text{in } \mathbb{R}^N_+ = \{(x', x_N), x_N > 0\} \\ \frac{\partial\varphi_0}{\partial x_N} &= \frac{N-2}{2}\frac{\sum_{i=1}^{N-1} k_i x_i^2}{(1+|x|^2)^{\frac{N}{2}}} & \text{on } \partial\mathbb{R}^N_+, \\ \varphi_0(x) &\to 0 & \text{as } |x| \to +\infty. \end{cases} \tag{5.3}$$

Using Green's reprensentation, φ_0 writes as

$$\varphi_0(x) = \frac{1}{\omega_{N-1}} \sum_{i=1}^{N-1} k_i \int_{\mathbb{R}^{N-1}} \frac{y_i^2}{(1+|y'|^2)^{\frac{N}{2}}} \frac{1}{|x-y'|^{N-2}} dy' \tag{5.4}$$

where ω_{N-1} denotes the measure of the unit sphere in $\mathbb{R}^N$. From (5.3) we deduce that

$$|\varphi_0(x)| \le \frac{C}{(1+|x|)^{N-3}} \tag{5.5}$$

and

$$|\nabla\varphi_0(x)| \le \frac{C}{(1+|x|)^{N-2}} \qquad |D^2\varphi_0(x)| \le \frac{C}{(1+|x|)^{N-1}}. \tag{5.6}$$

From now on, we consider φ_0 as a smooth continuation in $\mathbb{R}^N$ of the previous function defined in $\mathbb{R}^N_+$, such that (5.5) (5.6) hold in whole $\mathbb{R}^N$. We state

Lemma 5.1 *For $N \geq 4$, we have the expansion*

$$\varphi_{\Lambda,a,\mu,\varepsilon}(x) = (\Lambda\varepsilon)^{\frac{4-N}{2}}\varphi_0(\frac{x-a}{\Lambda\varepsilon}) + O(\varepsilon^{\frac{6-N}{2}}|\ln\varepsilon|^m)$$

with $m = 1$ for $N = 4$ and $m = 0$ for $N \geq 5$. Moreover,

$$|\varphi_{\Lambda,a,\mu,\varepsilon}(x)| \leq \frac{\varepsilon^{\frac{4-N}{2}}|\ln\varepsilon|^n}{(1+|\frac{x-a}{\Lambda\varepsilon}|)^{N-3}}$$

with $n = 1$ for $N = 4, 5$ and $n = 0$ for $N \neq 6$.

Proof. We decompose

$$\varphi = \varphi^1 + \varphi^2$$

where φ^1 satisfies

$$\begin{cases} -\Delta\varphi^1_{\Lambda,a,\mu,\varepsilon} + \mu\varphi^1_{\Lambda,a,\mu,\varepsilon} & = 0 \qquad \text{in } \Omega \\ \frac{\partial\varphi^1_{\Lambda,a,\mu,\varepsilon}}{\partial n} & = \frac{\partial U_{\frac{1}{\Lambda\varepsilon},a}}{\partial n} \qquad \text{on } \partial\Omega \end{cases}$$

and φ^2 satisfies

$$\begin{cases} -\Delta\varphi^2_{\Lambda,a,\mu,\varepsilon} + \mu\varphi^2_{\Lambda,a,\mu,\varepsilon} & = \mu U_{\frac{1}{\Lambda\varepsilon},a} \qquad \text{in } \Omega \\ \frac{\partial\varphi^2_{\Lambda,a,\mu,\varepsilon}}{\partial n} & = 0 \qquad \text{on } \partial\Omega. \end{cases}$$

Let us estimate φ^2 first. Let

$$\hat{\varphi}^j(x) = \varepsilon^{\frac{N-2}{2}}\varphi^j(\varepsilon x), j = 1, 2.$$

Then $\hat{\varphi}^2$ satisfies

$$\begin{cases} -\Delta\hat{\varphi}^2_{\Lambda,a,\mu,\varepsilon} + \mu\varepsilon^2\hat{\varphi}^2_{\Lambda,a,\mu,\varepsilon} & = \mu\varepsilon^2 U_{\frac{1}{\Lambda},\xi} \qquad \text{in } \Omega_\varepsilon \\ \frac{\partial\hat{\varphi}^2_{\Lambda,a,\mu,\varepsilon}}{\partial n} & = 0 \qquad \text{on } \partial\Omega_\varepsilon. \end{cases}$$

Inequality (3.6) of Lemma 3.2 provides us with

$$|\hat{\varphi}^2(x)| \leq C\varepsilon^2\int_{\Omega_\varepsilon}\frac{U_{\frac{1}{\Lambda},\xi}}{|x-y|^{N-2}}dy \leq C\varepsilon^2\int_{\Omega_\varepsilon}\frac{dy}{(1+|y-\xi|)^{N-2}|x-y|^{N-2}}$$

whence

$$|\hat{\varphi}^2(x)| \leq C\frac{\varepsilon^2|\ln\varepsilon|^m}{(1+|x-\xi|)^{N-4}}$$

with $m = 1$ for $N = 4$ and $m = 0$ for $N \geq 5$. (For $N \geq 5$, see Lemma 2.3 of [23].) Consequently

$$\varphi^2(x) = O(\varepsilon^{\frac{6-N}{2}} |\ln \varepsilon|^m) \qquad \text{and} \qquad |\varphi^2(x)| \leq C \frac{\varepsilon^{\frac{4-N}{2}} |\ln \varepsilon|^m}{(1 + |\frac{x-a}{\Lambda\varepsilon}|)^{N-3}}.$$

This finishes the estimate for φ^2. Next we estimate φ^1. To this end, we write

$$\varphi^1_{\Lambda,a,\mu,\varepsilon} = (\Lambda\varepsilon)^{\frac{4-N}{2}} \varphi_0(\frac{x-a}{\Lambda\varepsilon}) + \varphi^3_{\Lambda,a,\mu,\varepsilon}(x) + \varphi^4_{\Lambda,a,\mu,\varepsilon}(x)$$

where $\varphi^3_{\Lambda,a,\mu,\varepsilon}$ satisfies

$$\begin{cases} -\Delta\varphi^3_{\Lambda,a,\mu,\varepsilon} + \mu\varphi^3_{\Lambda,a,\mu,\varepsilon} = 0 & \text{in } \Omega \\ \frac{\partial\varphi^3_{\Lambda,a,\mu,\varepsilon}}{\partial n} = \frac{\partial U_{\frac{1}{\Lambda\varepsilon},a}}{\partial n} - \frac{\partial}{\partial n}\Big((\Lambda\varepsilon)^{\frac{4-N}{2}}\varphi_0(\frac{x-a}{\Lambda\varepsilon})\Big) & \text{on } \partial\Omega \end{cases}$$

and $\varphi^4_{\Lambda,a,\mu,\varepsilon}$ satisfies

$$\begin{cases} -\Delta\varphi^4_{\Lambda,a,\mu,\varepsilon} + \mu\varphi^4_{\Lambda,a,\mu,\varepsilon} = (\Delta - \mu)\Big((\Lambda\varepsilon)^{\frac{4-N}{2}}\varphi_0(\frac{x-a}{\Lambda\varepsilon})\Big) & \text{in } \Omega \\ \frac{\partial\varphi^4_{\Lambda,a,\mu,\varepsilon}}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

The estimate for φ^4 is similar to that of φ^2. Namely, in view of (5.3) and (5.4), inequality (3.6) of Lemma 3.2 gives

$$\begin{aligned} |\hat{\varphi}^4(x)| &\leq C\varepsilon^3\Big(\frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon\setminus\mathbb{R}^N_+} \frac{dy}{(1+|y-\xi|)^{N-1}|x-y|^{N-2}} \\ &\qquad\qquad + \int_{\Omega_\varepsilon} \frac{dy}{(1+|y-\xi|)^{N-3}|x-y|^{N-2}} dy\Big) \\ &\leq C\varepsilon^3\left(\frac{1}{\varepsilon(1+|x-\xi|)^{N-3}} + \frac{|\ln\varepsilon|^p}{(1+|x-\xi|)^{N-5}}\right) \end{aligned}$$

with $p = 1$ for $N = 5$ and $p = 0$ for $N \neq 5$, whence

$$\varphi^4(x) = O(\varepsilon^{\frac{6-N}{2}}) \qquad \text{and} \qquad |\varphi^4(x)| \leq C\frac{\varepsilon^{\frac{4-N}{2}}|\ln\varepsilon|^p}{(1+|\frac{x-a}{\Lambda\varepsilon}|)^{N-3}}.$$

It only remains to estimate φ^3. For $x \in \partial\Omega \cap B(a,\delta)$, we consider the following change of variable (still assuming $a = 0$)

$$\Lambda\varepsilon y' = x' \qquad \Lambda\varepsilon y_N = x_N - \rho(x').$$

According to the definition of U and (2.13), we have

$$\begin{aligned}\frac{\partial U_{\frac{1}{\Lambda\varepsilon},a}}{\partial n} &= -(N-2)(\Lambda\varepsilon)^{\frac{N-2}{2}}\frac{\langle x-a,n\rangle}{((\Lambda\varepsilon)^2+|x-a|^2)^{\frac{N}{2}}}\\ &= -\frac{N-2}{2}\frac{(\Lambda\varepsilon)^{\frac{N-2}{2}}}{((\Lambda\varepsilon)^2+|x-a|^2)^{\frac{N}{2}}}\Big(\sum_{i=1}^{N-1}k_i x_i^2+O(|x'|^3)\Big)\\ &= -\frac{N-2}{2}\frac{(\Lambda\varepsilon)^{\frac{2-N}{2}}}{(1+|y'|^2)^{\frac{N}{2}}}\Big(\sum_{i=1}^{N-1}k_i y_i^2+O(\varepsilon|y'|^3)\Big)\end{aligned}$$

and, using (5.3) and (5.6)

$$\begin{aligned}\frac{\partial}{\partial n}\Big((\Lambda\varepsilon)^{\frac{4-N}{2}}\varphi_0(\frac{x-a}{\Lambda\varepsilon})\Big) &= (\Lambda\varepsilon)^{\frac{2-N}{2}}\Big(\nabla'\varphi_0(\frac{x-a}{\Lambda\varepsilon}).\nabla'\rho(x)-\frac{\partial\varphi_0}{\partial x_N}(\frac{x-a}{\Lambda\varepsilon})\Big)\\ &= -\frac{N-2}{2}\frac{(\Lambda\varepsilon)^{\frac{2-N}{2}}}{(1+|y'|^2)^{\frac{N}{2}}}\sum_{i=1}^{N-1}k_i y_i^2+O\Big(\frac{\varepsilon^{\frac{4-N}{2}}|y'|}{(1+|y'|)^{N-2}}\Big).\end{aligned}$$

Therefore

$$\frac{\partial\hat{\varphi}^3}{\partial n}(x)=O\Big(\frac{\varepsilon^2|x'|}{(1+|x'|)^{N-2}}\Big)\qquad \text{for } x\in\partial\Omega_\varepsilon\cap B(a,\frac{\delta}{\varepsilon}). \tag{5.7}$$

On the other hand we have clearly, from (5.6) and the definition of U

$$\frac{\partial\hat{\varphi}^3}{\partial n}(x)=O(\varepsilon^{N-1})\qquad \text{for } x\in\partial\Omega_\varepsilon\cap B^c(a,\frac{\delta}{\varepsilon}). \tag{5.8}$$

Then, standard elliptic theory shows that $\hat{\varphi}^3=O(\varepsilon^2)$ uniformly in Ω_ε, whence $\varphi^3(x)=O(\varepsilon^{\frac{6-N}{2}})$ uniformly in Ω. Moreover, (5.7) and (5.8) lead, through Green's reprensentation, to the estimate

$$|\hat{\varphi}^3(x)|\le C\frac{\varepsilon^2}{(1+|x-\xi|)^{N-4}}$$

whence

$$|\varphi^3(x)|\le C\frac{\varepsilon^{\frac{4-N}{2}}}{(1+|\frac{x-a}{\Lambda\varepsilon}|)^{N-3}}.$$

This concludes the proof of Lemma 5.1. $\square$

5.2 Integral estimates

Omitting, for sake of simplicity, the indices $\Lambda, a, \mu, \varepsilon$, we state :

Proposition 5.1 *Assuming that Λ satisfies (2.3), we have the uniform expansions as ε goes to zero*

$$J_\varepsilon(W) = A - B\Lambda|\varepsilon|H(a) + \frac{(N-2)^2 A}{4}\varepsilon \ln\Lambda + (C + \frac{(N-2)^2 A}{4N})\varepsilon + O(\varepsilon^{2-\tau})$$

$$\frac{\partial J_\varepsilon}{\partial\Lambda} = \frac{(N-2)^2 A\varepsilon}{4\Lambda} - BH(a)|\varepsilon| + O(\varepsilon^{2-\tau})$$

with

$$A = (N-2)\int_{\mathbb{R}^N_+} U_{1,0}^{\frac{2N}{N-2}} \qquad C = -\frac{(N-2)^2}{2}\int_{\mathbb{R}^N_+} U_{1,0}^{\frac{2N}{N-2}} \ln U_{1,0} > 0 \tag{5.9}$$

and

$$B = \frac{(N-2)^2}{N-3}\int_{\partial\mathbb{R}^N_+} U_{1,0}^{\frac{2N}{N-2}}|y|^2. \tag{5.10}$$

Proof. For sake of simplicity, we assume that $\varepsilon > 0$ (the computations are equivalent as $\varepsilon < 0$). In view of (5.2) and (2.16), we write

$$\begin{aligned}\int_{\Omega_\varepsilon}(|\nabla W|^2 + \mu\varepsilon^2 W^2) &= \int_{\Omega_\varepsilon}(-\Delta W + \mu\varepsilon^2 W)W = \int_{\Omega_\varepsilon}\Big(\alpha_N U^{\frac{N+2}{N-2}})W \\ &= \alpha_N \int_{\Omega_\varepsilon} U^{\frac{2N}{N-2}} - \alpha_N\int_{\Omega_\varepsilon} U^{\frac{N+2}{N-2}}\hat\varphi.\end{aligned}$$

with $U = U_{\frac{1}{\Lambda},\xi}$. On the other hand

$$\begin{aligned}\int_{\Omega_\varepsilon} W^{\frac{2N}{N-2}+\varepsilon} &= \int_{\Omega_\varepsilon} W^{\frac{2N}{N-2}} + \int_{\Omega_\varepsilon} W^{\frac{2N}{N-2}}(W^\varepsilon - 1) \\ &= \int_{\Omega_\varepsilon}(U-\hat\varphi)^{\frac{2N}{N-2}} + \varepsilon\int_{\Omega_\varepsilon}(U-\hat\varphi)^{\frac{2N}{N-2}}\ln(U-\hat\varphi) + O(\varepsilon^2|\ln\varepsilon|) \\ &= \int_{\Omega_\varepsilon} U^{\frac{2N}{N-2}} - \frac{2N}{N-2}\int_{\Omega_\varepsilon} U^{\frac{N+2}{N-2}}\hat\varphi + \varepsilon\int_{\Omega_\varepsilon}(U-\hat\varphi)^{\frac{2N}{N-2}}\ln(U-\hat\varphi) + O(\varepsilon^2|\ln\varepsilon|).\end{aligned}$$

The validity of this expansion can be verified by Lebesgue's Dominated Convergence Theorem and the fact that $|W-U| \le C\varepsilon|\ln\varepsilon|^n U_{\frac{1}{\Lambda},a}^{\frac{N-3}{N-2}}$ (see Lemma 5.1 and similar arguments in Section 5 of [36]). Note also that

$$\int_{\Omega_\varepsilon}(U-\hat{\varphi})^{\frac{2N}{N-2}}\ln(U-\hat{\varphi}) = -\frac{N-2}{2}\ln\Lambda\int_{\mathbb{R}^N_+}U_{1,0}^{\frac{2N}{N-2}} + \int_{\mathbb{R}^N_+}U_{1,0}^{\frac{2N}{N-2}}\ln U_{1,0} + O(\varepsilon).$$

Then, according to the definition (3.28) of J_ε and $\alpha_N = N(N-2)$

$$\begin{aligned}J_\varepsilon[W] &= \big((N-2)+\frac{(N-2)^3}{4N}\varepsilon\big)\int_{\Omega_\varepsilon}U^{\frac{2N}{N-2}} + \frac{N(N-2)}{2}\int_{\Omega_\varepsilon}U^{\frac{N+2}{N-2}}\hat{\varphi}\\ &\quad+\frac{(N-2)^3}{4}\varepsilon\ln\Lambda\int_{\mathbb{R}^N_+}U_{1,0}^{\frac{2N}{N-2}} - \varepsilon\frac{(N-2)^2}{2}\int_{\mathbb{R}^N_+}U_{1,0}^{\frac{2N}{N-2}}\ln U_{1,0} + O(\varepsilon^{2-\tau})\end{aligned} \tag{5.11}$$

noticing (see estimates below), that $\int_{\Omega_\varepsilon}U^{\frac{2N}{N-2}} = O(1)$ and $\int_{\Omega_\varepsilon}U^{\frac{N+2}{N-2}}\hat{\varphi} = O(\varepsilon)$. We observe that

$$\begin{aligned}\int_{\Omega_\varepsilon}U^{\frac{2N}{N-2}} &= \int_{\mathbb{R}^N_+}U_{1,0}^{\frac{2N}{N-2}}(y', y_N + \frac{\rho(\varepsilon y')}{\varepsilon}) + O(\varepsilon^{2-\tau})\\ &= \int_{\mathbb{R}^N_+}U_{1,0}^{\frac{2N}{N-2}}(y', y_N) + \int_{\mathbb{R}^N_+}\frac{\partial U_{1,0}^{\frac{2N}{N-2}}}{\partial y_N}\left(\frac{\rho(\varepsilon y')}{\varepsilon}\right) + O(\varepsilon^{2-\tau})\end{aligned}$$

whence

$$\int_{\Omega_\varepsilon}U^{\frac{2N}{N-2}} = \int_{\mathbb{R}^N_+}U_{1,0}^{\frac{2N}{N-2}} - \frac{1}{2}(N-1)\Lambda\varepsilon H(a)\int_{\partial\mathbb{R}^N_+}U_{1,0}^{\frac{2N}{N-2}}|y|^2dy + O(\varepsilon^{2-\tau}). \tag{5.12}$$

On the other hand, in view of the expansion of $\varphi_{\Lambda,a,\mu,\varepsilon}$ in Lemma 5.1, we also have

$$\begin{aligned}\alpha_N\int_{\Omega_\varepsilon}U^{\frac{N+2}{N-2}}\hat{\varphi}_{\Lambda,a,\mu,\varepsilon} &= \Lambda\varepsilon\alpha_N\int_{\Omega_\varepsilon}U_{1,0}^{\frac{N+2}{N-2}}\varphi_0 + O(\varepsilon^{2-\tau})\\ &= \Lambda\varepsilon\alpha_N\int_{\mathbb{R}^N_+}U_{1,0}^{\frac{N+2}{N-2}}\varphi_0 + O(\varepsilon^{2-\tau})\\ &= \Lambda\varepsilon\int_{\mathbb{R}^N_+}(-\Delta U_{1,0}\varphi_0 + U_{1,0}\Delta\varphi_0) + O(\varepsilon^{2-\tau})\\ &= \Lambda\varepsilon\int_{\partial\mathbb{R}^N_+}(-\frac{\partial\varphi_0}{\partial y_N}U_{1,0}) + O(\varepsilon^{2-\tau})\\ &= -\Lambda\varepsilon\frac{N-2}{2}\sum_{j=1}^{N-1}k_j\int_{\partial\mathbb{R}^N_+}U_{1,0}\frac{y_j^2}{(1+|y|^2)^{\frac{N}{2}}} + O(\varepsilon^{2-\tau}).\end{aligned}$$

Therefore

$$\alpha_N \int_{\Omega_\varepsilon} U^{\frac{N+2}{N-2}} \hat{\varphi}_{\Lambda,a,\mu,\varepsilon} = -\Lambda\varepsilon \frac{N-2}{2} H(a) \int_{\partial\mathbb{R}^N_+} \frac{|y|^2}{(1+|y|^2)^{N-1}} + O(\varepsilon^{2-\tau}). \quad (5.13)$$

Substituting (5.12) and (5.13) into (5.11), we obtain

$$J_\varepsilon[W] = A - B^* \Lambda\varepsilon H(a) + \frac{(N-2)^2}{4} A\varepsilon \ln(\Lambda) + \varepsilon(\frac{(N-2)^2}{4N} A + C) + O(\varepsilon^{2-\tau})$$

where A, C are given in (5.9) and

$$B^* = \frac{N-2}{2} \int_{\partial\mathbb{R}^N_+} U_{1,0}^{\frac{2N}{N-2}} |y|^2 + \frac{N-2}{4} \int_{\partial\mathbb{R}^N_+} \frac{|y|^2}{(1+|y|^2)^{N-1}}.$$

To make the proof of Proposition 5.1 complete, it only remains to show that $B^* = B$ defined by (5.10). In fact, it is easily seen that

$$\int_{\partial\mathbb{R}^N_+} U_{1,0}^{\frac{2N}{N-2}} |y|^2 = \omega_{N-2} \int_0^\infty \frac{r^N}{(1+r^2)^N} dr = \frac{N-3}{2(N-1)} \omega_{N-2} \int_0^\infty \frac{r^N}{(1+r^2)^{N-1}} dr$$

where ω_{N-2} is the area of the unit sphere in R^{N-1}. The last equality follows from simple integration by parts. Then, we can rewrite B^* as

$$B^* = B = \frac{(N-2)^2}{N-3} \int_{\partial\mathbb{R}^N_+} U_{1,0}^{\frac{2N}{N-2}} |y|^2.$$

The expansions for the derivatives of J_ε are obtained exactly in the same way. □

5.3 Proof of Lemma 3.2

We prove (3.6) first. Through scaling, we may assume that $\varepsilon = 1$. Let $G(x, y)$ be the Green's function satisfying

$$-\Delta G(x,y) + \mu G(x,y) = \delta_y \ \text{ in } \Omega, \quad \frac{\partial G(x,y)}{\partial n} = 0 \ \text{ on } \partial\Omega.$$

Then we have for $x \in \Omega$,

$$u(x) = \int_\Omega G(x,y) f(y) dy.$$

So it is enough to show that there exists a constant C, independent of x and y, such that

$$|G(x,y)| \leq \frac{C}{|x-y|^{N-2}}.$$

To this end, we decompose G into two parts:

$$G(x,y) = K(|x-y|) + H(x,y)$$

where $K(|x-y|)$ is the singular part of G and $H(x,y)$ is the regular part of G. Certainly we have $|K(|x-y|)| \leq \frac{C}{|x-y|^{N-2}}$. It remains to show that

$$|H(x,y)| \leq \frac{C}{|x-y|^{N-2}}. \tag{5.14}$$

Note that, if $d(x,\partial\Omega) > d_0 > 0$ or $d(y,\partial\Omega) > d_0 > 0$, then $|H(x,y)| \leq C$ and hence (5.14) also holds. So we just need to estimate $H(x,y)$ for $d(x,\partial\Omega)$ and $d(y,\partial\Omega)$ small. Let $y \in \Omega$ be such that $d = d(y,\partial\Omega)$ is small. So there exists a unique point $\bar{y} \in \partial\Omega$ such that $d = |y-\bar{y}|$. Without loss of generality, we may assume $\bar{y} = 0$ and the outer normal at $\bar{y}$ is pointing toward x_N-direction. Let y^* be the reflection point $y^* = (0,...,0,-d)$ and consider the following auxiliary function

$$H^*(x,y) = K(|x-y^*|)$$

Then H^* satisfies $\Delta H^* - \mu H^* = 0$ in Ω and on $\partial\Omega$

$$\frac{\partial}{\partial n}(H^*(x,y)) = -\frac{\partial}{\partial n}(K(|x-y|)) + O(\frac{1}{d^{N-3}}).$$

Hence we derive that

$$H(x,y) = -H^*(x,y) + O(\frac{1}{d^{N-3}})$$

which proves (5.14) for $x, y \in \Omega$. This implies that for $x \in \Omega$

$$|u(x)| \leq C \int_\Omega \frac{1}{|x-y|^{N-2}} |f(y)| dy. \tag{5.15}$$

If $x \in \partial\Omega$, we consider a sequence of points $x_i \in \Omega, x_i \to x \in \partial\Omega$ and take the limit in (5.15). Lebesgue's Dominated Convergence Theorem applies and (3.6) is proved.

We turn now to the proof of (3.7). By Lemma 3.1, we have

$$\|u\|_{L^t_\beta(\Omega_\varepsilon)} \le C\|f\|_{L^t_{\beta+2}(\Omega_\varepsilon)}$$

hence

$$\|\varepsilon^2 u\|_{L^t_{\beta+2}(\Omega_\varepsilon)} \le C\|u\|_{L^t_\beta(\Omega_\varepsilon)} \le C\|f\|_{L^t_{\beta+2}(\Omega_\varepsilon)}.$$

By a usual transformation and extension (as done in Step 2 of Proof of Theorem 2.1 in [32]) and interpolation, one can show that

$$\|u\|_{W^{2,t}_\beta(B_{\frac{\delta}{\varepsilon}}(\xi))} \le C\|\varepsilon^2 u\|_{L^t_{\beta+2}(\Omega_\varepsilon)} + C\|f\|_{L^t_{\beta+2}(\Omega_\varepsilon)} \le C\|f\|_{L^t_{\beta+2}(\Omega_\varepsilon)}. \tag{5.16}$$

where δ is a small fixed constant. Next we take a cut-off function $\chi(x)$ such that $\chi(x) = 1$ for $|x| \le \frac{\delta}{2}$ and $\chi(x) = 0$ for $|x| > \delta$, and we consider the function

$$u^1(x) = u(y)(1 - \chi(\varepsilon y - \xi))$$

which satisfies

$$-\Delta_x u^1 + \mu\varepsilon^2 u^1 = 2\varepsilon\nabla_y u.\nabla_x\chi + \varepsilon^2 u\Delta_x\chi + f(1-\chi)$$

in $\tilde\Omega = \Omega\backslash\{|x-a| < \delta\}$. Applying the well-known elliptic regularity theory, we obtain

$$\|u^1\|_{W^{2,t}(\tilde\Omega)} \le C\|2\varepsilon\nabla_y u\nabla_x\chi + \varepsilon^2\Delta_x\chi u + f(1-\chi)\|_{L^t(\tilde\Omega)}$$

whence, taking account of (5.16)

$$\|u^1\|_{W^{2,t}_\beta(\Omega_\varepsilon\backslash B_{\frac{\delta}{\varepsilon}}(\xi))} \le C\|f\|_{L^t(\tilde\Omega)} + C\varepsilon^{\beta+2}\|f\|_{L^t_{\beta+2}(\Omega_\varepsilon)}. \tag{5.17}$$

Combining (5.16) and (5.17), we obtain (3.7).

□

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