Spikes in Two-component Systems of Nonlinear Schrödinger Equations with Trapping Potentials

Tai-Chia Lin *†and Juncheng Wei ‡§

November 8, 2005

Abstract

Recently, two-component systems of nonlinear Schrödinger equations with trap potentials have been well-known to describe a binary mixture of Bose-Einstein condensates called a double condensate. In a double condensate, the locations of spikes can be influenced by the interspecies scattering length and trap potentials so the interaction of spikes becomes complicated, and the locations of spikes are difficult to be determined. Here we study spikes of a double condensate by analyzing least energy (ground state) solutions of two-component systems of nonlinear Schrödinger equations with trap potentials. Our mathematical arguments may prove how trap potentials and the interspecies scattering length affect the locations of spikes. We use Nehari's manifold to construct least energy solutions and derive their asymptotic behaviors by some techniques of singular perturbation problems.

1 Introduction

In Bose-Einstein condensates, spikes may occur when the s-wave scattering length is negative and large. Due to Feshbach resonance, the s-wave scattering length of a single condensate can be tuned over a very large range by adjusting the externally applied magnetic field. As the s-wave scattering length of a single condensate is negative and large enough, the interactions of atoms are strongly attractive and the associated condensate tends to increase its density at the centre of the trap potential in order to lower the interaction energy (cf. [21]). Donley et al (cf. [6]) observed anisotropic atom bursts that explode from a single condensate, atoms leaving the condensate in undetected forms, and spikes appearing in the condensate wavefunction.

In a double condensate i.e. a binary mixture of Bose-Einstein condensates in two different hyperfine states $|1\rangle$ and $|2\rangle$, spikes may appear when the intraspecies scattering lengths $a_{jj}, j = 1, 2$ are negative and large enough. In this case each state tends to contract at its spike center. However, spike centers are difficult to be determined since the interspecies scattering length and trap potentials may affect the locations of spikes. Without the effect of trap potentials, the interaction of spikes are determined by the interspecies scattering length a_{12}. When a_{12} is positive and large enough, the states $|1\rangle$ and $|2\rangle$ may repel each other and form segregated domains called phase separation (cf. [24]) so spikes of states $|j\rangle$'s may repel each other and behave like two separate spikes. In contrast, if a_{12} is negative and large enough, spikes of states $|j\rangle$'s may attract each other and behave like one single spike. On the other hand, to reduce the interaction energy, the centres and configurations of trap potentials on states $|j\rangle$'s may change the locations of spikes. In this paper, we study how trap potentials and the interspecies scattering length a_{12} influence the locations of spikes in a double condensate. Our results may provide information for physical experiments to observe spikes in a double condensate.

From the Hartree-Fock theory for a double condensate (cf. [11]), the mathematical model of a double condensate is a two-component system of nonlinear Schrödinger equations given by

$$\begin{cases} \frac{\hbar^2}{2m}\triangle u_j(x) - \tilde{V}_j(x)u_j(x) + \displaystyle\sum_{i=1}^{2} \tilde{\beta}_{ij}u_i^2(x)u_j(x) = \tilde{\lambda}_j u_j & \text{in } \Omega \subseteq \mathbb{R}^n,\ n = 2, 3\,, \\ u_j > 0 \text{ in } \Omega\,, \quad u_j = 0 & \text{on } \partial\Omega\,,\ j = 1, 2, \end{cases} \tag{1.1}$$

where Ω is the region for condensate dwelling, u_j's are corresponding condensate amplitudes, $\hbar$ is Planck constant, m is the atom mass, $\tilde{\lambda}_j$'s are chemical potentials, $\tilde{\beta}_{jj} = -(N_j - 1)U_{jj}$ and $\tilde{\beta}_{12} = \tilde{\beta}_{21} = -N_2U_{12}$. Here

*Department of Mathematics, National Taiwan University, Taipei, Taiwan 106. email : tclin@math.ntu.edu.tw
†National Center of Theoretical Sciences, National Tsing Hua University, Hsinchu, Taiwan
‡Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong. email: wei@math.cuhk.edu.hk
§Supported by an Earmarked Grant of RGC of Hong Kong (Proj. No. 402503)

each $N_j \geq 1$ is a fixed number of atoms in the hyperfine state $|j\rangle$, and $U_{ij} = 4\pi\frac{\hbar^2}{2m}a_{ij}$, where a_{jj}'s and a_{12} are the intraspecies and interspecies scattering lengths. $\tilde{V}_j$ is the trapping potential for the j-th species. In physics, the usual trapping potential is given by

$$\tilde{V}_j(x) = \sum_{k=1}^{n} \tilde{a}_{j,k}(x_k - \tilde{z}_{j,k})^2 \quad \text{for } x = (x_1, \cdots, x_n) \in \Omega, j = 1, 2\,, \tag{1.2}$$

where $\tilde{a}_{j,k} \geq 0$ is the associated axial frequency, and $\tilde{z}_j = (\tilde{z}_{j,1}, \cdots, \tilde{z}_{j,n})$ is the center of the trapping potential $\tilde{V}_j$.

To obtain spikes in a double condensate, we may use Feshbach resonance to let $\tilde{\beta}_{jj}$'s, $\tilde{\lambda}_j$'s and $\tilde{a}_{j,k}$'s be very large quantities. By rescaling and some simple assumptions, the problem (1.1) with very large $\tilde{\beta}_{jj}$'s, $\tilde{\lambda}_j$'s and $\tilde{a}_{j,k}$'s is equivalent to the following singularly perturbed problem:

$$\begin{cases} \epsilon^2 \triangle u_j - \quad \hat{V}_j(x)u_j + \sum_{i=1}^{2} \beta_{ij} u_i^2 u_j = 0 \quad \text{in } \Omega, \\ \qquad\qquad u_j > 0 \text{ in } \Omega\,, \qquad u_j = 0 \quad \text{on } \partial\Omega\,,\ j = 1, 2, \end{cases} \tag{1.3}$$

where $0 < \epsilon \ll 1$ is a small parameter, and Ω is a bounded smooth domain in $\mathbb{R}^n$, and

$$\hat{V}_j(x) = \lambda_j + \sum_{k=1}^{n} a_{j,k}(x_k - z_{j,k})^2, \quad \text{for } x = (x_1, \cdots, x_n) \in \Omega\,, \tag{1.4}$$

with trap centers $z_j = (z_{j,1}, \cdots, z_{j,n}) \in \Omega, j = 1, 2$. Besides, $\lambda_j > 0$, $a_{j,k} \geq 0$ and β_{ij}'s are constants independent of ϵ. In particular, as $\hat{V}_j \equiv \lambda_j, j = 1, 2$, problem (1.3) becomes

$$\begin{cases} \epsilon^2 \triangle u_j - \quad \lambda_j u_j + \sum_{i=1}^{2} \beta_{ij} u_i^2 u_j = 0 \quad \text{in } \Omega, \\ \qquad\qquad u_j > 0 \text{ in } \Omega\,, \quad u_j = 0 \quad \text{on } \partial\Omega,\ j = 1, 2\,. \end{cases} \tag{1.5}$$

The ground states i.e. least-energy solutions of problem (1.5) have been investigated (cf. [19]). We showed that as $\beta_{12} = \beta_{21} < 0$, the spikes of ground states repel each other and reach a sphere-packing position; as $\beta_{12} = \beta_{21} > 0$, the spikes of ground states attract each other and reach at the most centered part of the domain. In this paper, we shall study the problem (1.3) with nonconstant trapping potential $\hat{V}_j$.

To solve problem (1.3), we firstly consider the domain Ω as an entire space $\mathbb{R}^n$ and formulate another problem given by

$$\begin{cases} \epsilon^2 \triangle u_j - V_j(x)u_j + \sum_{i=1}^{2} \beta_{ij} u_i^2 u_j = 0 \text{ in } \mathbb{R}^n, \\ u_j > 0,\ u_j \in H^1(\mathbb{R}^n),\ j = 1, 2, \end{cases} \tag{1.6}$$

where $n = 2, 3$, $0 < \epsilon \ll 1$ is a small parameter, β_{ij}'s are constants, and V_j's are positive and smooth functions satisfying

$$\lim_{|x| \to \infty} V_j(x) = b_j^\infty,\ j = 1, 2,\ 0 < b_j^\infty \leq +\infty, \tag{1.7}$$

$$\inf_{x\ \in \mathbb{R}^n} V_j(x) = b_j^0,\ j = 1, 2,\ 0 < b_j^0 < +\infty. \tag{1.8}$$

Certainly

$$0 < b_j^0 \leq b_j^\infty \leq +\infty, j = 1, 2. \tag{1.9}$$

Here (1.7)-(1.9) includes more general cases of trapping potentials than those of (1.4). Throughout the paper, we also assume that

$$\beta_{11} = \mu_1 > 0,\ \beta_{22} = \mu_2 > 0,\ \beta_{12} = \beta_{21} = \beta \in (-\infty, \beta_0), \tag{1.10}$$

where $\beta_0 > 0$ will be specified later. In particular, as $V_j \equiv \lambda_j, j = 1, 2$, the existence and non-existence of ground states are proved in [20]. Here we investigate the role of the two potentials V_j's on the existence of least energy solutions i.e. ground states. We also study the asymptotic behavior of ground states (when they exist) as $\epsilon \to 0$. Notice that in the rest of this paper, all the convergence with respect to $\epsilon \to 0$ may be up to a subsequence, and we may omit it for notational convenience.

2 Main Results

Now we state our main results. To this end, we need to define some energy functionals as follows:

$$E^j_{\epsilon,\Omega}[u_j] = \frac{\epsilon^2}{2}\int_\Omega |\nabla u_j|^2 + \frac{1}{2}\int_\Omega V_j(x)u_j^2 - \frac{1}{4}\int_\Omega \mu_j u_j^4,$$

$$E^{j,0}_{\epsilon,\Omega}[u_j] = \frac{\epsilon^2}{2}\int_\Omega |\nabla u_j|^2 + \frac{1}{2}\int_\Omega b_j^0 u_j^2 - \frac{1}{4}\int_\Omega \mu_j u_j^4,$$

$$E^{j,\infty}_{\epsilon,\Omega}[u_j] = \frac{\epsilon^2}{2}\int_\Omega |\nabla u_j|^2 + \frac{1}{2}\int_\Omega b_j^\infty u_j^2 - \frac{1}{4}\int_\Omega \mu_j u_j^4,$$

$$E_{\epsilon,\Omega}[u_1,u_2] = E^1_{\epsilon,\Omega}[u_1] + E^2_{\epsilon,\Omega}[u_2] - \frac{\beta}{2}\int_\Omega u_1^2 u_2^2,$$

$$E^0_{\epsilon,\Omega}[u_1,u_2] = E^{1,0}_{\epsilon,\Omega}[u_1] + E^{2,0}_{\epsilon,\Omega}[u_2] - \frac{\beta}{2}\int_\Omega u_1^2 u_2^2,$$

$$E^\infty_{\epsilon,\Omega}[u_1,u_2] = E^{1,\infty}_{\epsilon,\Omega}[u_1] + E^{2,\infty}_{\epsilon,\Omega}[u_2] - \frac{\beta}{2}\int_\Omega u_1^2 u_2^2,$$

for $j = 1,2$, $\Omega \subseteq \mathbb{R}^n$ is a domain and $(u_1,u_2) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$. By Sobolev embedding, $E^j_{\epsilon,\Omega}$, $E^{j,0}_{\epsilon,\Omega}$, $E^{j,\infty}_{\epsilon,\Omega}$, $E_{\epsilon,\Omega}$, $E^0_{\epsilon,\Omega}$ and $E^\infty_{\epsilon,\Omega}$ are well-defined.

Let

$$N^j_{\epsilon,\Omega} \equiv \left\{u_j \geq 0,\ u_j \in H^1(\mathbb{R}^n) : \epsilon^2\int_\Omega |\nabla u_j|^2 + \int_\Omega V_j u_j^2 = \int_\Omega \mu_j u_j^4\right\},\ j = 1,2, \tag{2.1}$$

$$N_{\epsilon,\Omega} \equiv \left\{(u_1,u_2) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) : \epsilon^2\int_\Omega |\nabla u_j|^2 + \int_\Omega V_j u_j^2 = \int_\Omega \mu_j u_j^4 + \beta\int_\Omega u_1^2u_2^2,\ j = 1,2.\right\}. \tag{2.2}$$

Similarly, we can also define $N^{j,0}_{\epsilon,\Omega}$, $N^{j,\infty}_{\epsilon,\Omega}$, $N^0_{\epsilon,\Omega}$ and $N^\infty_{\epsilon,\Omega}$. Again, let

$$C^j_{\epsilon,\Omega} = \inf_{u_j \in N^j_{\epsilon,\Omega}} E^j_{\epsilon,\Omega}[u_j], \tag{2.3}$$

$$C_{\epsilon,\Omega} = \inf_{\substack{(u_1,u_2)\in N_{\epsilon,\Omega}\\ u_1,u_2\geq 0}} E_{\epsilon,\Omega}[u_1,u_2]. \tag{2.4}$$

Furthermore, we may define $C^{j,0}_{\epsilon,\Omega}$, $C^{j,\infty}_{\epsilon,\Omega}$, $C^0_{\epsilon,\Omega}$ and $C^\infty_{\epsilon,\Omega}$, respectively. When $\Omega = \mathbb{R}^n$, we omit the index $\mathbb{R}^n$; when $\Omega = B_k(0)$, we replace $B_k(0)$ by k. Our main concern is the existence and asymptotic behavior of ground states. Hereafter, $(u_{\epsilon,1},u_{\epsilon,2})$ is called a ground state if $(u_{\epsilon,1},u_{\epsilon,2})$ satisfies (1.6) and

$$C_\epsilon = E[u_{\epsilon,1},u_{\epsilon,2}] = \inf_{\substack{(u_1,u_2)\in N_\epsilon\\ u_1,u_2\geq 0}} E_\epsilon[u_1,u_2]. \tag{2.5}$$

It is easy to see that such a $(u_{\epsilon,1},u_{\epsilon,2})$ has the smallest energy among all solutions of (1.6).

Let w be the unique solution of

$$\begin{cases} \triangle w - w + w^3 = 0 \text{ in } \mathbb{R}^n \\ w(0) = \max\limits_{y\in\mathbb{R}^n} w(y),\ w > 0 \text{ in } \mathbb{R}^n,\ w(y) \to 0 \text{ as } |y| \to +\infty. \end{cases} \tag{2.6}$$

Then $w_{\lambda_j,\mu_j}(y) = \sqrt{\frac{\lambda_j}{\mu_j}}w(\sqrt{\lambda_j}y)$ satisfies

$$\triangle w_{\lambda_j,\mu_j} - \lambda_j w_{\lambda_j,\mu_j} + \mu_j w^3_{\lambda_j,\mu_j} = 0 \text{ in } \mathbb{R}^n. \tag{2.7}$$

We recall the following basic theorem.

THEOREM A (Theorem 3.3 of [19]). *Suppose $0 < b_j^\infty < +\infty$. Consider the following problem*

$$C_1^\infty = \inf_{\substack{(u_1,u_2)\in N_1^\infty\\ u_1\not\equiv 0,u_2\not\equiv 0}} E_1^\infty[u_1,u_2]. \tag{2.8}$$

(1) If $0<\beta<\beta_0<\sqrt{\mu_1\mu_2}$, then C_1^∞ is attained. Let (u_1^∞, u_2^∞) be a minimizer, then (u_1^∞, u_2^∞) are positive, radially symmetric and strictly decreasing.

(2) If $\beta<0$, then C_1^∞ is never attained and

$$C_1^\infty=\left((b_1^\infty)^{\frac{4-n}{2}}\mu_1^{-1}+(b_2^\infty)^{\frac{4-n}{2}}\mu_2^{-1}\right)I[w]. \tag{2.9}$$

Hereafter, the quantity $I[w]$ is defined by

$$I[w]=\int_{\mathbb{R}^n}\frac{1}{2}|\nabla w|^2+\frac{1}{2}w^2-\frac{1}{4}w^4\,dx\,.$$

We are ready to state our theorems. We divide our results into two parts. The first part is on the existence of ground state solution for *any* $\epsilon>0$. The second part concerns the asymptotic behavior of ground state solutions as $\epsilon\to 0$. Our first theorem concerns the case $b_1^\infty=+\infty$ or $b_2^\infty=+\infty$.

THEOREM 2.1.

(a) If $b_1^\infty=b_2^\infty=+\infty$, then a ground state solution $(u_{\epsilon,1},u_{\epsilon,2})$ to (1.6) always exists.

(b) If $b_1^\infty=+\infty$, $b_2^\infty<+\infty$ and

$$C_\epsilon<C_\epsilon^1+\epsilon^n(b_2^\infty)^{\frac{4-n}{2}}\mu_2^{-1}I[w]\,, \tag{2.10}$$

then a ground state solution $(u_{\epsilon,1},u_{\epsilon,2})$ to (1.6) exists.

(c) If $b_2^\infty=\infty$, $b_1^{+\infty}<+\infty$ and

$$C_\epsilon<C_\epsilon^2+\epsilon^n(b_1^\infty)^{\frac{4-n}{2}}\mu_1^{-1}I[w]\,, \tag{2.11}$$

then a ground state solution $(u_{\epsilon,1},u_{\epsilon,2})$ to (1.6) exists.

REMARK. *Trapping potentials given by (1.4) satisfy (a).*

The inequality (2.10) is a sharp condition for the existence of ground state solutions. A counterexample can be given by

COROLLARY 2.2. *Assume $V_1(x)=\lambda_1+|x|^2$ and $V_2(x)=\lambda_2$, $\forall x\in\mathbb{R}^n$, where λ_j'x are positive constants. Then*

(i) (2.10) holds if $0<\beta<\beta_0$, where $\beta_0\in(0,\sqrt{\mu_1\mu_2})$ is defined in Theorem A.

(ii) If $\beta<0$, then there is no ground state solution of (1.6), and

$$C_\epsilon=C_\epsilon^1+\epsilon^n(b_2^\infty)^{\frac{4-n}{2}}\mu_2^{-1}I[w]+o(\epsilon^n)\ \text{as}\ \epsilon\to 0, \tag{2.12}$$

where $o(1)$ is a small quantity tending to zero as ϵ goes to zero.

Our second theorem concerns the case $0<b_j^\infty<+\infty$, $j=1,2$ as follows:

THEOREM 2.3. *Suppose $0<b_j^\infty<+\infty$, $j=1,2$. Assume that*

$$C_\epsilon<\min\left\{C_\epsilon^1+\epsilon^n(b_2^\infty)^{\frac{4-n}{2}}\mu_2^{-1}I[w],C_\epsilon^2+\epsilon^n(b_1^\infty)^{\frac{4-n}{2}}\mu_1^{-1}I[w],C_\epsilon^\infty\right\}, \tag{2.13}$$

then a ground state solution $(u_{\epsilon,1},u_{\epsilon,2})$ of (1.6) exists.

When b_j^0's and b_j^∞'s have suitable control, and $0<\epsilon\ll 1$ is a small parameter, all conditions of Theorems 2.1 and 2.3 can be verified by

THEOREM 2.4. *Assume that $\beta<0$ and*

$$0<b_j^0<b_j^\infty\leq+\infty,\ j=1,2. \tag{2.14}$$

Then for ϵ sufficiently small, a ground state solution $(u_{\epsilon,1},u_{\epsilon,2})$ of (1.6) exists. Furthermore, $u_{\epsilon,j}$ has only one local maximum point P_j^ϵ, $j=1,2$ satisfying

$$V_j(P_j^\epsilon)\ \to\ \inf_{x\in\mathbb{R}^n}V_j(x)=b_j^0,\quad j=1,2, \tag{2.15}$$

$$\frac{|P_1^\epsilon-P_2^\epsilon|}{\epsilon}\ \to\ +\infty\,, \tag{2.16}$$

$$U_{\epsilon,j}(y)=u_{\epsilon,j}(P_j^\epsilon+\epsilon y)\ \to\ w_{b_j^0,\mu_j}(y)\quad \text{as}\ \epsilon\to 0\,. \tag{2.17}$$

REMARK.

(a) *(2.15) shows the spikes are trapped at the minimum points of $V_j(x)$. (2.16) says that on the $O(\epsilon)$-scale, the spikes are separated.*

(b) *The inequality (2.14) is crucial for the existence of ground state solutions. Corollary2.2(ii) may give a counterexample for the case $b_2^0 = b_2^\infty < +\infty$.*

Theorem 2.4 can be generalized to the case of bounded smooth domains given by

COROLLARY 2.5. *Assume that $\beta < 0$, Ω is a bounded smooth domain in $\mathbb{R}^n$, and*

$$0 < \inf_{x\in\Omega} V_j(x) < \inf_{x\in\partial\Omega} V_j(x) \le \sup_{x\in\partial\Omega} V_j(x) < +\infty, \ j = 1, 2. \tag{2.18}$$

Then for ϵ sufficiently small,

$$\begin{cases} \epsilon^2 \triangle u_j - & V_j(x) u_j + \sum_{i=1}^{2} \beta_{ij} u_i^2 u_j = 0 \quad \text{in } \Omega, \\ & u_j > 0 \text{ in } \Omega, \qquad u_j = 0 \quad \text{on } \partial\Omega\ ,\ j = 1, 2, \end{cases} \tag{2.19}$$

has a ground state solution $(u_{\epsilon,1}, u_{\epsilon,2})$. Each $u_{\epsilon,j}$ has only one local maximum point P_j^ϵ, satisfying

$$V_j(P_j^\epsilon) \to \inf_{x\in\Omega} V_j(x), \ j = 1, 2\,, \tag{2.20}$$

(2.16) and (2.17).

If $\beta > 0$, things become more complicated. To state the results, we may define

$$N_{\lambda_1,\lambda_2} = \left\{ (u_1, u_2) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\nabla u_j|^2 + \lambda_j \int_{\mathbb{R}^n} u_j^2 = \mu_j \int_{\mathbb{R}^n} u_j^4 + \beta \int_{\mathbb{R}^n} u_1^2 u_2^2, \ j = 1, 2. \right\}, \tag{2.21}$$

where λ_1, λ_2 are two positive numbers. By Theorem 3.3 of [19], The following minimization problem attains a solution:

$$\rho(\lambda_1, \lambda_2) = \inf_{\substack{(u_1,u_2)\in N_{\lambda_1,\lambda_2} \\ u_1 \ge 0, u_2 \ge 0}} \left\{ \sum_{j=1}^{2} \left[\frac{1}{2} \int_{\mathbb{R}^n} |\nabla u_j|^2 + \frac{1}{2} \lambda_j \int_{\mathbb{R}^n} u_j^2 - \frac{1}{4} \mu_j \int_{\mathbb{R}^n} u_j^4 \right] - \frac{1}{2} \beta \int_{\mathbb{R}^n} u_1^2 u_2^2 \right\}. \tag{2.22}$$

Then we have

THEOREM 2.6. *Suppose $0 < \beta < \beta_0$. Assume that*

$$\inf_{x\in\mathbb{R}^n} \rho(V_1(x), V_2(x)) < \min \left\{ \left[(b_1^0)^{\frac{4-n}{2}} \mu_1^{-1} + (b_2^\infty)^{\frac{4-n}{2}} \mu_2^{-1} \right] I[w], \left[(b_1^\infty)^{\frac{4-n}{2}} \mu_1^{-1} + (b_2^0)^{\frac{4-n}{2}} \mu_2^{-1} \right] I[w], C_1^\infty \right\}, \tag{2.23}$$

then

(i) *$(u_{\epsilon,1}, u_{\epsilon,2})$ is attained.*

(ii) *Suppose*

$$\inf_{x\in\mathbb{R}^n} \rho(V_1(x), V_2(x)) < \left[(b_1^0)^{\frac{4-n}{2}} \mu_1^{-1} + (b_2^0)^{\frac{4-n}{2}} \mu_2^{-1} \right] I[w]. \tag{2.24}$$

Let P_j^ϵ be the unique local maximum points of $u_{\epsilon,j}$ and $u_{\epsilon,j}(P_j^\epsilon + \epsilon y) := U_{\epsilon,j}(y)$. Then as $\epsilon \to 0$, $(U_{\epsilon,1}, U_{\epsilon,2}) \to (U_{0,1}, U_{0,2})$, where $(U_{0,1}, U_{0,2})$ satisfies $\triangle U_{0,j} - V_j(P^0) U_{0,j} + \sum_{i=1}^{2} \beta_{ij} U_{0,j} (U_{0,i})^2 = 0$,

$$\rho(V_1(P_1^\epsilon), V_2(P_2^\epsilon)) \to \inf_{x\in\mathbb{R}^n} \rho(V_1(x), V_2(x)) = \rho(V_1(P^0), V_2(P^0))\,, \tag{2.25}$$

and

$$\frac{|P_1^\epsilon - P_2^\epsilon|}{\epsilon} \to 0\,, \quad P_j^\epsilon \to P^0\,, \quad j = 1, 2\,. \tag{2.26}$$

(iii) Suppose

$$\left[(b_1^0)^{\frac{4-n}{2}}\mu_1^{-1}+(b_2^0)^{\frac{4-n}{2}}\mu_2^{-1}\right]I[w]<\inf_{x\in\mathbb{R}^n}\rho(V_1(x),V_2(x)). \tag{2.27}$$

Then we have

$$V_j(P_j^\epsilon)\to b_j^0, \tag{2.28}$$

$$U_{\epsilon,j}(y)=u_{\epsilon,j}(P_j^\epsilon+\epsilon y)\to w_{b_j^0,\mu_j}(y)\,,\quad j=1,2\,. \tag{2.29}$$

REMARK.

(a) *In general, condition (2.23) is difficult to check. However, if $V_1(x)=V_2(x)=V(x)$ and $\inf\limits_{x\in\mathbb{R}^n}V(x)<\lim\limits_{|x|\to+\infty}V(x)$, then (2.23) is satisfied.*

(b) *Assume V_j's are independent of β. Then by our results in [19] and [20], we have*

$$\forall x\in\mathbb{R}^n\,,\rho(V_1(x),V_2(x))=\left(V_1(x)^{\frac{4-n}{2}}\mu_1^{-1}+V_2(x)^{\frac{4-n}{2}}\mu_2^{-1}\right)I[w]+O(\beta)\ as\ \beta\to0+\,.$$

Hence as β sufficiently small, $\inf\limits_{x\in\mathbb{R}^n}\rho(V_1(x),V_2(x))$ is determined by $\inf\limits_{x\in\mathbb{R}^n}\left(V_1(x)^{\frac{4-n}{2}}\mu_1^{-1}+V_2(x)^{\frac{4-n}{2}}\mu_2^{-1}\right)$. Then by (1.4) and (1.8), we may obtain (2.27) if $z_1\neq z_2$, where z_j's are defined in (1.4).

Theorem 2.6 can also be generalized to the case of bounded smooth domains as follows:

COROLLARY 2.7. *Assume $0<\beta<\beta_0$, Ω is a bounded smooth domain in $\mathbb{R}^n$, (2.18) holds, and*

$$\inf_{x\in\Omega}\rho(V_1(x),V_2(x))<\min\left\{\left[(b_1^0)^{\frac{4-n}{2}}\mu_1^{-1}+(b_2^\infty)^{\frac{4-n}{2}}\mu_2^{-1}\right]I[w],\left[(b_1^\infty)^{\frac{4-n}{2}}\mu_1^{-1}+(b_2^0)^{\frac{4-n}{2}}\mu_2^{-1}\right]I[w],C_1^\infty\right\}, \tag{2.30}$$

where $b_j^0=\inf\limits_{x\in\Omega}V_j(x)$ and $b_j^\infty=\inf\limits_{x\in\partial\Omega}V_j(x)$ for $j=1,2$. Then

(i) *for ϵ sufficiently small, the problem (2.19) has a ground state solution $(u_{\epsilon,1},u_{\epsilon,2})$. Each $u_{\epsilon,j}$ has only one local maximum point $P_j^\epsilon\in\Omega$.*

(ii) *Suppose*

$$\inf_{x\in\Omega}\rho(V_1(x),V_2(x))<\left[(b_1^0)^{\frac{4-n}{2}}\mu_1^{-1}+(b_2^0)^{\frac{4-n}{2}}\mu_2^{-1}\right]I[w]. \tag{2.31}$$

Let $u_{\epsilon,j}(P_j^\epsilon+\epsilon y):=U_{\epsilon,j}(y)$. Then as $\epsilon\to0$, $(U_{\epsilon,1},U_{\epsilon,2})\to(U_{0,1},U_{0,2})$, where $(U_{0,1},U_{0,2})$ satisfies $\triangle U_{0,j}-V_j(P^0)U_{0,j}+\sum\limits_{i=1}^{2}\beta_{ij}U_{0,j}(U_{0,i})^2=0$,

$$\rho(V_1(P_1^\epsilon),V_2(P_2^\epsilon))\to\inf_{x\in\Omega}\rho(V_1(x),V_2(x))=\rho(V_1(P^0),V_2(P^0))\,, \tag{2.32}$$

and

$$\frac{|P_1^\epsilon-P_2^\epsilon|}{\epsilon}\to0\,,\quad P_j^\epsilon\to P^0\in\Omega\,,\quad j=1,2\,. \tag{2.33}$$

(iii) *Suppose*

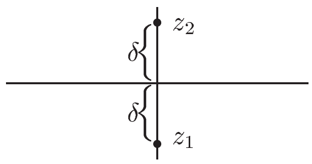
$$\left[(b_1^0)^{\frac{4-n}{2}}\mu_1^{-1}+(b_2^0)^{\frac{4-n}{2}}\mu_2^{-1}\right]I[w]<\inf_{x\in\Omega}\rho(V_1(x),V_2(x)). \tag{2.34}$$

Then (2.28) and (2.29) hold.

REMARK. *Both Corollary 2.5 and 2.7 are applicable to solve the problem (1.3).*

When $\beta>0$ and $V_1(x)\neq V_2(x)$, there is a competition between attractions of spikes and trap potential wells. If attraction of potential wells is strong enough, (2.27) holds. However, if attraction of potential wells is not strong enough, (2.24) holds. In the following example, we show both (2.24) and (2.27) can happen.

EXAMPLE 2.1. *We take the example*
$V_j(x)=\lambda_j+\sum\limits_{k=1}^{n}a_{j,k}(x_k-z_{j,k})^2$, *j=1,2,*
where $z_1=-\delta e_n$, $z_2=\delta e_n$.
Now let $\delta=l\sqrt{\beta}$, $z_{j,k}=0$, $k=1,...,n-1$.
Then we have



THEOREM 2.8. *There exists a l_0 such that for $\beta > 0$ sufficiently small,*

$$\begin{aligned} &if\ l < l_0,\ (2.24)\ holds\ and\ the\ spikes\ come\ together,\\ &if\ l > l_0,\ (2.27)\ holds\ and\ the\ spikes\ separate. \end{aligned}$$

For single Schrödinger equations,

$$\epsilon^2 \Delta u - V(x)u + f(u) = 0, u > 0 \text{ in} \mathbb{R}^n, \tag{2.35}$$

there have been many investigations in the past decade. Various results on existence and concentration phenomenon have been obtained. We refer the reader for instance to [1, 3, 2, 4, 5, 8, 9, 10, 12, 14, 15, 16, 17, 18, 22, 23, 25, 26] and the references therein. In particular, Rabinowitz [22] showed that (2.35) has a positive ground state for "every $\epsilon > 0$" if $\limsup_{|x|\to\infty} V(x) = \sup_{x\in\mathbb{R}^n} V(x)$ or if $\liminf_{|x|\to\infty} V(x) > \inf_{x\in\mathbb{R}^n} V(x)$. (Similar results were also obtained in [7].) Theorem 2.1 and 2.3 generalize some results of Rabinowitz [22] and Ding-Ni [7] from a single nonlinear Schrödinger equation to the system (1.6). On the other hand, X. Wang [25] studied the asymptotic behavior of ground states of (2.35) and showed that the ground states concentrate at a global minimum point of $V(x)$. Theorem 2.4 and 2.6 extend results of X.Wang [25] to two-component systems of nonlinear Schrödinger equations. The main difficulty in our proofs is that the two components u_1 and u_2 interact with each other. It will be interesting (and more difficult) to study systems of nonlinear Schrödinger equations of three or more components.

The rest of this paper is organized as follows:
In Section 3, we develop some results for equations on B_k and use these results to approximate equations on the entire space $\mathbb{R}^n$. Then we prove Theorem 2.3 in Section 4. In Section 5 and 6, we show Theorem 2.4 and 2.6, respectively for $b_j^\infty < +\infty, j = 1, 2$. In Section 7, we complete the proof of Theorem 2.1, 2.4 and 2.6. Finally, we prove Theorem 2.8, Corollary 2.2, 2.5 and 2.7 in Section 8.

3 Equations on B_k

In this section, we study $C_{\epsilon,B_k} = C_{\epsilon,k}$ and problem (1.6) on B_k

$$\begin{cases} \epsilon^2 \triangle u_j(x) - V_j(x)u_j(x) + \sum_{i=1}^{2} \beta_{ij} u_i^2(x) u_j(x) = 0 \text{ in } B_k,\ u_j > 0,\ j = 1, 2, \\ u_j = 0 \text{ on } \partial B_k. \end{cases} \tag{3.1}$$

Combining Lemma 2.1-Lemma 2.3 of [19], we have

LEMMA 3.1. *There exists a $\beta_0 \in (0, \sqrt{\mu_1 \mu_2})$ such that for $-\infty < \beta < \beta_0$, $C_{\epsilon,k}$ is obtained by some $(u_{1,k}, u_{2,k})$ with the following properties*

(1) $u_{1,k} > 0$, $u_{2,k} > 0$ and satisfies (3.1),

(2) $(u_{1,k}, u_{2,k})$ has the least energy among all possible positive solutions of (3.1), and

$$c_1 \epsilon^n \le \int_{B_k} u_{j,k}^4 \le c_2 \epsilon^n,\ j = 1, 2, \tag{3.2}$$

where c_1 and c_2 are independent of $\epsilon \le 1$, $k \ge 1$.

(3) If $\beta < 0$, then

$$C_{\epsilon,k} = \inf_{\substack{(u_1,u_2)\in H_0^1(B_k)\times H_0^1(B_k)\\ u_1\not\equiv 0, u_2\not\equiv 0}} \sup_{s,t>0} E_{\epsilon,k}[\sqrt{s}u_1, \sqrt{t}u_2]. \tag{3.3}$$

If $0 < \beta < \beta_0$, then

$$C_{\epsilon,k} \ge \inf_{\substack{(u_1,u_2)\in H_0^1(B_k)\times H_0^1(B_k)\\ u_1\not\equiv 0, u_2\not\equiv 0}} \sup_{s,t>0} E_{\epsilon,k}[\sqrt{s}u_1, \sqrt{t}u_2]. \tag{3.4}$$

REMARK. *The β_0 in Lemma 3.1 can be chosen to be independent of $\epsilon \le 1$ and $k \ge 1$. From now on we fix this β_0 for all $\epsilon \le 1, k \ge 1$.*

Before we state the main results of this section, we need a technical lemma which will be used throughout the paper.

Let $\beta_{(u_1,u_2)}(t_1, t_2) = E_{\epsilon,\Omega}[\sqrt{t_1}u_1, \sqrt{t_2}u_2]$, $\beta_{u_j}(t_j) = E_{\epsilon,0}[\sqrt{t_j}u_j, 0]$, $j = 1, 2$.

LEMMA 3.2.

(a) $\beta_{u_j}(t_j)$ has a unique critical point $t_j > 0$ and

$$|t_j - 1| \le \frac{\left|\epsilon^2 \int_\Omega |\nabla u_j|^2 + \int_\Omega V_j u_j^2 - \int_\Omega \mu_j u_j^4\right|}{\mu_j \int_\Omega u_j^4}, \ j = 1, 2. \tag{3.5}$$

(b) $\beta_{(u_1,u_2)}(t_1, t_2)$ has a unique critical point $(\widetilde{t}_1, \widetilde{t}_2)$ such that $t_j > 0, j = 1, 2$. Moreover, we have

$$|\widetilde{t}_1 - 1| + |\widetilde{t}_2 - 1| \le c\left(\sum_{i=1}^{2} \frac{1}{\int_\Omega u_i^4}\right) \sum_{j=1}^{2} \left| \int_\Omega \left(\epsilon^2 |\nabla u_j|^2 + V_j u_j^2 - \mu_j u_j^4 - \beta u_1^2 u_2^2\right) \right|, \tag{3.6}$$

where c is a positive constant independent of ϵ.

PROOF. We only need to prove (b). The existence and uniqueness of $(\widetilde{t}_1, \widetilde{t}_2)$ have been proved in Claim 2 of [19]. We just need to prove (3.6). In fact, we have

$$\epsilon^2 \int_\Omega |\nabla u_1|^2 + \int_\Omega V_1 u_1^2 - \int_\Omega \mu_1 u_1^4 - \beta \int_\Omega u_1^2 u_2^2 = \mu_1(\widetilde{t}_1 - 1) \int_\Omega u_1^4 + \beta(\widetilde{t}_2 - 1) \int_\Omega u_1^2 u_2^2, \tag{3.7}$$

$$\epsilon^2 \int_\Omega |\nabla u_2|^2 + \int_\Omega V_2 u_2^2 - \int_\Omega \mu_2 u_2^4 - \beta \int_\Omega u_1^2 u_2^2 = \mu_2(\widetilde{t}_2 - 1) \int_\Omega u_2^4 + \beta(\widetilde{t}_1 - 1) \int_\Omega u_1^2 u_2^2. \tag{3.8}$$

Since $\beta < \sqrt{\mu_1 \mu_2}$, we have

$$\mu_1 \mu_2 \int_\Omega u_1^4 \int_\Omega u_2^4 - \beta^2 \left(\int_\Omega u_1^2 u_2^2\right)^2 \ge (\mu_1 \mu_2 - \beta^2) \int_\Omega u_1^4 \int_\Omega u_2^4. \tag{3.9}$$

(3.6) then follows (3.7)-(3.9). □

From (3.2) and using the system (3.1), we may obtain

$$\epsilon^2 \int_{B_k} |\nabla u_{j,k}|^2 + \int_{B_k} V_j u_{j,k}^2 \le c_3 \epsilon^n$$

Now we extend $u_{j,k}$ equal to 0 outside B_k. Hence $||u_{j,k}||_{H^1(\mathbb{R}^n)} \le c_4(\epsilon)$, $j = 1, 2$.
We now study the asymptotic behavior of $u_{j,k}$ as $k \to \infty$. Since $||u_{j,k}||_{H^1(\mathbb{R}^n)} \le c_4(\epsilon)$, we obtain that as $k \to \infty$, $u_{j,k} \rightharpoonup \bar{u}_j$, $j = 1, 2$, where $\bar{u}_j \ge 0$ and $\bar{u}_j \in H^1(\mathbb{R}^n)$.

The following is the main result of this section.

THEOREM 3.3.

(a) As $k \to \infty$, $C_{\epsilon,k} \to C_\epsilon$;

(b) If $\bar{u}_j \not\equiv 0$, then $(\bar{u}_1, \bar{u}_2)$ is a solution of (1.6) and attains C_ϵ, i.e. $(\bar{u}_1, \bar{u}_2)$ is a ground state.

PROOF. (a) The proof of (a) is actually standard. We include it for reader's convenience. By our definition, we have that $C_{\epsilon,k}$ is a decreasing function in k and $C_\epsilon \le C_{\epsilon,k}$. Therefore

$$C_\epsilon \le \widetilde{C}_\epsilon := \lim_{k \to \infty} C_{\epsilon,k} \le C_{\epsilon,k} \tag{3.10}$$

with $E[u_1, u_2] < C_\epsilon + \frac{1}{M}$ and M large.
Next, for each $(u_1, u_2) \in N_\epsilon$, since $\overline{\mathcal{C}_0^\infty(\mathbb{R}^n)} = H^1(\mathbb{R}^n)$, we obtain two functions $(\varphi_1, \varphi_2) \in \left(\mathcal{C}_0^\infty(\mathbb{R}^n)\right)^2$ such that

$$||\varphi_j - u_j||_{H^1(\mathbb{R}^n)} \le \frac{1}{M}, \ j = 1, 2, \ |E[\varphi_1, \varphi_2] - C_\epsilon| \le \frac{1}{M} \tag{3.11}$$

By (3.11), we also have

$$\left|\epsilon^2 \int_{\mathbb{R}^n} |\nabla \varphi_j|^2 + \int_{\mathbb{R}^n} V_j \varphi_j^2 - \int_{\mathbb{R}^n} \mu_j \varphi_j^4 - \beta \int_{\mathbb{R}^n} \varphi_1^2 \varphi_2^2\right| < \frac{c}{M}, \ j = 1, 2. \tag{3.12}$$

Since $C_\epsilon \le E[u_1,u_2] \le C_\epsilon + \frac{1}{M}$, we see that $c_5 \le \int_{\mathbb{R}^n} u_j^4 \le c_6$, where c_5, c_6 are independent of M.
By Claim 2 of Lemma 2.1 of [19] or Lemma2.2, the function $E[\sqrt{s}\varphi_1, \sqrt{t}\varphi_2]$ has a unique maximum, denoted by $(\widetilde{s}, \widetilde{t})$, then by Lemma 2.2,

$$|\widetilde{s}-1| < \frac{c_7}{M},\ |\widetilde{t}-1| < \frac{c_8}{M}.$$

Let $\mathrm{supp}(\varphi_1) \cup \mathrm{supp}(\varphi_2) \subset B_k(0)$ for some $k \le 1$, then $(\sqrt{\widetilde{s}}\varphi_1, \sqrt{\widetilde{t}}\varphi_2) \in N_{\epsilon, B_k(0)}$ and hence

$$\begin{aligned} C_{\epsilon,k} &\le E[\sqrt{s}\varphi_1, \sqrt{t}\varphi_2] \\ &\le E[\varphi_1,\varphi_2] + O\Big(\frac{1}{M}\Big) \\ &\le E[u_1,u_2] + O\Big(\frac{1}{M}\Big) \\ &\le C_\epsilon + O\Big(\frac{1}{M}\Big) \end{aligned}$$

Letting $k \to \infty$, we have $\widetilde{C}_\epsilon \le C_\epsilon$.

This proves (a).

(b) Let $\overline{u}_j \not\equiv 0$. Then by standard elliptic regularity theory and the Maximum Principle, $\overline{u}_j > 0, \overline{u}_j \in H^1(\mathbb{R}^n)$. Hence $C_\epsilon \le E[\overline{u}_1, \overline{u}_2]$. To show the other inequality, we let K_M be such that $C_\epsilon \le C_{\epsilon,k} \le C_\epsilon + \frac{1}{M}, \forall k \ge K_M$. By Fatou's Lemma

$$C_{\epsilon,k} = E_k[u_{1,k}, u_{2,k}] = \frac{1}{4}\sum_{j=1}^{2}\left(\epsilon^2\int_{B_k}|\nabla u_{j,k}|^2 + \int_{B_k} V_j u_{j,k}^2\right)$$

$$\begin{aligned} C_\epsilon \ge \lim_{k\to+\infty} C_{\epsilon,k} &= \frac{1}{4}\sum_{j=1}^{2}\lim_{k\to+\infty}\left(\epsilon^2\int_{B_k}|\nabla u_{j,k}|^2 + \int_{B_k} V_j u_{j,k}^2\right) \\ &= \frac{1}{4}\sum_{j=1}^{2}\lim_{k\to+\infty}\left(\epsilon^2\int_{\mathbb{R}^n}|\nabla u_{j,k}|^2 + \int_{\mathbb{R}^n} V_j u_{j,k}^2\right) \\ &\ge \frac{1}{4}\sum_{j=1}^{2}\left(\epsilon^2\int_{\mathbb{R}^n}\liminf_{k\to+\infty}|\nabla u_{j,k}|^2 + \int_{\mathbb{R}^n}\liminf_{k\to+\infty} V_j u_{j,k}^2\right) \\ &= \frac{1}{4}\sum_{j=1}^{2}\left(\epsilon^2\int_{\mathbb{R}^n}|\nabla \overline{u}_j|^2 + \int_{\mathbb{R}^n} V_j \overline{u}_j^2\right) \\ &= E[\overline{u}_1, \overline{u}_2]. \end{aligned}$$

Hence $C_\epsilon \ge E[\overline{u}_1, \overline{u}_2] \ge C_\epsilon$ and (b) is thus proved. □

4 Proof of Theorem 2.3

Let us first assume that

$$0 < b_j^0 \le b_j^\infty < +\infty,\ j = 1, 2. \tag{4.1}$$

By Theorem 3.3, we just need to show that $\overline{u}_j \not\equiv 0,\ j = 1,2$. We prove it by contradiction. We exclude three cases:
Case 1: $\overline{u}_1 \equiv 0, \overline{u}_2 \equiv 0$
Case 2: $\overline{u}_1 \not\equiv 0, \overline{u}_2 \equiv 0$
Case 3: $\overline{u}_1 \equiv 0, \overline{u}_2 \not\equiv 0$
Since the proof of Case 3 is similar to Case 2, we just need to prove Case 1 and Case 2.

LEMMA 4.1. *Case 1 is impossible under (2.13).*

PROOF. Suppose $\overline{u}_1 \equiv \overline{u}_2 \equiv 0$. This then implies that

$$u_{j,k} \to 0 \text{ in } C^2_{loc}(\mathbb{R}^n). \tag{4.2}$$

Let M, R be such that

$$|V_j(x) - b_j^\infty| < \frac{1}{M}, \quad \text{for } |x| \ge R. \tag{4.3}$$

Let $\chi_R(x)$ be such that $\chi_R(x) = 1$ for $|x| \le R$, $\chi_R(x) = 0$ for $|x| \ge 2R$. Now let us consider

$$\widetilde{u}_{j,k} = u_{j,k}(1 - \chi_R). \tag{4.4}$$

Then we have

$$\int_{\mathbb{R}^n} |\nabla \widetilde{u}_{j,k}|^2 = \int_{\mathbb{R}^n} |\nabla u_{j,k}|^2 - 2\int_{\mathbb{R}^n} \nabla u_{j,k} \cdot \nabla(u_{j,k}\chi_R) + \int_{\mathbb{R}^n} |\nabla u_{j,k}\chi_R|^2,$$
$$\lim_{k\to+\infty} \left(\left| \int_{\mathbb{R}^n} \nabla u_{j,k} \cdot \nabla(u_{j,k}\chi_R) \right| + \int_{\mathbb{R}^n} |\nabla u_{j,k}\chi_R|^2 \right) = 0.$$

We now use $o(1)$ to denote the terms that approach zero as $k \to \infty$. Thus we can write

$$\int_{\mathbb{R}^n} |\nabla \widetilde{u}_{j,k}|^2 = \int_{\mathbb{R}^n} |\nabla u_{j,k}|^2 + o(1). \tag{4.5}$$

Similarly,

$$\int_{\mathbb{R}^n} V_j \widetilde{u}_{j,k}^p = \int_{\mathbb{R}^n} V_j u_{j,k}^p + o(1), \ \forall\ 2 \le p \le 4.$$

Hence $E_{\epsilon,k}[u_{1,k}, u_{2,k}] = C_{\epsilon,k} = E_{\epsilon,k}[\widetilde{u}_{1,k}, \widetilde{u}_{2,k}] + o(1)$. Moreover,

$$\begin{aligned}
&\epsilon^2 \int_{\mathbb{R}^n} |\nabla \widetilde{u}_{j,k}|^2 + \int_{\mathbb{R}^n} b_j^\infty \widetilde{u}_{j,k}^2 + \int_{\mathbb{R}^n} \mu_j \widetilde{u}_{j,k}^4 - \beta \int_{\mathbb{R}^n} \widetilde{u}_{1,k}^2 \widetilde{u}_{2,k}^2 \\
&= \int_{\mathbb{R}^n} (V_j(x) - b_j^\infty) \widetilde{u}_{j,k}^2 + o(1) \\
&= O\Big(\frac{1}{M} \int_{\mathbb{R}^n} \widetilde{u}_{j,k}^2\Big) + o(1) \\
&= O\Big(\frac{1}{M}\Big) + o(1), \ j = 1, 2.
\end{aligned} \tag{4.6}$$

By (3.6) of Lemma 3.2, we see that the unique critical point $(\widetilde{t}_1, \widetilde{t}_2)$ of the function $E_\epsilon^\infty[\sqrt{\widetilde{t}_1}\widetilde{u}_{1,k}, \sqrt{\widetilde{t}_2}\widetilde{u}_{2,k}]$ satisfies

$$|\widetilde{t}_1 - 1| + |\widetilde{t}_2 - 1| = O\Big(\frac{1}{M}\Big) + o(1), \tag{4.7}$$

which yields

$$\begin{aligned}
E_\epsilon^\infty\Big[\sqrt{\widetilde{t}_1}\widetilde{u}_{1,k}, \sqrt{\widetilde{t}_2}\widetilde{u}_{2,k}\Big] &= E_\epsilon^\infty[\widetilde{u}_{1,k}, \widetilde{u}_{2,k}] + O\Big(\frac{1}{M}\Big) + o(1) \\
&= E_\epsilon[\widetilde{u}_{1,k}, \widetilde{u}_{2,k}] + O\Big(\frac{1}{M}\Big) + o(1) \\
&= E_\epsilon[u_{1,k}, u_{2,k}] + O\Big(\frac{1}{M}\Big) + o(1) \\
&= C_{\epsilon,k} + O\Big(\frac{1}{M}\Big) + o(1).
\end{aligned}$$

On the other hand,

$$\Big(\sqrt{\widetilde{t}_1}\widetilde{u}_{1,k}, \sqrt{\widetilde{t}_2}\widetilde{u}_{2,k}\Big) \in N_\epsilon^\infty \tag{4.8}$$

and hence

$$E_\epsilon^\infty\Big[\sqrt{\widetilde{t}_1}\widetilde{u}_{1,k}, \sqrt{\widetilde{t}_2}\widetilde{u}_{2,k}\Big] \ge C_\epsilon^\infty. \tag{4.9}$$

Consequently, $C_\epsilon^\infty \le C_{\epsilon,k} + O\big(\frac{1}{M}\big) + o(1)$. Letting $M \to +\infty$ and $k \to +\infty$, we obtain $C_\epsilon^\infty \le C_\epsilon$ which may contradict with (2.13). □

LEMMA 4.2. *Case 2 is impossible.*

PROOF. Suppose $\overline{u}_1 \not\equiv 0$, $\overline{u}_2 \equiv 0$, $\overline{u}_1 \in H^1(\mathbb{R}^n)$. Then $\overline{u}_1 > 0$ and satisfies

$$\epsilon^2 \triangle \overline{u}_1 - V_1(x)\overline{u}_1 + \mu_1 \overline{u}_1^3 = 0 \text{ in } \mathbb{R}^n. \tag{4.10}$$

Hence $\overline{u}_1 \in N_{\epsilon,1}$ and $E_{\epsilon,1}[\overline{u}_1] \ge c_1$. On the other hand, due to $(u_{1,k}, u_{2,k}) \in N_{\epsilon,k}$, we obtain

$$C_{\epsilon,k} = \frac{1}{4} \sum_{j=1}^{2} (\epsilon^2 \int_{B_k} |\nabla u_{j,k}|^2 + \int_{B_k} V_j u_{j,k}^2).$$

Consequently,

$$\lim_{k\to\infty}\left(\epsilon^2\int_{B_k\cap B_R}|\nabla u_{1,k}|^2+\int_{B_k\cap B_R}V_1u_{1,k}^2\right)\geq\epsilon^2\int_{B_R}|\nabla\overline{u}_1|^2+\int_{B_R}V_1\overline{u}_1^2$$
$$\geq c_1+O\Big(\frac{1}{R}\Big)\,.$$

The last inequality follows from the fact that $\overline{u}_1\leq e^{-c|x|}$ for some $c>0$. Notice that R may depend on ϵ.

Let χ_R be defined as in Lemma 4.1 and $\widetilde{u}_{j,k}=u_{j,k}(1-\chi_R)$. We discuss two cases as follows:

Case 2.1. $\int_{B_k}u_{1,k}^2u_{2,k}^2=o(1)$
As for the proof of Lemma 4.1, we obtain

$$\begin{aligned}\epsilon^2\int_{B_k}|\nabla\widetilde{u}_{2,k}^2|+b_2^\infty\int_{B_k}\widetilde{u}_{2,k}^2-\mu_2\int_{B_k}\widetilde{u}_{2,k}^4&=\int_{B_k}(b_2^\infty-V_2(x))\widetilde{u}_{2,k}^2+\epsilon^2\int_{B_k}|\nabla\widetilde{u}_{2,k}^2|+\int_{B_k}V_2(x)\widetilde{u}_{2,k}^2-\mu_2\int_{B_k}\widetilde{u}_{2,k}^4\\
&=O\Big(\frac{1}{M}\Big)+\epsilon^2\int_{B_k}|\nabla u_{2,k}|^2-2\epsilon^2\int_{B_k}\nabla u_{2,k}\cdot\nabla(u_{2,k}\chi_R)\\
&\quad+\int_{B_k}\epsilon^2|\nabla u_{2,k}\chi_R|^2+\int_{B_k}V_2u_{2,k}^2+\int_{B_k}V_2u_{2,k}^2[(1-\chi_R)^2-1]\\
&\quad-\mu_2\int_{B_k}u_{2,k}^4+\int_{B_k}\mu_2u_{2,k}^4[1-(1-\chi_R)^4]\\
&=O\Big(\frac{1}{M}\Big)+\beta\int_{B_k}u_{1,k}^2u_{2,k}^2+o(1)\\
&=O\Big(\frac{1}{M}\Big)+o(1).\end{aligned}\tag{4.11}$$

By Lemma 3.2 (a) and (4.11), we have

$$\widetilde{t}\left(\frac{\epsilon^2}{2}\int_{B_k}|\nabla\widetilde{u}_{2,k}|^2+b_2^\infty\int_{B_k}\widetilde{u}_{2,k}^2\right)-\frac{\mu_2\widetilde{t}^2}{4}\int_{B_k}\widetilde{u}_{2,k}^4\geq C_\epsilon^{2,\infty}=\epsilon^n\Big((b_2^\infty)^{\frac{4-n}{2}}\mu_2^{-1}I[w]\Big),$$

where $\widetilde{t}$ is such that $\widetilde{t}\widetilde{u}_{2,k}\in N_\epsilon^{2,\infty}$ satisfying

$$|\widetilde{t}-1|=O\Big(\frac{1}{M}\Big)+o(1)\,.\tag{4.12}$$

Hence

$$\begin{aligned}&\frac{\epsilon^2}{2}\int_{B_k}|\nabla\widetilde{u}_{2,k}|^2+\frac{1}{2}\int_{B_k}V_2\widetilde{u}_{2,k}^2-\frac{1}{4}\int_{B_k}\mu_2\widetilde{u}_{2,k}^4\\
=&\frac{\epsilon^2}{2}\int_{B_k}|\nabla\widetilde{u}_{2,k}|^2+\frac{1}{2}\int_{B_k}b_2^\infty\widetilde{u}_{2,k}^2-\frac{1}{4}\int_{B_k}\mu_2\widetilde{u}_{2,k}^4+O\Big(\frac{1}{M}\Big)\\
\geq&C_\epsilon^{2,\infty}+O\Big(\frac{1}{M}\Big)+o(1)\,.\end{aligned}$$

Thus

$$\Big(\frac{1}{2}-\frac{1}{4}\Big)\left(\epsilon^2\int_{B_k}|\nabla u_{2,k}|^2+\int_{B_k}V_2u_{2,k}^2\right)\geq C_\epsilon^{2,\infty}+O\Big(\frac{1}{M}\Big)+o(1)\,,$$

and

$$\begin{aligned}C_{\epsilon,k}=&\sum_{j=1}^2\frac{1}{4}\left[\epsilon^2\int_{B_k}|\nabla u_{j,k}|^2+\int_{B_k}V_ju_{j,k}^2\right]\\
\geq&C_\epsilon^{2,\infty}+o(1)+O\Big(\frac{1}{M}\Big)+C_\epsilon^1.\end{aligned}$$

Letting $k\,,M\to\infty$, we obtain

$$C_\epsilon\geq C_\epsilon^1+C_\epsilon^{2,\infty}\,,\tag{4.13}$$

which may contradict with (2.13).

Case 2.2. $\int_{B_k} u_{1,k}^2 u_{2,k}^2 \geq c > 0$

In this case, we will have

$$\int_{\mathbb{R}^n} \widetilde{u}_{1,k}^4 \geq c > 0 \tag{4.14}$$

Let us compute the equation for $\widetilde{u}_{1,k}$. Using the equation of u_k, it is easy to see that

$$\begin{aligned}&\epsilon^2 \triangle \widetilde{u}_{1,k} - b_1^\infty \widetilde{u}_{1,k} + \mu_1 \widetilde{u}_{1,k}^3 + \beta \widetilde{u}_{1,k} \widetilde{u}_{2,k}^2 \\ =&(V_1 - b_1^\infty) u_{1,k}(1-\chi_R) + \mu_1 u_{1,k}^3 [(1-\chi_R)^3 - (1-\chi_R)] + \beta u_{1,k} u_{2,k}^2 [(1-\chi_R)^3 - (1-\chi_R)] \\ &- 2\epsilon^2 \nabla u_{1,k} \cdot \nabla \chi_R - \epsilon^2 (\triangle \chi_R) u_{1,k}.\end{aligned}$$

Hence

$$\begin{aligned}&\epsilon^2 \int_{\mathbb{R}^n} |\nabla \widetilde{u}_{1,k}|^2 + b_1^\infty \int_{\mathbb{R}^n} \widetilde{u}_{1,k}^2 - \mu_1 \int_{\mathbb{R}^n} \widetilde{u}_{1,k}^4 - \beta \int_{\mathbb{R}^n} \widetilde{u}_{1,k}^2 \widetilde{u}_{2,k}^2 \\ =&O\Big(\frac{1}{M}\Big) + O\bigg(\int_{\mathbb{R}^n} u_{1,k}^4 (1-\chi_R)\chi_R + \int_{\mathbb{R}^n} u_{1,k}^2 u_{2,k}^2 (1-\chi_R)\chi_R \bigg) + \epsilon^2 \int_{\mathbb{R}^n} (\nabla u_{1,k} \cdot \nabla \chi_R) u_{1,k} + \epsilon^2 \int_{\mathbb{R}^n} u_{1,k}^2 \triangle \chi_R.\end{aligned} \tag{4.15}$$

If we first let $k \to \infty$, we obtain the right hand side of (4.15) equals to

$$O\Big(\frac{1}{M}\Big) + O\bigg(\int_{\mathbb{R}^n} \overline{u}_1^4 (1-\chi_R)\chi_R \bigg) + \epsilon^2 \int_{\mathbb{R}^n} [(\nabla \overline{u}_1 \cdot \nabla \chi_R) \overline{u}_1 + \overline{u}_1^2 \triangle \chi_R]. \tag{4.16}$$

Now we let $R \to \infty$, then all the terms of (4.16) approach zero. Similarly, we have

$$\begin{aligned}&\int_{\mathbb{R}^n} |\nabla \widetilde{u}_{2,k}|^2 + b_2^\infty \int_{\mathbb{R}^n} \widetilde{u}_{1,k}^2 - \mu_2 \int_{\mathbb{R}^n} \widetilde{u}_{2,k}^4 - \beta \int_{\mathbb{R}^n} \widetilde{u}_{1,k}^2 \widetilde{u}_{2,k}^2 \\ =&O\Big(\frac{1}{M}\Big) + o(1),\end{aligned} \tag{4.17}$$

where $o(1) \to 0$ as $k \to \infty$ first and $R \to \infty$ second.

Let $(\widetilde{t}_1, \widetilde{t}_2)$ be the unique critical point of $E_\epsilon^\infty[\sqrt{t_1}\widetilde{u}_{1,k}, \sqrt{t_2}\widetilde{u}_{2,k}]$. Then by (4.14) and Lemma 3.2, we have

$$|\widetilde{t}_1 - 1| + |\widetilde{t}_2 - 1| = O\big(\frac{1}{M}\big) + o(1)\,.$$

Hence

$$\begin{aligned}E_\epsilon^\infty[\widetilde{u}_{1,k}, \widetilde{u}_{2,k}] =& \frac{1}{4} \sum_{j=1}^2 \bigg(\int_{\mathbb{R}^n} |\nabla \widetilde{u}_{j,k}|^2 + \int_{\mathbb{R}^n} b_j^\infty \widetilde{u}_{j,k}^2 \bigg) + O\Big(\frac{1}{M}\Big) + o(1) \\ =&E_\epsilon^\infty[\sqrt{t_1}\widetilde{u}_{1,k}, \sqrt{t_2}\widetilde{u}_{2,k}] + O\Big(\frac{1}{M}\Big) + o(1) \\ \geq& C_\epsilon^\infty + O\Big(\frac{1}{M}\Big) + o(1).\end{aligned}$$

Consequently,

$$\sum_{j=1}^2 \frac{1}{4} \bigg(\int_{\mathbb{R}^n} |\nabla \widetilde{u}_{j,k}|^2 + \int_{\mathbb{R}^n} V_j \widetilde{u}_{j,k}^2 \bigg) \geq C_\epsilon^\infty + O\Big(\frac{1}{M}\Big) + o(1)\,.$$

However,

$$\begin{aligned}C_{\epsilon,k} =& \sum_{j=1}^2 \frac{1}{4} \int_{\mathbb{R}^n} (\epsilon^2 |\nabla u_{j,k}|^2 + V_j u_{j,k}^2) \\ \geq& \sum_{j=1}^2 \frac{1}{4} \int_{B_R(0)} (\epsilon^2 |\nabla u_{j,k}|^2 + V_j u_{j,k}^2) + \sum_{j=1}^2 \frac{1}{4} \int_{\mathbb{R}^n} (\epsilon^2 |\nabla \widetilde{u}_{j,k}|^2 + V_j \widetilde{u}_{j,k}^2) \\ \geq& C_\epsilon^1 + C_\epsilon^\infty + O\Big(\frac{1}{M}\Big) + o(1).\end{aligned}$$

Therefore

$$\lim_{k,M\to+\infty} C_{\epsilon,k} \geq C_\epsilon^1 + C_\epsilon^\infty > C_\epsilon^\infty.$$

which may contradict with (2.13), and we complete the proof of Lemma 4.2. □

5 Proof of Theorem2.4

In this section, we study the asymptotic behavior of $(u_{\epsilon,1}, u_{\epsilon,2})$ as $\epsilon \to 0$, when $\beta < 0$. First we need an upper bound.

LEMMA 5.1. *Suppose $\beta < 0$. Then for ϵ sufficiently small we have*

$$C_\epsilon \leq \epsilon^n \Big((b_1^0)^{\frac{4-n}{2}} \mu_1^{-1} + (b_2^0)^{\frac{4-n}{2}} \mu_2^{-1} \Big) I[w] + o(\epsilon^n) \tag{5.1}$$

PROOF. Let $V_1(x_1^0) = \min\limits_{x\in\mathbb{R}^n} V_1(x) = b_1^0$, $V_2(x_2^0) = \min\limits_{x\in\mathbb{R}^n} V_2(x) = b_2^0$. By (2.14), x_1^0, x_2^0 exist. Let

$$P_1 = x_1^0 - \big(\epsilon \log \frac{1}{\epsilon}\big)\vec{e}_1,\ P_2 = x_2^0 + \big(\epsilon \log \frac{1}{\epsilon}\big)\vec{e}_1.$$

Then it is obvious that

$$|P_1 - P_2| \geq \epsilon \log \frac{1}{\epsilon}.$$

Let δ be a small number such that $0 < \delta \ll \log \frac{1}{\epsilon}$. Set

$$u_j(x) = \sqrt{\frac{b_j^0}{\mu_j}}\, \chi_\delta\, w\Big(\sqrt{b_j^0} \cdot \frac{x - P_j}{\epsilon}\Big).$$

Then we have

$$\epsilon^2 \int_{\mathbb{R}^n} |\nabla u_j|^2 + \int_{\mathbb{R}^n} V_j u_j^2 - \int_{\mathbb{R}^n} \mu_j u_j^4 - \beta \int_{\mathbb{R}^n} u_1^2 u_2^2 = o(\epsilon^n), \tag{5.2}$$

and

$$\int_{\mathbb{R}^n} u_j^4 \geq c\epsilon^n.$$

Let $(\tilde{t}_1, \tilde{t}_2)$ be the unique critical point of $E[\sqrt{\tilde{t}_1} u_1, \sqrt{\tilde{t}_2} u_2]$. Then by Lemma 3.2 and (5.2), we obtain

$$|\tilde{t}_1 - 1| + |\tilde{t}_2 - 1| = o(1).$$

Hence

$$\begin{aligned} C_\epsilon \leq& E[\sqrt{\tilde{t}_1} u_1, \sqrt{\tilde{t}_2} u_2] \\ =& E[u_1, u_2] + o(\epsilon^n) \\ \leq& \epsilon^n \Big[(b_1^0)^{\frac{4-n}{2}} \mu_1^{-1} + (b_2^0)^{\frac{4-n}{2}} \mu_2^{-1} \Big] I[w] + o(\epsilon^n). \end{aligned}$$

□

COROLLARY 5.2. *Assume (2.14) holds. Then for ϵ sufficiently small, condition (2.13) holds. As a consequence, $(u_{\epsilon,1}, u_{\epsilon,2})$ exists.*

PROOF. When $\beta < 0$, it is easy to see that by Theorem A,

$$C_\epsilon^\infty = \epsilon^n C_1^\infty = \epsilon^n \Big[(b_1^\infty)^{\frac{4-n}{2}} \mu_1^{-1} + (b_2^\infty)^{\frac{4-n}{2}} \mu_2^{-1} \Big] I[w]. \tag{5.3}$$

By Lemma 5.1,

$$\lim_{\epsilon\to 0} \epsilon^{-n} C_\epsilon \leq \Big[(b_1^0)^{\frac{4-n}{2}} \mu_1^{-1} + (b_2^0)^{\frac{4-n}{2}} \mu_2^{-1} \Big] I[w].$$

By (2.14) and (5.3), (2.13) holds trivially. □

From Lemma 5.1, we see that

$$C_\epsilon = \frac{1}{4}\sum_{j=1}^{2}\left[\epsilon^2\int_{\mathbb{R}^n}|\nabla u_{\epsilon,j}|^2 + \int_{\mathbb{R}^n} V_j u_{\epsilon,j}^2\right] \le c\epsilon^n,$$

and hence

$$\epsilon^2\int_{\mathbb{R}^n}|\nabla u_{\epsilon,j}|^2 + \int_{\mathbb{R}^n} V_j u_{\epsilon,j}^2 \le c\epsilon^n. \tag{5.4}$$

Let x_1^ϵ be a local maximum point of $u_{\epsilon,1}$ and x_2^ϵ be a local maximum point of $u_{\epsilon,2}$. By the equations of $u_{\epsilon,j}$'s,

$$u_{\epsilon,j}(x_j^\epsilon) \ge \sqrt{\mu_j^{-1}V_j(x_j^\epsilon)} \ge c_0 > 0. \tag{5.5}$$

LEMMA 5.3. *$|x_j^\epsilon| \to \infty$ as $\epsilon \to \infty$, $j = 1,2$, is impossible.*

PROOF. Let

$$\begin{cases}(U_{\epsilon,1}^1, U_{\epsilon,2}^1) = (u_{\epsilon,1}(x_1^\epsilon + \epsilon y), u_{\epsilon,2}(x_1^\epsilon + \epsilon y)),\\ (U_{\epsilon,1}^2, U_{\epsilon,2}^2) = (u_{\epsilon,1}(x_2^\epsilon + \epsilon y), u_{\epsilon,2}(x_2^\epsilon + \epsilon y)).\end{cases} \tag{5.6}$$

Then $(U_{\epsilon,1}^l, U_{\epsilon,2}^l) \to (U_1^l, U_2^l)$, $l = 1,2$, and U_j^1's satisfy

$$\begin{cases}\triangle U_1^1 - b_1^\infty U_1^1 + \mu_1 (U_1^1)^3 + \beta U_1^1 (U_2^1)^2 = 0,\\ \triangle U_2^1 - b_2^\infty U_2^1 + \mu_2 (U_2^1)^3 + \beta U_2^1 (U_1^1)^2 = 0,\\ U_1^1(0) \ge c_0,\ U_1^1, U_2^1 \in H^1(\mathbb{R}^n).\end{cases} \tag{5.7}$$

If $U_2^1 \not\equiv 0$, then $U_1^1, U_2^1 > 0$ and

$$E_1^\infty[U_1^1, U_2^1] \ge C_1^\infty = \left[(b_1^\infty)^{\frac{4-n}{2}}\mu_1^{-1} + (b_2^\infty)^{\frac{4-n}{2}}\mu_2^{-1}\right] I[w].$$

Hence

$$\begin{aligned} C_\epsilon &\ge \frac{1}{4}\int_{B_{\epsilon R}(x_1^\epsilon)}\sum_{j=1}^{2}[|\nabla u_{j,\epsilon}|^2 + V_j u_{j,\epsilon}^2] \\ &\ge \epsilon^n E_1^\infty[U_1^1, U_2^1] + O\Big(\frac{1}{R}\Big) \\ &\ge \epsilon^n C_1^\infty + O\Big(\frac{1}{R}\Big), \end{aligned}$$

which may contradict with (5.1). Consequently, $U_2^1 \equiv 0$ and $U_1^1 = w_{b_1^\infty,\mu_1}$. Moreover, $\frac{|x_1^\epsilon - x_2^\epsilon|}{\epsilon} \to +\infty$, and

$$\frac{1}{4}\int_{B_{\epsilon R}(x_1^\epsilon)}\sum_{j=1}^{2}[\epsilon^n|\nabla u_{j,\epsilon}|^2 + V_j u_{j,\epsilon}^2] \ge \epsilon^n\left[(b_1^\infty)^{\frac{4-n}{2}}\mu_1^{-1} I[w] + O\Big(\frac{1}{R}\Big)\right]. \tag{5.8}$$

Similarly, we also have

$$\frac{1}{4}\int_{B_{\epsilon R}(x_2^\epsilon)}\sum_{j=1}^{2}[\epsilon^n|\nabla u_{j,\epsilon}|^2 + V_j u_{j,\epsilon}^2] \ge \epsilon^n\left[(b_2^\infty)^{\frac{4-n}{2}}\mu_2^{-1} I[w] + O\Big(\frac{1}{R}\Big)\right]. \tag{5.9}$$

Combining (5.8) and (5.9), and letting $R \to +\infty$, we obtain

$$\lim_{\epsilon\to 0}\epsilon^{-n}C_\epsilon \ge \left[(b_1^\infty)^{\frac{4-n}{2}}\mu_1^{-1} + (b_2^\infty)^{\frac{4-n}{2}}\mu_2^{-1}\right] I[w],$$

which may contradict with (5.1). $\square$

LEMMA 5.4. *Both $\sup_{\epsilon>0}|x_1^\epsilon| < +\infty, |x_2^\epsilon| \to +\infty$ and $|x_1^\epsilon| \to +\infty, \sup_{\epsilon>0}|x_2^\epsilon| < +\infty$ are impossible.*

PROOF. Without loss of generality, we suppose $\sup_{\epsilon>0} |x_1^\epsilon| < +\infty, |x_2^\epsilon| \to +\infty$. As in Lemma 5.3, we have

$$\frac{1}{4}\int_{B_{\epsilon R}(x_2^\epsilon)} \sum_{j=1}^{2} [\epsilon^2 |\nabla u_{j,\epsilon}|^2 + V_j u_{j,\epsilon}^2] \geq \epsilon^n \Big[(b_2^\infty)^{\frac{4-n}{2}} \mu_2^{-1} I[w] + O\Big(\frac{1}{R}\Big)\Big]. \tag{5.10}$$

Recall $(U_{\epsilon,1}^1(y), U_{\epsilon,2}^1(y)) = (u_{\epsilon,1}(x_1^\epsilon + \epsilon y), u_{\epsilon,2}(x_1^\epsilon + \epsilon y)) \to (U_1^1, U_2^1)$. If $U_2^1 \not\equiv 0$, then

$$\frac{1}{4}\int_{B_{\epsilon R}(x_1^\epsilon)} \sum_{j=1}^{2} [\epsilon^2 |\nabla u_{j,\epsilon}|^2 + V_j u_{j,\epsilon}^2] \geq \epsilon^n \Big[(V_1(x_1^0))^{\frac{4-n}{2}} \mu_1^{-1} I[w] + (V_2(x_1^0))^{\frac{4-n}{2}} \mu_2^{-1} I[w] + O\Big(\frac{1}{R}\Big)\Big], \tag{5.11}$$

where $x_1^\epsilon \to x_1^0$. If $U_2^1 \equiv 0$, then

$$\frac{1}{4}\int_{B_{\epsilon R}(x_1^\epsilon)} \sum_{j=1}^{2} [\epsilon^2 |\nabla u_{j,\epsilon}|^2 + V_j u_{j,\epsilon}^2] \geq \epsilon^n \Big[(V_1(x_1^0))^{\frac{4-n}{2}} \mu_1^{-1} I[w] + O\Big(\frac{1}{R}\Big)\Big]. \tag{5.12}$$

Combining (5.10)-(5.12), we obtain a contradiction to (5.1) again. □

LEMMA 5.5. $V_1(x_1^0) = b_1^0, V_2(x_2^0) = b_2^0$, $\frac{|x_1^\epsilon - x_2^\epsilon|}{\epsilon} \to +\infty$.

PROOF. The proof of $V_1(x_1^0) = b_1^0$ and $V_2(x_2^0) = b_2^0$ is similar to that of Lemma 5.4. We omit the details. Suppose $\sup_{\epsilon>0} \frac{|x_1^\epsilon - x_2^\epsilon|}{\epsilon} < +\infty$, then $x_1^0 = x_2^0$. Hence $(U_{\epsilon,1}^1(y), U_{\epsilon,2}^1(y)) \to (U_1^1, U_2^1)$ satisfies

$$\begin{cases} \triangle U_1^1 - b_1^0 U_1^1 + \mu_1 (U_1^1)^3 + \beta U_1^1 (U_2^1)^2 = 0, & \text{in } \mathbb{R}^n, \\ \triangle U_2^1 - b_2^0 U_2^1 + \mu_2 (U_2^1)^3 + \beta U_2^1 (U_1^1)^2 = 0, & \text{in } \mathbb{R}^n, \\ U_1^1, U_2^1 > 0, \ U_1^1, U_2^1 \in H^1(\mathbb{R}^n). \end{cases}$$

By Theorem A and $\beta < 0$,

$$\frac{1}{4}\sum_{j=1}^{2} \int_{\mathbb{R}^n} (|\nabla U_j^1|^2 + b_j^0 (U_j^1)^2) > \Big[(b_1^0)^{\frac{4-n}{2}} \mu_1^{-1} + (b_2^0)^{\frac{4-n}{2}} \mu_2^{-1} \Big] I[w]$$

which then implies that

$$\lim_{\epsilon\to 0} \epsilon^{-n} C_\epsilon > \Big[(b_1^0)^{\frac{4-n}{2}} \mu_1^{-1} + (b_2^0)^{\frac{4-n}{2}} \mu_2^{-1} \Big] I[w].$$

A contradiction. □

LEMMA 5.6. *$u_{\epsilon,1}(x) u_{\epsilon,2}(x) \to 0$ uniformly in $\mathbb{R}^n$ and $x_1^\epsilon, x_2^\epsilon$ are unique,* $|u_{\epsilon,j}(x)| \leq c e^{-\frac{a|x - x_j^\epsilon|}{\epsilon}}$, $j = 1, 2$.

PROOF. This follows from the same proof as that of Claim 3, Claim 4 and Claim 5 of [19]. □

6 Proof of Theorem 2.6

In this section, we study the asymptotic behavior of $(u_{\epsilon,1}, u_{\epsilon,2})$ as $\epsilon \to \infty$, for $\beta > 0$. As before, we let $u_{\epsilon,j}(x_j^\epsilon) = \max_{x\in\mathbb{R}^n} u_{\epsilon,j}(x)$.
First, we obtain an upper bound for the energy:

LEMMA 6.1. *For $\beta > 0$ and $\epsilon << 1$, we have*

$$C_\epsilon \leq \epsilon^n \min\Big\{ \Big[(b_1^0)^{\frac{4-n}{2}} \mu_1^{-1} + (b_2^0)^{\frac{4-n}{2}} \mu_2^{-1} \Big] I[w], \inf_{x\in\mathbb{R}^n} \rho(V_1(x), V_2(x)) \Big\} + o(\epsilon^n). \tag{6.1}$$

PROOF. The proof is simple and thus omitted. □

LEMMA 6.2. $\sup_{\epsilon>0} (|x_1^\epsilon| + |x_2^\epsilon|) < +\infty$.

PROOF. Suppose not. We have
Case 5.1: $|x_1^\epsilon| \to +\infty$, $|x_2^\epsilon| \to +\infty$.
In this case, we let $(U_{\epsilon,1}^1(y), U_{\epsilon,2}^1(y)) = (u_{\epsilon,1}(x_1^\epsilon + \epsilon y), u_{\epsilon,2}(x_1^\epsilon + \epsilon y)) \to (U_1^1, U_2^1)$.
If $U_2^1 \not\equiv 0$, then as in the proof of Lemma 5.3, we have $\lim_{\epsilon\to 0} \epsilon^{-n} C_\epsilon \geq C_1^\infty$ which may contradict with (2.23) and (6.1). Consequently, $U_2^1 \equiv 0$, $\frac{|x_1^\epsilon - x_2^\epsilon|}{\epsilon} \to +\infty$ and

$$\frac{1}{4}\int_{B_{\epsilon R}(x_1^\epsilon)} \sum_{j=1}^{2} [\epsilon^2 |\nabla u_{\epsilon,j}|^2 + V_j u_{\epsilon,j}^2] \geq \epsilon^n \Big[(b_1^\infty)^{\frac{4-n}{2}} \mu_1^{-1} + o(1) \Big] I[w]. \tag{6.2}$$

On the other hand, we may consider $(U_{\epsilon,1}^2(y), U_{\epsilon,2}^2(y)) = (u_{\epsilon,1}(x_2^\epsilon + \epsilon y), u_{\epsilon,2}(x_2^\epsilon + \epsilon y)) \to (U_1^2, U_2^2)$.
Then as for the proof of (6.2), we obtain $U_1^2 \equiv 0$ and

$$\frac{1}{4}\int_{B_{\epsilon R}(x_2^\epsilon)} \sum_{j=1}^{2} [\epsilon^2 |\nabla u_{\epsilon,j}|^2 + V_j u_{\epsilon,j}^2] \geq \epsilon^n \Big[(b_2^\infty)^{\frac{4-n}{2}} \mu_2^{-1} + o(1) \Big] I[w]. \tag{6.3}$$

Combining (6.2) and (6.3), we obtain a contradiction to (2.23) and (6.1).
Case 5.2: $|x_1^\epsilon| \to +\infty$, $\sup_{\epsilon>0} |x_2^\epsilon| < +\infty$.
In this case, we have $(U_{\epsilon,1}^1, U_{\epsilon,2}^1) \to (U_1^1, U_2^1)$, as in Case 5.1, $U_2^1 \equiv 0$ and

$$\frac{1}{4}\int_{B_{\epsilon R}(x_1^\epsilon)} \sum_{j=1}^{2} [\epsilon^2 |\nabla u_{\epsilon,j}|^2 + V_j u_{\epsilon,j}^2] \geq \epsilon^n \Big[(b_1^\infty)^{\frac{4-n}{2}} \mu_1^{-1} + o(1) \Big] I[w]. \tag{6.4}$$

If $U_1^2 \not\equiv 0$, then

$$\frac{1}{4}\int_{B_{\epsilon R}(x_2^\epsilon)} \sum_{j=1}^{2} [\epsilon^2 |\nabla u_{\epsilon,j}|^2 + V_j u_{\epsilon,j}^2] \geq \epsilon^n \rho(V_1(x_2^0), V_2(x_2^0)) \geq \epsilon^n \inf_{x\in\mathbb{R}^n} \rho(V_1(x), V_2(x)). \tag{6.5}$$

If $U_1^2 \equiv 0$, then

$$\frac{1}{4}\int_{B_{\epsilon R}(x_2^\epsilon)} \sum_{j=1}^{2} [\epsilon^2 |\nabla u_{\epsilon,j}|^2 + V_j u_{\epsilon,j}^2] \geq \epsilon^n (b_2^0)^{\frac{4-n}{2}} \mu_2^{-1} I[w]. \tag{6.6}$$

Combining (6.4)-(6.6), we obtain a contradiction. Similarly, $\sup_{\epsilon>0} |x_1^\epsilon| < +\infty, |x_2^\epsilon| \to +\infty$ is impossible. $\square$

Let $(x_1^\epsilon, x_2^\epsilon) \to (x_1^0, x_2^0)$.

LEMMA 6.3. *If* $\big[(b_1^0)^{\frac{4-n}{2}} \mu_1^{-1} + (b_2^0)^{\frac{4-n}{2}} \mu_2^{-1}\big] I[w] < \inf_{x\in\mathbb{R}^n} \rho(V_1(x), V_2(x))$, *then we have* $x_1^0 \neq x_2^0$ *and*

$$V_1(x_1^\epsilon) \to b_1^0, \quad V_2(x_2^\epsilon) \to b_2^0.$$

PROOF. We first show that $\frac{|x_1^\epsilon - x_2^\epsilon|}{\epsilon} \to +\infty$ and $x_1^0 \neq x_2^0$. In fact, suppose not. Then $(U_{\epsilon,1}^1, U_{\epsilon,2}^1) \to (U_1^1, U_2^1)$. Hence we have

$$\lim_{\epsilon\to 0} \epsilon^{-n} C_\epsilon \geq \rho(V_1(x_1^0), V_2(x_1^0)) \geq \inf_{x\in\mathbb{R}^n} \rho(V_1(x), V_2(x)), \tag{6.7}$$

which may contradict with (6.1). Thus we obtain $\frac{|x_1^\epsilon - x_2^\epsilon|}{\epsilon} \to +\infty$. Moreover, $(U_{\epsilon,1}^1, U_{\epsilon,2}^1) \to (U_1^1, 0)$, $(U_{\epsilon,1}^2, U_{\epsilon,2}^2) \to (0, U_2^2)$ and

$$\lim_{\epsilon\to 0} \epsilon^{-n} C_\epsilon \geq \Big[\big(V_1(x_1^0)\big)^{\frac{4-n}{2}} \mu_1^{-1} + \big(V_2(x_2^0)\big)^{\frac{4-n}{2}} \mu_2^{-1} \Big] I[w].$$

By Lemma 6.1, we see that $V_1(x_1^\epsilon) \to \inf_{x\in\mathbb{R}^n} V_1(x)$, $V_2(x_2^\epsilon) \to \inf_{x\in\mathbb{R}^n} V_2(x)$. $\square$

LEMMA 6.4. *If* $\inf_{x\in\mathbb{R}^n} \rho(V_1(x), V_2(x)) < \big[(b_1^0)^{\frac{4-n}{2}} \mu_1^{-1} + (b_2^0)^{\frac{4-n}{2}} \mu_2^{-1}\big] I[w]$, *then* $x_1^0 = x_2^0$, $\frac{|x_1^\epsilon - x_2^\epsilon|}{\epsilon} \to 0$ *and*

$$\rho(V_1(x_1^0), V_2(x_1^0)) = \inf_{x\in\mathbb{R}^n} \rho(V_1(x), V_2(x)).$$

PROOF. We first show that $\frac{|x_1^\epsilon - x_2^\epsilon|}{\epsilon} \leq c$. Suppose not. Then as in the proof of Lemma 6.2,

$$\lim_{\epsilon\to 0} \epsilon^{-n} C_\epsilon \geq \left[(b_1^0)^{\frac{4-n}{2}} \mu_1^{-1} + (b_2^0)^{\frac{4-n}{2}} \mu_2^{-1}\right] I[w],$$

which may contradict with Lemma 6.1.

Now we may let $(U_{\epsilon,1}^1, U_{\epsilon,2}^1) \to (U_1^1, U_2^1)$, $U_j^1 > 0,\ j = 1,2$, and satisfy

$$\begin{cases} \triangle U_1^1 - V_1(x_1^0)U_1^1 + \mu_1 (U_1^1)^3 + \beta U_1^1 (U_2^1)^2 = 0, \\ \triangle U_2^1 - V_2(x_1^0)U_2^1 + \mu_2 (U_2^1)^3 + \beta U_2^1 (U_1^1)^2 = 0. \end{cases}$$

Then by the Maximum Principle, $U_j^1 > 0,\ j = 1,2$. Moreover, we have

$$\lim_{\epsilon\to 0} \epsilon^{-n} C_\epsilon \geq \rho(V_1(x_1^0), V_2(x_1^0)).$$

By Lemma 6.1,

$$\rho(V_1(x_1^0), V_2(x_1^0)) \leq \inf_{x\in\mathbb{R}^n} \rho(V_1(x), V_2(x)),$$

so we have $\rho(V_1(x_1^0), V_2(x_1^0)) = \inf_{x\in\mathbb{R}^n} \rho(V_1(x), V_2(x))$. It remains to show that $\frac{|x_1^\epsilon - x_2^\epsilon|}{\epsilon} \to 0$. In fact, this follows from the fact that (U_1^1, U_2^1) is radially symmetric and strictly decreasing. □

LEMMA 6.5. *$x_1^\epsilon, x_2^\epsilon$ are unique.*

PROOF. This follows from Claim 8 of [19]. □

LEMMA 6.6. $|u_{\epsilon,j}| \leq c e^{-\frac{a|x - x_j^\epsilon|}{\epsilon}},\ j = 1,2.$

PROOF. See Claim 9 of [19]. □

7 Proof of Theorem 2.1, 2.4 and 2.6 when $\mathbf{b}_\mathbf{j}^\infty = +\infty$ for some $j = 1,2$

We only discuss the case of $b_1^\infty = +\infty$. The proof of the other case is similar. To prove Theorem 2.1, we note that the inequality

$$c_1 \epsilon^n \leq \int_{B_k} u_{j,k}^4 \leq c_2 \epsilon^n,\ j = 1,2 \tag{7.1}$$

is true, where c_1, c_2 are independent of $\epsilon \leq 1,\ k \geq 1$. Then as for the proof of Claim 1 in [19], we have

$$\begin{aligned} C_{\epsilon,k} &= \frac{1}{4}\left[\mu_1 \int_{B_k} u_{1,k}^4 + 2\beta \int_{B_k} u_{1,k}^2 u_{2,k}^2 + \mu_2 \int_{B_k} u_{2,k}^4\right] \\ &\leq c_3 \epsilon^n, \end{aligned}$$

and

$$\begin{aligned} C_{\epsilon,k} &= \frac{1}{4}\left[\epsilon^2 \int_{B_k} |\nabla u_{1,k}|^2 + \int_{B_k} V_1 u_{1,k}^2\right] + \frac{1}{4}\left[\epsilon^2 \int_{B_k} |\nabla u_{2,k}|^2 + \int_{B_k} V_2 u_{2,k}^2\right] \\ &\geq c_4 \epsilon^{n/2} \left(\sqrt{\int_{B_k} u_{1,k}^4} + \sqrt{\int_{B_k} u_{2,k}^4}\right). \end{aligned}$$

Consequently,

$$c_5 \epsilon^n \leq C_{\epsilon,k} \leq c_6 \epsilon^n, \tag{7.2}$$

where c_5, c_6 are independent of $\epsilon \leq 1,\ k \geq 1$. This may give

$$\epsilon^2 \int_{B_k} |\nabla u_{j,k}|^2 + \int_{B_k} V_j u_{j,k}^2 \leq c_7 \epsilon^n.$$

By Sobolev's embedding (due to $n = 2,3$),

$$\int_{B_k} u_{1,k}^6 \leq c_8 \epsilon^n, \quad \int_{B_k \cap \{|x| \geq R\}} u_{1,k}^2 \leq c_7 \epsilon^n \cdot \frac{1}{\min\limits_{|x|\geq R} V_1(x)}. \tag{7.3}$$

Hence

$$\int_{B_k\cap\{|x|\geq R\}} u_{1,k}^4 \leq \left(\int_{B_k\cap\{|x|\geq R\}} u_{1,k}^2\right)^{1/2}\left(\int_{B_k\cap\{|x|\geq R\}} u_{1,k}^6\right)^{1/2}$$
$$\leq c_9\epsilon^n\cdot\left(\frac{1}{\min\limits_{|x|\geq R} V_1(x)}\right)^{1/2}. \tag{7.4}$$

By (7.1) and (7.4), we have

$$\int_{B_k\cap\{|x|\leq R\}} u_{1,k}^4 \geq \left(c_1 - \frac{c_9}{\sqrt{\min\limits_{|x|\geq R} V_1(x)}}\right)\epsilon^n. \tag{7.5}$$

Thus if $u_{1,k}\rightharpoonup \overline{u}_1$, then $\overline{u}_1\geq 0$ and

$$\int_{B_k\cap\{|x|\leq R\}} \overline{u}_1^4 \geq \left(c_1 - \frac{c_9}{\sqrt{\min\limits_{|x|\geq R} V_1(x)}}\right)\epsilon^n. \tag{7.6}$$

Since $b_1^\infty=+\infty$, we may choose R large enough such that $c_1-\frac{c_9}{\sqrt{\min\limits_{|x|\geq R}V_1(x)}}\geq\frac{1}{2}c_1$, then $\int_{B_k\cap\{|x|\leq R\}}\overline{u}_1^4\geq\frac{1}{2}c_1\epsilon^n$ and hence $\overline{u}_1\not\equiv 0$.
Moreover, (7.1), (7.4) and Holder inequality may imply

$$\int_{B_k\cap\{|x|\geq R\}} u_{1,k}^2u_{2,k}^2 \leq c_{10}\epsilon^n\left(\frac{1}{\min\limits_{|x|\geq R} V_1(x)}\right)^{1/4},$$

and then we have

$$\lim_{\substack{R\to+\infty\\ k\to+\infty}}\int_{B_k\cap\{|x|\geq R\}} u_{1,k}^2u_{2,k}^2\to 0\,.$$

Here we have used the fact that $b_1^\infty=+\infty$. Follow the arguments for the case 2.1 of Lemma 4.2, we may complete the proof of Theorem 2.1.

Let us now prove Theorem 2.4 and 2.6. Without loss of generality, we assume that $b_1^\infty=+\infty$. All we need to show is that $|x_1^\epsilon|\leq C_0$, where C_0 is a positive constant independent of ϵ. To this end, we first prove a uniform bound of $u_{\epsilon,1}$ and $u_{\epsilon,2}$. In fact, as for the proof of (7.3), we have

$$\int_{\mathbb{R}^n} u_{\epsilon,j}^q\leq c\epsilon^n,\ 2\leq q\leq 6,\ j=1,2. \tag{7.7}$$

The equation of $u_{\epsilon,1}$ may give

$$\begin{aligned}\epsilon^2\triangle u_{\epsilon,1} =& V_1u_{\epsilon,1}-\mu_1u_{\epsilon,1}^3-\beta u_{\epsilon,1}u_{\epsilon,2}^2\\ \geq& -(\mu_1u_{\epsilon,1}^2+\beta u_{\epsilon,2}^2)u_{\epsilon,1}\\ =& -C(x)u_{\epsilon,1}.\end{aligned}$$

Let $\widetilde{U}_{\epsilon,1}(y)=u_{\epsilon,1}(\epsilon y)$, and $C_\epsilon(y)=C(\epsilon y)$, then

$$\triangle\widetilde{U}_{\epsilon,1}+C_\epsilon(y)\widetilde{U}_{\epsilon,1}\geq 0,\quad \text{and } C_\epsilon\in L^3(\mathbb{R}^n)\,. \tag{7.8}$$

By the subsolution estimate (Theorem 8.17 of [13])

$$|\widetilde{U}_{\epsilon,1}(y)|\leq C\left(\int_{B(y,1)}|\widetilde{U}_{\epsilon,1}|^2\right)^{1/2}, \tag{7.9}$$

where C doesn't depend on ϵ. From (7.7) and (7.9), we see that $||\widetilde{U}_{\epsilon,1}||_{L^\infty}\leq C$ and hence $0<u_{\epsilon,1}\leq C$. Similarly, we may obtain $0<u_{\epsilon,2}\leq C$.

Now we claim $|x_1^\epsilon| \le C_0$, where C_0 is a positive constant independent of ϵ. Since x_1^ϵ is a local maximum point of $u_{\epsilon,1}$, then $\triangle u_{\epsilon,1}(x_1^\epsilon) \le 0$. Hence by the equation of $u_{\epsilon,1}$, we may obtain

$$\epsilon^2 \triangle u_{\epsilon,1}(x_1^\epsilon) = V_1(x_1^\epsilon) u_{\epsilon,1}(x_1^\epsilon) - \mu_1 u_{\epsilon,1}^3(x_1^\epsilon) - \beta u_{\epsilon,1}(x_1^\epsilon) u_{\epsilon,2}^2(x_1^\epsilon) \le 0,$$

which then implies that

$$V_1(x_1^\epsilon) \le \mu_1 u_{\epsilon,1}^2(x_1^\epsilon) + \beta u_{\epsilon,2}^2(x_1^\epsilon) \le C, \tag{7.10}$$

and hence

$$|x_1^\epsilon| \le C_0 \,. \tag{7.11}$$

Here both C and C_0 are positive constants independent of ϵ. Besides, we have used the fact that $\lim_{|x|\to\infty} V_1(x) = +\infty$. Once we have (7.11), the rest of the proofs for Theorem 1.4 and 1.5 are similar to the proofs in Section 4 and 5, respectively.

8 Proof of Theorem 2.8, Corollary 2.2,2.5 and 2.7

By Theorem A, $\rho(\lambda_1^0, \lambda_2^0)$ can be achieved by some (u_1, u_2) satisfying

$$\begin{cases} \triangle u_1 - \lambda_1^0 u_1 + \mu_1 u_1^3 + \beta u_1 u_2^2 = 0, & \text{in } \mathbb{R}^n \,, \\ \triangle u_2 - \lambda_2^0 u_2 + \mu_2 u_2^3 + \beta u_1^2 u_1^2 = 0, & \text{in } \mathbb{R}^n \,, \\ u_1 = u_1(r), \ u_2 = u_2(r), \ u_1' < 0, \ u_2' < 0. \end{cases} \tag{8.1}$$

We may set λ_j^0's as small perturbations of λ_j's i.e.

$$\lambda_j^0 = \lambda_j + \delta_j, \quad |\delta_j| << 1, \ j = 1, 2\,. \tag{8.2}$$

Moreover, $\beta > 0$ is a small parameter i.e. $0 < \beta << 1$. Then it is easy to check that

$$u_j = w_{\lambda_j,\mu_j} + \phi_j \,, \quad \phi_j = O(|\delta_1| + |\delta_2| + \beta)\,, \ j = 1, 2\,.$$

Hence

$$\begin{aligned} &\sum_{j=1}^2 \left(\int_{\mathbb{R}^n} \frac{1}{2}|\nabla u_j|^2 + \frac{\lambda_j^0}{2} u_j^2 - \frac{\mu_j}{4} u_j^4 \right) - \frac{\beta}{2} \int_{\mathbb{R}^n} u_1^2 u_2^2 \\ =& \sum_{j=1}^2 \left(\int_{\mathbb{R}^n} \frac{1}{2}|\nabla (w_{\lambda_j,\mu_j} + \phi_j)|^2 + \frac{\lambda_j + \delta_j}{2}(w_{\lambda_j,\mu_j} + \phi_j)^2 - \frac{\mu_j}{4}(w_{\lambda_j,\mu_j} + \phi_j)^4 \right) \\ &- \frac{\beta}{2} \int_{\mathbb{R}^n} w_{\lambda_1,\mu_1}^2 w_{\lambda_2,\mu_2}^2 + O((|\delta_1| + |\delta_2| + \beta)\beta) \\ =& \sum_{j=1}^2 \left[\left(\lambda_j^{\frac{4-n}{2}} \mu_j^{-1} \right) I[w] + \frac{\delta_j}{2} w_{\lambda_j,\mu_j}^2 \right] - \frac{\beta}{2} \int_{\mathbb{R}^n} w_{\lambda_1,\mu_1}^2 w_{\lambda_2,\mu_2}^2 + O((|\delta_1| + |\delta_2| + \beta)^2)\,. \end{aligned}$$

i.e.

$$\begin{aligned} \sum_{j=1}^2 \left(\int_{\mathbb{R}^n} \frac{1}{2}|\nabla u_j|^2 + \frac{\lambda_j^0}{2} u_j^2 - \frac{\mu_j}{4} u_j^4 \right) - \frac{\beta}{2} \int_{\mathbb{R}^n} u_1^2 u_2^2 = & \sum_{j=1}^2 \left[\left(\lambda_j^{\frac{4-n}{2}} \mu_j^{-1} \right) I[w] + \frac{\delta_j}{2} w_{\lambda_j,\mu_j}^2 \right] \\ & -\tfrac{\beta}{2} \int_{\mathbb{R}^n} w_{\lambda_1,\mu_1}^2 w_{\lambda_2,\mu_2}^2 + O((|\delta_1| + |\delta_2| + \beta)^2)\,. \end{aligned} \tag{8.3}$$

Now we take our example

$$V_j(x) = \lambda_j + \sum_{k=1}^n a_{j,k}(x_k - z_{j,k})^2\,, \quad a_{j,n} > 0\,, \, a_{j,k} = 0\,, \quad j = 1, 2\,, k = 1, ..., n-1\,.$$

Let $x = t\sqrt{\beta} e_n$, $z_1 = -l\sqrt{\beta} e_n$ and $z_2 = l\sqrt{\beta} e_n$. Then we have

$$V_1(x) = \lambda_1 + a_{1,n}(t+l)^2 \beta, \quad V_2(x) = \lambda_2 + a_{2,n}(t-l)^2 \beta, \tag{8.4}$$

which may have the same form as (8.2) by setting $\delta_1 = a_{1,n}(t+l)^2\beta$ and $\delta_2 = a_{2,n}(t-l)^2\beta$. Thus (8.3) and (8.4) may give

$$\rho(V_1(x),V_2(x)) = \sum_{j=1}^{2}\left(\lambda_j^{\frac{4-n}{2}}\mu_j^{-1}\right)I[w] + \frac{1}{2}\left[a_{1,n}(t+l)^2\int_{\mathbb{R}^n} w^2_{\lambda_1,\mu_1} + a_{2,n}(t-l)^2\int_{\mathbb{R}^n} w^2_{\lambda_2,\mu_2}\right]\beta$$
$$-\frac{\beta}{2}\int_{\mathbb{R}^n} w^2_{\lambda_1,\mu_1}w^2_{\lambda_2,\mu_2} + O(\beta^2).$$

Let

$$\alpha_1 = a_{1,n}\int_{\mathbb{R}^n} w^2_{\lambda_1,\mu_1}, \quad \text{and} \quad \alpha_2 = a_{2,n}\int_{\mathbb{R}^n} w^2_{\lambda_2,\mu_2}.$$

Then

$$\min_{t\in\mathbb{R}}\left(\alpha_1(t+l)^2+\alpha_2(t-l)^2\right) = \frac{4\alpha_1\alpha_2}{\alpha_1+\alpha_2}l.$$

Hence if

$$\frac{4\alpha_1\alpha_2}{\alpha_1+\alpha_2}l < \int_{\mathbb{R}^n} w^2_{\lambda_1,\mu_1}w^2_{\lambda_2,\mu_2},$$

then

$$\min_{x\in\mathbb{R}^n}\rho(V_1(x),V_2(x)) < \sum_{j=1}^{2}\left(\lambda_j^{\frac{4-n}{2}}\mu_j^{-1}\right)I[w],$$

i.e. (2.24) holds. However, if

$$\frac{4\alpha_1\alpha_2}{\alpha_1+\alpha_2}l > \int_{\mathbb{R}^n} w^2_{\lambda_1,\mu_1}w^2_{\lambda_2,\mu_2},$$

then

$$\min_{x\in\mathbb{R}^n}\rho(V_1(x),V_2(x)) > \sum_{j=1}^{2}\left(\lambda_j^{\frac{4-n}{2}}\mu_j^{-1}\right)I[w],$$

i.e. (2.27) holds. Setting

$$l_0 = \frac{\alpha_1+\alpha_2}{4\alpha_1\alpha_2}\int w^2_{\lambda_1,\mu_1}w^2_{\lambda_2,\mu_2},$$

we may complete the proof of Theorem 2.8.

PROOF of COROLLARY 2.2.

For the proof of (i), we set $(u_0^\epsilon, v_0^\epsilon)$ as the least energy solution of

$$\begin{cases}\epsilon^2\triangle u - \lambda_1 u + \mu_1 u^3 + \beta v^2 u = 0 \text{ in } B_1(0),\\ \epsilon^2\triangle v - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0 \text{ in } B_1(0),\\ u = v = 0 \text{ on } \partial B_1(0).\end{cases} \tag{8.5}$$

Here we have used the assumption $0<\beta<\beta_0$ which ensures the existence of $(u_0^\epsilon, v_0^\epsilon)$ (cf. [19]). Moreover,

$$\begin{aligned}E^0_{\epsilon,1}[u_0^\epsilon, v_0^\epsilon] =& \inf_{\substack{(u_1,u_2)\in H_0^1(B_1)\times H_0^1(B_1)\\ u_1\not\equiv 0,u_2\not\equiv 0}}\ \sup_{t_1,t_2>0} E^0_{\epsilon,1}[\sqrt{t}_1u_1,\sqrt{t}_2u_2]\\ <&\epsilon^n\left(\lambda_1^{\frac{4-n}{2}}\mu_1^{-1}+\lambda_2^{\frac{4-n}{2}}\mu_2^{-1}\right)I[w]\\ \le& C^1_\epsilon + \epsilon^n(b_2^\infty)^{\frac{4-n}{2}}\mu_2^{-1}I[w],\end{aligned}$$

i.e.

$$E^0_{\epsilon,1}[u_0^\epsilon, v_0^\epsilon] < C^1_\epsilon + \epsilon^n(b_2^\infty)^{\frac{4-n}{2}}\mu_2^{-1}I[w]. \tag{8.6}$$

By Lemma 3.2, it is easy to check that

$$\sup_{t_1,t_2>0} E_{\epsilon,1}[\sqrt{t_1}u_0^\epsilon, \sqrt{t_2}v_0^\epsilon] = E_{\epsilon,1}\left[\sqrt{\tilde{t}_1}u_0^\epsilon, \sqrt{\tilde{t}_2}v_0^\epsilon\right] \tag{8.7}$$

for some $(\tilde{t}_1, \tilde{t}_2)$ satisfying

$$|\tilde{t}_1 - 1| + |\tilde{t}_2 - 1| = O(\epsilon^2). \tag{8.8}$$

Here we have used (3.6) and the definition of V_j's. Furthermore, by (8.8), we may obtain

$$E_{\epsilon,1}\left[\sqrt{\tilde{t}_1}u_0^\epsilon, \sqrt{\tilde{t}_2}v_0^\epsilon\right] = E_{\epsilon,1}^0[u_0^\epsilon, v_0^\epsilon] + O(\epsilon^{n+2}). \tag{8.9}$$

Thus by (8.6), (8.7), (8.9) and Lemma 3.1,

$$C_\epsilon \leq C_{\epsilon,1} \leq \sup_{t_1,t_2>0} E_{\epsilon,1}\left[\sqrt{\tilde{t}_1}u_0^\epsilon, \sqrt{\tilde{t}_2}v_0^\epsilon\right] < C_\epsilon^1 + \epsilon^n (b_2^\infty)^{\frac{4-n}{2}} \mu_2^{-1} I[w],$$

and we may complete the proof of (i). For the proof of (ii), one may follow the proof of Theorem 1 in [20] to get the nonexistence of ground state solution, and

$$C_\epsilon = \epsilon^n \left(\lambda_1^{\frac{4-n}{2}} \mu_1^{-1} + \lambda_2^{\frac{4-n}{2}} \mu_2^{-1}\right) I[w]. \tag{8.10}$$

From [25],

$$C_\epsilon^1 = \epsilon^n \lambda_1^{\frac{4-n}{2}} \mu_1^{-1} I[w] + o(\epsilon^n), \tag{8.11}$$

where $o(1)$ is a small quantity tending to zero as ϵ goes to zero. Therefore by (8.10) and (8.11), we obtain (2.12) and complete the proof of Corollary 2.2. □

PROOF of COROLLARY 2.5 and 2.7.

By (2.18), we may extend V_j's to the entire space $\mathbb{R}^n$ satisfying $\inf_{x\in\mathbb{R}^n\setminus\Omega} V_j(x) = \inf_{x\in\partial\Omega} V_j(x) > \inf_{x\in\Omega} V_j(x)$ and $\lim_{|x|\to\infty} V_j(x) = b_j^\infty < +\infty$ for $j = 1, 2$. Then we may apply Theorem 2.4 and 2.6 to get the existence and asymptotic behaviors of ground state solution $(u_{\epsilon,1}, u_{\epsilon,2})$ on the entire space $\mathbb{R}^n$.

Now we may choose a smooth cut-off function ϕ such that $\tilde{u}_{\epsilon,j} = u_{\epsilon,j} \cdot \phi \in H_0^1(\Omega)$ for $j = 1, 2$, and $\phi \equiv 1$ in $\tilde{\Omega}$, where $\tilde{\Omega} \subset\subset \Omega$ is a bounded open subset containing P_j^ϵ's the local maximum points of $u_{\epsilon,j}$'s. Hence by Lemma 3.2, we may obtain

$$\sup_{s,t>0} E_{\epsilon,\Omega}[\sqrt{s}\tilde{u}_{\epsilon,1}, \sqrt{t}\tilde{u}_{\epsilon,2}] = E_{\epsilon,\Omega}[\sqrt{\tilde{t}_1}\tilde{u}_{\epsilon,1}, \sqrt{\tilde{t}_2}\tilde{u}_{\epsilon,2}] = E_\epsilon[u_{\epsilon,1}, u_{\epsilon,2}] + o(\epsilon^n) = C_\epsilon + o(\epsilon^n)\,, \tag{8.12}$$

where $|\tilde{t}_1 - 1| + |\tilde{t}_2 - 1| = o(1)$. Here $o(1)$ is a small quantity tending to zero as ϵ goes to zero. On the other hand,

$$C_\epsilon \leq C_{\epsilon,\Omega} \leq \sup_{s,t>0} E_{\epsilon,\Omega}[\sqrt{s}\tilde{u}_{\epsilon,1}, \sqrt{t}\tilde{u}_{\epsilon,2}]\,. \tag{8.13}$$

Thus (8.12) and (8.13) may give

$$C_{\epsilon,\Omega} = C_\epsilon + o(\epsilon^n)\,. \tag{8.14}$$

Therefore by (8.14), we may complete the proof of Corollary 2.5 and 2.7.

□

References

[1] Ambrosetti, A.; Badiale, M.; Cingolani, S. Semiclassical states of nonlinear Schrödinger equations. *Arch. Rational Mech. Anal.* **140** (1997), 285–300.

[2] Ambrosetti, A.; Malchiodi, A.; Secchi, S. Multiplicity results for some nonlinear singularly perturbed elliptic problems on R^n. *Arch. Rat. Mech. Anal.* **159** (2001), 253–271.

[3] Ambrosetti, A.; Malchiodi, A.; Ni, W.-M. Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres, Part I. *Comm. Math. Phys.* **235** (2003), 427-466.

[4] Byeon, J.; Wang, Z.-Q. Standing waves with a critical frequency for nonlinear Schrödinger equations, *Arch. Rat. Mech. Anal.* **165** (2002), 295–316.

[5] Cingolani, S.; Lazzo M. Multiple positive solutions to nonlinear Schrödinger equations with competeing potential functions. *J. Differential Equations* **160** (2000) 118–138.

[6] E. A. Donley, N. R. Claussen, S. L. Cornish, J. L. Roberts, E. A. Cornell, C. E. Wieman Dynamics of collapsing and exploding Bose-Einstein condensates, *Nature* **19:412**(2001), 295-299.

[7] W. Y. Ding and W. M. Ni,On the existence of positive entire solutions of a semilinear elliptic equations, *Arch. Rational Mech. Anal.* **91**(1986) 283-308.

[8] del Pino, M.; Felmer, P. Semi-classcal states for nonlinear Schrödinger equations, *J. Funct. Anal.* **149** (1997), 245–265.

[9] del Pino, M.; Felmer, P. Semi-classical states of nonlinear Schrödinger equations: a variational reduction method. *Math. Ann.* **324** (2002), no. 1, 1-32.

[10] del Pino, M; Kowalczyk, M; Wei, J. Concentrations on geodesics for nonlinear Schrödinger equations. *Comm. Pure Appl. Math.*, to appear.

[11] B. D. Esry, C. H. Greene, J. P. Burke Jr., J. L. Bohn, Hartree-Fock theory for double condensates, *Phys. Rev. Lett.* **78** (1997) 3594-3597.

[12] Floer, A.; Weinstein, A. Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential. *J. Funct. Anal.* **69** (1986), 397-408.

[13] D. Gilbarg and N. S. Trudinger,*Elliptic Partial Differential Equations of Second order*, 2nd.ed, Springer-Verlag,1983.

[14] Grossi, M. On the number of single-peak solutions of the nonlinear Schrödinger equation. *Ann. I. H. Poincare, Anal. non lineaire* **19** (2002) 261–280.

[15] Gui, C. Existence of multi-bump solutions for nonlinear Schrödinger equations via variational method. *Comm. in P.D.E.* **21** (1996), 787-820.

[16] Jeanjean, L.; Tanaka K., Singularly perturbed elliptic problems with superlinear or asymptotically linear nonlinearities. *Calc. Var. Partial Differential Equations* **21** (2004), no. 3, 287-318.

[17] Kang, X.; Wei, J. On interacting bumps of semi-classical states of nonlinear Schrödinger equations. *Adv. Diff. Eqn.* **5**(7-9) (2000), 899-928.

[18] Li, Y. Y. On a singularly perturbed elliptic equation. *Adv. Diff. Eqn.* **2**(6) (1997), 955-980.

[19] T. C. Lin and J. Wei, Spikes in two coupled nonlinear Schrödinger equations, *Ann. I. H. Poincaré Analyse. Non.* **22**, no.4(2005) 403-439.

[20] T. C. Lin and J. Wei, Ground state of N coupled nonlinear Schrödinger equations in $\mathbb{R}^n, n \leq 3$, *Comm. Math. Phys.* **255**, (2005)629-653.

[21] L. Pitaevskii and S. Stringari, *Bose-Einstein Condensation*, Oxford, 2003.

[22] P. H. Rabinowitz,On a class of nonlinear Schrödinger equation,*Z. Angew. Math. Phys.* **43**,no.2(1992)270-291.

[23] Sirakov, B. Standing wave solutions of the nonlinear Schrödinger equation in $\mathbb{R}^N$. *Ann. Mat. Pura Appl.* **4**(181) (2002), no. 1, 73–83.

[24] E. Timmermans, Phase separation of Bose-Einstein condensates, *Phys. Rev. Lett.* **81** (1998) 5718-5721.

[25] X. Wang,On conentration of positive bound states of nonlinear Schrödinger equations, *Comm. Math. Phys.* (1993)229-244.

[26] Wang, Z.-Q. Existence and symmetry of multi-bump solutions for nonlinear Schrödinger equations. *J. Differential Equations* **159** (1999), no. 1, 102–137.