INFINITE TIME BLOW-UP FOR CRITICAL HEAT EQUATION WITH DRIFT TERMS

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Abstract. We construct infinite time blow-up solution to the following heat equation with Sobolev critical exponent and drift terms

$$\begin{cases} u_t = \Delta u + \nabla b(x) \cdot \nabla u + u^{\frac{n+2}{n-2}} & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^n, \end{cases}$$

where $b(x)$ is a smooth bounded function in $\mathbb{R}^n$ with $n \geq 5$ and the initial datum u_0 is positive and smooth. Let $q_j \in \mathbb{R}^n, j = 1, \cdots, k$, be distinct nondegenerate local minimum points of $b(x)$. Assume that an eigenvalue condition (1.6) is satisfied. We prove the existence of a positive smooth solution $u(x,t)$ which blows up at infinite time near those points with the form

$$u(x,t) \approx \sum_{j=1}^{k} \alpha_n \left(\frac{\mu_j(t)}{\mu_j(t)^2 + |x - \xi_j(t)|^2} \right)^{\frac{n-2}{2}}, \quad \text{as } t \to +\infty.$$

Here $\xi_j(t) \to q_j$ and $0 < \mu_j(t) \to 0$ exponentially as $t \to +\infty$.

1. Introduction and main results

In this paper, we consider the following anisotropic heat equation with Sobolev critical exponent

$$\begin{cases} u_t = \Delta u + \nabla b(x) \cdot \nabla u + u^{\frac{n+2}{n-2}} & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where we write $b(x) = \log a(x)$ and $a(x)$ is a positive smooth bounded function in $\mathbb{R}^n (n \geq 5)$, u_0 is a positive smooth function. Anisotropic elliptic and parabolic equations have attracted much attention in recent years. The anisotropic differential operator of the divergence form is defined by

$$\Delta_a u = \frac{1}{a(x)} \operatorname{div}(a(x) \nabla u) = \Delta u + \nabla \log a(x) \cdot \nabla u.$$

For instance, the Green's function of Δ_a in a smooth bounded domain was investigated in [30]. Anisotropic equations have a wide range of applications in mathematical modeling of physical and mechanical processes in anisotropic continuous medium. Concentration phenomena have been found in many anisotropic elliptic problems. The role of the anisotropic coefficient $a(x)$ in elliptic bubbling phenomena has been known for a long time. Generally speaking, the bubbling location is determined by the anisotropic coefficient $a(x)$. See for example [24, 25, 29, 55, 57–59] and the references therein. Inspired by these results, we consider the related problem in a parabolic setting, namely the problem (1.1). We will construct *infinite time blow-up* solutions to problem (1.1), which rely on the anisotropic coefficient $a(x)$.

For the semilinear parabolic problem with gradient term

$$\begin{cases} u_t = \Delta u + u^p + g(x, t, u, \nabla u), & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u(\cdot, 0) = u_0, & \text{in } \mathbb{R}^n, \end{cases}$$

where $p > 1$, the asymptotic behavior and the blow-up rate may be influenced by the gradient perturbations. Related results can be found in the survey [52] and the book [46]. For a special case

$g(x,t,u,\nabla u) = \mathbf{a}\cdot\nabla(u^q)$, in [1], Aguirre and Escobedo gave conditions which guarantee the existence of finite time blow-up solutions.

When $a(x)$ is a constant, problem (1.1) becomes a special case of the Fujita equation

$$\begin{cases} u_t = \Delta u + u^p, & \text{in } \mathbb{R}^n \times (0,+\infty), \\ u(\cdot,0) = u_0, & \text{in } \mathbb{R}^n, \end{cases} \tag{1.2}$$

with $p > 1$. After Fujita's seminal work [27], a lot of literatures have been devoted to studying this problem about the blow-up rates, sets and profiles. See, for example, [8, 33, 34, 36–39, 53] and the references therein. Solutions with multiple type I blow-up were first built in the real line in [40]. In particular, the values of the exponent p in problem (1.2) have fundamental effect on the blow-up phenomena. The critical case $p = \frac{n+2}{n-2}$ is very special in various ways. For the subcritical case $p < \frac{n+2}{n-2}$, in [42] Merle and Zaag found multiple-point, finite time type I blow-up solution and studied its stability. For the supercritical case $p > \frac{n+2}{n-2}$, Matano and Merle classified the radial blow-up solutions in [39]. Define the Joseph-Lundgren exponent

$$p_{JL}(n) := \begin{cases} \infty & \text{for } \ 3 \le n \le 10, \\ 1 + \frac{4}{n-4-2\sqrt{n-1}} & \text{for } \ n \ge 11. \end{cases}$$

For $\frac{n+2}{n-2} < p < p_{JL}$, no type II blow-up is present for radial solutions in the case of a ball or in entire space under additional assumptions [37, 38, 43]. In [21], del Pino, Musso and Wei constructed *non-radial* type II blow-up solutions in the range $\frac{n+2}{n-2} < p < p_{JL}$. For the critical case $p = \frac{n+2}{n-2}$, Collot, Merle and Raphaël proved classification results near the ground state of the energy critical heat equation in $\mathbb{R}^n$ with $n \ge 7$ in [7]. In [51], by using the energy method, Schweyer constructed the radial, type II finite time blow-up solution to the energy critical heat equation in $\mathbb{R}^4$. In [22], del Pino, Musso and Wei found the existence of finite time type II blow-up solution for the energy critical heat equation in $\mathbb{R}^5$. Concerning infinite time blow-up, in a very interesting paper [26], Fila and King studied problem (1.2) with $p = \frac{n+2}{n-2}$ and provided insight on the question of infinite time blow-up in the case of a radially symmetric, positive initial condition with an exact power decay rate. Using formal matching asymptotic analysis, they demonstrated that the power decay determines the blow-up rate in a precise manner. Intriguingly enough, their analysis leads them to conjecture that infinite time blow-up should only happen in low dimensions 3 and 4, see Conjecture 1.1 in [26]. Recently this is confirmed and rigorously proved in [20]. Bubbling phenomena are present in many other critical contexts, for example, Keller-Segel chemotaxis system, harmonic map heat flow, Schrödinger map and various geometric flows. We refer the readers for instance to [10–13, 28, 31, 32, 35, 41] and the references therein.

In [9], Cortázar, del Pino and Musso investigated the energy critical heat equation

$$\begin{cases} u_t = \Delta u + u^{\frac{n+2}{n-2}}, & \text{in } \Omega \times (0,+\infty), \\ u = 0, & \text{on } \partial\Omega \times (0,+\infty), \\ u(\cdot,0) = u_0, & \text{in } \Omega, \end{cases} \tag{1.3}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$ with $n \ge 5$, and the initial datum u_0 is positive and smooth. Cortázar, del Pino and Musso constructed solutions exhibiting infinite time blow-up at prescribed points $q_1, \cdots, q_k$ such that the matrix

$$\mathcal{G}(q) = \begin{bmatrix} H(q_1,q_1) & -G(q_1,q_2) & \cdots & -G(q_1,q_k) \\ -G(q_2,q_1) & H(q_2,q_2) & \cdots & -G(q_2,q_k) \\ \vdots & \vdots & \ddots & \vdots \\ -G(q_k,q_1) & -G(q_k,q_2) & \cdots & H(q_k,q_k) \end{bmatrix} \tag{1.4}$$

is positively definite, where $G(x,y)$ is the Dirichlet Green's function of $-\Delta$ in Ω and $H(x,y)$ is its regular part. More precisely, they proved the existence of an initial datum u_0 and smooth parameters

$\xi_j(t) \to q_j$, $0 < \mu_j(t) \to 0$, as $t \to +\infty$, $j = 1, \cdots, k$, such that there exists an infinite time blow-up solution u_q to (1.3) of the following approximate form

$$u_q \approx \sum_{j=1}^{k} \alpha_n \left(\frac{\mu_j(t)}{\mu_j(t)^2 + |x - \xi_j(t)|^2} \right)^{\frac{n-2}{2}}$$

with $\alpha_n = [n(n-2)]^{\frac{n-2}{4}}$ and $\mu_j(t) = \beta_j t^{-\frac{1}{n-4}}(1 + o(1))$ for certain positive constant β_j. We remark that, after the pioneering works [4,5], there has been a lot of literature devoted to studying the role of Green's function and its regular part in elliptic bubbling phenomena for perturbations of the critical problem. See [15,45,47,48] and the references therein.

In this paper, we consider the anisotropic heat equation (1.1) in $\mathbb{R}^n$ $(n \geq 5)$. As mentioned before, in the absence of the vector fields, infinite time blow-up may not exist for dimensions $n \geq 5$ ([26]). The main aim of this paper is to show that the existence of vector fields can produce infinite time blow-up in all dimensions $n \geq 5$. It turns out that the anisotropic coefficient $a(x)$ will play an important role in the sense that it basically determines the location of blow-up points and the blow-up rates. More precisely, the blow-up points $q_1, \cdots, q_k$ are distinct critical points of $a(x)$ such that the Hessian matrix

$$\nabla^2 \log a(q_j) := \mathrm{A}(q_j)$$

is positively definite for $j = 1, \cdots, k$. The Hessian matrix $\mathrm{A}(q_j)$ will play a similar role as $\mathcal{G}(q)$ given in (1.4). We denote P_j by the invertible matrix such that

$$\mathrm{P}_j \mathrm{A}(q_j) {\mathrm{P}_j}^T = \mathrm{diag}\Big(\sigma_1^{(j)}, \sigma_2^{(j)}, \cdots, \sigma_n^{(j)}\Big), \quad \text{for} \quad j = 1, \cdots, k, \tag{1.5}$$

where all the eigenvalues $\sigma_1^{(j)}, \cdots, \sigma_n^{(j)}$ of the matrix $\mathrm{A}(q_j)$ are positive. Define

$$\bar{\sigma}_j := \sum_{i=1}^{n} \frac{\sigma_i^{(j)}}{n}, \qquad j = 1, \cdots, k.$$

We shall assume that there exists a small positive number $\sigma \ll 1$, such that eigenvalues $\sigma_i^{(j)}$ and $\bar{\sigma}_j$ satisfy the following condition

$$\sigma_i^{(j)} < (1 + \sigma) \frac{3n}{n+2} \bar{\sigma}_j, \qquad i = 1, \cdots, n, \ j = 1, \cdots, k. \tag{1.6}$$

The above restriction (1.6) is required to guarantee the solvability of our final reduced equations for parameter functions $\xi_j(t)$, $j = 1, \cdots, k$. See Section 5 for details.

Our main result is stated as follows.

Theorem 1. *Assume that $n \geq 5$, $q_1, \cdots, q_k$ are distinct critical points of $a(x)$ such that the Hessian matrix $\mathrm{A}(q_j)$ is positively definite and the eigenvalues $\sigma_i^{(j)}$ $(i = 1, \cdots, n, \ j = 1, \cdots, k)$ satisfy the condition (1.6). Then there exist smooth functions $\mu_j(t)$, $\xi_j(t)$, $j = 1, \cdots, k$ and an initial datum u_0, such that problem (1.1) has a solution of the form*

$$u(x,t) = \sum_{j=1}^{k} \alpha_n \left(\frac{\mu_j(t)}{\mu_j(t)^2 + |x - \xi_j(t)|^2} \right)^{\frac{n-2}{2}} \big(1 + o(1)\big),$$

where $\alpha_n = [n(n-2)]^{\frac{n-2}{4}}$ and $o(1) \to 0$ as $t \to +\infty$, uniformly away from the points q_j, and

$$\mu_j(t) = e^{-\kappa_j t}(1 + o(1)), \quad |\xi_j(t) - q_j| = O(e^{-\kappa_j (1+\sigma) t}), \quad \text{as} \ \ t \to +\infty,$$

for certain positive constant κ_j defined in (2.25). Moreover, there exists a codimension-k submanifold $\mathcal{M}$ in $X := \big\{u \in C^1(\mathbb{R}^n) : \lim_{|x| \to \infty} u(x) = 0\big\}$ containing $u_q(x,0)$ such that, if u_0 is a small perturbation *of $u_q(x,0)$ in $\mathcal{M}$, then the solution $u(x,t)$ of (1.1) still takes the form*

$$u(x,t) = \sum_{j=1}^{k} \alpha_n \left(\frac{\tilde{\mu}_j(t)}{\tilde{\mu}_j(t)^2 + |x - \tilde{\xi}_j(t)|^2} \right)^{\frac{n-2}{2}} (1 + o(1)),$$

where the point $\tilde{q}_j$ is close to q_j for $j = 1, \cdots, k$.

We make several remarks as follows.

Remark 1.1.

(1) *A specific example of the anisotropic function $a(x)$ is $a(x) = e^{-e^{-|x|^2}}$. In this case, the infinite blow-up occurs at the origin $\mathcal{O} = (0, \cdots, 0)$, which is the local minimum of $b(x) = \log a(x)$. Moreover, the eigenvalue condition (1.6) is satisfied since the Hessian is a diagonal matrix $\mathrm{A}(\mathcal{O}) = \mathrm{diag}(2, \cdots, 2)$ and $(1+\sigma)\frac{3n}{n+2} > 1$ for $n \geq 5$.*

(2) *In fact, the case in which the anisotropic function $a(x,t)$ depends also on time can be dealt with similarly. More precisely, the dynamics for $\mu_{0j}(t)$ (c.f. (2.23)) now become*

$$c_0\, \dot{\mu}_{0j}(t) + c_j(t)\, \mu_{0j}(t) = 0, \qquad j = 1, \cdots, k,$$

where $c_j(t)$ is a function depending on $a(x,t)$ and also the blow-up points. Then suitable scaling parameter $\mu_{0j}(t)$ can be chosen by the above ODE. Another aspect is that the key estimates in the linear heat equation (Section 4.1) can be obtained similarly due to the work of Aronson [2] as long as similar regularity assumptions are imposed on $a(x,t)$.

(3) *We believe the infinite time blow-up also exists for the low dimensions $n = 3, 4$. But difficulties may arise when solving the scaling parameter μ_{0j} due to the slow decay of the error. We will return to this topic in a future work.*

The proof of Theorem 1 is mainly based on the *inner–outer gluing procedure*. The inner–outer gluing procedure has been a very powerful tool in constructing solutions in various elliptic problems, see for instance [13,16–18] and the references therein. Also, this method has been successfully applied to many parabolic equations recently, such as the harmonic map flow from $\mathbb{R}^2$ to $\mathbb{S}^2$ [12], the infinite time blow-up [9,20,22,23] and infinite time bubble towers [19] in energy critical heat equations, type II finite time blow-up along curve for supercritical heat equation [21], vortex dynamics in Euler flows [11], and others arsing from geometry and fractional context [10,44,49,50]. We refer the readers to a survey by del Pino [14] for more results in parabolic settings.

Before we proceed to the proof, we sketch some of the main ideas used in our analysis. In Section 2, we shall construct the first approximation of the form (2.3) and compute the error. In order to improve the approximation, solvability conditions are required for the elliptic linearized operator around the bubble, which will imply the scaling parameter functions $\mu_j(t)$ at main order. After the correction has been added, we set up the inner–outer gluing scheme in Section 3, in which we decompose the small perturbation in the form $\sum_{j=1}^{k} \eta_{j,R}\tilde{\phi}_j + \psi$ where $\eta_{j,R}$ is a smooth cut-off function supported near the concentration point q_j. The tuple $(\tilde{\phi}_j, \psi)$ will satisfy a coupled nonlinear system: the outer problem for ψ and the inner problem for $\tilde{\phi}_j$. Basically, the outer problem is a heat equation with coupling from the inner solution $\tilde{\phi}_j$, while the inner problem is the linearized equation around the bubble. In Section 4, we will use the contraction mapping theorem to solve the outer problem (4.1) for ψ. The key ingredient of the proof is to derive a priori estimates for associated linear problem of the outer problem. In [9] bounded domain case, the a priori estimates are established by a well-chosen comparison function and the parabolic estimates. Due to the extra gradient term in our case, the a priori estimates are achieved by the Duhamel's formula. As a consequence, the a priori estimates we get appear more in parabolic nature. More precisely, the solution to the linear outer problem behaves differently inside and outside the self-similar region. From the linear theory developed in [9, Section 7], the inner problem can be solved, provided that certain orthogonality conditions are satisfied, by means of the contraction mapping theorem. This will be the context of Section 6. The orthogonality conditions will be achieved by adjusting higher order terms of the parameter functions $\mu_j(t)$ and $\xi_j(t)$ in Section 5.2.

Notation. In the sequel, we shall use the symbol " $\lesssim$ " to denote " $\leq C$" for a positive constant C independent of t and t_0, and C may change from line to line. Here $t_0 > 0$ is a constant fixed sufficiently large.

2. Construction of the approximate solution and error estimates

The aim of this Section is to construct the approximate solution $u^*_{\mu,\xi}$ (see (2.28)) and then evaluate its error $\mathbf{S}[u^*_{\mu,\xi}]$, where the error operator $\mathbf{S}$ is defined as

$$\mathbf{S}[u] := -u_t + \Delta u + \nabla \log a(x) \cdot \nabla u + u^p$$

with $p = \frac{n+2}{n-2}$.

It is well known that all positive entire solutions of the equation

$$\Delta U + U^{\frac{n+2}{n-2}} = 0 \quad \text{in} \ \ \mathbb{R}^n$$

are given by the Aubin-Talenti bubbles

$$U_{\mu,\xi}(x) = \mu^{-\frac{n-2}{2}} U\left(\frac{x-\xi}{\mu}\right) = \alpha_n \left(\frac{\mu}{\mu^2 + |x-\xi|^2}\right)^{\frac{n-2}{2}},$$

which are extremals of the Sobolev's embedding (see [3,54]), where $\alpha_n = [n(n-2)]^{\frac{n-2}{4}}$.

To explain the idea, we sketch the major steps of Section 2 here:

Step 1 (First approximate solution)

Our first approximate solution to problem (1.1) is

$$u_{\mu,\xi}(x,t) \ = \ \sum_{j=1}^{k} U_{\mu_j(t),\xi_j(t)}(x)$$

with $\xi_j(t) \to q_j$ and $0 < \mu_j(t) \to 0$ as $t \to \infty$. Near each concentration point q_j for $j = 1, \cdots, k$, the error of $u_{\mu,\xi}$ can be computed as

$$\mathbf{S}[u_{\mu,\xi}](x,t) \ = \ \mu_j^{-\frac{n+2}{2}} \mathcal{E}_{0j}[\mu_j, \dot{\mu}_j](y_j,t) \ + \ \text{h.o.t.}, \quad y_j = \frac{x - \xi_j(t)}{\mu_j(t)},$$

with the main error

$$\mathcal{E}_{0j}[\mu_j, \dot{\mu}_j](y_j,t) \ = \ \mu_j \dot{\mu}_j Z_{n+1}(y_j) \ + \ \mu_j^2 \mathrm{A}(q_j) y_j \cdot \nabla U(y_j). \tag{2.1}$$

Here $Z_{n+1} = \frac{n-2}{2} U(y) + y \cdot \nabla U(y)$. We assume that

$$\mu_j(t) \ = \ b_j \mu_{0j}(t) + \lambda_j(t), \quad |\xi_j(t) - q_j| \ = \ O(\mu_{0j}^{1+\sigma}(t)),$$

where b_j are positive constants (we shall take $b_j = 1$ in the sequel) and the function $\lambda_j(t) \to 0$ as $t \to \infty$. Then we see that the main order of (2.1) is given by

$$\mathcal{E}_{0j}[\mu_{0j}, \dot{\mu}_{0j}](y_j,t) := b_j^2 \mu_{0j} \dot{\mu}_{0j} Z_{n+1}(y_j) \ + \ b_j^2 \mu_{0j}^2 \mathrm{A}(q_j) y_j \cdot \nabla U(y_j).$$

Step 2 (Second approximate solution with correction)

In order to improve the approximate solution, we add a correction

$$\check{\Phi}(x,t) \ := \ \sum_{j=1}^{k} \mu_j^{-\frac{n-2}{2}} \Phi_j \left(\frac{x-\xi_j}{\mu_j}, t\right)$$

to cancel out the main order of the error $\mu_j^{-\frac{n+2}{2}} \mathcal{E}_{0j}[\mu_{0j}, \dot{\mu}_{0j}](y_j,t)$, namely, Φ_j solves the following elliptic equation

$$\Delta \Phi_j \ + \ pU^{p-1} \Phi_j \ = \ -\mathcal{E}_{0j}[\mu_{0j}, \dot{\mu}_{0j}] \quad \text{in} \ \ \mathbb{R}^n, \quad \Phi_j(y_j,t) \to 0 \ \ \text{as} \ \ |y_j| \to \infty. \tag{2.2}$$

Problem (2.2) is solvable if and only if the orthogonality conditions

$$\int_{\mathbb{R}^n} \mathcal{E}_{0j}[\mu_{0j}, \dot{\mu}_{0j}](y_j, t)\, Z_{n+1}(y_j)\, \mathrm{d}y_j \;=\; 0,\;\; j = 1, \cdots, k$$

hold. The scaling parameters $\mu_{0j}(t)$ at main order are derived from the above orthogonality conditions. So we choose a better approximate solution

$$u^*_{\mu,\xi}(x,t) \;:=\; u_{\mu,\xi}(x,t) \;+\; \check{\Phi}(x,t).$$

Step 3 (Estimates of the new error $\mathbf{S}[u^*_{\mu,\xi}]$)

Near the concentration point q_j for fixed $j \in \{1, \cdots, k\}$, we shall evaluate the new error

$$\begin{aligned}\mathbf{S}[u^*_{\mu,\xi}] \;=\mu_j^{-\frac{n+2}{2}} \Big\{ & \big(\mu_{0j}\dot{\lambda}_j + \lambda_j \dot{\mu}_{0j} + \lambda_j \dot{\lambda}_j\big) Z_{n+1}(y_j) + \;\big(2\mu_{0j}\lambda_j + \lambda_j^2\big) \mathrm{A}(q_j) y_j \cdot \nabla U(y_j) \\ & +\; \mu_j \big(\dot{\xi}_j \cdot \nabla U(y_j) + \mathrm{A}(q_j)(\xi_j - q_j) \cdot \nabla U(y_j)\big) \Big\} + \text{h.o.t.},\end{aligned}$$

while in the region away from each concentration point, the error $\mathbf{S}[u^*_{\mu,\xi}]$ is of smaller size compared to the error in the region near q_j.

2.1. The first approximate solution.

Given k distinct points $q_1, \cdots, q_k \in \mathbb{R}^n$, our first approximate solution of problem (1.1) is

$$u_{\mu,\xi}(x,t) = \sum_{j=1}^{k} U_{\mu_j(t),\xi_j(t)}(x) = \sum_{j=1}^{k} \mu_j^{-\frac{n-2}{2}}(t) U\Big(\frac{x - \xi_j(t)}{\mu_j(t)}\Big) \tag{2.3}$$

with $\xi_j(t) \to q_j$ and $0 < \mu_j(t) \to 0$ as $t \to \infty$ for each $j = 1, \cdots, k$. The functions ξ_j and μ_j cannot of course be arbitrary. More precisely, we assume that $\mu_j(t)$ and $\xi_j(t)$ take the following forms respectively

$$\mu_j(t) = b_j \mu_{0j}(t) + \lambda_j(t), \qquad \big|\xi_j(t) - q_j\big| = O(\mu_{0j}^{1+\sigma}(t)), \tag{2.4}$$

where b_j are positive constants (in the following, we will choose b_j to be 1). Moreover, we assume that λ_j satisfies $|\lambda_j(t)| + |\dot{\lambda}_j(t)| \ll \mu_{0j}(t)$ and $\lambda_j(t) \to 0$ as $t \to \infty$.

The error of the first approximation $u_{\mu,\xi}(x,t)$ is

$$\mathbf{S}[u_{\mu,\xi}] = -\sum_{j=1}^{k} \partial_t U_{\mu_j,\xi_j} + \sum_{j=1}^{k} \nabla_x \log a(x) \cdot \nabla_x U_{\mu_j,\xi_j} + \Big(\sum_{j=1}^{k} U_{\mu_j,\xi_j}\Big)^p - \sum_{j=1}^{k} U^p_{\mu_j,\xi_j}.$$

We define

$$B_j \;:=\; \Big\{x \in \mathbb{R}^N,\, |x - \xi_j(t)| \le \delta\Big\},\; j = 1, \cdots, k, \tag{2.5}$$

where $0 < \delta \ll \frac{1}{2} \min\limits_{i \neq j} |q_i - q_j|$ is a small fixed number. In the following lemma, we will give a description of the error $\mathbf{S}[u_{\mu,\xi}]$.

Lemma 2.1. *Let $y_j = \frac{x - \xi_j(t)}{\mu_j(t)}$ and $0 < \alpha < 1$. The error $\mathbf{S}[u_{\mu,\xi}](x,t)$ can be estimated as*

$$\mathbf{S}[u_{\mu,\xi}] = \begin{cases} \mu_j^{-\frac{n+2}{2}} \Big\{ \mathcal{E}_{0j}[\mu_j, \dot{\mu}_j] + \mathcal{E}_{1j}[\mu_j, \xi_j, \dot{\xi}_j] + \mathcal{R}_j \Big\}, \; \text{if } x \in B_j, \\ \displaystyle\sum_{j=1}^{k} \left(\frac{\mu_{0j}^{-\frac{n}{2}+1-\alpha} \dot{\mu}_j g_j}{1 + |y_j|^{n-3+\alpha}} + \frac{\mu_{0j}^{-\frac{n}{2}+2-\alpha} \dot{\xi}_j \cdot \vec{g}_j}{1 + |y_j|^{n-3+\alpha}} + \frac{\mu_{0j}^{-\frac{n}{2}+2-\alpha} \tilde{g}_j}{1 + |y_j|^{n-3+\alpha}} \right), \; \text{if } x \notin \cup_{j=1}^{k} B_j, \end{cases} \tag{2.6}$$

with

$$\mathcal{E}_{0j}[\mu_j, \dot{\mu}_j] \;=\; \mu_j \dot{\mu}_j Z_{n+1}(y_j) + \mu_j^2 \mathrm{A}(q_j) y_j \cdot \nabla U(y_j), \tag{2.7}$$

$$\mathcal{E}_{1j}[\mu_j, \xi_j, \dot{\xi}_j] = \mu_j \dot{\xi}_j \cdot \nabla U(y_j) + \mu_j \mathrm{A}(q_j)(\xi_j - q_j) \cdot \nabla U(y_j), \tag{2.8}$$

and

$$\begin{aligned}\mathcal{R}_j =& \sum_{i\neq j}\frac{(\mu_{0i}\mu_{0j})^{\frac{n-2}{2}}O(|q_j-q_i|^{2-n})}{1+|y_j|^4} + \sum_{i\neq j}(\mu_{0i}\mu_{0j})^{\frac{n+2}{2}}O\big(|q_j-q_i|^{-(n+2)}\big)\\ &+\sum_{i\neq j}\mu_{0j}^{\frac{n+2}{2}}\Big(\mu_i^{\frac{n}{2}-2}\dot\mu_i O(|q_j-q_i|^{2-n}) + \mu_i^{\frac{n}{2}-1}\dot\xi_i\cdot\vec f_{i1}\Big) + \frac{\mu_{0j}^3 O(1)}{1+|y_j|^{n-3}}\\ &+\sum_{i\neq j}\mu_{0i}^{\frac{n-2}{2}}\mu_{0j}^{\frac{n+2}{2}}O(|q_j-q_i|^{1-n}),\end{aligned} \tag{2.9}$$

where $\vec f_{i1}$ *is a smooth and bounded function of* $(y,\mu_{0j}^{-1}\mu,\xi,\mu_j y_j)$, g_j, $\vec g_j$, $\tilde g_j$ *are smooth and bounded functions of* $(x,\ \mu_{0j}^{-1}\mu,\ \xi)$. *Here* $Z_{n+1}(y) := \frac{n-2}{2}U(y) + y\cdot\nabla U(y)$.

Proof. We discuss two different cases.

Case 1: $x\in B_j$ for any fixed $j\in\{1,\cdots,k\}$.

Recalling the definition of $u_{\mu,\xi}$ as in (2.3) and $y_j = \frac{x-\xi_j(t)}{\mu_j(t)}$, we can write $u_{\mu,\xi}$ as

$$u_{\mu,\xi}(x,t) = \sum_{j=1}^k \mu_j^{-\frac{n-2}{2}}U(y_j).$$

We decompose

$$\mathbf{S}[u_{\mu,\xi}] := \mathbf{S}_1+\mathbf{S}_2+\mathbf{S}_3,$$

where

$$\begin{aligned}\mathbf{S}_1 =&\ \mu_j^{-\frac{n+2}{2}}\Bigg\{\Big(\mu_j\dot\mu_j Z_{n+1}(y_j) + \mu_j\dot\xi_j\cdot\nabla U(y_j)\Big)\\ &\quad+\sum_{i\neq j}(\mu_i\mu_j^{-1})^{-\frac{n+2}{2}}\Big(\mu_i\dot\mu_i Z_{n+1}(y_i)+\mu_i\dot\xi_i\cdot\nabla U(y_i)\Big)\Bigg\},\\ \mathbf{S}_2 =&\ \mu_j^{-\frac{n}{2}}\Bigg\{\nabla_x\log a(x)\cdot\nabla U(y_j) + \sum_{i\neq j}\big(\mu_i\mu_j^{-1}\big)^{-\frac{n}{2}}\nabla_x\log a(x)\cdot\nabla U(y_i)\Bigg\},\\ \mathbf{S}_3 =&\ \left(\sum_{j=1}^k\mu_j^{-\frac{n-2}{2}}U(y_j)\right)^p - \sum_{j=1}^k\mu_j^{-\frac{n+2}{2}}U^p(y_j),\end{aligned}$$

and

$$Z_{n+1}(y) := \frac{n-2}{2}U(y) + y\cdot\nabla U(y).$$

For $i\neq j$, we write

$$\begin{aligned}U(y_i) =& U\Big(\frac{x-\xi_i}{\mu_i}\Big) = U\Big(\frac{\mu_j y_j+\xi_j-\xi_i}{\mu_i}\Big)\\ =& \frac{\alpha_n\mu_i^{n-2}}{\big[\mu_i^2+|\mu_j y_j+\xi_j-\xi_i|^2\big]^{\frac{n-2}{2}}}\\ =& \frac{\alpha_n\mu_i^{n-2}}{|\mu_j y_j+\xi_j-\xi_i|^{n-2}}\frac{1}{\Big(1+\frac{\mu_i^2}{|\mu_j y_j+\xi_j-\xi_i|^2}\Big)^{\frac{n-2}{2}}}.\end{aligned}$$

Then by direct Taylor expansion and $y_j = \frac{x-\xi_j}{\mu_j}$, we get the following estimates

$$U(y_i) = \mu_i^{n-2}O(|q_j-q_i|^{2-n}). \tag{2.10}$$

Similarly, we can get

$$y_i \cdot \nabla U(y_i) = \mu_i^{n-2} O\big(|q_j - q_i|^{2-n}\big). \tag{2.11}$$

First, we will give the estimate of the terms in $\mathbf{S}_1$. Using (2.10) and (2.11), for $i \neq j$, we get

$$\begin{aligned}\mu_i\dot{\mu}_i Z_{n+1}(y_i) + \mu_i\dot{\xi}_i \cdot \nabla U(y_i) &= \mu_i\dot{\mu}_i \left(\frac{n-2}{2}U(y_i) + y_i \cdot \nabla U(y_i)\right) + \mu_i\dot{\xi}_i \cdot \nabla U(y_i) \\ &= \dot{\mu}_i\mu_i^{n-1}O(|q_j - q_i|^{2-n}) + \mu_i^n \dot{\xi}_i \cdot \vec{f}_{i1},\end{aligned}$$

where $\vec{f}_{i1}$ is a smooth function with order $|\vec{f}_{i1}| = O\big(|q_j - q_i|^{1-n}\big)$ and thus

$$\begin{aligned}\mathbf{S}_1 = \mu_j^{-\frac{n+2}{2}}\Bigg\{&\mu_j\dot{\mu}_j Z_{n+1}(y_j) + \mu_j\dot{\xi}_j \cdot \nabla U(y_j) \\ &+ \sum_{i\neq j} \mu_{0j}^{\frac{n+2}{2}} \left(\mu_i^{\frac{n}{2}-2}\dot{\mu}_i O(|q_j - q_i|^{2-n}) + \mu_i^{\frac{n}{2}-1}\dot{\xi}_i \cdot \vec{f}_{i1}\right)\Bigg\}.\end{aligned}$$

Next we consider $\mathbf{S}_2$. Using Taylor expansion, we can rewrite $\nabla_x \log a(\mu_j y_j + \xi_j)$ in the following form

$$\nabla_x \log a(\mu_j y_j + \xi_j) = \mu_j \mathrm{A}(q_j)y_j + \mathrm{A}(q_j)(\xi_j - q_j) + \vec{f}_{j2},$$

where $\vec{f}_{j2}$ is a smooth function with order $|\vec{f}_{j2}| = O\big(|\mu_j y_j + \xi_j - q_j|^2\big)$ and we have used the assumption that $\nabla_x \log a(q_j) = 0$ and the Hessian matrix $\mathrm{A}(q_j)$ is positively definite. Then, the component $\mathbf{S}_2$ can be written in the form by using (2.4)

$$\begin{aligned}\mathbf{S}_2 = \ \mu_j^{-\frac{n+2}{2}}\Bigg\{&\mu_j^2\mathrm{A}(q_j)y_j \cdot \nabla U(y_j) + \mu_j\mathrm{A}(q_j)(\xi_j - q_j) \cdot \nabla U(y_j) \\ &+ \frac{\mu_j^3}{1 + |y_j|^{n-3}}O(1)\Bigg\} + \sum_{i\neq j} \mu_i^{\frac{n-2}{2}} O(|q_j - q_i|^{1-n}).\end{aligned}$$

Finally we estimate $\mathbf{S}_3$. We further write

$$\begin{aligned}\mathbf{S}_3 &= \mu_j^{-\frac{n+2}{2}}\Big\{\Big[\big(U(y_j) + \vartheta_j\big)^p - U^p(y_j)\Big] - \sum_{i\neq j}\big(\mu_i\mu_j^{-1}\big)^{-\frac{n+2}{2}}U^p(y_i)\Big\} \\ &:= \mu_j^{-\frac{n+2}{2}}\Big(\mathbf{S}_{31} + \mathbf{S}_{32}\Big),\end{aligned}$$

where

$$\vartheta_j = \sum_{i\neq j}\big(\mu_j\mu_i^{-1}\big)^{\frac{n-2}{2}}U(y_i).$$

By Taylor expansion, we have

$$\begin{aligned}\mathbf{S}_{31} &= \big[U(y_j) + \vartheta_j\big]^p - U^p(y_j) \\ &= pU^{p-1}(y_j)\vartheta_j + p(p-1)\vartheta_j^2\int_0^1 (1-s)\big[U(y_j) + s\vartheta_j\big]^{p-2}\mathrm{d}s.\end{aligned}$$

From (2.10) and (2.11), we obtain

$$\vartheta_j = \sum_{i\neq j}\big(\mu_j\mu_i\big)^{\frac{n-2}{2}} O\big(|q_j - q_i|^{2-n}\big). \tag{2.12}$$

Since $x \in B_j$, we notice that $|\vartheta_j| \lesssim (\mu_{0j}\mu_{0i})^{\frac{n-2}{2}}$ uniformly in small δ, where δ is the radius given in (2.5). Thus the second term in $\mathbf{S}_{31}$ is of smaller order compared with $pU^{p-1}(y_j)\vartheta_j$. On the other hand, we have that

$$\mathbf{S}_{32} = -\sum_{i\neq j}\big(\mu_i\mu_j^{-1}\big)^{-\frac{n+2}{2}}U^p(y_i) = \sum_{i\neq j}\big(\mu_{0i}\mu_{0j}\big)^{\frac{n+2}{2}}O\big(|q_j - q_i|^{-(n+2)}\big). \tag{2.13}$$

Therefore, by (2.12) and (2.13), we obtain

$$\mathbf{S}_3 = \mu_j^{-\frac{n+2}{2}} \left[\sum_{i\neq j} \frac{(\mu_{0i}\mu_{0j})^{\frac{n-2}{2}}}{1+|y_j|^4} O\big(|q_j - q_i|^{2-n}\big) + \sum_{i\neq j} (\mu_{0i}\mu_{0j})^{\frac{n+2}{2}} O\big(|q_j - q_i|^{-(n+2)}\big) \right].$$

Collecting the above estimates for $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{S}_3$, we get the estimate of $\mathbf{S}[u_{\mu,\xi}]$ for $x \in B_j$.

Case 2: $x \notin \cup_{j=1}^k B_j$.

Recall that

$$\begin{aligned}\mathbf{S}[u_{\mu,\xi}] =& \sum_{j=1}^k \mu_j^{-\frac{n+2}{2}} \Big(\mu_j \dot{\mu}_j Z_{n+1}(y_j) + \mu_j \dot{\xi}_j \cdot \nabla U(y_j)\Big) + \sum_{j=1}^k \mu_j^{-\frac{n}{2}} \nabla_x \log a(x) \cdot \nabla U(y_j) \\ &+ \left(\sum_{j=1}^k \mu_j^{-\frac{n-2}{2}} U(y_j) \right)^p - \sum_{j=1}^k \mu_j^{-\frac{n+2}{2}} U^p(y_j).\end{aligned}$$

From (2.10) and (2.11), the error estimate of the approximation $u_{\mu,\xi}$ in the region far away from the concentration point q_j $(j = 1, \cdots, k)$ is a direct consequence of Taylor expansion similar to the first case. Indeed, we take the first term in $\mathbf{S}[u_{\mu,\xi}]$ as an example

$$\sum_{j=1}^k \mu_j^{-\frac{n}{2}} \dot{\mu}_j Z_{n+1}(y_j) = \sum_{j=1}^k \frac{\mu_{0j}^{-\frac{n}{2}+1-\alpha} \dot{\mu}_j}{1+|y_j|^{n-3+\alpha}} O(|x - q_j|^{\alpha-1}),$$

where we have used $Z_{n+1}(y) \sim \frac{1}{1+|y|^{n-2}}$. The estimates for the rest terms can be carried out in a similar manner. □

For a solution with the following form

$$u(x,t) = u_{\mu,\xi}(x,t) + \check{\varphi}(x,t),$$

we now derive some useful formulas needed later on. The new error of $u(x,t)$ is

$$\mathbf{S}[u_{\mu,\xi} + \check{\varphi}] = -\partial_t \check{\varphi} + \Delta \check{\varphi} + \nabla \log a(x) \cdot \nabla \check{\varphi} + p u_{\mu,\xi}^{p-1} \check{\varphi} + \mathbf{S}[u_{\mu,\xi}] + \check{N}_{\mu,\xi}(\check{\varphi}),$$

where $\check{N}_{\mu,\xi}(\check{\varphi}) = (u_{\mu,\xi} + \check{\varphi})^p - u_{\mu,\xi}^p - p u_{\mu,\xi}^{p-1} \check{\varphi}$. We write

$$\check{\varphi}(x,t) := \sum_{j=1}^k \mu_j^{-\frac{n-2}{2}} \varphi_j \left(\frac{x - \xi_j}{\mu_j}, t \right) = \sum_{j=1}^k \mu_j^{-\frac{n-2}{2}} \varphi_j(y_j, t). \tag{2.14}$$

Then, it follows that

$$\mathbf{S}[u_{\mu,\xi} + \check{\varphi}] = \mathbf{S}[u_{\mu,\xi}] + \sum_{j=1}^k \mu_j^{-\frac{n+2}{2}} [\Delta \varphi_j(y_j,t) + p U^{p-1}(y_j) \varphi_j(y_j,t)] + \mathbb{A}[\check{\varphi}], \tag{2.15}$$

where

$$\begin{aligned}\mathbb{A}[\check{\varphi}] =& \sum_{j=1}^k \mu_j^{-\frac{n+2}{2}} \Bigg\{ -\mu_j^2 \partial_t \varphi_j(y_j,t) + \mu_j \dot{\mu}_j \Big[\frac{n-2}{2} \varphi_j(y_j,t) + y_j \cdot \nabla \varphi_j(y_j,t) \Big] \\ &+ \mu_j \dot{\xi}_j \cdot \nabla \varphi_j(y_j,t) \Bigg\} + \sum_{j=1}^k \mu_j^{-\frac{n}{2}} \nabla_x \log a(x) \cdot \nabla \varphi_j(y_j,t) \\ &+ \Big(u_{\mu,\xi} + \sum_{j=1}^k \mu_j^{-\frac{n-2}{2}} \varphi_j \Big)^p - u_{\mu,\xi}^p - p \sum_{j=1}^k \mu_j^{-\frac{n+2}{2}} U^{p-1}(y_j) \varphi_j(y_j,t).\end{aligned} \tag{2.16}$$

In order to reduce the size of the error $\mathbf{S}[u_{\mu,\xi}]$, it is reasonable to assume that the correction term $\varphi_j(y_j,t)$, $j = 1, \cdots, k$ decays in the y_j variable and for large t the terms in $\mathbb{A}[\check{\varphi}]$ are comparatively small.

2.2. **The second approximate solution with correction.** In order to improve the approximation, we should cancel out the main order components of the largest term in the expansion of the error $\mu_j^{\frac{n+2}{2}}\mathbf{S}[u_{\mu,\xi}]$ given in (2.6), i.e., $\mathcal{E}_{0j}[\mu_j,\dot\mu_j]$ defined in (2.7).

Recall that

$$\mu_j(t)=b_j\mu_{0j}(t)+\lambda_j(t),\ |\xi_j(t)-q_j|=O\big(\mu_{0j}^{1+\sigma}(t)\big),$$

where $\lambda_j(t)$ is a small perturbation term of $\mu_{0j}(t)$. Then, from (2.7) we obtain that

$$\begin{aligned}\mathcal{E}_{0j}[\mu_j,\dot\mu_j]&=\mathcal{E}_{0j}[\mu_{0j},\dot\mu_{0j}]+(b_j\mu_{0j}\dot\lambda_j+b_j\lambda_j\dot\mu_{0j}+\lambda_j\dot\lambda_j)Z_{n+1}(y_j)\\&\quad+(2b_j\mu_{0j}\lambda_j+\lambda_j^2)\mathrm{A}(q_j)y_j\cdot\nabla U(y_j),\end{aligned}\tag{2.17}$$

where $\mathcal{E}_{0j}[\mu_{0j},\dot\mu_{0j}]$ is the leading order term of $\mathcal{E}_{0j}[\mu_j,\dot\mu_j]$

$$\mathcal{E}_{0j}[\mu_{0j},\dot\mu_{0j}]\ :=\ b_j^2\mu_{0j}\dot\mu_{0j}Z_{n+1}(y_j)+b_j^2\mu_{0j}^2\mathrm{A}(q_j)y_j\cdot\nabla U(y_j).\tag{2.18}$$

An improvement of the approximation can be obtained if we solve the elliptic equation

$$\Delta\phi+pU^{p-1}\phi\ =-\mathcal{E}_{0j}[\mu_{0j},\dot\mu_{0j}]\quad\text{in }\mathbb{R}^n,\quad \phi(y,t)\to0\quad\text{as}\quad|y|\to\infty.\tag{2.19}$$

The decay condition is added in order not to essentially modify the size of the error far away from the q_j's.

We first consider the associated linear problem of (2.19)

$$L_0(\phi):=\ \Delta\phi+pU^{p-1}\phi=h\quad\text{in }\mathbb{R}^n,\quad\phi(y,t)\to0\quad\text{as}\quad|y|\to\infty.\tag{2.20}$$

It is well known (see for example [6]) that all bounded solutions to $L_0(\psi)=0$ in $\mathbb{R}^n$ consist of the linear combinations of the functions $Z_1,\cdots,Z_{n+1}$ defined by

$$Z_i(y)\ :=\ \frac{\partial U}{\partial y_i}(y),\quad i=1,\cdots,n,\qquad Z_{n+1}(y)\ :=\ \frac{n-2}{2}U(y)+y\cdot\nabla U(y),\tag{2.21}$$

and for a function $h(y)=O(|y|^{-m})$, $m>2$, the problem (2.20) is solvable if and only if the following orthogonality conditions hold

$$\int_{\mathbb{R}^n}h(y)Z_\ell(y)\,\mathrm{d}y=0\quad\text{for all}\quad\ell=1,\cdots,n+1.$$

By the definition (2.18), $\mathcal{E}_{0j}[\mu_{0j},\dot\mu_{0j}]$ is even in y_j. Notice from (2.21) that $Z_1(y),\cdots,Z_n(y)$ are odd in y, while $Z_{n+1}(y)$ is even in y. Thus, the orthogonality conditions

$$\int_{\mathbb{R}^n}\mathcal{E}_{0j}[\mu_{0j},\dot\mu_{0j}]\,Z_\ell(y_j)\,\mathrm{d}y_j=0,\ \ell=1,\cdots,n+1,\ j=1,\cdots,k\tag{2.22}$$

imply

$$c_0\,\dot\mu_{0j}(t)+c_j\,\mu_{0j}(t)=0,\qquad j=1,\cdots,k,\tag{2.23}$$

where

$$c_0=\int_{\mathbb{R}^n}|Z_{n+1}(y)|^2\,\mathrm{d}y,\quad c_j=\int_{\mathbb{R}^n}\mathrm{A}(q_j)y\cdot\nabla U(y)Z_{n+1}(y)\,\mathrm{d}y,\qquad j=1,\cdots,k.\tag{2.24}$$

Note that $c_0<+\infty$ thanks to our assumption $n\geq5$. By direct computations, we have

$$\begin{aligned}c_j&=\int_{\mathbb{R}^n}\nabla U(y)\cdot(\mathrm{A}(q_j)y)Z_{n+1}(y)\,\mathrm{d}y\\&=\int_{\mathbb{R}^n}\nabla U(y)\cdot\big(\sigma_1^{(j)}y_1,\cdots,\sigma_n^{(j)}\,y_n\big)Z_{n+1}(y)\,\mathrm{d}y\\&=\sum_i\frac{\sigma_i^{(j)}}{n}\int_{\mathbb{R}^n}\nabla U(y)\cdot y\,Z_{n+1}(y)\,\mathrm{d}y:=\sum_i\frac{\sigma_i^{(j)}}{n}\tilde c,\end{aligned}$$

where $\tilde c:=\int_{\mathbb{R}^n}\nabla U(y)\cdot y\,Z_{n+1}(y)\,\mathrm{d}y$.

Define $I_p^q := \int_0^{+\infty} \frac{r^q}{(1+r)^p} dr$ with $p-q>1$. By using the properties (see for instance [13, Remark 4.1])

$$I_{p+1}^q = \frac{p-q-1}{p} I_p^q \quad \text{and} \quad I_{p+1}^{q+1} = \frac{q+1}{p-q-1} I_{p+1}^q,$$

we obtain

$$\tilde{c} = \alpha_n^2 \frac{3(n-2)^2}{2(n-4)} \omega_n I_n^{n/2} \quad \text{and} \quad c_0 = \alpha_n^2 \frac{(n-2)^2(n+2)}{2n(n-4)} \omega_n I_n^{n/2},$$

where $\alpha_n = [n(n-2)]^{\frac{n-2}{4}}$ and ω_n is the area of the sphere $\mathbb{S}^{n-1}$. It then follows that

$$\tilde{c} = \frac{3n}{n+2} c_0.$$

Therefore, we obtain that

$$\kappa_j := \frac{c_j}{c_0} = \frac{3n}{n+2} \bar{\sigma}_j, \qquad j = 1, \cdots, k, \tag{2.25}$$

where $\bar{\sigma}_j := \sum_{i=1}^n \frac{\sigma_i^{(j)}}{n} > 0$ by the assumption. Therefore, we can choose

$$\mu_{0j}(t) = e^{-\kappa_j t}, \qquad j = 1, \cdots, k, \tag{2.26}$$

such that the orthogonality conditions (2.22) are satisfied. From the choice of the parameter function $\mu_{0j}(t)$, we have

$$-\mathcal{E}_{0j}[\mu_{0j}, \dot{\mu}_{0j}] = \mu_{0j}^2(t) \big[\kappa_j \, b_j^2 \, Z_{n+1}(y_j) \, - \, b_j^2 \, \mathrm{A}(q_j) \, y_j \cdot \nabla U(y_j) \big] := \mu_{0j}^2(t) \, \omega_j(y_j)$$

with $\int_{\mathbb{R}^n} \omega_j(y_j) \, Z_\ell(y_j) \, \mathrm{d}y_j = 0$, $\ell = 1, \cdots, n+1$, $j = 1, \cdots, k$. Here $\omega_j(y_j)$ is even in y_j.

By (2.18), the orthogonality conditions (2.22) hold for any $b_j > 0$. So we can simply let $b_j = 1$ for $j = 1, \cdots, k$. Let $\mathbf{p}_j(y_j)$ be a decaying solution to

$$\Delta_y \mathbf{p}_j(y_j) + pU(y_j)^{p-1} \mathbf{p}_j(y_j) = \omega_j(y_j) \quad \text{in } \mathbb{R}^n, \quad \mathbf{p}_j(y_j) \to 0 \quad \text{as} \quad |y_j| \to \infty.$$

From (2.21), we have

$$|\omega_j(y_j)| \lesssim \frac{1}{1+|y_j|^{n-2}}.$$

Then from standard elliptic theory, it holds that

$$|\mathbf{p}_j(y_j)| \lesssim \frac{1}{1+|y_j|^{n-4}} \quad \text{as} \quad |y_j| \to \infty.$$

Therefore

$$\Phi_j(y_j, t) \; := \; \mu_{0j}^2(t) \, \mathbf{p}_j(y_j) \tag{2.27}$$

is a solution to (2.19) with

$$|\Phi_j(y_j, t)| \lesssim \frac{\mu_{0j}^2(t)}{1+|y_j|^{n-4}} \quad \text{and} \quad |\nabla \Phi_j(y_j, t)| \lesssim \frac{\mu_{0j}^2(t)}{1+|y_j|^{n-3}}.$$

Thus, to reduce the size of the error $\mathbf{S}[u_{\mu,\xi}]$, in (2.14) we can choose $\varphi_j = \Phi_j$. More precisely, we define the corrected approximate solution as

$$u_{\mu,\xi}^*(x,t) \; := \; u_{\mu,\xi}(x,t) + \check{\Phi}(x,t) \quad \text{with} \quad \check{\Phi}(x,t) \; := \; \sum_{j=1}^k \mu_j^{-\frac{n-2}{2}} \Phi_j \left(\frac{x-\xi_j}{\mu_j}, t \right). \tag{2.28}$$

2.3. **Estimating the new error $\mathbf{S}[u^*_{\mu,\xi}]$.** By the previous computations (2.15) and (2.16), we obtain the new error of approximate solution $u^*_{\mu,\xi}$ in the form

$$\begin{aligned}\mathbf{S}[u^*_{\mu,\xi}] &= \mathbf{S}[u_{\mu,\xi}] - \mu_j^{-\frac{n+2}{2}}\mathcal{E}_{0j}[\mu_{0j},\dot\mu_{0j}] - \sum_{i\neq j}^{k}\mu_i^{-\frac{n+2}{2}}\mathcal{E}_{0i}[\mu_{0i},\dot\mu_{0i}] + \mathbb{A}[\check\Phi]\\ &:= \mathbf{A}_1 + \mathbf{A}_2,\end{aligned} \tag{2.29}$$

where $\mathbb{A}[\check\Phi]$ is defined in (2.16),

$$\mathbf{A}_1 = \mathbf{S}[u_{\mu,\xi}] - \mu_j^{-\frac{n+2}{2}}\mathcal{E}_{0j}[\mu_{0j},\dot\mu_{0j}],$$

and

$$\mathbf{A}_2 = -\sum_{i\neq j}^{k}\mu_i^{-\frac{n+2}{2}}\mathcal{E}_{0i}[\mu_{0i},\dot\mu_{0i}] + \mathbb{A}[\check\Phi].$$

The expansion for the new error $\mathbf{S}[u^*_{\mu,\xi}]$ is given by the following lemma.

Lemma 2.2. *Assume $\mu_j(t) = \mu_{0j}(t) + \lambda_j(t)$ with $|\lambda_j(t)| \lesssim \mu_{0j}^{1+\sigma}(t)$ for some $\sigma > 0$ small and $|\xi_j(t) - q_j| \lesssim \mu_{0j}^{1+\sigma}(t)$. It holds that for t large*

$$\mathbf{S}[u^*_{\mu,\xi}] = \begin{cases}\mathbf{S}_{\mu,\xi,j} + \mathbf{S}^{(2)}_{\mu,\xi}, & \text{if } x\in B_j,\\ \mathbf{S}^{(3)}_{\mu,\xi}, & \text{if } x\notin \cup_{j=1}^k B_j,\end{cases} \tag{2.30}$$

with

$$\begin{aligned}\mathbf{S}_{\mu,\xi,j} = \mu_j^{-\frac{n+2}{2}}\Big\{&(\mu_{0j}\dot\lambda_j + \lambda_j\dot\mu_{0j} + \lambda_j\dot\lambda_j)Z_{n+1}(y_j) + (2\mu_{0j}\lambda_j + \lambda_j^2)\mathrm{A}(q_j)y_j\cdot\nabla U(y_j)\\ &+ \mu_j\Big(\dot\xi_j\cdot\nabla U(y_j) + \mathrm{A}(q_j)(\xi_j - q_j)\cdot\nabla U(y_j)\Big)\Big\},\end{aligned} \tag{2.31}$$

$$\begin{aligned}\mathbf{S}^{(2)}_{\mu,\xi} =& \mu_j^{-\frac{n+2}{2}}\mathcal{R}_j + \sum_{j=1}^{k}\mu_j^{-\frac{n+2}{2}}\left\{\frac{\mu_{0j}^3\dot\mu_{0j}O(1)}{1+|y_j|^{n-4}} + \frac{\mu_{0j}^3|\dot\xi_j|O(1)}{1+|y_j|^{n-3}} + \frac{\mu_{0j}^3 O(1)}{1+|y_j|^{n-3}}\right\}\\ &+ \sum_{i\neq j}^{k}\mu_{0i}^{\frac{n-2}{2}}O(|q_j - q_i|^{2-n}) + L_2,\end{aligned} \tag{2.32}$$

and

$$\begin{aligned}\mathbf{S}^{(3)}_{\mu,\xi} =& \sum_{j=1}^{k}\left(\frac{\mu_{0j}^{-\frac{n}{2}-\alpha}(\mu_{0j}\dot\lambda_j + \lambda_j\dot\mu_{0j} + \lambda_j\dot\lambda_j)h_j}{1+|y_j|^{n-3+\alpha}} + \frac{\mu_{0j}^{-\frac{n}{2}+2-\alpha}\dot\xi_j\cdot\vec{h}_j}{1+|y_j|^{n-3+\alpha}} + \frac{\mu_{0j}^{-\frac{n}{2}+2-\alpha}\tilde h_j}{1+|y_j|^{n-3+\alpha}}\right)\\ &+ \sum_{j=1}^{k}\left(\frac{\mu_{0j}^{-\frac{n}{2}+2}\dot\mu_j h_j}{1+|y_j|^{n-4}} + \frac{\mu_{0j}^{-\frac{n}{2}+2}\dot\xi_j\cdot\vec{h}_j}{1+|y_j|^{n-3}} + \frac{\mu_{0j}^{-\frac{n}{2}+2}\tilde h_j}{1+|y_j|^{n-3}}\right),\end{aligned} \tag{2.33}$$

where $x = \xi_j + \mu_j y_j$, $0 < \alpha < 1$ and L_2, $\mathcal{R}_j$ are defined in (2.37)and (2.9), respectively. Moreover, h_j, $\vec{h}_j$, $\tilde h_j$ are smooth, bounded functions of $(x,\ \mu_{0j}^{-1}\mu,\ \xi)$.

Proof. We consider two cases.

Case 1: $x \in B_j$ for any fixed $j \in \{1,\cdots,k\}$.

We need to estimate $\mathbf{S}[u^*_{\mu,\xi}] = \mathbf{A}_1 + \mathbf{A}_2$. By (2.6), (2.29) and (2.17), one has

$$\begin{aligned}\mathbf{A}_1 &= \mu_j^{-\frac{n+2}{2}}\Big\{\mathcal{E}_{0j}[\mu_j,\dot\mu_j] - \mathcal{E}_{0j}[\mu_{0j},\dot\mu_{0j}] + \mathcal{E}_{1j}[\mu_j,\xi_j,\dot\xi_j] + \mathcal{R}_j\Big\}\\ &= \mu_j^{-\frac{n+2}{2}}\Big\{(\mu_{0j}\dot\lambda_j + \lambda_j\dot\mu_{0j} + \lambda_j\dot\lambda_j)Z_{n+1}(y_j) + (2\mu_{0j}\lambda_j + \lambda_j^2)\mathrm{A}(q_j)y_j\cdot\nabla U(y_j)\\ &\qquad + \mu_j\Big(\dot\xi_j\cdot\nabla U(y_j) + \mathrm{A}(q_j)(\xi_j - q_j)\cdot\nabla U(y_j)\Big) + \mathcal{R}_j\Big\},\end{aligned} \tag{2.34}$$

We then estimate $\mathbf{A}_2$ in the region $x \in B_j$. By (2.10) and (2.11), we get

$$\mathcal{E}_{0i}[\mu_{0i}, \dot{\mu}_{0i}] = \mu_{0i}^n O(|q_j - q_i|^{2-n}).$$

According to the expression of $\mathbb{A}$ as in (2.16), we express

$$\mathbb{A}[\check{\Phi}] := L_1 + L_2,$$

where

$$L_1 = \sum_{j=1}^k \mu_j^{-\frac{n+2}{2}} \Big\{ - \mu_j^2 \partial_t \Phi_j(y_j, t) + \mu_j \dot{\mu}_j \Big[\frac{n-2}{2} \Phi_j(y_j, t) + y_j \cdot \nabla \Phi_j(y_j, t) \Big] + \mu_j \dot{\xi}_j \cdot \nabla \Phi_j(y_j, t) \Big\} + \sum_{j=1}^k \mu_j^{-\frac{n}{2}} \nabla_x \log a(x) \cdot \nabla \Phi_j(y_j, t)$$

and

$$L_2 = \Big(u_{\mu,\xi} + \sum_{j=1}^k \mu_j^{-\frac{n-2}{2}} \Phi_j \Big)^p - u_{\mu,\xi}^p - p \sum_{j=1}^k \mu_j^{-\frac{n+2}{2}} U(y_j)^{p-1} \Phi_j(y_j, t).$$

By (2.27), we obtain

$$L_1 = \sum_{j=1}^k \mu_j^{-\frac{n+2}{2}} \left(\frac{\mu_{0j}^3 \dot{\mu}_{0j} O(1)}{1 + |y_j|^{n-4}} + \frac{\mu_{0j}^3 |\dot{\xi}_j| O(1)}{1 + |y_j|^{n-3}} + \frac{\mu_{0j}^3 O(1)}{1 + |y_j|^{n-3}} \right).$$

We can rewrite L_2 as

$$\begin{aligned} L_2 &= \Big(u_{\mu,\xi} + \sum_{j=1}^k \mu_j^{-\frac{n-2}{2}} \Phi_j \Big)^p - u_{\mu,\xi}^p - p (u_{\mu,\xi})^{p-1} \sum_{j=1}^k \mu_j^{-\frac{n-2}{2}} \Phi_j \\ &\quad + p (u_{\mu,\xi})^{p-1} \sum_{j=1}^k \mu_j^{-\frac{n-2}{2}} \Phi_j - p \sum_{j=1}^k \mu_j^{-\frac{n+2}{2}} U(y_j)^{p-1} \Phi_j \\ &= L_{21} + L_{22} \end{aligned} \tag{2.35}$$

with

$$\begin{aligned} L_{21} &:= \Big(u_{\mu,\xi} + \sum_{j=1}^k \mu_j^{-\frac{n-2}{2}} \Phi_j \Big)^p - u_{\mu,\xi}^p - p (u_{\mu,\xi})^{p-1} \sum_{j=1}^k \mu_j^{-\frac{n-2}{2}} \Phi_j \\ &= O(1) \begin{cases} u_{\mu,\xi}^{p-2} \Big(\sum_{j=1}^k \mu_j^{-\frac{n-2}{2}} \Phi_j \Big)^2, & \text{for } \ n = 5, 6, \\ \Big(\sum_{j=1}^k \mu_j^{-\frac{n-2}{2}} \Phi_j \Big)^p, & \text{for } \ n \geq 7, \end{cases} \end{aligned} \tag{2.36}$$

where we have used the fact that $\sum_{j=1}^k \mu_j^{-\frac{n-2}{2}} \Phi_j \lesssim u_{\mu,\xi}$ for $x \in B_j$. For L_{22}, we write

$$\begin{aligned} & p (u_{\mu,\xi})^{p-1} \sum_{j=1}^k \mu_j^{-\frac{n-2}{2}} \Phi_j - p \sum_{j=1}^k \mu_j^{-\frac{n+2}{2}} U(y_j)^{p-1} \Phi_j \\ &= p \left(\mu_j^{-\frac{n-2}{2}} U(y_j) + \sum_{i \neq j}^k \mu_i^{-\frac{n-2}{2}} U(y_i) \right)^{p-1} \left(\mu_j^{-\frac{n-2}{2}} \Phi_j + \sum_{i \neq j}^k \mu_i^{-\frac{n-2}{2}} \Phi_i \right) \\ &\quad - p \left(\mu_j^{-\frac{n+2}{2}} U(y_j)^{p-1} \Phi_j + \sum_{i \neq j}^k \mu_i^{-\frac{n+2}{2}} U(y_i)^{p-1} \Phi_i \right) \end{aligned}$$

which turns out to be small by a similar argument as in Lemma 2.1. By (2.27), (2.35) and (2.36), we obtain

$$L_2 = O(1)\begin{cases} \displaystyle\sum_{i=1}^{k} \frac{\mu_i^{\frac{6-n}{2}}}{1+|y_i|^{n-2}}, & \text{for } \ n=5,6,\\ \displaystyle\sum_{i=1}^{k} \frac{\mu_i^{-\frac{n+2}{2}+\frac{2(n+2)}{n-2}}}{1+|y_i|^{\frac{(n+2)(n-4)}{n-2}}}, & \text{for } \ n\geq 7. \end{cases} \tag{2.37}$$

Note that in (2.37) for the case $n=5$, we have used the fact that $x\in B_j$ such that all terms in the summation $i\neq j$ are essentially of smaller order. Therefore, we conclude that for $x\in B_j$

$$\begin{aligned} \mathbf{A}_2 =& \sum_{j=1}^{k} \mu_j^{-\frac{n+2}{2}} O(1)\left(\frac{\mu_{0j}^3\dot{\mu}_{0j}}{1+|y_j|^{n-4}} + \frac{\mu_{0j}^3|\dot{\xi}_j|}{1+|y_j|^{n-3}} + \frac{\mu_{0j}^3}{1+|y_j|^{n-3}}\right)\\ &+\sum_{i\neq j}^{k} \mu_{0i}^{\frac{n-2}{2}} O(|q_j-q_i|^{2-n}) + L_2. \end{aligned} \tag{2.38}$$

Collecting (2.34) and (2.38), we get the desired estimate.

Case 2: $x\notin \cup_{j=1}^k B_j$.

Due to the spatial decay of Φ_j, the size of $\check{\Phi}$ is μ_{0j}^2-times smaller than that of $u_{\mu,\xi}$ in the region far away from q_j. A direct consequence of Lemma 2.1 and (2.27) is that, if $x\notin \cup_{j=1}^k B_j$, the error $\mathbf{S}[u^*_{\mu,\xi}]$ can be described as follows

$$\begin{aligned} \mathbf{S}[u^*_{\mu,\xi}] =& \sum_{j=1}^{k}\left(\frac{\mu_{0j}^{-\frac{n}{2}-\alpha}(\mu_{0j}\dot{\lambda}_j+\lambda_j\dot{\mu}_{0j}+\lambda_j\dot{\lambda}_j)h_j}{1+|y_j|^{n-3+\alpha}} + \frac{\mu_{0j}^{-\frac{n}{2}+2-\alpha}\dot{\xi}_j\cdot\vec{h}_j}{1+|y_j|^{n-3+\alpha}} + \frac{\mu_{0j}^{-\frac{n}{2}+2-\alpha}\tilde{h}_j}{1+|y_j|^{n-3+\alpha}}\right)\\ &+\sum_{j=1}^{k}\left(\frac{\mu_{0j}^{-\frac{n}{2}+2}\dot{\mu}_j h_j}{1+|y_j|^{n-4}} + \frac{\mu_{0j}^{-\frac{n}{2}}\dot{\xi}_j\cdot\vec{h}_j}{1+|y_j|^{n-3}} + \frac{\mu_{0j}^{-\frac{n}{2}+2}\tilde{h}_j}{1+|y_j|^{n-3}}\right), \end{aligned}$$

where h_j, $\vec{h}_j$, $\tilde{h}_j$ are smooth, bounded functions of $(x,\,\mu_{0j}^{-1}\mu,\,\xi)$. Indeed, for $x\notin\cup_{j=1}^k B_j$, by (2.10) and (2.11), we have

$$\begin{aligned} &\mathbf{S}[u_{\mu,\xi}] - \mu_j^{-\frac{n+2}{2}}\mathcal{E}_{0j}[\mu_{0j},\dot{\mu}_{0j}] - \sum_{i\neq j}^{k}\mu_i^{-\frac{n+2}{2}}\mathcal{E}_{0i}[\mu_{0i},\dot{\mu}_{0i}]\\ =& \sum_{j=1}^{k}\left(\frac{\mu_{0j}^{-\frac{n}{2}-\alpha}(\mu_{0j}\dot{\lambda}_j+\lambda_j\dot{\mu}_{0j}+\lambda_j\dot{\lambda}_j)h_j}{1+|y_j|^{n-3+\alpha}} + \frac{\mu_{0j}^{-\frac{n}{2}+2-\alpha}\dot{\xi}_j\cdot\vec{h}_j}{1+|y_j|^{n-3+\alpha}} + \frac{\mu_{0j}^{-\frac{n}{2}+2-\alpha}\tilde{h}_j}{1+|y_j|^{n-3+\alpha}}\right), \end{aligned}$$

where $0<\alpha<1$, and

$$\mathbb{A}[\check{\Phi}] = \sum_{j=1}^{k}\left(\frac{\mu_{0j}^{-\frac{n}{2}+2}\dot{\mu}_j h_j}{1+|y_j|^{n-4}} + \frac{\mu_{0j}^{-\frac{n}{2}+2}\dot{\xi}_j\cdot\vec{h}_j}{1+|y_j|^{n-3}} + \frac{\mu_{0j}^{-\frac{n}{2}+2}\tilde{h}_j}{1+|y_j|^{n-3}}\right),$$

where we have used Lemma 2.1, and h_j, $\vec{h}_j$, $\tilde{h}_j$ are smooth and bounded in the expansions.

Collecting all the estimates for the two different cases, we complete the proof. □

3. The inner–outer gluing procedure

Let $t_0 > 0$. We consider the problem

$$u_t = \Delta u + \nabla \log a(x) \cdot \nabla u + u^p, \quad \text{in } \mathbb{R}^n \times [t_0, +\infty). \tag{3.1}$$

In this Section, we shall set up the *inner–outer gluing scheme* to find a solution u to problem (3.1). Then $u(x, t - t_0)$ is a solution to the original problem (1.1) with suitable initial condition to be determined later.

We introduce a smooth cut-off function η with $\eta(s) = 1$ for $s < 1$ and $\eta(s) = 0$ for $s > 2$. Define

$$\eta_{j,R}(x, t) = \eta\left(\frac{x - \xi_j(t)}{R\mu_{0j}(t)}\right), \tag{3.2}$$

where R is sufficiently large and independent of t. For convenience, we take $R := e^{\rho t_0}$, where $\rho > 0$ is small enough and t_0 is the initial time.

For a small perturbation $\mathbf{w}(x, t)$, the function $u(x, t) = u^*_{\mu,\xi}(x, t) + \mathbf{w}(x, t)$ solves problem (3.1) if

$$\partial_t \mathbf{w} \;=\; \Delta \mathbf{w} \;+\; \nabla \log a(x) \cdot \nabla \mathbf{w} \;+\; p(u^*_{\mu,\xi})^{p-1}\mathbf{w} \;+\; \mathbf{N}(\mathbf{w}) \;+ \mathbf{S}[u^*_{\mu,\xi}] \quad \text{in } \mathbb{R}^n \times [t_0, \infty) \tag{3.3}$$

where

$$\mathbf{N}(\mathbf{w}) := (u^*_{\mu,\xi} + \mathbf{w})^p - (u^*_{\mu,\xi})^p - p(u^*_{\mu,\xi})^{p-1}\mathbf{w}, \tag{3.4}$$

$$\mathbf{S}[u^*_{\mu,\xi}] \;=\; -\,\partial_t u^*_{\mu,\xi} \;+\; \Delta u^*_{\mu,\xi} \;+ \nabla \log a(x) \cdot \nabla u^*_{\mu,\xi} \;+\; (u^*_{\mu,\xi})^p. \tag{3.5}$$

According to the expression of the error $\mathbf{S}[u^*_{\mu,\xi}]$ as in Lemma 2.2, for $x \in B_j$, we can decompose $\mathbf{S}[u^*_{\mu,\xi}]$ into

$$\mathbf{S}[u^*_{\mu,\xi}] \;=\; \eta_{j,R}\mathbf{S}_{\mu,\xi,j} \;+\; \mathbf{S}^{out}_{\mu,\xi,j},\;\; x \in B_j,$$

where $\mathbf{S}_{\mu,\xi,j}$ defined in (2.31) is the leading part in $\mathbf{S}[u^*_{\mu,\xi}]$, and

$$\mathbf{S}^{out}_{\mu,\xi,j} = (1 - \eta_{j,R})\mathbf{S}_{\mu,\xi,j} + \mathbf{S}^{(2)}_{\mu,\xi},\;\; x \in B_j$$

with $\mathbf{S}^{(2)}_{\mu,\xi}$ defined in (2.32). In this way, $\mathbf{S}^{out}_{\mu,\xi}$ encodes the information of the error of $\mathbf{S}[u^*_{\mu,\xi}]$ regarding the smaller order terms and the part in the region far away from the concentrating points q_j, $j = 1, \cdots, k$. Note that for $x \notin \cup_{j=1}^k B_j$, the error $\mathbf{S}[u^*_{\mu,\xi}]$ given in Lemma 2.2 is smaller compared with $\mathbf{S}_{\mu,\xi,j}$, and it also carries part of the information for the region away from q_j, $j = 1, \cdots k$. In the notation of Lemma 2.2, we denote

$$\mathbf{S}^{out}_{\mu,\xi} = \begin{cases} \mathbf{S}^{out}_{\mu,\xi,j}, \;\; x \in B_j, \\ \mathbf{S}^{(3)}_{\mu,\xi}, \;\; x \notin \cup_{j=1}^k B_j, \end{cases} \tag{3.6}$$

where $\mathbf{S}^{(3)}_{\mu,\xi}$ is defined in (2.33).

We look for $\mathbf{w}$ in the following inner and outer profiles

$$\mathbf{w}(x, t) = \psi(x, t) + \phi^{in}(x, t)$$

with

$$\phi^{in}(x, t) \;:=\; \sum_{j=1}^k \eta_{j,R}(x, t)\tilde{\phi}_j(x, t),\;\; \tilde{\phi}_j(x, t) := \mu_{0j}^{-\frac{n-2}{2}}\phi_j\left(\frac{x - \xi_j(t)}{\mu_{0j}(t)}, t\right). \tag{3.7}$$

A main observation we make is that $\mathbf{w}(x, t)$ solves problem (3.3) if $(\psi, \tilde{\phi}_j)$ solves the following coupled system:

- ψ solves the so-called *outer problem*

$$\begin{aligned} \partial_t \psi = \Delta \psi \;+ \nabla \log a(x) \cdot \nabla \psi + V_{\mu,\xi}\psi + \sum_{j=1}^k \nabla \log a(x) \cdot \nabla \eta_{j,R}\tilde{\phi}_j \\ + \sum_{j=1}^k \left[2\nabla \eta_{j,R} \cdot \nabla \tilde{\phi}_j + \tilde{\phi}_j(\Delta - \partial_t)\eta_{j,R}\right] + \mathbf{N}(\mathbf{w}) + \mathbf{S}^{out}_{\mu,\xi} \quad \text{in } \mathbb{R}^n \times [t_0, +\infty), \end{aligned} \tag{3.8}$$

with

$$V_{\mu,\xi} := p\sum_{j=1}^{k}\Big(\big|u^*_{\mu,\xi}\big|^{p-1} - \Big|\mu_j^{-\frac{n-2}{2}}U\Big(\frac{x-\xi_j}{\mu_j}\Big)\Big|^{p-1}\Big)\eta_{j,R} + p\Big(1-\sum_{j=1}^{k}\eta_{j,R}\Big)\big|u^*_{\mu,\xi}\big|^{p-1}. \tag{3.9}$$

- $\tilde\phi_j$ solves the so-called *inner problem* for $j=1,\cdots,k$,

$$\begin{aligned}\partial_t\tilde\phi_j &= \Delta\tilde\phi_j + pU_j^{p-1}\tilde\phi_j + pU_j^{p-1}\psi + \nabla\log a(x)\cdot\nabla\tilde\phi_j + \mathbf{S}_{\mu,\xi,j}\\ &\quad + p\big[(u^*_{\mu,\xi})^{p-1} - U_j^{p-1}\big]\tilde\phi_j, \quad \text{in } B_{2R\mu_{0j}}(\xi_j)\times[t_0,+\infty),\end{aligned} \tag{3.10}$$

where $U_j = \mu_j^{-\frac{n-2}{2}}U(\frac{x-\xi_j}{\mu_j})$.

In terms of $\phi_j(y,t)$ as in (3.7), the equation (3.10) becomes

$$\begin{aligned}\mu_{0j}^2\partial_t\phi_j &= \Delta_y\phi_j + pU(y)^{p-1}\phi_j + \mu_{0j}^{\frac{n+2}{2}}\mathbf{S}_{\mu,\xi,j}(\xi_j+\mu_{0j}y,\,t)\\ &\quad + p\mu_{0j}^{\frac{n-2}{2}}\,\frac{\mu_{0j}^2}{\mu_j^2}\left|U\Big(\frac{\mu_{0j}}{\mu_j}y\Big)\right|^{p-1}\psi(\xi_j+\mu_{0j}y,\,t) + B_j^1[\phi_j] + B_j^2[\phi_j] + B_j^3[\phi_j],\end{aligned} \tag{3.11}$$

where

$$B_j^1[\phi_j] := \mu_{0j}\dot\mu_{0j}\Big(\frac{n-2}{2}\phi_j + y\cdot\nabla_y\phi_j\Big) + \mu_{0j}\dot\xi_j\cdot\nabla_y\phi_j, \tag{3.12}$$

$$B_j^2[\phi_j] := p\Big[\Big|U\Big(\frac{\mu_{0j}}{\mu_j}y\Big)\Big|^{p-1} - U^{p-1}(y)\Big]\phi_j + p\left(\frac{\mu_{0j}^2}{\mu_j^2}-1\right)\Big|U\Big(\frac{\mu_{0j}}{\mu_j}y\Big)\Big|^{p-1}\phi_j, \tag{3.13}$$

$$B_j^3[\phi_j] := p\mu_{0j}^2\big[(u^*_{\mu,\xi})^{p-1} - U_j^{p-1}\big]\phi_j + \mu_{0j}\nabla_x\log a(\xi_j+\mu_{0j}y)\cdot\nabla_y\phi_j. \tag{3.14}$$

The reason for choosing such scaled spatial variable $\frac{x-\xi_j(t)}{\mu_{0j}(t)}$ is the following. In Section 4 and Section 5, when we develop the linear theories for the outer and inner problems (see Lemma 4.1 and Proposition 5.1), the behavior of the scaling parameter $\mu_j(t)$ is needed. At this stage, we only know the leading order $\mu_{0j}(t)$ of $\mu_j(t)$ as we shall solve the remainder $\lambda_j(t)$ in Section 5 (see Proposition 5.2). Observe that

$$\frac{x-\xi_j}{\mu_j} = \frac{x-\xi_j}{\mu_{0j}}(1+o(1))$$

so that the remainder term is essentially of smaller order. It is then reasonable to rescale the spatial variable as $\frac{x-\xi_j}{\mu_{0j}}$. Moreover, in Proposition 5.2, we shall show that the remainder $\lambda_j(t)$ is indeed of smaller order

$$\lambda_j(t)\sim\mu_{0j}^{1+\sigma}(t) \ \text{ for some } \ \sigma>0.$$

We next describe precisely our strategy to solve the *outer problem* (3.8) and *inner problems* (3.10). For given parameters $\lambda,\xi,\dot\lambda,\dot\xi$ and functions ϕ_j fixed in proper range, we first solve for ψ in problem (3.8), in the form of a nonlocal operator $\psi = \Psi(\lambda,\xi,\dot\lambda,\dot\xi,\phi)$. We will solve it by means of a priori estimates of the associated linear problem and fixed point arguments. This will be done in full details in Section 4. After we solve the outer problem, the inner problem is then reduced to a nonlinear and nonlocal problem. In order to solve the reduced inner problem, a linear theory concerning the solvability and estimates of the associated linear problem with certain orthogonality conditions is required. The orthogonality conditions will be achieved by adjusting the parameter functions λ and ξ. Finally we shall solve the inner problem by the linear theory and the fixed point argument. See Section 5 and Section 6 for full details.

4. Solving the outer problem

The aim of this Section is to solve the *outer problem* (3.8) for given parameters $\lambda, \xi, \dot{\lambda}, \dot{\xi}$ satisfying (4.8) and (4.9), and for small function ϕ satisfies (4.11). We consider the problem

$$\begin{cases} \partial_t \psi = \Delta\psi + \nabla \log a(x) \cdot \nabla\psi + f(\psi, \phi, \lambda, \xi) \quad \text{in } \mathbb{R}^n \times [t_0, +\infty), \\ \psi(\cdot, t_0) = 0 \quad \text{in } \mathbb{R}^n, \end{cases} \tag{4.1}$$

where

$$f := \sum_{j=1}^{k} \Big[2\nabla\eta_{j,R} \cdot \nabla\tilde{\phi}_j + \tilde{\phi}_j(\Delta - \partial_t)\eta_{j,R} + \nabla \log a(x) \cdot \nabla\eta_{j,R}\tilde{\phi}_j \Big] + V_{\mu,\xi}\psi + \mathbf{N}(\mathbf{w}) + \mathbf{S}^{out}_{\mu,\xi}. \tag{4.2}$$

We will use the contraction mapping theorem to solve the nonlinear equation (4.1). For our purpose, we first consider a linear model problem in Subsection 4.1.

4.1. The linear heat equation.

In this Subsection, we consider the following linear heat equation

$$\begin{cases} \psi_t(x,t) = \Delta\psi(x,t) + \nabla \log a(x) \cdot \nabla\psi(x,t) + \mathbf{f}(x,t) \quad \text{in } \mathbb{R}^n \times [t_0, +\infty), \\ \psi(\cdot, t_0) = 0 \quad \text{in } \mathbb{R}^n, \end{cases} \tag{4.3}$$

where function $\mathbf{f}(x,t)$ is smooth. We assume that for two real numbers β, γ, the nonhomogeneous term $\mathbf{f}(x,t)$ satisfies

$$|\mathbf{f}(x,t)| \le M \sum_{j=1}^{k} \frac{\mu_{0j}^{-2}(t)\, \mu_{0j}^{\beta}(t)}{1 + |y_j|^{2+\gamma}}, \qquad y_j = \frac{x - \xi_j(t)}{\mu_j(t)}. \tag{4.4}$$

The norm $\|\mathbf{f}\|_{*,\beta,2+\gamma}$ is defined as the least number $M > 0$ such that (4.4) holds. By the ansatz of $\mu_j(t)$ in (2.4) we know

$$\frac{x - \xi_j(t)}{\mu_j(t)} = \frac{x - \xi_j(t)}{\mu_{0j}(t)}(1 + o(1)).$$

Then it is reasonable to use the rescaled spatial variable $\frac{x-\xi_j(t)}{\mu_{0j}(t)}$ since the remainder produced by $\lambda_j(t)$ is essentially of smaller order. In the sequel, if there is no confusion, we write

$$y_j = \frac{x - \xi_j(t)}{\mu_{0j}(t)}$$

due to the discussion above. Using the heat kernel (see [2]), we know that the solution $\psi(x,t)$ of problem (4.3) satisfies

$$|\psi(x,t)| \lesssim \left| \int_t^{+\infty} \int_{\mathbb{R}^n} \frac{1}{(s-t)^{n/2}} e^{-\frac{|x-y|^2}{s-t}} \mathbf{f}(y,s)\, \mathrm{d}y\, \mathrm{d}s \right|. \tag{4.5}$$

Remark 4.1. *Note that the heat kernel bounds given in [2] are global and independent of time T. More precisely, the upper bound does not depend on time T, while the lower bound dependence on T can be removed by choosing a constant in the argument. See [2, Section 5]. In our case, we only need the upper Gaussian bound for the heat kernel.*

Then, we have the following estimate of the solution $\psi(x,t)$ to the linear heat equation (4.3).

Lemma 4.1. *Assume that $\|\mathbf{f}\|_{*,\beta,2+\gamma} < +\infty$, $0 < \gamma < n-2$ and $\beta + \gamma > 0$. Let $\psi(x,t)$ be the solution to problem (4.3). Then, for all $(x,t) \in \mathbb{R}^n \times [t_0, +\infty)$, it holds that for $y_j = \frac{x-\xi_j(t)}{\mu_{0j}(t)}$*

$$|\psi(x,t)| \lesssim \|\mathbf{f}\|_{*,\beta,2+\gamma} \begin{cases} \sum_{j=1}^{k} \frac{\mu_{0j}^{\beta}(t)}{1+|y_j|^{\gamma}} & \text{if } |y_j| < \mu_{0j}^{-1}(t), \\ \sum_{j=1}^{k} \frac{\mu_{0j}^{-2}(t)\, \mu_{0j}^{\beta}(t)}{1+|y_j|^{2+\gamma}} & \text{if } |y_j| > \mu_{0j}^{-1}(t). \end{cases}$$

Proof. Since $y_j = \frac{x-\xi_j(t)}{\mu_{0j}(t)}$ and $\frac{\xi_j(t)-q_j}{\mu_{0j}(t)} = o(1)$, we have

$$\begin{aligned} |\mathbf{f}(x,t)| &\le \|\mathbf{f}\|_{*,\beta,2+\gamma} \sum_{j=1}^{k} \frac{\mu_{0j}^{-2}(t)\,\mu_{0j}^{\beta}(t)}{1+|y_j|^{2+\gamma}} \\ &\lesssim \|\mathbf{f}\|_{*,\beta,2+\gamma} \sum_{j=1}^{k} \frac{\mu_{0j}^{\beta+\gamma}(t)}{\mu_{0j}^{2+\gamma}(t)+|x-q_j|^{2+\gamma}}. \end{aligned} \tag{4.6}$$

Then, if we define $\tilde{x} = x - q_j$ and $\tilde{y} = y - q_j$, from (4.5) and (4.6), we obtain

$$\begin{aligned} |\psi(x,t)| &\lesssim \|\mathbf{f}\|_{*,\beta,2+\gamma} \sum_{j=1}^{k} \int_t^{+\infty} \int_{\mathbb{R}^n} \frac{1}{(s-t)^{n/2}} e^{-\frac{|\tilde{x}-\tilde{y}|^2}{s-t}} \frac{\mu_{0j}^{\beta+\gamma}(s)}{\mu_{0j}^{2+\gamma}(s)+|\tilde{y}|^{2+\gamma}}\, \mathrm{d}\tilde{y}\, \mathrm{d}s \\ &= \|\mathbf{f}\|_{*,\beta,2+\gamma} \sum_{j=1}^{k} \left\{ \int_t^{t+1} + \int_{t+1}^{+\infty} \right\} \int_{\mathbb{R}^n} \frac{1}{(s-t)^{n/2}} e^{-\frac{|\tilde{x}-\tilde{y}|^2}{s-t}} \frac{\mu_{0j}^{\beta+\gamma}(s)}{\mu_{0j}^{2+\gamma}(s)+|\tilde{y}|^{2+\gamma}}\, \mathrm{d}\tilde{y}\, \mathrm{d}s \\ &:= \|\mathbf{f}\|_{*,\beta,2+\gamma}\,(I_1+I_2). \end{aligned}$$

First, we consider the case $|y_j| < \mu_{0j}^{-1}(t)$, for all $j = 1, \cdots, k$. In order to estimate the term I_1, we perform the change of variable $p = \frac{\tilde{x}-\tilde{y}}{\sqrt{s-t}}$, $\frac{\mathrm{d}s}{2(s-t)} = -\frac{1}{p}\mathrm{d}p$ and we get

$$\begin{aligned} I_1 &\lesssim \sum_{j=1}^{k} \mu_{0j}^{\beta+\gamma}(t) \int_t^{t+1} \int_{\mathbb{R}^n} \frac{1}{(s-t)^{n/2}} e^{-\frac{|\tilde{x}-\tilde{y}|^2}{s-t}} \frac{1}{\mu_{0j}^{2+\gamma}(t+1)+|\tilde{y}|^{2+\gamma}}\, \mathrm{d}\tilde{y}\, \mathrm{d}s \\ &\lesssim \sum_{j=1}^{k} \mu_{0j}^{\beta+\gamma}(t) \int_{\mathbb{R}^n} \frac{1}{|\tilde{x}-\tilde{y}|^{n-2}} \frac{1}{\mu_{0j}^{2+\gamma}(t+1)+|\tilde{y}|^{2+\gamma}}\, \mathrm{d}\tilde{y} \int_{|\tilde{x}-\tilde{y}|}^{+\infty} p^{n-3} e^{-p^2}\, \mathrm{d}p \\ &\lesssim \sum_{j=1}^{k} \mu_{0j}^{\beta+\gamma}(t) \int_{\mathbb{R}^n} \frac{1}{|\tilde{x}-\tilde{y}|^{n-2}} \frac{1}{\mu_{0j}^{2+\gamma}(t+1)+|\tilde{y}|^{2+\gamma}}\, \mathrm{d}\tilde{y} \\ &= \sum_{j=1}^{k} \frac{\mu_{0j}^{\beta+\gamma}(t)}{\mu_{0j}^{2+\gamma}(t+1)} \int_{\mathbb{R}^n} \frac{1}{\mu_{0j}^{n-2}(t+1)|\frac{\tilde{x}-\tilde{y}}{\mu_{0j}(t+1)}|^{n-2}} \frac{1}{1+|\frac{\tilde{y}}{\mu_{0j}(t+1)}|^{2+\gamma}}\, \mathrm{d}\tilde{y} \\ &= \sum_{j=1}^{k} \frac{\mu_{0j}^{\beta+\gamma}(t)\mu_{0j}^{2}(t+1)}{\mu_{0j}^{2+\gamma}(t+1)} \int_{\mathbb{R}^n} \frac{1}{|\check{x}-\check{y}|^{n-2}} \frac{1}{1+|\check{y}|^{2+\gamma}}\, \mathrm{d}\check{y} \\ &\lesssim \sum_{j=1}^{k} \frac{\mu_{0j}^{\beta+\gamma}(t)\mu_{0j}^{2}(t+1)}{\mu_{0j}^{2+\gamma}(t+1)} \frac{1}{1+|\check{x}|^{\gamma}} \lesssim \sum_{j=1}^{k} \frac{\mu_{0j}^{\beta}(t)}{1+|y_j|^{\gamma}}, \end{aligned}$$

where $\check{x} := \frac{\tilde{x}}{\mu_{0j}(t+1)}$, $\check{y} := \frac{\tilde{y}}{\mu_{0j}(t+1)}$ and we have used the fact that for $0 < \gamma < n-2$

$$\int_{\mathbb{R}^n} \frac{1}{|\check{x}-\check{y}|^{n-2}} \frac{1}{1+|\check{y}|^{2+\gamma}}\, \mathrm{d}\check{y} \lesssim \frac{1}{(1+|\check{x}|)^{\gamma}}.$$

See [56, Lemma B.2] for instance.

Next, we compute the term I_2. If we define $\bar{x} = \tilde{x}(s-t)^{-\frac{1}{2}}$, $\bar{y} = \tilde{y}(s-t)^{-\frac{1}{2}}$, we then obtain

$$\begin{aligned} I_2 &\lesssim \sum_{j=1}^{k} \int_{t+1}^{+\infty} \int_{\mathbb{R}^n} e^{-|\bar{x}-\bar{y}|^2} \frac{\mu_{0j}^{\beta+\gamma}(s)}{\mu_{0j}^{2+\gamma}(s)+(\sqrt{s-t})^{2+\gamma}|\bar{y}|^{2+\gamma}}\, \mathrm{d}\bar{y}\, \mathrm{d}s \\ &= \sum_{j=1}^{k} \int_{t+1}^{+\infty} \Big(\int_{|\bar{y}|>2|\bar{x}|} + \int_{|\bar{y}|\le 2|\bar{x}|} \Big) e^{-|\bar{x}-\bar{y}|^2} \frac{\mu_{0j}^{\beta+\gamma}(s)}{\mu_{0j}^{2+\gamma}(s)+(\sqrt{s-t})^{2+\gamma}|\bar{y}|^{2+\gamma}}\, \mathrm{d}\bar{y}\, \mathrm{d}s \\ &:= I_{21} + I_{22}. \end{aligned}$$

For I_{21}, we have the following estimate

$$
\begin{aligned}
I_{21} &\lesssim \sum_{j=1}^{k} \int_{t+1}^{+\infty} \mu_{0j}^{\beta+\gamma}(s) \int_{|\bar{y}|\geq 2|\bar{x}|} e^{-\frac{|\bar{y}|^2}{4}} \frac{1}{|\bar{y}|^{2+\gamma}} \,\mathrm{d}\bar{y}\,\mathrm{d}s \\
&\lesssim \sum_{j=1}^{k} \mu_{0j}^{\beta+\gamma}(t) \lesssim \sum_{j=1}^{k} \frac{\mu_{0j}^{\beta}(t)}{1+|y_j|^{\gamma}}.
\end{aligned}
$$

For I_{22}, we have

$$
\begin{aligned}
I_{22} &\leq \sum_{j=1}^{k} \int_{t+1}^{+\infty} \mu_{0j}^{\beta+\gamma}(s) \int_{|\bar{y}|\leq 2|\bar{x}|} \frac{1}{|\bar{y}\sqrt{s-t}|^{2+\gamma}} \,\mathrm{d}\bar{y}\,\mathrm{d}s \\
&\lesssim \sum_{j=1}^{k} \int_{t+1}^{+\infty} \mu_{0j}^{\beta+\gamma}(s) \frac{|\bar{x}|^{n-2-\gamma}}{|\sqrt{s-t}|^{2+\gamma}} \mathrm{d}s \\
&\lesssim \sum_{j=1}^{k} \mu_{0j}^{\beta+\gamma}(t) \int_{t+1}^{+\infty} \frac{1}{(\sqrt{s-t})^{n}} \,\mathrm{d}s \\
&\lesssim \sum_{j=1}^{k} \mu_{0j}^{\beta+\gamma}(t) \lesssim \sum_{j=1}^{k} \frac{\mu_{0j}^{\beta}(t)}{1+|y_j|^{\gamma}}.
\end{aligned}
$$

Next, we will consider the case $|y_j| > \mu_{0j}^{-1}(t)$, for all $j = 1, \cdots, k$. We compute

$$
\begin{aligned}
|\psi(x,t)| &\lesssim \|\mathbf{f}\|_{*,\beta,2+\gamma} \sum_{j=1}^{k} \int_{t}^{+\infty} \int_{\mathbb{R}^n} \frac{1}{(s-t)^{n/2}} e^{-\frac{|\tilde{x}-\tilde{y}|^2}{s-t}} \frac{\mu_{0j}^{\beta+\gamma}(s)}{\mu_{0j}^{2+\gamma}(s)+|\tilde{y}|^{2+\gamma}} \,\mathrm{d}\tilde{y}\,\mathrm{d}s \\
&= \|\mathbf{f}\|_{*,\beta,2+\gamma} \sum_{j=1}^{k} \int_{t}^{+\infty} \Big(\int_{B_{\frac{|\tilde{x}|}{2}}(\tilde{x})} + \int_{B^{C}_{\frac{|\tilde{x}|}{2}}(\tilde{x})} \Big) \frac{1}{(s-t)^{n/2}} e^{-\frac{|\tilde{x}-\tilde{y}|^2}{s-t}} \frac{\mu_{0j}^{\beta+\gamma}(s)}{\mu_{0j}^{2+\gamma}(s)+|\tilde{y}|^{2+\gamma}} \,\mathrm{d}\tilde{y}\,\mathrm{d}s \\
&:= \|\mathbf{f}\|_{*,\beta,2+\gamma} \big(J_1 + J_2\big).
\end{aligned}
$$

Denote $\bar{x} = \tilde{x}(s-t)^{-\frac{1}{2}}$, $\bar{y} = \tilde{y}(s-t)^{-\frac{1}{2}}$. Straightforward computations imply that

$$
\begin{aligned}
J_1 &\lesssim \sum_{j=1}^{k} \frac{1}{|\tilde{x}|^{2+\gamma}} \int_{t}^{+\infty} \mu_{0j}^{\beta+\gamma}(s) \int_{B_{\frac{|\tilde{x}|}{2}}(\tilde{x})} \frac{1}{(s-t)^{\frac{n}{2}}} e^{-\frac{|\tilde{x}-\tilde{y}|^2}{s-t}} \,\mathrm{d}\tilde{y}\,\mathrm{d}s \\
&\lesssim \sum_{j=1}^{k} \frac{1}{|\tilde{x}|^{2+\gamma}} \int_{t}^{+\infty} \mu_{0j}^{\beta+\gamma}(s) \int_{B_{\frac{|\bar{x}|}{2}}(\bar{x})} e^{-|\bar{x}-\bar{y}|^2} \,\mathrm{d}\bar{y}\,\mathrm{d}s \\
&\lesssim \sum_{j=1}^{k} \frac{1}{|\tilde{x}|^{2+\gamma}} \int_{t}^{+\infty} \mu_{0j}^{\beta+\gamma}(s) \int_{0}^{+\infty} e^{-r^2} r^{n-1} \,\mathrm{d}r\,\mathrm{d}s \\
&\lesssim \sum_{j=1}^{k} \frac{\mu_{0j}^{\beta+\gamma}(t)}{|\tilde{x}|^{2+\gamma}} \lesssim \sum_{j=1}^{k} \frac{\mu_{0j}^{-2}(t)\, \mu_{0j}^{\beta}(t)}{1+|y_j|^{2+\gamma}},
\end{aligned}
$$

where we have used the facts that $\int_0^{+\infty} e^{-r^2} r^{n-1}\,\mathrm{d}r < +\infty$, $\beta+\gamma > 0$ and (2.26). Similarly, we have

$$
\begin{aligned}
J_2 &= \sum_{j=1}^{k} \int_{t}^{+\infty} \Big(\int_{B_{2|\bar{x}|}(\bar{x}) \setminus B_{\frac{|\bar{x}|}{2}}(\bar{x})} + \int_{B^{C}_{2|\bar{x}|}(\bar{x})} \Big) \frac{e^{-|\bar{x}-\bar{y}|^2} \mu_{0j}^{\beta+\gamma}(s)}{\mu_{0j}^{2+\gamma}(s) + (\sqrt{s-t})^{2+\gamma} |\bar{y}|^{2+\gamma}} \,\mathrm{d}\bar{y}\,\mathrm{d}s \\
&:= J_{21} + J_{22}.
\end{aligned}
$$

Moreover, one has

$$
\begin{aligned}
J_{21} &\lesssim \sum_{j=1}^{k} \int_t^{+\infty} \int_0^{3|\bar{x}|} e^{-\frac{|\bar{x}|^2}{4}} \frac{\mu_{0j}^{\beta+\gamma}(s)}{\mu_{0j}^{2+\gamma}(s) + (\sqrt{s-t})^{2+\gamma} r^{2+\gamma}} r^{n-1}\, \mathrm{d}r\, \mathrm{d}s \\
&= \sum_{j=1}^{k} \int_t^{+\infty} \mu_{0j}^{\beta+\gamma}(s) \frac{1}{|\tilde{x}|^{2+\gamma}} \frac{|\tilde{x}|^{2+\gamma}}{(\sqrt{s-t})^{2+\gamma}}\, e^{-\frac{|\bar{x}|^2}{4}}\, |\bar{x}|^{n-2-\gamma}\, \mathrm{d}s \\
&\lesssim \sum_{j=1}^{k} \int_t^{+\infty} \mu_{0j}^{\beta+\gamma}(s) \frac{1}{|\tilde{x}|^{2+\gamma}}\, e^{-\frac{|\bar{x}|^2}{4}}\, |\bar{x}|^{n}\, \mathrm{d}s \\
&\lesssim \sum_{j=1}^{k} \frac{\mu_{0j}^{\beta+\gamma}(t)}{|\tilde{x}|^{2+\gamma}} \lesssim \sum_{j=1}^{k} \frac{\mu_{0j}^{-2}(t)\, \mu_{0j}^{\beta}(t)}{1+|y_j|^{2+\gamma}}
\end{aligned}
$$

and

$$
\begin{aligned}
J_{22} &\leq \sum_{j=1}^{k} \int_t^{+\infty} \frac{\mu_{0j}^{\beta+\gamma}(s)}{(\sqrt{s-t})^{2+\gamma}|\bar{x}|^{2+\gamma}}\, \mathrm{d}s \int_{B^C_{2|\bar{x}|}(\bar{x})} e^{-|\bar{x}-\bar{y}|^2}\, \mathrm{d}\bar{y} \\
&\lesssim \sum_{j=1}^{k} \frac{\mu_{0j}^{\beta+\gamma}(t)}{|\tilde{x}|^{2+\gamma}} \lesssim \sum_{j=1}^{k} \frac{\mu_{0j}^{-2}(t)\mu_{0j}^{\beta}(t)}{1+|y_j|^{2+\gamma}}.
\end{aligned}
$$

Combining all the estimates above, we complete the proof of Lemma 4.1. □

4.2. **Solving the outer problem.** Let $\sigma \ll 1$ be a small positive constant satisfying the constraint (1.6). For a given function $h(t) = (h_1(t), \cdots, h_k(t)) : (t_0, \infty) \to \mathbb{R}^k$ and $\delta > 0$, we define the weighted L^∞-norm as

$$
\|h(t)\|_\delta := \max_{j=1,\cdots,k} \|\mu_{0j}^{-\delta}(t)\, h_j(t)\|_{L^\infty(t_0,+\infty)}. \tag{4.7}
$$

In what follows we assume that the parameter functions $\lambda(t)$, $\xi(t)$, $\dot{\lambda}(t)$, $\dot{\xi}(t)$ satisfy the constraints

$$
\|\dot{\lambda}(t)\|_{1+\sigma} + \|\dot{\xi}(t)\|_{1+\sigma} \leq C, \tag{4.8}
$$

$$
\|\lambda(t)\|_{1+\sigma} + \|\xi(t) - q\|_{1+\sigma} \leq C \tag{4.9}
$$

with some positive constant C independent of t, t_0 and R.

Define

$$
\|\phi\|_{2+\sigma,n-5+a} = \max_{j=1,\cdots,k} \|\phi_j\|_{2+\sigma,n-5+a},
$$

where $\|\phi_j\|_{2+\sigma,n-5+a}$ is the least number $M > 0$ such that

$$
(1+|y|)|\nabla\phi_j(y,t)| + |\phi_j(y,t)| \leq M \frac{\mu_{0j}^{2+\sigma}(t)}{1+|y|^{n-5+a}}, \quad j = 1, \cdots, k, \tag{4.10}
$$

holds, where $0 < a < 1$.

We assume that $\phi = (\phi_1, \cdots, \phi_k)$ satisfies the constraint

$$
\|\phi\|_{2+\sigma,n-5+a} \leq c\, e^{-\varepsilon t_0}, \tag{4.11}
$$

for some constant $\varepsilon > 0$ small enough. Then we have the following result.

Proposition 4.1. *Assume that the parameter functions λ, ξ, $\dot{\lambda}$, $\dot{\xi}$ satisfy (4.8) and (4.9), and the vector function $\phi = (\phi_1, \cdots, \phi_k)$ satisfies (4.11). Then there exists t_0 sufficiently large such that the outer problem (4.1) has a unique solution $\psi(x,t) = \Psi(\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi)\,(x,t)$. Moreover, for $y_j = \frac{x-\xi_j(t)}{\mu_{0j}(t)}$*

and $1/2 < \alpha < a < 1$*, there exists* $\sigma \in (0,1)$ *sufficiently small such that the solution* ψ *satisfies the following estimates,*

$$|\psi(x,t)| \lesssim \begin{cases} \sum_{j=1}^{k} \frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-5+\alpha}}, & \text{if} \quad |y_j| < \mu_{0j}^{-1}(t), \\ \sum_{j=1}^{k} \frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}, & \text{if} \quad |y_j| > \mu_{0j}^{-1}(t), \end{cases} \tag{4.12}$$

and

$$|\nabla\psi(x,t)| \lesssim \begin{cases} \sum_{j=1}^{k} \frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{2+\sigma}(t)\mu_{0j}^{-1}(t)}{1+|y_j|^{n-4+\alpha}}, & \text{if} \quad |y_j| < \mu_{0j}^{-1}(t), \\ \sum_{j=1}^{k} \frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)\mu_{0j}^{-1}(t)}{1+|y_j|^{n-2+\alpha}}, & \text{if} \quad |y_j| > \mu_{0j}^{-1}(t). \end{cases} \tag{4.13}$$

Proof. Recall that Lemma 4.1 defines a linear operator T such that $\psi = T(\mathbf{f})$ is the solution to the linear problem (4.3). We establish the existence of a solution ψ to problem (4.1), as a fixed point problem

$$\psi = \mathcal{A}(\psi), \qquad \mathcal{A}(\psi) := T(f(\psi)), \tag{4.14}$$

where $f(\psi)$ is defined in (4.2). By the contraction mapping theorem, we will prove that there exists a fixed point ψ for $\mathcal{A}$ in the following function space

$$\mathcal{B} = \Big\{ \psi \,:\, \|\psi\|_* \,\le\, C\, e^{-\varepsilon t_0} \Big\} \tag{4.15}$$

for some sufficiently large $C > 0$. Here ε is some positive number. We denote $\|\psi\|_*$ as the least number $M > 0$ such that the following inequality holds

$$|\psi(x,t)| \le M \begin{cases} \sum_{j=1}^{k} \frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-5+\alpha}}, & \text{if} \quad |y_j| < \mu_{0j}^{-1}(t), \\ \sum_{j=1}^{k} \frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}, & \text{if} \quad |y_j| > \mu_{0j}^{-1}(t). \end{cases} \tag{4.16}$$

In order to prove $\mathcal{A}$ is a contraction map, we will estimate

$$f = \sum_{j=1}^{k} \Big[2\nabla\eta_{j,R}\cdot\nabla\tilde{\phi}_j + \tilde{\phi}_j(\Delta - \partial_t)\eta_{j,R} + \nabla \log a(x)\cdot\nabla\eta_{j,R}\tilde{\phi}_j \Big] + V_{\mu,\xi}\psi + \mathbf{N}(\mathbf{w}) + \mathbf{S}_{\mu,\xi}^{out}$$

term by term.

We first consider the term $\nabla\eta_{j,R}\cdot\nabla\tilde{\phi}_j$. Using the definition of $\tilde{\phi}_j(x,t)$ in (3.7) and the assumption (4.11), we obtain

$$\begin{aligned} \big|(\nabla\eta_{j,R}\cdot\nabla\tilde{\phi}_j)(x,t)\big| &\lesssim \mu_{0j}^{-\frac{n-2}{2}}(t)\frac{\eta'\Big(\Big|\frac{x-\xi_j(t)}{R\mu_{0j}(t)}\Big|\Big)}{R\mu_{0j}(t)}\frac{|\nabla_y\phi_j|}{\mu_{0j}(t)} \\ &\lesssim \frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{R(1+|y_j|^{n-4+a})}\|\phi\|_{2+\sigma,n-5+a}\chi_{\{R\le|y_j|\le 2R\}} \\ &\lesssim R^{\alpha-a}\|\phi\|_{2+\sigma,n-5+a}\sum_{j=1}^{k}\frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}} \\ &\lesssim e^{-\varepsilon t_0}\|\phi\|_{2+\sigma,n-5+a}\sum_{j=1}^{k}\frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}, \end{aligned} \tag{4.17}$$

where we have used the fact that $\alpha < a$. Now, we consider the term $\tilde{\phi}_j(\Delta - \partial_t)\eta_{j,R}$. A direct computation gives that

$$\begin{aligned} \big|\tilde{\phi}_j(\Delta - \partial_t)\eta_{j,R}\big| &\lesssim \left\{\Big|\frac{\eta''}{R^2\mu_{0j}^2(t)}\Big| + \Big|\frac{\eta'[-\dot{\xi}_j(t)R\mu_{0j}(t) - (x-\xi_j(t))R\dot{\mu}_{0j}(t)]}{R^2\mu_{0j}^2(t)}\Big|\right\}|\tilde{\phi}_j| \\ &\lesssim R^{\alpha-a}\|\phi\|_{2+\sigma,n-5+a}\sum_{j=1}^k \frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}} \\ &\lesssim e^{-\varepsilon t_0}\|\phi\|_{2+\sigma,n-5+a}\sum_{j=1}^k \frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}, \end{aligned} \tag{4.18}$$

where we used the assumption (4.11) and the fact that only in the region $R \le |y_j| \le 2R$, $\eta'(|\frac{x-\xi_j}{R\mu_{0j}}|) \neq 0$.

Similarly, we can get the estimate of $\nabla \log a(x)\cdot \nabla\eta_{j,R}\tilde{\phi}_j$

$$\begin{aligned} |\nabla \log a(x)\cdot\nabla\eta_{j,R}\tilde{\phi}_j| &\lesssim R^{\alpha-a}\|\phi\|_{2+\sigma,n-5+a}\sum_{j=1}^k \frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}} \\ &\lesssim e^{-\varepsilon t_0}\|\phi\|_{2+\sigma,n-5+a}\sum_{j=1}^k \frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}. \end{aligned} \tag{4.19}$$

For the term $V_{\mu,\xi}\psi$ defined in (3.9), we evaluate

$$\begin{aligned} |V_{\mu,\xi}\psi| &\lesssim \sum_{j=1}^k (1-\eta_{j,R})\frac{\mu_{0j}^{-2}(t)}{1+|y_j|^4}|\psi| \\ &\lesssim R^{-2}\|\psi\|_*\sum_{j=1}^k \frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}} \\ &\lesssim e^{-\varepsilon t_0}\|\psi\|_*\sum_{j=1}^k \frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}. \end{aligned} \tag{4.20}$$

For the nonlinear term $\mathbf{N}(\mathbf{w})$ defined in (3.4), we get

$$|\mathbf{N}(\mathbf{w})| = \left|\mathbf{N}\Big(\psi + \sum_{j=1}^k \eta_{j,R}\tilde{\phi}_j\Big)\right| \lesssim \begin{cases} (u_{\mu,\xi}^*)^{p-2}\Big[|\psi|^2 + \sum\limits_{j=1}^k |\eta_{j,R}\tilde{\phi}_j|^2\Big], & \text{if} \quad n = 5,6, \\ |\psi|^p + \sum\limits_{j=1}^k |\eta_{j,R}\tilde{\phi}_j|^p, & \text{if} \quad n \ge 7. \end{cases}$$

First, we consider the case $n = 5,6$. Direct computation implies

$$\begin{aligned} \big|(u_{\mu,\xi}^*)^{p-2}(\eta_{j,R}\tilde{\phi}_j)^2\big| &\lesssim \|\phi\|_{2+\sigma,n-5+a}^2\sum_{j=1}^k \frac{\mu_{0j}^{\frac{n-6}{2}}(t)}{(1+|y_j|^2)^{\frac{6-n}{2}}}\frac{\mu_{0j}^{-(n-2)}(t)\mu_{0j}^{4+2\sigma}(t)}{(1+|y_j|^{n-5+a})^2}\chi_{\{|y_j|\le 2R\}} \\ &\lesssim e^{-\varepsilon t_0}\,\|\phi\|_{2+\sigma,n-5+a}^2\sum_{j=1}^k \frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}. \end{aligned} \tag{4.21}$$

Recall the definition of $\|\psi\|_*$ as in (4.16).

- If $|y_j| < \mu_{0j}^{-1}(t)$ for $j = 1, \cdots, k$, then we have the following estimate

$$\begin{aligned}\left|(u^*_{\mu,\xi})^{p-2}\psi^2\right| &\lesssim (u^*_{\mu,\xi})^{p-2}\|\psi\|_*^2 \sum_{j=1}^{k}\Big(\frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-5+\alpha}}\Big)^2\\ &\lesssim \|\psi\|_*^2\sum_{j=1}^{k}\frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}\underbrace{\frac{\mu_{0j}^{2+\sigma}(t)}{(1+|y_j|^2)^{\frac{6-n}{2}}}\frac{1+|y_j|^{n-3+\alpha}}{(1+|y_j|^{n-5+\alpha})^2}}_{\lesssim\mu_{0j}^{1+\sigma+\alpha}(t)}\\ &\lesssim e^{-\varepsilon t_0}\,\|\psi\|_*^2\sum_{j=1}^{k}\frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}. \end{aligned}\tag{4.22}$$

- If $|y_j| > \mu_{0j}^{-1}(t)$ for $j = 1, \cdots, k$, then we have the following estimate

$$\begin{aligned}\left|(u^*_{\mu,\xi})^{p-2}\psi^2\right| &\lesssim (u^*_{\mu,\xi})^{p-2}\|\psi\|_*^2 \sum_{j=1}^{k}\Big(\frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}\Big)^2\\ &\lesssim \|\psi\|_*^2\sum_{j=1}^{k}\frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}\underbrace{\frac{\mu_{0j}^{\frac{n-6}{2}}(t)}{(1+|y_j|^2)^{\frac{6-n}{2}}}\frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}}_{\lesssim\mu_{0j}^{1+\sigma+\alpha}(t)}\\ &\lesssim e^{-\varepsilon t_0}\sum_{j=1}^{k}\frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}. \end{aligned}\tag{4.23}$$

Then we consider the case $n \geq 7$. Due to the cut-off function $\eta_{j,R}$, we get

$$\begin{aligned}\left|\eta_{j,R}\tilde{\phi}_j\right|^p &\lesssim \|\phi\|^p_{2+\sigma,n-5+a}\sum_{j=1}^{k}\frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{(2+\sigma)p}(t)}{(1+|y_j|^{n-5+a})^p}\,\chi_{\{|y_j|\leq 2R\}}\\ &\lesssim \|\phi\|^p_{2+\sigma,n-5+a}\frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}\,\frac{\big(1+|y_j|^{n-3+\alpha}\big)\mu_{0j}^{(2+\sigma)(p-1)}(t)}{(1+|y_j|^{n-5+a})^p}\,\chi_{\{|y_j|\leq 2R\}}\\ &\lesssim e^{-\varepsilon t_0}\,\|\phi\|^p_{2+\sigma,n-5+a}\sum_{j=1}^{k}\frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}. \end{aligned}\tag{4.24}$$

Indeed, if we choose $\alpha \in (\frac{1}{2}, 1)$ in the case $n \geq 7$, then we have $n-3+\alpha \leq (n-5+\alpha)p$ and

$$\frac{\big(1+|y_j|^{n-3+\alpha}\big)\mu_{0j}^{(2+\sigma)(p-1)}(t)}{(1+|y_j|^{n-5+a})^p} \lesssim \mu_{0j}^{(2+\sigma)(p-1)}(t)$$

since $\alpha < a$.

Next, we shall estimate the term $|\psi|^p$. We discuss two cases.

- If $|y_j| < \mu_{0j}^{-1}(t)$ for $j = 1, \cdots, k$, then we get

$$\begin{aligned}|\psi|^p &\lesssim \|\psi\|_*^p\sum_{j=1}^{k}\frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\,\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}\underbrace{\frac{\mu_{0j}^{(2+\sigma)(p-1)}(t)(1+|y_j|^{n-3+\alpha})}{(1+|y_j|^{n-5+\alpha})^p}}_{\lesssim\mu_{0j}^{(2+\sigma)(p-1)}(t)}\\ &\lesssim e^{-\varepsilon t_0}\,\|\psi\|_*^p\sum_{j=1}^{k}\frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\,\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}. \end{aligned}\tag{4.25}$$

- If $|y_j| > \mu_{0j}^{-1}(t)$ for $j = 1, \cdots, k$, then similarly we obtain

$$
\begin{aligned}
|\psi|^p &\lesssim \|\psi\|_*^p \sum_{j=1}^k \frac{\mu_{0j}^{-\frac{n-2}{2}} \mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}} \underbrace{\frac{\mu_{0j}^{-2p}(t)\mu_{0j}^{(2+\sigma)(p-1)}(t)}{(1+|y_j|^{n-3+\alpha})^{p-1}}}_{\lesssim \mu_{0j}^{\frac{2(n-4+2\alpha+2\sigma)}{n-2}}(t)} \\
&\lesssim e^{-\varepsilon t_0} \|\psi\|_*^p \sum_{j=1}^k \frac{\mu_{0j}^{-\frac{n-2}{2}} \mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}.
\end{aligned} \tag{4.26}
$$

In conclusion, we get from (4.21)–(4.26) that

$$
|\mathbf{N}(\mathbf{w})| \lesssim e^{-\varepsilon t_0} \begin{cases} \left(\|\phi\|_{2+\sigma,n-5+a}^2 + \|\psi\|_*^2\right) \sum_{j=1}^k \frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}, & \text{if } n = 5, 6, \\ \left(\|\phi\|_{2+\sigma,n-5+a}^p + \|\psi\|_*^p\right) \sum_{j=1}^k \frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}, & \text{if } n \geq 7. \end{cases} \tag{4.27}
$$

To estimate $\mathbf{S}_{\mu,\xi}^{out}$, we first consider the case $x \in B_j$ for fixed $j \in \{1, \cdots, k\}$, namely $|y_j| < \delta\mu_{0j}^{-1}(t)$. In this case, we recall from (3.6) that

$$
\mathbf{S}_{\mu,\xi}^{out} = (1-\eta_{j,R})\mathbf{S}_{\mu,\xi,j} + \mathbf{S}_{\mu,\xi}^{(2)}
$$

with $\mathbf{S}_{\mu,\xi}^{(2)}$ and $\mathbf{S}_{\mu,\xi,j}$ defined in (2.32) and (2.31) respectively. For $\mathbf{S}_{\mu,\xi}^{(2)}$, using (2.9) and the fact that $|y_j| < \delta\mu_{0j}^{-1}(t)$, we have

$$
\begin{aligned}
\left|\mu_j^{-\frac{n+2}{2}}\mathcal{R}_j\right| &\lesssim e^{-\varepsilon t_0} \sum_{j=1}^k \frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}} \mu_{0j}^{1-\alpha-\sigma}(t) \\
&\lesssim e^{-\varepsilon t_0} \sum_{j=1}^k \frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}
\end{aligned} \tag{4.28}
$$

for small σ such that $\sigma < 1-\alpha$. Moreover, according to the estimate of L_2 as in (2.37), we have in the case $n = 5, 6$,

$$
\begin{aligned}
|L_2| &\lesssim \sum_{j=1}^k \frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}} \mu_{0j}^{2-\sigma}(t) \\
&\lesssim e^{-\varepsilon t_0} \sum_{j=1}^k \frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}},
\end{aligned} \tag{4.29}
$$

while in the case $n \geq 7$,

$$
\begin{aligned}
|L_2| &\lesssim \sum_{j=1}^k \frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}} \underbrace{\frac{\mu_{0j}^{\frac{8}{n-2}-\sigma}(t)(1+|y_j|^{n-3+\alpha})}{1+|y_j|^{\frac{(n+2)(n-4)}{n-2}}}}_{\lesssim \mu_{0j}^{\frac{8}{n-2}-\sigma}(t)} \\
&\lesssim e^{-\varepsilon t_0} \sum_{j=1}^k \frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}
\end{aligned} \tag{4.30}
$$

for $\sigma < \frac{8}{n-2}$. Therefore, from (4.28)–(4.30) we obtain

$$\begin{aligned}\left|\mathbf{S}^{(2)}_{\mu,\xi}(x,t)\right| &= \Bigg|\sum_{j=1}^{k}\mu_j^{-\frac{n+2}{2}}\left\{\mathcal{R}_j+\frac{\mu_{0j}^3\dot\mu_{0j}O(1)}{1+|y_j|^{n-4}}+\frac{\mu_{0j}^3|\dot\xi_j|O(1)}{1+|y_j|^{n-3}}+\frac{\mu_{0j}^3O(1)}{1+|y_j|^{n-3}}\right\}\\ &\quad+\sum_{i\neq j}^{k}\mu_{0i}^{\frac{n-2}{2}}O(|q_j-q_i|^{2-n})+L_2\Bigg|\\ &\lesssim e^{-\varepsilon t_0}\sum_{j=1}^{k}\frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}.\end{aligned} \tag{4.31}$$

From the definition of the cut-off function $\eta_{j,R}$ in (3.2), we know that $1-\eta_{j,R}\not\equiv 0$ only for $|x-\xi_j|>\mu_{0j}(t)R$, namely $|y_j|>R$. Therefore, in the region $x\in B_j$, using assumptions (4.8) and (4.9), we have

$$\begin{aligned}\left|(1-\eta_{j,R})\mathbf{S}_{\mu,\xi,j}\right| &\lesssim R^{\alpha-1}\sum_{j=1}^{k}\frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}\\ &\lesssim e^{-\varepsilon t_0}\sum_{j=1}^{k}\frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}.\end{aligned} \tag{4.32}$$

For the case $x\notin\cup_{j=1}^{k}B_j$, namely $|y_j|>\delta\mu_{0j}^{-1}(t)$, we can get the estimate of $\mathbf{S}^{out}_{\mu,\xi}$ by Lemma 2.2 similarly

$$\left|\mathbf{S}^{out}_{\mu,\xi}(x,y)\right| \lesssim e^{-\varepsilon t_0}\sum_{j=1}^{k}\frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}. \tag{4.33}$$

By Lemma 4.1, for function $\mathbf{f}$ with $\|\mathbf{f}\|_{*,-\frac{n-2}{2}+2+\sigma,n-3-\alpha}<+\infty$, we have

$$|\psi(x,t)|=|T(\mathbf{f})| \lesssim \|\mathbf{f}\|_{*,-\frac{n-2}{2}+2+\sigma,n-3+\alpha}\begin{cases}\sum\limits_{j=1}^{k}\frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-5+\alpha}}, & \text{if } |y_j|<\mu_{0j}^{-1}(t),\\ \sum\limits_{j=1}^{k}\frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}, & \text{if } |y_j|>\mu_{0j}^{-1}(t).\end{cases}$$

Here the norm $\|\cdot\|_{*,-\frac{n-2}{2}+2+\sigma,n-3+\alpha}$ is defined in (4.4) by replacing β by $-\frac{n-2}{2}+2+\sigma$ and γ by $n-5+\alpha$. Therefore, by the estimates (4.17), (4.18), (4.19), (4.20), (4.27), (4.31), (4.32), (4.33) and Lemma 4.1, it follows that the mapping $\mathcal{A}$ maps the set $\mathcal{B}$ to itself.

Then it suffices to show that $\mathcal{A}$ is a contraction mapping. We claim that for any ψ_1, $\psi_2\in\mathcal{B}$,

$$\|\mathcal{A}(\psi_1)-\mathcal{A}(\psi_2)\|_* \le c\,\|\psi_1-\psi_2\|_*,$$

where $0<c<1$. Indeed, we observe that

$$\mathcal{A}(\psi_1)-\mathcal{A}(\psi_2)=T\Big(\mathbf{N}(\psi_1+\phi^{in})-\mathbf{N}(\psi_2+\phi^{in})+V_{\mu,\xi}(\psi_1-\psi_2)\Big)$$

(see (4.14)). Recalling the definition of $\mathbf{N}$ as in (3.4), we get

$$\left|\mathbf{N}\big(\psi_1+\phi^{in}\big)-\mathbf{N}\big(\psi_2+\phi^{in}\big)\right| \lesssim \begin{cases}(u^*_{\mu,\xi})^{p-2}\,|\phi^{in}|\,\big|\psi_1-\psi_2\big|, & \text{if } n=5,6,\\ |\phi^{in}|^{p-1}\,\big|\psi_1-\psi_2\big|, & \text{if } n\ge 7.\end{cases}$$

In the case $n=5,6$, we get

$$\left|\mathbf{N}\big(\psi_1+\phi^{in}\big)-\mathbf{N}\big(\psi_2+\phi^{in}\big)\right|\lesssim e^{-\varepsilon t_0}\,\|\phi\|_{2+\sigma,n-5+a}\|\psi_1-\psi_2\|_*\sum_{j=1}^{k}\frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}},$$

while in the case $n \geq 7$, we have

$$\left|\mathbf{N}\left(\psi_1+\phi^{in}\right)-\mathbf{N}\left(\psi_2+\phi^{in}\right)\right| \lesssim e^{-\varepsilon t_0}\|\phi\|_{2+\sigma, n-5+a}^{p-1}\|\psi_1-\psi_2\|_* \sum_{j=1}^k \frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}.$$

For the term $V_{\mu,\xi}\psi$ defined in (3.9), it is easy to derive that

$$|V_{\mu,\xi}(\psi_1-\psi_2)| \leq e^{-\varepsilon t_0}\|\psi_1-\psi_2\|_* \sum_{j=1}^k \frac{\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{-2}(t)\mu_{0j}^{2+\sigma}(t)}{1+|y_j|^{n-3+\alpha}}.$$

Hence

$$\|\mathcal{A}(\psi_1)-\mathcal{A}(\psi_2)\|_* \leq c\,\|\psi_1-\psi_2\|_*$$

holds with $0<c<1$ when t_0 is sufficiently large. Therefore, if t_0 is sufficiently large, the operator $\mathcal{A}$ is a contraction mapping in $\mathcal{B}$. By the contraction mapping theorem, we get the existence of desired solution in $\mathcal{B}$. Estimate (4.13) follows similarly as (4.12) by standard parabolic theory. The proof is completed. □

4.3. **Lipschitz dependence of ψ on $\lambda, \xi, \dot\lambda, \dot\xi$ and ϕ.** The function $\psi = \Psi(\lambda, \xi, \dot\lambda, \dot\xi, \phi)$ is a solution of problem (4.1), which also depends on the parameter functions λ, ξ, $\dot\lambda$, $\dot\xi$, and ϕ. Next we want to clarify this dependence, which is done by estimating for example $\partial_\phi\Psi(\lambda,\xi,\dot\lambda,\dot\xi,\phi)[\bar\phi] = \partial_s\Psi\big[\lambda,\xi,\dot\lambda,\dot\xi,\phi+s\bar\phi\big]\Big|_{s=0}$ as a linear operator between Banach spaces. We have the following proposition, whose proof can be carried out after some minor modifications as in [9, Proposition 4.2]. We omit the details.

Proposition 4.2. *Assume the validity of the hypotheses in Proposition 4.1. Then,* Ψ *depends smoothly on* λ, ξ, $\dot\lambda$, $\dot\xi$, ϕ, *and we have*

$$\|\Psi[\lambda_1,\xi,\dot\lambda,\dot\xi,\phi]-\Psi[\lambda_2,\xi,\dot\lambda,\dot\xi,\phi]\|_* \lesssim e^{-\varepsilon t_0}\,\|\lambda_1-\lambda_2\|_{1+\sigma}, \tag{4.34}$$

$$\|\Psi[\lambda,\xi_1,\dot\lambda,\dot\xi,\phi]-\Psi[\lambda,\xi_2,\dot\lambda,\dot\xi,\phi]\|_* \lesssim e^{-\varepsilon t_0}\,\|\xi_1-\xi_2\|_{1+\sigma}, \tag{4.35}$$

$$\|\Psi[\lambda,\xi,\dot\lambda_1,\dot\xi,\phi]-\Psi[\lambda,\xi,\dot\lambda_2,\dot\xi,\phi]\|_* \lesssim e^{-\varepsilon t_0}\,\|\dot\lambda_1-\dot\lambda_2\|_{1+\sigma}, \tag{4.36}$$

$$\|\Psi[\lambda,\xi,\dot\lambda,\dot\xi_1,\phi]-\Psi[\lambda,\xi,\dot\lambda,\dot\xi_2,\phi]\|_* \lesssim e^{-\varepsilon t_0}\,\|\dot\xi_1-\dot\xi_2\|_{1+\sigma}, \tag{4.37}$$

$$\|\Psi[\lambda,\xi,\dot\lambda,\dot\xi,\phi^{(1)}]-\Psi[\lambda,\xi,\dot\lambda,\dot\xi,\phi^{(2)}]\|_* \lesssim e^{-\varepsilon t_0}\,\|\phi^{(1)}-\phi^{(2)}\|_{2+\sigma,n-5+a}, \tag{4.38}$$

where the above norms are defined in (4.7), (4.10) *and* (4.16), *respectively.*

5. Linear theory and choices of parameter functions $\lambda(t)$, $\xi(t)$

After we get the outer solution $\psi = \Psi[\lambda,\xi,\dot\lambda,\dot\xi,\phi]$ as in Proposition 4.1 and Proposition 4.2, we substitute the function ψ into the inner problem (3.11) and get a nonlinear and nonlocal equation of ϕ_j. We perform a further change of variable in equation (3.11)

$$t = t(\tau_j), \qquad \frac{dt}{d\tau_j} = \mu_{0j}^2(t), \quad j = 1,\cdots,k,$$

namely

$$\tau_j(t) = (2\kappa_j)^{-1}e^{2\kappa_j t} = (2\kappa_j)^{-1}\mu_{0j}^{-2}(t). \tag{5.1}$$

For simplicity, we write τ_j as τ in the following if there is no confusion. By the above change of variable, equation (3.11) becomes

$$\partial_\tau\phi_j = \Delta_y\phi_j + pU(y)^{p-1}\phi_j + H_j[\lambda,\xi,\dot\lambda,\dot\xi,\psi,\phi](y,\tau) \quad \text{in} \quad B_{2R}(0)\times[\tau_0,+\infty) \tag{5.2}$$

with $t(\tau_0) = t_0$ and

$$\begin{aligned} H_j[\lambda, \xi, \dot\lambda, \dot\xi, \psi, \phi](y, \tau) = \mu_{0j}^{\frac{n+2}{2}} \mathbf{S}_{\mu,\xi,j}\big(\xi_j + \mu_{0j} y,\, t(\tau)\big) + B_j^1[\phi_j] + B_j^2[\phi_j] + B_j^3[\phi_j] \\ + p\mu_{0j}^{\frac{n-2}{2}} \frac{\mu_{0j}^2}{\mu_j^2} \Big| U\Big(\frac{\mu_{0j}}{\mu_j} y\Big)\Big|^{p-1} \psi\big(\xi_j + \mu_{0j} y,\, t(\tau)\big). \end{aligned} \tag{5.3}$$

Next, let us explain formally how we solve problem (5.2). The linear operator $L_\tau(\phi) := -\phi_\tau + \Delta\phi + pU^{p-1}\phi$ is certainly not invertible since all τ-independent elements of the kernel of $L_0(\phi) := \Delta\phi + pU^{p-1}\phi$ are also the elements of the kernel of L_τ. Thus, for solvability, we expect some *orthogonality conditions* to hold. Moreover, the solution ϕ_j we look for cannot grow exponentially in time. Recall that the operator L_0 as in (2.20) has a positive radially symmetric bounded eigenfunction Z_0 associated to the only negative eigenvalue λ_0 to the problem

$$L_0(\phi) + \lambda\phi = 0, \quad \phi \in L^\infty(\mathbb{R}^n). \tag{5.4}$$

Furthermore, λ_0 is simple and Z_0 has the asymptotic behavior $Z_0(y) \sim |y|^{-\frac{n-2}{2}} e^{-\sqrt{|\lambda_0|}\,|y|}$ as $|y| \to \infty$. To avoid exponential growth in time due to the instability, we construct a solution to problem in the class of functions that are parallel to Z_0 in the initial time τ_0.

The above formal argument leads us to construct a solution $\phi = (\phi_1, \cdots, \phi_k)$ of the system

$$\begin{cases} \partial_\tau \phi_j = \Delta_y \phi_j + pU(y)^{p-1}\phi_j + H_j[\lambda, \xi, \dot\lambda, \dot\xi, \psi, \phi](y, \tau), \text{ in } B_{2R}(0) \times [\tau_0, +\infty), \\ \phi_j(y, \tau_0) = e_0 Z_0(y), \quad y \in B_{2R}(0), \\ \displaystyle\int_{B_{2R}} H_j[\lambda, \xi, \dot\lambda, \dot\xi, \psi, \phi](y, \tau)\, Z_\ell(y)\, \mathrm{d}y = 0, \ \forall \tau > \tau_0, \ j = 1, \cdots, k, \ \ell = 1, \cdots, n+1, \end{cases} \tag{5.5}$$

for some constant e_0. For $\ell = 1, \cdots, n+1$, $Z_\ell(y)$ are the only bounded elements in the kernel of the operator L_0 . Moreover, the parameters λ and ξ (as functions of the given ϕ) will be chosen such that the orthogonality conditions

$$\int_{B_{2R}(0)} H_j[\lambda, \xi, \dot\lambda, \dot\xi, \psi, \phi](y, \tau) Z_\ell(y) \mathrm{d}y = 0, \ \forall \tau > \tau_0, \ j = 1, \cdots, k, \ \ell = 1, \cdots, n+1 \tag{5.6}$$

are satisfied. The above $k \times (n+1)$ orthogonality conditions imply a nonlinear nonlocal system of $k \times (n+1)$ ODEs. In Section 5.2, we will prove that the ODE system is solvable. After we solve the ODEs of parameters λ, ξ, we will prove that problem (5.5) is solvable in the class of functions ϕ_j satisfying (4.11). A central point of the construction is a linear theory developed in [9, Section 7], which allows us to solve system (5.5) by means of the contraction mapping theorem. This will be the context of Section 6.

5.1. **The linear theory of inner problem.** The key ingredient to solve the inner problem for function ϕ satisfying (4.11) is the resolution of the linear problem: For a large number $R > 0$, we shall construct a solution to an initial value problem of the form

$$\begin{cases} \phi_\tau = \Delta\phi + pU(y)^{p-1}\phi + h(y, \tau) & \text{in } B_{2R}(0) \times (\tau_0, +\infty), \\ \phi(y, \tau_0) = e_0\, Z_0(y) & \text{in } B_{2R}(0), \end{cases} \tag{5.7}$$

provided that h satisfies certain space-time decay and certain orthogonality conditions. Here Z_0 is the positive radially symmetric bounded eigenfunction associated to the only negative eigenvalue to the eigenvalue problem (5.4).

We recall that $\tau_j = \tau_j(t)$ is given in (5.1), namely $\tau_j(t) = \frac{1}{2\kappa_j}\, \mu_{0j}^{-2}(t)$ for $j = 1, \cdots, k$. In the τ-variable, we define

$$\|h\|_{\nu,a} := \sup_{\tau > \tau_0} \sup_{y \in B_{2R}(0)} \tau^\nu (1 + |y|^a)\, |h(y, \tau)|, \tag{5.8}$$

where $\nu = 1 + \frac{\sigma}{2}$, so that we have $\tau_j^{-\nu}(t) \sim \mu_{0j}^{2+\sigma}(t)$.

Decompose $h(y,\tau)$ into the following spherical harmonic modes

$$h(y,\tau)=\sum_{j=0}^{\infty}h_j(|y|,\tau)\Theta_j(y/|y|)\quad\text{with}\quad h_j(|y|,\tau)=\int_{\mathbb{S}^{n-1}}h(y,\tau)\Theta_j(\theta)d\theta,$$

where $\theta=y/|y|$, and Θ_j $(j\in\mathbb{N})$ are orthogonal basis of $L^2(\mathbb{S}^{n-1})$ made up of spherical harmonics, namely eigenfunctions of the problem

$$\Delta_{\mathbb{S}^{n-1}}\Theta_j+\lambda_j\Theta_j=0\quad\text{in }\mathbb{S}^{n-1}.$$

We denote $h=h^0+h^1+h^\perp$ with

$$h^0=h_0(|y|,\tau),\ \ h^1=\sum_{j=1}^{n}h_j(|y|,\tau)\Theta_j\ \ \text{and}\ \ h^\perp=\sum_{j=n+1}^{\infty}h_j(|y|,\tau)\Theta_j.$$

We have the following proposition concerning the estimate of solution ϕ to the linear problem (5.7), which can be proved by similar arguments as in [9, Proposition 7.1].

Proposition 5.1. *Let ν,a be given positive numbers with $\nu=1+\frac{\sigma}{2}$ and $0<a<1$. Then, for all sufficiently large $R>0$ and any $h=h(y,\tau)$ with $\|h\|_{\nu,n-3+a}<+\infty$ that satisfies*

$$\int_{B_{2R}(0)}h(y,\tau)\,Z_\ell(y)\,\mathrm{d}y=0\quad\textit{for all}\quad\tau\in(\tau_0,+\infty),\quad\ell=1,\cdots,n+1,\tag{5.9}$$

there exist $\phi=\phi[h]$ and $e_0=e_0[h]$ which solve problem (5.7). Moreover, they define linear operators of h that satisfy the estimates

$$\begin{aligned}(1+|y|)\big|\nabla\phi(y,\tau)\big|+\big|\phi(y,\tau)\big|\ &\lesssim\tau^{-\nu}\frac{R^{5-a}}{1+|y|^n}\|h^0\|_{\nu,n-3+a}+\tau^{-\nu}\frac{R^{6-a}}{1+|y|^{n+1}}\|h^1\|_{\nu,n-3+a}\\&\quad+\frac{\tau^{-\nu}}{1+|y|^{n-5+a}}\|h\|_{\nu,n-3+a},\end{aligned}\tag{5.10}$$

and

$$\big|e_0[h]\big|\lesssim\|h\|_{\nu,n-3+a}.\tag{5.11}$$

5.2. **Adjusting the parameter functions.** In this Subsection, we first derive the ODE system of λ and ξ such that the orthogonality conditions (5.6) are satisfied. For convenience, we use the following notation

$$\lambda(t)=\begin{pmatrix}\lambda_1(t)\\\vdots\\\lambda_k(t)\end{pmatrix},\ \dot\lambda(t)=\begin{pmatrix}\dot\lambda_1(t)\\\vdots\\\dot\lambda_k(t)\end{pmatrix},\ \xi(t)=\begin{pmatrix}\xi_1(t)\\\vdots\\\xi_k(t)\end{pmatrix},\ \dot\xi(t)=\begin{pmatrix}\dot\xi_1(t)\\\vdots\\\dot\xi_k(t)\end{pmatrix},\ q=\begin{pmatrix}q_1\\\vdots\\q_k\end{pmatrix}.$$

First, we describe (5.6) when $\ell=n+1$.

Lemma 5.1. *For fixed $j\in\{1,\cdots,k\}$, there exists a positive constant $\varepsilon>0$ such that (5.6) with $\ell=n+1$ is equivalent to*

$$\dot\lambda_j(t)+\kappa_j\lambda_j(t)=\Pi_{1,j}[\lambda,\xi,\dot\lambda,\dot\xi,\phi](t)\tag{5.12}$$

with

$$\Pi_{1,j}[\lambda,\xi,\dot\lambda,\dot\xi,\phi](t)=\mu_{0j}^{1+\sigma}(t)\mathfrak{f}(t)+e^{-\varepsilon t_0}\,\Upsilon\big[\dot\lambda,\,\dot\xi,\,\lambda,\,\mu_{0j}(\xi-q),\,\mu_{0j}^{1+\sigma}\phi\big](t),\tag{5.13}$$

where κ_j $(j=1,\cdots,k)$ is a positive constant as in (2.25), $\mathfrak{f}(t)$ and $\Upsilon\big[\dot\lambda,\,\dot\xi,\,\lambda,\,\mu_{0j}(\xi-q),\,\mu_{0j}^{1+\sigma}\phi\big](t)$ are smooth and bounded functions for $t\in[t_0,+\infty)$. Moreover, the following Lipschitz dependence of

$\Upsilon[\cdots](t)$ *on its parameters hold*

$$|\,\Upsilon[\dot\lambda^{(1)}](t)-\Upsilon[\dot\lambda^{(2)}](t)\,| \lesssim e^{-\varepsilon t_0}\,|\dot\lambda^{(1)}(t)-\dot\lambda^{(2)}(t)|, \tag{5.14}$$

$$|\,\Upsilon[\dot\xi^{(1)}](t)-\Upsilon[\dot\xi^{(2)}](t)\,| \lesssim e^{-\varepsilon t_0}\,|\dot\xi^{(1)}(t)-\dot\xi^{(2)}(t)| \tag{5.15}$$

$$|\,\Upsilon[\lambda^{(1)}](t)-\Upsilon[\lambda^{(2)}](t)| \lesssim e^{-\varepsilon t_0}\,|\lambda^{(1)}(t)-\lambda^{(2)}(t)|, \tag{5.16}$$

$$|\,\Upsilon[\mu_{0j}(\xi^{(1)}-q)](t)-\Upsilon[\mu_{0j}(\xi^{(2)}-q)](t)| \lesssim e^{-\varepsilon t_0}\,|\xi^{(1)}(t)-\xi^{(2)}(t)|, \tag{5.17}$$

$$|\Upsilon[\mu_{0j}^{1+\sigma}\phi^{(1)}](t)-\Upsilon[\mu_{0j}^{1+\sigma}\phi^{(2)}](t)| \lesssim e^{-\varepsilon t_0}\,\|\phi^{(1)}-\phi^{(2)}\|_{2+\sigma,n-5+a}. \tag{5.18}$$

Proof. Let σ be the positive number fixed sufficiently small as in Proposition 4.1. For ϕ_j satisfying (4.10) and any fixed $j\in\{1,\cdots,k\}$, we want to compute

$$\int_{B_{2R}(0)} H_j[\lambda,\xi,\dot\lambda,\dot\xi,\psi,\phi](y,t(\tau))\,Z_{n+1}(y)\,\mathrm{d}y,$$

where H_j is given in (5.3). Recalling the expression of $\mathbf{S}_{\mu,\xi,j}$ in (2.31), we can write

$$\begin{aligned}\mu_{0j}^{\frac{n+2}{2}}\mathbf{S}_{\mu,\xi,j}(\xi_j+\mu_{0j}y,\,t) &= (\mu_{0j}\mu_j^{-1})^{\frac{n+2}{2}}\big[\,\mu_{0j}S_1(z,t)+\lambda_jS_2(z,t)+\mu_jS_3(z,t)\,\big]_{z=\xi_j+\mu_jy}\\ &\quad+(\mu_{0j}\mu_j^{-1})^{\frac{n+2}{2}}\mu_{0j}\big[\,S_1(\xi_j+\mu_{0j}y,t)-S_1(\xi_j+\mu_jy,\,t)\,\big]\\ &\quad+(\mu_{0j}\mu_j^{-1})^{\frac{n+2}{2}}\lambda_j\big[\,S_2(\xi_j+\mu_{0j}y,t)-S_2(\xi_j+\mu_jy,\,t)\,\big]\\ &\quad+(\mu_{0j}\mu_j^{-1})^{\frac{n+2}{2}}\mu_j\big[\,S_3(\xi_j+\mu_{0j}y,t)-S_3(\xi_j+\mu_jy,\,t)\,\big],\end{aligned}$$

where

$$\begin{aligned}S_1(z,t)&=\big(\dot\lambda_j-\kappa_j\lambda_j\big)Z_{n+1}\Big(\frac{z-\xi_j}{\mu_j}\Big)+2\lambda_j\mathrm{A}(q_j)\frac{z-\xi_j}{\mu_j}\cdot\nabla U\Big(\frac{z-\xi_j}{\mu_j}\Big),\\ S_2(z,t)&=\dot\lambda_j\,Z_{n+1}\Big(\frac{z-\xi_j}{\mu_j}\Big)+\lambda_j\mathrm{A}(q_j)\frac{z-\xi_j}{\mu_j}\cdot\nabla U\Big(\frac{z-\xi_j}{\mu_j}\Big),\\ S_3(z,t)&=\dot\xi_j\cdot\nabla U\Big(\frac{z-\xi_j}{\mu_j}\Big)+\mathrm{A}(q_j)(\xi_j-q_j)\cdot\nabla U\Big(\frac{z-\xi_j}{\mu_j}\Big).\end{aligned}$$

It is direct to check that

$$\begin{aligned}&\int_{B_{2R}(0)} Z_{n+1}^2(y)\mathrm{d}y=c_0\big(1+O(R^{4-n})\big),\\ &\int_{B_{2R}(0)}\big(\mathrm{A}(q_j)\,y\big)\cdot\nabla U(y)Z_{n+1}(y)\,\mathrm{d}y=c_j\,\big(1+O(R^{4-n})\big)\end{aligned}$$

with c_0 and c_j defined in (2.24). Therefore, we obtain

$$\begin{aligned}&\int_{B_{2R}(0)} S_1(\xi_j+\mu_jy,t)Z_{n+1}(y)\,\mathrm{d}y=c_0\big(\dot\lambda_j+\kappa_j\lambda_j\big)\big(1+O(R^{4-n})\big),\\ &\int_{B_{2R}(0)} S_2(\xi_j+\mu_jy,t)Z_{n+1}(y)\,\mathrm{d}y=\ c_0\dot\lambda_j\big(1+O(R^{4-n})\big)+c_j\lambda_j\big(1+O(R^{4-n})\big),\\ &\int_{B_{2R}(0)} S_3(\xi_j+\mu_jy,t)Z_{n+1}(y)\,\mathrm{d}y=0,\end{aligned}$$

where we have used symmetry for the third integral above. Since $\frac{\mu_{0j}}{\mu_j}=\Big(1+\frac{\lambda_j}{\mu_{0j}}\Big)^{-1}$, we get, for any $\ell=1,2,3$

$$\begin{aligned}&\int_{B_{2R}(0)}\big[\,S_\ell(\xi_j+\mu_{0j}y,t)-S_\ell(\xi_j+\mu_jy,t)\big]\,Z_{n+1}(y)\,\mathrm{d}y\\ &=\mathfrak{g}\Big(t,\frac{\lambda_j}{\mu_{0j}}\Big)+\mathfrak{g}\Big(t,\frac{\lambda_j}{\mu_{0j}}\Big)\lambda_j+\mathfrak{g}\Big(t,\frac{\lambda_j}{\mu_{0j}}\Big)\dot\xi_j+\mathfrak{g}\Big(t,\frac{\lambda_j}{\mu_{0j}}\Big)(\xi_j-q_j)+\mu_{0j}^{1+\sigma}\mathfrak{f}(t),\end{aligned}$$

where $\mathfrak{f}$, $\mathfrak{g}$ are smooth and bounded functions, and $\mathfrak{g}(\cdot, s) \sim s$ as $s \to 0$. Thus we conclude that

$$c_0^{-1}\mu_{0j}^{-1}\left(\frac{\mu_j}{\mu_{0j}}\right)^{\frac{n+2}{2}}\int_{B_{2R}(0)}\mu_{0j}^{\frac{n+2}{2}}\mathbf{S}_{\mu,\xi,j}(\xi_j+\mu_{0j}y,\,t)\,Z_{n+1}(y)\,\mathrm{d}y$$
$$=(\dot\lambda_j+\kappa_j\lambda_j)+e^{-\varepsilon t_0}\mathfrak{g}\Big(t,\frac{\lambda_j}{\mu_{0j}}\Big)(\dot\lambda_j+\dot\xi_j)+e^{-\varepsilon t_0}\mathfrak{g}\Big(t,\frac{\lambda_j}{\mu_{0j}}\Big)\lambda_j+\mu_{0j}^{1+\sigma}\mathfrak{f}(t),$$

where $\mathfrak{g}$ is smooth and bounded in its argument and $\mathfrak{g}(\cdot, s) \sim s$ as $s \to 0$.

Next we consider the term

$$p\mu_{0j}^{\frac{n-2}{2}}\big(1+\frac{\lambda_j}{\mu_{0j}}\big)^{-2}\int_{B_{2R}(0)}\big|U\big(\frac{\mu_{0j}}{\mu_j}y\big)\big|^{p-1}\psi(\xi_j+\mu_{0j}y,t)Z_{n+1}(y)\mathrm{d}y.$$

The principal part is

$$G:=\int_{B_{2R}(0)}|U(y)|^{p-1}\psi(\xi_j+\mu_{0j}y,t)Z_{n+1}(y)\mathrm{d}y.$$

Since $\psi=\Psi[\lambda,\xi,\dot\lambda,\dot\xi,\phi](y,t)$, we can write

$$\begin{aligned}G&=\Psi[0,q,0,0,0](q_j,t)\int_{B_{2R}(0)}|U(y)|^{p-1}\,Z_{n+1}(y)\,\mathrm{d}y\\&\quad+\int_{B_{2R}(0)}|U(y)|^{p-1}\,Z_{n+1}(y)\,\Big(\Psi[0,q,0,0,0](\xi_j+\mu_{0j}y,t)-\Psi[0,q,0,0,0](q_j,t)\Big)\,\mathrm{d}y\\&\quad+\int_{B_{2R}(0)}|U(y)|^{p-1}\,Z_{n+1}(y)\,\Big(\Psi[\lambda,\xi,\dot\lambda,\dot\xi,\phi]-\Psi[0,q,0,0,0]\Big)\,(\xi_j+\mu_{0j}y,t)\,\mathrm{d}y\\&:=G_1+G_2+G_3.\end{aligned}$$

By Proposition 4.1, we obtain $G_1=e^{-\varepsilon t_0}\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{1+\sigma}(t)\mathfrak{f}(t)$ with smooth and bounded $\mathfrak{f}$. By the mean value theorem, we get that

$$G_2=e^{-\varepsilon t_0}\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{1+\sigma}(t)\mathfrak{g}\Big(t,\frac{\lambda_j}{\mu_{0j}},\xi_j-q_j\Big)$$

for a smooth function $\mathfrak{g}$ with $\mathfrak{g}(\cdot,s,\cdot)\sim s$ as $s\to 0$. The mean value theorem gives that, for some $s\in(0,1)$,

$$\begin{aligned}G_3=\int_{B_{2R}(0)}&U^{p-1}(y)\,Z_{n+1}(y)\,\Big[\partial_\lambda\Psi[0,q,0,0,0][s\lambda](\xi_j+\mu_{0j}y,t)\\&+\partial_\xi\Psi[0,q,0,0,0][s(\xi_j-q_j)](\xi_j+\mu_{0j}y,t)+\partial_{\dot\lambda}\Psi[0,q,0,0,0][s\dot\lambda](\xi_j+\mu_{0j}y,t)\\&+\partial_{\dot\xi}\Psi[0,q,0,0,0][s\dot\xi]((\xi_j+\mu_{0j}y,t)+\partial_\phi\Psi[0,q,0,0,0][s\phi]((\xi_j+\mu_{0j}y,t)\Big]\,dy.\end{aligned}$$

By Proposition 4.2, we obtain

$$G_3=e^{-\varepsilon t_0}\mu_{0j}^{-\frac{n-2}{2}}(t)\mu_{0j}^{1+\sigma}(t)\,\mathfrak{f}(t)(\dot\lambda_j+\dot\xi_j+\lambda_j+\xi_j)F(\lambda,\xi,\dot\lambda,\dot\xi,\phi)(t),$$

where $\mathfrak{f}$ is smooth and bounded, and F is a nonlocal, nonlinear smooth operator in its parameters with $F(0,q,0,0,0)(t)$ bounded.

Now we consider the terms $B_j^1[\phi_j], B_j^2[\phi_j]$ and $B_j^3[\phi_j]$ defined respectively by (3.12), (3.13) and (3.14). We obtain that

$$\begin{aligned}&\sum_{\ell=1}^{3}\int_{B_{2R}(0)}B_j^\ell[\phi_j](y,t)\,Z_{n+1}(y)\,\mathrm{d}y\\&=e^{-\varepsilon t_0}\Big\{\mu_{0j}^{1+\sigma}\mathfrak{q}[\phi](t)+\dot\xi_j\mathfrak{q}[\phi](t)+\mu_{0j}^{2+\sigma}(t)\,\mathfrak{g}\Big(\frac{\lambda_j}{\mu_{0j}}\Big)\,\mathfrak{q}[\phi](t)+\mu_{0j}^{1+\sigma}(t)\mathfrak{q}[\phi](t)\Big\},\end{aligned}$$

where $\mathfrak{q}[\phi](t)$ is a smooth and bounded function in t, while the function $\mathfrak{g}(s)$ is smooth with $\mathfrak{g}(s)\sim s$ as $s\to 0$. Collecting the above terms, we get the validity of (5.12). □

Similarly, up to some minor modifications as in Lemma 5.1, we can compute (5.6) for $\ell = 1, \cdots, n$. This is the content of next Lemma. Here we omit the proof.

Lemma 5.2. *For fixed $j \in \{1, \cdots, k\}$, there exists a positive constant $\varepsilon > 0$, such that the relation (5.6) for $\ell = 1, \cdots, n$ is equivalent to*

$$\dot{\xi}_j + \mathrm{A}(q_j)(\xi_j - q_j) = \Pi_{2,j}[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](t), \tag{5.19}$$

where $\mathrm{A}(q_j) = \nabla^2 \log a(q_j)$ is the Hessian matrix and

$$\Pi_{2,j}[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](t) = \mu_{0j}^{1+\sigma}(t)\, \vec{f}_j(t) + e^{-\varepsilon t_0}\, \Upsilon\big[\, \dot{\lambda},\, \dot{\xi},\, \lambda,\, \mu_{0j}(\xi - q),\, \mu_{0j}^{1+\sigma}\phi\,\big](t). \tag{5.20}$$

Here the function $\vec{f}_j = \vec{f}_j(t)$ is an explicit n dimensional vector function, and it is smooth and bounded for $t \in [t_0, +\infty)$. Moreover, $\Upsilon[\cdots](t)$ has the same Lipschitz properties as described in Lemma 5.1.

In summary, from Lemma 5.1 and Lemma 5.2, we have proved that solving (5.6) is equivalent to solving the system of ODEs of λ and ξ

$$\begin{cases} \dot{\lambda}_j \ + \kappa_j \lambda_j = \Pi_{1,j}[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](t), \\ \dot{\xi}_j + \mathrm{A}(q_j)(\xi_j - q_j) = \Pi_{2,j}[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](t). \end{cases} \tag{5.21}$$

We next show that, for any given ϕ satisfying (4.11), the system (5.21) is solvable and admits solution $\lambda = \lambda[\phi](t)$, $\xi = \xi[\phi](t)$ satisfying the restrictions (4.8)–(4.9). Moreover, we show the Lipschitz dependence of $\lambda = \lambda[\phi]$, $\xi = \xi[\phi]$ on ϕ, which is a crucial property to ensure the existence of the solution ϕ to problem (5.5).

Proposition 5.2. *Assume that ϕ satisfies (4.11). Then there exists a solution in the form $\lambda = \lambda[\phi](t)$, $\xi = \xi[\phi](t)$ to the nonlinear system of ODEs (5.21), which satisfies the bounds (4.8)–(4.9). Furthermore, for $t \in [t_0, +\infty)$, it holds that*

$$\mu_{0j}^{-(1+\sigma)}(t)\big|\, \lambda[\phi^{(1)}](t) - \lambda[\phi^{(2)}](t) \,\big| \ \lesssim\ e^{-\varepsilon t_0}\, \|\phi^{(1)} - \phi^{(2)}\|_{2+\sigma, n-5+a}, \tag{5.22}$$

$$\mu_{0j}^{-(1+\sigma)}(t)\big|\, \xi[\phi^{(1)}](t) - \xi[\phi^{(2)}](t) \,\big| \ \lesssim\ e^{-\varepsilon t_0}\, \|\phi^{(1)} - \phi^{(2)}\|_{2+\sigma, n-5+a}. \tag{5.23}$$

Proof. Let $h = (h_1, \cdots, h_k) : [t_0, +\infty) \to \mathbb{R}^k$ be a vector function with the bounded norm $\|h\|_{1+\sigma}$, where the norm $\|\cdot\|_{1+\sigma}$ is defined in (4.7). By the variation of parameters formula, the solution of

$$\dot{\lambda}_j(t) + \kappa_j\, \lambda_j(t) = h_j(t), \quad j = 1, \cdots, k \tag{5.24}$$

can be expressed as

$$\lambda_j(t) = e^{-\kappa_j t} \left[d_j + \int_{t_0}^{t} e^{\kappa_j s}\, h_j(s)\, ds \right],$$

where d_j $(j = 1, \cdots, k)$ are arbitrary constants. In order to ensure that λ_j decays to 0 as $t \to +\infty$, we choose

$$d_j = e^{\kappa_j t_0} \lambda_j(t_0) = -\int_{t_0}^{+\infty} e^{\kappa_j s}\, h_j(s)\, ds,$$

then

$$\begin{aligned} \left| e^{(1+\sigma)\kappa_j t} \lambda_j(t) \right| &= \left| -e^{\sigma\kappa_j t} \int_{t}^{+\infty} e^{\kappa_j s} h_j(s) ds \right| \\ &\le e^{\sigma\kappa_j t} \|h_j\|_{1+\sigma} \int_t^{+\infty} e^{-(1+\sigma)\kappa_j s} e^{\kappa_j s} ds \lesssim \|h_j\|_{1+\sigma}. \end{aligned}$$

Therefore we have

$$\|\lambda_j\|_{1+\sigma} \ \lesssim\ \|h_j\|_{1+\sigma}.$$

Letting $\Lambda(t) = \dot{\lambda}(t)$ and $\mathfrak{K} = \mathrm{diag}(\kappa_1, \cdots, \kappa_k)$, equation (5.24) becomes

$$\Lambda(t) + \mathfrak{K} \int_t^{\infty} \Lambda(s)\, \mathrm{d}s = h(t). \tag{5.25}$$

Then equation (5.25) defines a linear operator $\mathcal{L}_1 : h \to \Lambda$, which associates to any h with $\|h\|_{1+\sigma}$-bounded a solution to equation (5.24). This operator $\mathcal{L}_1$ is continuous in the Banach spaces $(L^\infty[t_0,+\infty))^k$ equipped with the $\|\cdot\|_{1+\sigma}$-topology.

Similarly, for vector function $\mathfrak{h} = (\mathfrak{h}_1,\cdots,\mathfrak{h}_k)$ with $\mathfrak{h}_j$: $[t_0,+\infty) \to \mathbb{R}^n$ and $\|\mathfrak{h}_j\|_{1+\sigma}$-bounded, we now consider the linear system of ODEs associated to (5.19)

$$\frac{d}{dt}(\xi_j(t)-q_j) + \mathrm{A}(q_j)(\xi_j(t)-q_j) = \mathfrak{h}_j(t),\ j=1,\cdots,k. \tag{5.26}$$

Recall from (1.5) that $\mathrm{P}_j\mathrm{A}(q_j){\mathrm{P}_j}^T = \mathrm{diag}(\sigma_1^{(j)},\sigma_2^{(j)},\cdots,\sigma_n^{(j)})$ for some orthogonal matrix P_j. If we denote that

$$\xi_j(t)-q_j = {\mathrm{P}_j}^T\, v(t),\ v(t) = (v_1(t),\cdots,v_n(t))^{\mathrm{T}},$$

then (5.26) becomes

$$v'(t) + \mathrm{diag}(\sigma_1^{(j)},\sigma_2^{(j)},\cdots,\sigma_n^{(j)})\, v(t) = \mathrm{P}_j\,\mathfrak{h}_j(t),$$

and each component of the solution $v(t)$ can be expressed as

$$v_i(t) = e^{-\sigma_i^{(j)}\,t}\Big[\tilde{d}_i + \int_{t_0}^t e^{\sigma_i^{(j)}\,s}\,(\mathrm{P}_j\,\mathfrak{h}_j)_i(s)\,\mathrm{d}s\Big], \tag{5.27}$$

where $\tilde{d}_i\,(i=1,\cdots,k)$ are arbitrary constants. In order to ensure that $v_i(t)$ decays as $t\to+\infty$, we choose the initial value $v_i(t_0)$ as

$$\tilde{d}_i = e^{\sigma_i^{(j)}\,t_0} v_i(t_0) = -\int_{t_0}^{+\infty} e^{\sigma_i^{(j)}\,s}\,(\mathrm{P}_j\,\mathfrak{h}_j)_i(s)\,\mathrm{d}s.$$

Therefore, we get

$$\begin{aligned}|e^{(1+\sigma)\kappa_j t}\, v_i(t)| &= \left| - e^{(1+\sigma)\kappa_j t} e^{-\sigma_i^{(j)}\,t}\int_t^{+\infty} e^{\sigma_i^{(j)}\,s}\,(\mathrm{P}_j\,\mathfrak{h}_j)_i(s)\,\mathrm{d}s\right| \\ &\le e^{(1+\sigma)\kappa_j t} e^{-\sigma_i^{(j)}\,t}\|\mathfrak{h}_j\|_{1+\sigma}\int_t^{+\infty} e^{-(1+\sigma)\kappa_j s}\, e^{\sigma_i^{(j)}\,s}\,\mathrm{d}s \\ &\le \|\mathfrak{h}_j\|_{1+\sigma},\end{aligned}$$

where the following condition

$$-(1+\sigma)\kappa_j + \sigma_i^{(j)} \,<\, 0 \tag{5.28}$$

is needed to avoid the exponential growth while integrating. By the relation (2.25), we see that (5.28) is equivalent to

$$\sigma_i^{(j)} \,<\, (1+\sigma)\frac{3n}{n+2}\bar{\sigma}_j,$$

where $\sigma_i^{(j)}$ is the i-th eigenvalue of the matrix $\mathrm{A}(q_j)$, σ is a small positive number and $\bar{\sigma}_j := \sum\limits_{i=1}^n \frac{\sigma_i^{(j)}}{n}$. Observe that $\|e^{(1+\sigma)\kappa_j t}\, v_i(t)\|_{L^\infty(t_0,+\infty)} \lesssim \|\mathfrak{h}_j\|_{1+\sigma}$. Then we have $\|\dot{\xi}_j\|_{1+\sigma} \lesssim \|\mathfrak{h}_j\|_{1+\sigma}$.

Let $\Xi(t) = \dot{\xi}(t)$, which is a $n\times k$-dimensional vector function. Thus (5.26) defines a linear operator $\mathcal{L}_2 : \mathfrak{h}_j \to \Xi$, which associates to any $n\times k$-dimensional vector function $\mathfrak{h}_j$ with $\|\mathfrak{h}_j\|_{1+\sigma}$-bounded a solution (5.27) to equation (5.26). This operator is continuous in the Banach space equipped with $\|\cdot\|_{1+\sigma}$-topology.

After introducing the linear operators $\mathcal{L}_i, i=1,2$, we observe that $(\lambda(t),\xi(t))$ is a solution to (5.21) if $\big(\Lambda(t),\Xi(t)\big) := \big(\dot{\lambda}(t),\dot{\xi}(t)\big)$ is a fixed point for the problem

$$(\Lambda,\Xi) = \mathcal{A}_4\,(\Lambda,\Xi)\,, \tag{5.29}$$

where

$$\begin{aligned}&\mathcal{A}_4\,(\Lambda,\Xi) \,:=\, \Big(\mathcal{L}_1\big(\hat{\Pi}_1[\Lambda,\Xi,\phi]\big),\, \mathcal{L}_2\big(\hat{\Pi}_2[\Lambda,\Xi,\phi]\big)\Big) \,=\, \big(\bar{A}_1(\Lambda,\Xi),\, \bar{A}_2(\Lambda,\Xi)\big)\,, \\ &\hat{\Pi}_l(\Lambda,\Xi,\phi) = \Pi_l\Big[\int_t^\infty \Lambda, q + \int_t^\infty \Xi,\Lambda,\Xi,\phi\Big], \quad l=1,2,\end{aligned}$$

and Π_1, Π_2 are defined respectively in (5.13) and (5.20). Let

$$K := \max_{j=1,\cdots,k}\Big\{ \|\mathfrak{f}\|_{1+\sigma},\, \|\vec{f}_j\|_{1+\sigma}\Big\},$$

where the functions $\mathfrak{f}$, $\vec{f}_j$, $j=1,\cdots,k$ are given in (5.13) and (5.20) respectively. We show that the problem (5.29) has a fixed point (Λ,Ξ) in the following space

$$\mathcal{C}=\Big\{\,(\Lambda,\Xi)\in L^\infty(t_0,+\infty)\times L^\infty(t_0,+\infty)\,:\,\|\Lambda\|_{1+\sigma}+\|\Xi\|_{1+\sigma}\,\le\, c\,K\,\Big\}$$

for some large constant $c>0$.

Indeed, we observe directly from (5.13) and (5.20) that

$$|\mu_{0j}^{-(1+\sigma)}(t)\bar{A}_l(\Lambda,\Xi)|\;\lesssim\;\|\phi\|_{2+\sigma,n-5+a}+K+e^{-\varepsilon t_0}\|\Lambda\|_{1+\sigma}+e^{-\varepsilon t_0}\|\Xi\|_{1+\sigma},\quad l=1,2.$$

Thus, we have that $\mathcal{A}_4(\mathcal{C})\subset\mathcal{C}$.

For the Lipschitz condition for $\mathcal{A}_4$, we have

$$\begin{aligned}
&e^{(1+\sigma)\kappa_j t}\left|\bar{A}_1(\Lambda_1,\Xi)-\bar{A}_1(\Lambda_2,\Xi)\right|\\
&=e^{(1+\sigma)\kappa_j t}\left|\mathcal{L}_1\Big(\hat{\Pi}_1[\Lambda_1,\Xi,\phi]-\hat{\Pi}_1[\Lambda_2,\Xi,\phi]\Big)\right|\\
&\lesssim e^{(1+\sigma)\kappa_j t}e^{-\varepsilon t_0}\left|\mathcal{L}_1\Big(\Upsilon\Big(\Lambda_1,\Xi,\int_t^\infty\Lambda_1\Big)-\Upsilon\Big(\Lambda_2,\Xi,\int_t^\infty\Lambda_2\Big)\Big)\right|\\
&\lesssim e^{-\varepsilon t_0}\|\Lambda_1-\Lambda_2\|_{1+\sigma},
\end{aligned}$$

as a direct consequence of (5.14) and (5.16). By the same argument, one can get a similar estimate for $|\bar{A}_2(\Lambda_1,\Xi)-\bar{A}_2(\Lambda_2,\Xi)|$. Thus, we have

$$\|\mathcal{A}_4(\Lambda_1,\Xi_1)-\mathcal{A}_4(\Lambda_2,\Xi_2)\|_{1+\sigma}\,\lesssim\,e^{-\varepsilon t_0}\,\|\Lambda_1-\Lambda_2\|_{1+\sigma}+e^{-\varepsilon t_0}\,\|\Xi_1-\Xi_2\|_{1+\sigma}.$$

Since $e^{-\varepsilon t_0}$ is small when t_0 is large enough, by the contraction mapping theorem, there exists a solution $(\lambda(t),\xi(t))$ to the system of ODEs (5.21) with λ and ξ satisfying (4.8) and (4.9).

Next, we want to prove (5.22) and (5.23). Let $\phi^{(1)}$ and $\phi^{(2)}$ satisfy (4.11). The functions $\bar{\lambda}=\lambda[\phi^{(1)}]-\lambda[\phi^{(2)}]$ and $\bar{\xi}=\xi[\phi^{(1)}]-\xi[\phi^{(2)}]$ solve the system of ODEs for $j=1,\cdots,k$

$$\dot{\bar{\lambda}}_j+\kappa_j\bar{\lambda}_j=\big(\bar{\Pi}_1(t)\big)_j\,,\qquad \dot{\bar{\xi}}_j+\mathrm{A}(q_j)\bar{\xi}_j=\big(\bar{\Pi}_2(t)\big)_j\,,$$

where

$$\begin{aligned}
\big(\bar{\Pi}_1(t)\big)_j\;=\;&c_0^{-1}\,p\,\mu_j^{\frac{n-2}{2}}\mu_{0j}^{-1}\int_{B_{2R}(0)}\left|U\Big(\frac{\mu_{0j}}{\mu_j}y\Big)\right|^{p-1}\big(\psi[\phi^{(1)}]-\psi[\phi^{(2)}]\big)(\xi_j+\mu_{0j}y,t)Z_{n+1}(y)\,\mathrm{d}y\\
&+c_0^{-1}\left(\frac{\mu_j}{\mu_{0j}}\right)^{\frac{n+2}{2}}\mu_{0j}^{-1}\int_{B_{2R}(0)}\sum_{\ell=1}^{3}\big(B_j^\ell[(\phi^{(1)})_j]-B_j^\ell[(\phi^{(2)})_j]\big)\,Z_{n+1}(y)\,\mathrm{d}y
\end{aligned}$$

and $\big(\bar{\Pi}_2(t)\big)_j$ has the similar form as $\big(\bar{\Pi}_1(t)\big)_j$. Thus, the validity of (5.22) and (5.23) follows from (5.18). This completes the proof. $\square$

6. Solving the inner problem

After we get the outer solution ψ as in Proposition 4.1 and Proposition 4.2 and the parameters $\lambda=\lambda(\phi)$ and $\xi=\xi(\phi)$ as in Proposition 5.2, the last step in the proof of our result is to solve the inner problem (5.2).

Proposition 5.1 concludes the existence of a linear operator $\mathcal{T}$ which associates a solution to the linear problem (5.7) for any function $h(y,t)$ with $\|h\|_{\nu,n-3+a}$-bounded and satisfying orthogonality condition (5.9), where the norm $\|\cdot\|_{\nu,n-3+a}$ is defined in (5.8). Moreover, it states that $\mathcal{T}$ is continuous between Banach spaces equipped with the topologies described by (5.10)–(5.11). Thus, the existence and properties of solutions ϕ and e_0 to problem (5.2) are reduced to the fixed point problem

$$\phi = (\phi_1, \cdots, \phi_k) = \mathcal{A}_5(\phi) := \Big(\mathcal{T}(H_1[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \psi, \phi]), \cdots, \mathcal{T}(H_k[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \psi, \phi])\Big)$$

in a proper set of functions. We recall that H_j $(j = 1, \cdots, k)$ is defined in (5.3).

Next, we shall prove that $\mathcal{A}_5$ is a contraction mapping from $\mathcal{D}$ to $\mathcal{D}$, where

$$\mathcal{D} := \{\phi : \|\phi\|_{\nu,n-5+a} \le \Lambda e^{-\varepsilon t_0}\}$$

with $\Lambda > 0$ fixed sufficiently large, where

$$\|\phi\|_{\nu,n-5+a} := \sup_{\tau>\tau_0} \sup_{y\in B_{2R}(0)} \tau^\nu (1 + |y|^{n-5+a}) \left(|\phi(y, \tau)| + (1 + |y|)|\nabla\phi(y, \tau)|\right).$$

To this end, our strategies are the linear theory given in Proposition 5.1 and the contraction mapping theorem. Note that here the $\|\cdot\|_{\nu,n-5+a}$-norm is defined in (y, τ) variables, and in the (y, t) variables, it is the same as (4.10) since we choose $\nu = 1 + \frac{\sigma}{2}$.

We claim that, for each $j = 1, \cdots, k$,

$$|H_j[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \psi, \phi](y, t)| \le e^{-\varepsilon t_0} \frac{\mu_{0j}^{2+\sigma}(t)}{1 + |y|^{n-3+a}}, \tag{6.1}$$

for some $a \in (0, 1)$, and for $\phi^{(1)}, \phi^{(2)} \in \mathcal{D}$

$$\|H_j[\phi^{(1)}] - H_j[\phi^{(2)}]\|_{\nu,n-3+a} \le c\|\phi^{(1)} - \phi^{(2)}\|_{\nu,n-5+a} \tag{6.2}$$

with $0 < c < 1$ when t_0 is sufficiently large. We recall the definition of $\mathbf{S}_{\mu,\xi,j}(x, t)$ as in (2.31). Then we can easily get

$$\left|\mu_{0j}^{\frac{n+2}{2}} \mathbf{S}_{\mu,\xi,j}(\xi_j + \mu_{0j} y, t)\right| \lesssim e^{-\varepsilon t_0} \frac{\mu_{0j}^{2+\sigma}(t)}{1 + |y|^{n-3+a}}. \tag{6.3}$$

Since the outer solution $\psi \in \mathcal{B}$ with $\mathcal{B}$ defined in (4.15), we obtain that

$$p\,\mu_{0j}^{\frac{n-2}{2}} \frac{\mu_{0j}^2}{\mu_j^2} \left|U\Big(\frac{\mu_{0j}}{\mu_j} y\Big)\right|^{p-1} |\psi(\xi_j + \mu_{0j} y,\, t)| \lesssim e^{-\varepsilon t_0} \frac{\mu_{0j}^{2+\sigma}(t)}{1 + |y|^{n-3+a}}. \tag{6.4}$$

According to the definitions of $B_j^1[\phi_j]$, $B_j^2[\phi_j]$, $B_j^3[\phi_j]$ as in (3.12)–(3.14), we can get

$$|B_j^\ell[\phi_j]| \lesssim e^{-\varepsilon t_0} \frac{\mu_{0j}^{2+\sigma}(t)}{1 + |y|^{n-3+a}} \|\phi\|_{\nu,n-5+a}, \quad \text{for all } \ell = 1, 2, 3. \tag{6.5}$$

Combining (6.3), (6.4) and (6.5), we conclude the validity of (6.1) for $\Lambda > 0$ fixed large.

We next prove that the map $\mathcal{A}_5$ is a contraction mapping. We should emphasize the fact that ψ depends on ϕ in a nonlinear and nonlocal way, recalling that

$$\psi = \Psi[\lambda(\phi), \xi(\phi), \dot{\lambda}(\phi), \dot{\xi}(\phi), \phi].$$

We claim that there exists $c \in (0, 1)$ such that, for any $\phi^{(1)}, \phi^{(2)} \in \mathcal{D}$

$$\|\mathcal{A}_5(\phi^{(1)}) - \mathcal{A}_5(\phi^{(2)})\|_{\nu,n-5+a} \le c\|\phi^{(1)} - \phi^{(2)}\|_{\nu,n-5+a}.$$

From the definition of $\mu_{0j}^{\frac{n+2}{2}} \mathbf{S}_{\mu,\xi,j}$ in (2.31) and the Lipschitz dependence of λ and ξ on ϕ as in (5.22) and (5.23), we have

$$\left\|\mu_{0j}^{\frac{n+2}{2}} \left(\mathbf{S}_{\mu,\xi,j}[\phi^{(1)}] - \mathbf{S}_{\mu,\xi,j}[\phi^{(2)}]\right)\right\|_{\nu,n-3+a} \le c\left\|\phi^{(1)} - \phi^{(2)}\right\|_{\nu,n-5+a} \tag{6.6}$$

with $0 < c < 1$ when t_0 is sufficiently large.

Next, we consider the term

$$p\mu_{0j}^{\frac{n-2}{2}} \frac{\mu_{0j}^2}{\mu_j^2} \left|U\Big(\frac{\mu_{0j}}{\mu_j} y\Big)\right|^{p-1} \psi(\xi_j + \mu_{0j} y,\, t)$$

and we compute

$$
\begin{aligned}
&p\mu_{0j}^{\frac{n+2}{2}}\Bigg|\frac{1}{\mu_j^2[\phi^{(1)}]}U^{p-1}\Big(\frac{\mu_{0j}}{\mu_j[\phi^{(1)}]}y\Big)\psi[\phi^{(1)}](\xi_j[\phi^{(1)}]+\mu_{0j}y,t)\\
&\qquad-\frac{1}{\mu_j^2[\phi^{(2)}]}U^{p-1}\Big(\frac{\mu_{0j}}{\mu_j[\phi^{(2)}]}y\Big)\psi[\phi^{(2)}](\xi_j[\phi^{(2)}]+\mu_{0j}y,t)\Bigg| \qquad (6.7)\\
&\lesssim e^{-\varepsilon t_0}\frac{\mu_{0j}^{2+\sigma}(t)}{1+|y|^{n-3+a}}\|\phi^{(1)}-\phi^{(2)}\|_{\nu,n-5+a}.
\end{aligned}
$$

Finally, it follows from the definitions (3.12), (3.13) and (3.14) respectively that

$$
\left|B_j^\ell[\phi^{(1)}]-B_j^\ell[\phi^{(2)}]\right|\lesssim e^{-\varepsilon t_0}\frac{\mu_{0j}^{2+\sigma}(t)}{1+|y|^{n-3+a}}\left\|\phi^{(1)}-\phi^{(2)}\right\|_{\nu,n-5+a},\quad \ell=1,2,3. \qquad (6.8)
$$

Therefore, by estimates (6.6)–(6.8) and Proposition 5.1, we obtain (6.2).

By (6.1), (6.2) and the contraction mapping theorem, we conclude that $\mathcal{A}_5$ has a fixed point ϕ satisfying $\|\phi\|_{\nu,n-5+a}<ce^{-\varepsilon t_0}$. The proof is complete. □

The stability part in Theorem 1 is similar to that of [9]. Here we omit it.

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