AARMS Summer School Pre-requisites

References:

[1] Chapter 5 of Gilbarg-Trudinger's book on "Elliptic Partial Differential Equations of Second Order"

[2] Appendix of Wei-Winter's book on "Mathematical Aspects of Pattern Formation in Biological Systems", Applied Mathematical Sciences Series, Vol. 189, Springer 2014 , ISBN: 978-4471-5525-6.

Sobolev spaces and linear operators are important tools throughout this monograph. Therefore we state their definition and most important results here. For a more detailed discussion we refer to the excellent book by Gilbarg-Trudinger.

Let Ω be a bounded, open, smooth domain in $\mathcal{R}^n$, where $n \geq 1$. For $p \geq 1$, let $L^p(\Omega)$ denote the Lebesgue space consisting of measurable functions defined on Ω such that

$$\|u\|_p := \|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \, dx\right)^{1/p} < \infty.$$

Then $L^p(\Omega)$ is a Banach space with the norm $\|u\|_p$. Further, the space $L^2(\Omega)$ is a Hilbert space with the scalar product

$$(u, v) := \int_\Omega uv \, dx.$$

For $k = 1, 2, \dots,$ we define

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) \, : \, D^\alpha u \in L^p(\Omega) \quad \text{for all } |\alpha| \leq k\},$$

where

$$\alpha = (\alpha_1, \dots, \alpha_n), \, \alpha_i \in \{0, 1, \dots\}, \, |\alpha| = \sum_{i=1}^n \alpha_i,$$

$$D^\alpha u := \frac{\partial^{|\alpha|} u}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}}.$$

We also denote $H^k(\Omega) := W^{k,2}(\Omega)$, and $H^k(\Omega)$ is a Hilbert space with the scalar product

$$(u, v)_k := \int_\Omega \sum_{|\alpha| \leq k} D^\alpha u D^\alpha v \, dx.$$

A Banach space $\mathcal{B}_\infty$ is said to be continuously embedded in a Banach space $\mathcal{B}_\in$ if there exists a bounded, linear, one-to-one mapping of $\mathcal{B}_\infty$ into $\mathcal{B}_\in$. Using the notation $\mathcal{B}_\infty \to \mathcal{B}_\in$, we have the following continuous Sobolev embeddings:

$$W_0^{k,p}(\Omega) \to L^{1/(1/p-k/n)}(\Omega) \quad \text{for } kp < n,$$

$$W_0^{k,p}(\Omega) \to C^m(\bar{\Omega}) \quad \text{for } 0 \leq m < k - \frac{n}{p},$$

where $W_0^{k,p}(\Omega)$ is the Banach space which is given by the closure of $C_0^k(\Omega)$ in $W^{k,p}(\Omega)$. Here $C_0^k(\Omega)$ is the set of continuous functions u defined in Ω with compact support in Ω for which also the partial derivatives $D^\alpha u$, $|\alpha| \leq k$ are continuous. Further, $C^k(\bar{\Omega})$ is the set of all functions in $C^k(\Omega)$ for which all derivatives $D^\alpha u$, $|\alpha| \leq k$ have continuous extensions to the closure $\bar{\Omega}$ of Ω. The space $C^k(\Omega)$ of functions which together with all derivatives up to order k is continuous, is a Banach space if it is endowed with the norm

$$\|u\|_{k,\infty} := \max_{|\alpha| \leq k}$$

Next we present two elliptic regularity theorems.

Theorem 1 *(Elliptic regularity-L^ptheory.) Let $u \in W^{2,p}(\Omega)$ solve the equation*

$$-\Delta u = f - \bar{f} \quad \text{in } \Omega,$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

where $\bar{f} = \frac{1}{|\Omega|} \int_\Omega f(x)\, dx$.

Assume that $f \in L^p(\Omega)$. Then there exists some $c > 0$ such that

$$\|u - \bar{u}\|_{2,p} \leq c\|f - \bar{f}\|_p. \tag{0.1}$$

Theorem 2 *(Elliptic regularity-Schauder Estimates.) Let u solve the equation*

$$-\Delta u = f \quad \text{in } \Omega.$$

Assume that $f \in C^\alpha(\Omega)$. Then there exists some $c > 0$ such that

$$\|u\|_{C^{2,\alpha}(B_r(x_0))} \leq c(\|f\|_{C^\alpha(B_{2r}(x_0)} + \|u\|_{C^\alpha(B_{2r}(x_0))}). \tag{0.2}$$

for any $B_{2r}(x_0) \subset \Omega$.

A map T from a normed linear space V into itself is called a contraction mapping if there exists $\theta < 1$ such that

$$\|Tx - Ty\| \leq \theta \|x - y\|, \forall x, y \in V$$

Contraction Mapping Principle states

Theorem 3 *A contraction mapping T in a Banach space V has a unique fixed point that is there exists a unique solution $x \in V$ such that $x = Tx$.*

The Fredholm Alternative holds for compact linear operators from a linear space into itself.

Theorem 4 *(Fredholm Alternative) A linear mapping T of a normed linear space into itself is called compact if L maps bounded sequences into sequences which contain converging subsequences. Let T be a compact linear mapping of a normed linear space L into itself. Then either (i) the homogeneous equation*

$$x - Tx = 0$$

has a nontrivial solution $x \in L$ or
(ii) for each $y \in L$ the equation

$$x - Tx = y$$

has a uniquely determined solution $x \in L$. Further, in case (ii) the "solution operator" $(I - T)^{-1}$ is bounded.

An example of compact operator is

$$T(f) = \int_{\Omega} G(x, y) f(y) dy$$

where $G(x, y)$ is the Green's function of $-\Delta$.

Next, let us state Brouwer's Fixed Theorem:

Theorem 5 *(Brouwer's Fixed Point Theorem.) Every continuous function from a closed ball of a Euclidean space to itself has a fixed point.*

Finally, we recall the mapping degree (see [**?**]). If $\Omega \subset \mathcal{R}^n$ is a bounded region, $f : \bar{\Omega} \to \mathcal{R}^n$ smooth, p a regular value of f and $p \notin f(\partial\Omega)$, then the degree $\deg(f, \Omega, p)$ is defined as follows:

$$\deg(f, \Omega, p) := \sum_{y \in f^{-1}(p)} \operatorname{sign} \det Df(y),$$

where $Df(y)$ is the Jacobi matrix of f in y. This definition of degree may be naturally extended to non-regular values p such that $\deg(f, \Omega, p) = \deg(f, \Omega, p')$, where p' is a point close to p.

The degree satisfies the following five properties and is uniquely characterised by them.

(i) If $\deg(f, \bar{\Omega}, p) \neq 0$, then there exists $x \in \Omega$ such that $f(x) = p$.

(ii) $\deg(Id, \Omega, y) = 1$ for all $y \in \Omega$.

(iii) *Decomposition property:*

$$\deg(f, \Omega, y) = \deg(f, \Omega_1, y) + \deg(f, \Omega_2, y),$$

where $\Omega_1 \cap \Omega_2 = \emptyset$, $\Omega = \Omega_1 \cup \Omega_2$ and $y \notin f(\bar{\Omega} \setminus (\Omega_1 \cup \Omega_2))$.

(iv) *Homotopy invariance:*

If f and g are homotopy equivalent via a continuous homotopy $F(t)$ such that $F(0) = f$, $F(1) = g$ and $p \notin F(t)(\partial\Omega)$ for all $0 < t < 1$, then $\deg(f, \Omega, p) = \deg(g, \Omega, p)$.

(v) The function $p \mapsto \deg(f, \Omega, p)$ is locally constant on $\mathcal{R}^n \setminus f(\partial\Omega)$.