**Course 2 - Homework Assignment 1 Solution**

1. (a) Using the equation

$$w'' = w - w^p$$

and its first integral

$$(w')^2 = w^2 - \frac{2}{p+1} w^{p+1},$$

we have

$$\begin{aligned}\phi_1'' &= \frac{p+1}{2}\left(w^{\frac{p-1}{2}} w'' + \frac{p-1}{2} w^{\frac{p-3}{2}} (w')^2\right) \\ &= \frac{p+1}{2}\left(w^{\frac{p+1}{2}} - w^{\frac{3p-1}{2}} + \frac{p-1}{2} w^{\frac{p+1}{2}} - \frac{p-1}{p+1} w^{\frac{3p-1}{2}}\right) \\ &= \left(\frac{p+1}{2}\right)^2 w^{\frac{p+1}{2}} - p w^{\frac{3p-1}{2}}.\end{aligned}$$

Hence,

$$L_0(\phi_1) = \left(\left(\frac{p+1}{2}\right)^2 - 1\right) w^{\frac{p+1}{2}} = \lambda_1 \phi_1.$$

Since the first eigenfunction is necessarily positive and other eigenfunctions are orthogonal to it, $\phi_1$ is the principal eigenfunction and $\lambda_1$ is the principal eigenvalue.

(b) From (a),

$$L_0(w^{r-1}) = \lambda_1 w^{r-1},$$

so

$$L_0^{-1}(w^{r-1}) = \lambda_1^{-1} w^{r-1}.$$

(c) It is clear that

$$\int w^{r-1} L_0^{-1}(w^{r-1}) = \frac{1}{\lambda_1} \int w^{p+1}.$$

2. First we show that $w$ decays exponentially. In fact since $w(\infty) = 0$ we see that for $r > r_1$ $w^{p-1} < \frac{3}{4}$ and hence for $r > r_1$ $w$ satisfies

$$\Delta w = (1 - w^{p-1}) w \geq \frac{1}{4} w$$

Next we note that the function $e^{-\frac{1}{2}r}$ satisfies

$$\Delta(e^{-\frac{1}{2}r}) \leq \frac{1}{4} e^{-\frac{1}{2}r}$$

By comparison principle for $r > r_1$

$$w(r) \leq w(r_1) e^{-\frac{1}{2}(r - r_1)}$$

Similarly since

$$\Delta w_r - w_r + p w^{p-1} w_r = \frac{N-1}{r^2} w_r$$

and $pw^{p-1}$ is small near $\infty$, $w_r$ decays exponentially at $\infty$. This fact implies that when we integrate by parts below, the boundary terms all vanish.

Following the hint and integrating on $(0, \infty)$, we have

$$\begin{aligned} 0 &= \int r w_r (r^{N-1} w_r)_r + \int r^N w_r(-w + w^p) \\ &= -\int r^{N-1} w_r (r w_{rr} + w_r) + \int r^N \left( -\frac{w^2}{2} + \frac{w^{p+1}}{p+1} \right)_r \\ &= -\int r^N \left( \frac{w_r}{2} \right)_r - \int r^{N-1} w_r^2 + \int N r^{N-1} \left( \frac{w^2}{2} - \frac{w^{p+1}}{p+1} \right) \\ &= \left( \frac{N}{2} - 1 \right) \int r^{N-1} w_r^2 + \frac{N}{2} \int r^{N-1} w^2 - \frac{N}{p+1} \int r^{N-1} w^{p+1}, \end{aligned}$$

and

$$\begin{aligned} 0 &= \int w (r^{N-1} w_r)_r + \int r^{N-1} w(-w + w^p) \\ &= -\int r^{N-1} w_r^2 - \int r^{N-1} w^2 + \int r^{N-1} w^{p+1}. \end{aligned}$$

Multiplying the second by $(N/2 - 1)$ and adding it to the first, we have

$$\begin{aligned} 0 &= \int r^{N-1} w^2 + \left( \frac{N}{2} - 1 - \frac{N}{p+1} \right) \int r^{N-1} w^{p+1} \\ &= \int r^{N-1} w^2 + \frac{(N-2)p - (N+2)}{2(p+1)} \int r^{N-1} w^{p+1}. \end{aligned}$$

Therefore, if $p \geq \frac{N+2}{N-2}$, then the right hand side is strictly positive, unless $w \equiv 0$.

3. Let

$$f(t) = \int (|\nabla u + t\phi|^2 + (u + t\phi)^2)$$

and

$$g(t) = \left( \int |u + t\phi|^{p+1} \right)^{-\frac{2}{p+1}}.$$

We compute

$$
\begin{aligned}
f'(t) &= 2\int(\nabla u\cdot\nabla\phi + u\phi) + 2t\int(|\nabla\phi|^2+\phi^2),\\
g'(t) &= -2\frac{\int|u+t\phi|^{p-1}(u+t\phi)\phi}{\left(\int|u+t\phi|^{p+1}\right)^{\frac{2}{p+1}+1}},\\
f(0) &= \int(|\nabla u|^2+u^2),\\
f'(0) &= 2\int(\nabla u\cdot\nabla\phi+u\phi),\\
f''(0) &= 2\int(|\nabla\phi|^2+\phi^2),\\
g(0) &= \left(\int|u|^{p+1}\right)^{-\frac{2}{p+1}},\\
g'(0) &= -2\frac{\int|u|^{p-1}u\phi}{\left(\int|u|^{p+1}\right)^{\frac{2}{p+1}+1}},\\
g''(0) &= 2(p+3)\frac{\left(\int|u|^{p-1}u\phi\right)^2}{\left(\int|u|^{p+1}\right)^{\frac{2}{p+1}+2}} - 2p\frac{\int|u|^{p-1}\phi^2}{\left(\int|u|^{p+1}\right)^{\frac{2}{p+1}+1}}.
\end{aligned}
$$

The first derivative is therefore

$$
\rho'(0) = 2\left(\int|u|^{p+1}\right)^{-\frac{2}{p+1}}\left(\left(\int(\nabla u\cdot\nabla\phi+u\phi)\right) - c_1\left(\int|u|^{p-1}u\phi\right)\right)
$$

where

$$
c_1 = \frac{\int(|\nabla u|^2+u^2)}{\int|u|^{p+1}}.
$$

Since $u$ is the minimizer of $E[u]$, we have $\rho'(0)$ and so

$$
\int(-\Delta u + u - c_1|u|^{p-1}u)\phi = 0
$$

for any test function $\phi$, from which we get the Euler-Lagrange equation

$$
\Delta u - u + c_1|u|^{p-1}u = 0.
$$

Before computing $\rho''(0)$, we wish to simply the calculations by choose $c_1 = 1$ by a scaling argument. In fact, $E[u]$ is scaling invariant, i.e. $E[u] = E[\lambda u]$ for any $\lambda > 0$. By going through the whole argument with $\lambda u$ instead, the Euler-Lagrange equation is

$$
\lambda\Delta u - \lambda u + c_1\lambda^p|u|^{p-1}u = 0.
$$

If we choose $\lambda$ such that $c_1\lambda^{p-1} = 1$, then we could have assumed, without loss of generality, that $c_1 = 1$. Note also the consequence

$$
\int(\nabla u\cdot\nabla\phi+u\phi) = \int|u|^{p-1}u\phi.
$$

Finally, we compute the second derivative by

$$
\begin{aligned}
f''(0)g(0) &= 2\frac{\int(|\nabla\phi|^2+\phi^2)}{\left(\int|u|^{p+1}\right)^{\frac{2}{p+1}}},\\
2f'(0)g'(0) &= -8\frac{\left(\int|u|^{p-1}\,u\phi\right)^2}{\left(\int|u|^{p+1}\right)^{\frac{2}{p+1}+1}},\\
f(0)g''(0) &= 2(p+3)\frac{\left(\int|u|^{p-1}\,u\phi\right)^2}{\left(\int|u|^{p+1}\right)^{\frac{2}{p+1}+1}} - 2p\frac{\int|u|^{p-1}\,\phi^2}{\left(\int|u|^{p+1}\right)^{\frac{2}{p+1}}},\\
\rho''(0) &= f''(0)g(0)+2f'(0)g'(0)+f(0)g''(0)\\
&= 2\left(\int|u|^{p+1}\right)^{-\frac{2}{p+1}}\left(\int(|\nabla\phi|^2+\phi^2) - p\int|u|^{p-1}\,\phi^2 + (p-1)\frac{\left(\int|u|^{p-1}\,u\phi\right)^2}{\left(\int|u|^{p+1}\right)^{\frac{2}{p+1}+1}}\right).
\end{aligned}
$$

4. (a)

$$
\begin{aligned}
H(t) &= \tanh\left(\frac{t}{\sqrt{2}}\right)\\
H'(t) &= \frac{1}{\sqrt{2}}\operatorname{sech}^2\left(\frac{t}{\sqrt{2}}\right)\\
H''(t) &= \operatorname{sech}^2\left(\frac{t}{\sqrt{2}}\right)\tanh\left(\frac{t}{\sqrt{2}}\right)\\
&= \left(1-\tanh^2\left(\frac{t}{\sqrt{2}}\right)\right)\tanh\left(\frac{t}{\sqrt{2}}\right)\\
&= H(t)-H^3(t)
\end{aligned}
$$

(b) Write $\rho(t)=E[u+t\phi]$.

$$
\begin{aligned}
\rho'(0) &= \int\left(\nabla u\cdot\nabla\phi + \frac{1}{2}(1-u^2)(-2u\phi)\right)\\
&= \int(-\Delta u - u + u^3)\phi.
\end{aligned}
$$

Hence the result follows.

(c) For steady states $H''=0$, so

$$
\begin{aligned}
-H + \frac{H^2}{1+aH^2} &= 0\\
-H(1+aH^2-H) &= 0\\
h_\pm &= \frac{1\pm\sqrt{1-4a}}{2a}.
\end{aligned}
$$

Now we compute the integral

$$\begin{aligned} F(h_+) &= \int_0^{h_+} \left( -H + \frac{H^2}{1+aH^2} \right) dH \\ &= \int_0^{h_+} \left( -H + \frac{1}{a} - \frac{1}{a(1+aH^2)} \right) dH \\ &= -\frac{h_+^2}{2} + \frac{h_+}{a} - \frac{1}{a\sqrt{a}} \tan^{-1}(\sqrt{a} h_+). \end{aligned}$$

Since $ah_+^2 = h_+ - 1$, the equation for which the integral is zero is

$$-\frac{h_+ - 1}{2} + h_+ - \frac{1}{\sqrt{a}} \tan^{-1}(\sqrt{a} h_+) = 0,$$

or

$$\frac{1}{\sqrt{a}} \tan^{-1}(\sqrt{a} h_+) = \frac{h_+ + 1}{2}.$$

For $a = a_0$, we integrate the equation as follows.

$$\frac{(H')^2}{2} + F(H) = 0$$
$$H' = \pm\sqrt{-2F(H)}$$

An implicit solution is given by

$$\begin{aligned} t &= \int_0^t \frac{H'}{\sqrt{-2F(H)}} \, dt \\ &= \int_{H(0)}^{H(t)} \frac{ds}{\sqrt{-2F(s)}}. \end{aligned}$$

5. (a) If $u(x) = H(a \cdot x + b)$, then $\Delta u = |a|^2 H''(a \cdot x + b)$. Therefore the condition is $|a| = 1$.

   (b) This is the same as 4(b).

   (c) Since $|a| = 1$, there is a $j$ such that $a_j \neq 0$. Let $\psi = \dfrac{\partial u}{\partial x_j} = a_j H'(a \cdot x + b) \neq 0$, which satisfies the equation $\Delta\psi = (3u^2 - 1)\psi$. Let $\phi$ be any smooth function with compact support. Testing the linearized equation with $\phi^2/\psi$ (tricky!), we have

$$\begin{aligned} \int (3u^2 - 1)\phi^2 &= \int \frac{\phi^2 \Delta\psi}{\psi} \\ &= -\int \nabla\psi \cdot \nabla \left( \frac{\phi^2}{\psi} \right) \\ &= -\int \nabla\psi \cdot \left( \frac{2\psi\phi\nabla\phi - \phi^2\nabla\psi}{\psi^2} \right) \end{aligned}$$

Notice that the integration by parts is justified by the (exponential) decay of $\psi$ near $\infty$. Now we complete the trick by

$$\begin{aligned}\int(|\nabla\phi|^2 + (3u^2-1)\phi^2) &= \int\left(|\nabla\phi|^2 - 2\nabla\phi\cdot\frac{\phi\nabla\psi}{\psi} + \frac{\phi^2\,|\nabla\psi|^2}{\psi^2}\right) \\ &= \int\left|\nabla\phi - \frac{\phi\nabla\psi}{\psi}\right|^2 \\ &\geq 0\end{aligned}$$

If equality holds, then

$$\nabla\phi - \frac{\phi\nabla\psi}{\psi} \equiv 0.$$

This implies

$$\nabla\left(\frac{\phi}{\psi}\right) = \frac{\psi\nabla\phi - \phi\nabla\psi}{\psi^2} \equiv 0,$$

which is impossible unless $\phi \equiv 0$ since $\psi$ does not have a compact support.