AARMS SUMMER SCHOOL–LECTURE VII: INTRODUCTION TO INFINITE DIMENSIONAL REDUCTION METHODS FOR SOLVING PDE'S

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1. Back to Allen Cahn in $\mathbb{R}^2$

We consider the functional

$$J(u)=\int_{\mathbb{R}^2}\left(\varepsilon^2\frac{|\nabla u|^2}{2}+\frac{(1-u^2)^2}{4}\right)a(x)dx.$$

Critical points of J are solutions of

$$\varepsilon^2\mathrm{div}(a(x)\nabla u)+a(x)(1-u^2)u=0,$$

where we suppose $0<\alpha\le a(x)\le\beta$. This equation is equal to

$$\varepsilon^2\Delta u+\varepsilon^2\frac{\nabla a}{a}(x)\nabla u+(1-u^2)u=0. \tag{1.1}$$

Using the change of variables $v(x)=u(\varepsilon x)$, we find the equation

$$\Delta v+\varepsilon\frac{\nabla a}{a}(x)\nabla v+(1-v^2)v=0. \tag{1.2}$$

We will study the problem: Given a curve Γ in $\mathbb{R}^2$ we want to find a solution $u_\varepsilon(x)$ to (1.1) such that $u_\varepsilon(x)\approx w(\frac{z}{\varepsilon})$, for points $x=y+z\nu(y)$, $y\in\Gamma$, $|z|<\delta$, where $\nu(y)$ is a vector perpendicular to the curve and $w(t)=\tanh(\frac{t}{\sqrt{2}})$, which solves the problem

$$w''+(1-w^2)w=0,\quad w(\pm\infty)=\pm1.$$

First issue: Laplacian near Γ, which we will consider as smooth as we need.

Assume: Γ is parametrized by arc-length

$$\gamma:[0,l]\to\mathbb{R}^2,\ s\to\gamma(s),\ |\dot\gamma(s)|=1, l=|\Gamma|.$$

Convention: $\nu(s)$ inner unit normal at $\gamma(s)$. We have that $|\nu(s)|^2=1$, which implies that $2\nu\dot\nu=0$, so we take $\dot\nu(s)=-k(s)\dot\gamma(s)$, where $k(s)$ is the curvature.

Coordinates: $x(s,t)=\gamma(s)+z\nu(s)$, $s\in(0,l)$ and $|z|<\delta$. If we take a compact supported function $\psi(x)$ near Γ, and we call $\tilde\psi(s,z)=\psi(\gamma(s)+z\nu(s))$, then $\frac{\partial\tilde\psi}{\partial s}=\nabla\psi\cdot[\dot\gamma+z\dot\nu]=(1-kz)\nabla\psi\cdot\dot\gamma$ and $\frac{\partial\tilde\psi}{\partial t}=$

$\nabla\psi\cdot\nu$. Observe that $\nabla\psi = (\nabla\psi\cdot\dot\gamma)\dot\gamma(\nabla\cdot\nu)\nu$. This means that $\nabla\psi = \frac{1}{1-kz}\frac{\partial\tilde\psi}{\partial s}\dot\gamma + \frac{\partial\tilde\psi}{\partial z}\nu$, and $|\nabla\psi|^2 = \frac{1}{(1-kz)^2}|\tilde\psi_s|^2 + |\tilde\psi_z|^2$. Then

$$\int_{\mathbb{R}^2} |\nabla\psi(x)|^2 dx = \iint \left(\frac{1}{(1-kz)^2}|\tilde\psi_s|^2 + |\tilde\psi_z|^2\right)(1-kz)dsdz$$

$\psi \to \psi + t\varphi$ and differentiating at $t=0$ we get

$$\int \nabla\psi\nabla\varphi dx = \iint \frac{1}{(1-kz)}\tilde\psi_s\tilde\varphi_s + \tilde\psi_z\tilde\varphi_z(1-kz)dsdz$$

So

$$-\int \Delta\psi\varphi dx = -\iint \frac{1}{(1-kz)}\left(\left(\frac{1}{(1-kz)}\tilde\psi_s\right)_s + (\tilde\psi_z(1-lz))_z\right)\tilde\varphi(1-kz)dsdz$$

then

$$\Delta\tilde\psi = \frac{1}{(1-kz)}\frac{\partial}{\partial s}(\frac{1}{1-kz}\tilde\psi_s) + \tilde\psi_{zz} - \frac{k}{1-kz}\tilde\psi_z$$

We just say

$$\Delta\tilde\psi = \frac{1}{1-kz}(\frac{1}{1-kz}\psi_s)_s + \psi_{zz} - \frac{k}{1-kz}\psi_z$$

Near Γ $(x = \gamma(s) + z\nu(s))$, we have the new equation for $u \to \tilde u(s,z)$

$$S[u] = \varepsilon^2\frac{1}{1-kz}(\frac{1}{1-kz}u_s)_s + \varepsilon^2 u_{zz} + (1-u^2)u - \frac{\varepsilon^2 k}{1-kz}u_z + \frac{\varepsilon^2}{1-kz}\frac{a_s}{a}u_s + \frac{\varepsilon^2}{1-kz}\frac{a_z}{a}u_z = 0$$

we want a solution $u(s,z) \approx w(\frac{z}{\varepsilon})$.

$$S[w(\frac{z}{\varepsilon})] = \varepsilon[\frac{a_z}{a} - \frac{k(s)}{1-k(s)z}]w'(\frac{z}{\varepsilon})$$

The condition we ask (geodesic condition) is $\frac{a_z}{a}(s,0) = k(s)$. In v language we want

$$\Delta v + \varepsilon\frac{\nabla a}{a}(\varepsilon x)\cdot\nabla v + f(v) = 0$$

transition on $\Gamma_\varepsilon = \frac{1}{\varepsilon}\Gamma$. we use coordinates relative to Γ_ε rather than Γ

$$X_\varepsilon(s,z) = \frac{1}{\varepsilon}\gamma(\varepsilon s) + z\nu(\varepsilon s), \quad |z| < \delta/\varepsilon$$

Laplacian for coordinates relative to Γ_ε are

$$\Delta\psi = \frac{1}{(1-\varepsilon k(\varepsilon s)z)}\left(\frac{1}{(1-\varepsilon k(\varepsilon s)z)}v_s\right)_s + \psi_{zz} - \frac{\varepsilon k(\varepsilon s)}{(1-\varepsilon k(\varepsilon s)z)} + \varepsilon\frac{a_s}{a}\frac{1}{(1-\varepsilon k(\varepsilon s)z)^2}v_s + \varepsilon\frac{a_z}{a}v_z +$$

where we use the computation $\frac{\partial\gamma(\varepsilon s)}{\partial s} = -k(\varepsilon)\dot\gamma_\varepsilon(s)$, where $k_\varepsilon = \varepsilon k(\varepsilon s)$

Hereafter we use $\tilde{s}$ instead of s and $\tilde{z}$ instead of $\tilde{z}$. Observation: The operator is closed to the Laplacian on $(\tilde{s}, \tilde{z})$ variables, at least on the curve Γ, if we assume the validity of the relation

$$a_{\tilde{z}}(\tilde{s}, 0) = k(\tilde{s})a(\tilde{s}, 0), \quad \forall \tilde{s} \in (0, l).$$

We can write this relation also like $\partial_\nu a = ka$ on Γ (Geodesic condition). This relation means that Γ is a critical point of curve length weighted by a. Let $L_a[\Gamma] = \int_\Gamma adl$. Consider a normal perturbation of Γ, say $\Gamma_h := \{\gamma(\tilde{s}) + h(\tilde{s})\nu(\tilde{s}) | \tilde{s} \in (0, l)\}$, $\|h\|_{C^2(\Gamma)} \ll 1$. We want: first variation along this type of perturbation be equal to zero. This is

$$DL_a[\Gamma_h]|_{h=0} = 0$$

This means

$$\frac{\partial}{\partial \lambda} L[\Gamma_{\lambda h}]|_{h=0} = 0$$

or just $\langle DL(\Gamma), h\rangle = 0$ for all h. Observe that

$$L(\Gamma_{\lambda h}) = \int_0^l a(\gamma(\tilde{s}) + h(\tilde{s})\nu(\tilde{s})) \cdot |\dot{\gamma}(\tilde{s})_{\lambda h}| d\tilde{s}$$

and also $\dot{\gamma}_{\lambda h}(\tilde{s}) = \dot{\gamma}(\tilde{s}) + \lambda \dot{h}\nu + \lambda h\dot{\nu}$, and $\dot{\nu} = -k\dot{\gamma}$. With the taylor expansion

$$(1-2k\lambda h+\lambda^2 k^2 h^2+\lambda^2 \dot{h}^2)^{1/2} = 1+\frac{1}{2}(-2k\lambda h+\lambda^2 k^2 h^2+\lambda^2 \dot{h}^2)-\frac{1}{8}4k^2\lambda^2 h^2+O(\lambda^2 h^3)$$

and

$$a(\gamma(\tilde{(s)})+\lambda h(\tilde{s}\nu(\tilde{s})) = a(\tilde{s}, \lambda h(\tilde{s})) = a(\tilde{s}, 0)+\lambda a_{\tilde{z}}(\tilde{s}, 0)h(\tilde{s})+\frac{1}{2}\lambda^2 a_{\tilde{z}\tilde{z}}(\tilde{s}, 0)h(\tilde{s})^2+O(\lambda^3 h^3).$$

we conclude

$$L_h[\Gamma_{\lambda h}] = L_a(\Gamma) = \lambda \int_0^l (-ka+a_{\tilde{z}})(\tilde{s}, 0)h(\tilde{s})d\tilde{s}+\lambda^2 \int_0^l (a\frac{\dot{h}^2}{2}+a_{\tilde{z}}k^2 h^2+\frac{1}{2}a_{\tilde{z}\tilde{z}}h^2)+O(\lambda^3 h^3)$$

This tells us:

$$\frac{\partial}{\partial \lambda} L_h[\Gamma_{\lambda h}]|_{\lambda=0} = 0 \Leftrightarrow k(\tilde{s})a(\tilde{s}, 0) = a_{\tilde{z}}(\tilde{s}, 0),$$

the geodesic condition. Also we conclude that

$$\frac{\partial^2}{\partial \lambda^2} L(\Gamma_{\lambda h})|_{\lambda=0} = \int_0^l (a\dot{h}^2-2k^2 a+a_{\tilde{z}\tilde{z}}h^2)d\tilde{s} = -\int_0^l (a(\tilde{s}, 0)\dot{h}\tilde{s})' h+(2a(\tilde{s}, 0)k^2-a_{\tilde{z}\tilde{z}}(\tilde{s}, 0)h)h$$

This can be expressed as $D^2L(\Gamma) = J_a$, which means $D^2L(\Gamma)[h]^2 = -\int_0^l J_a[h]h$. $J_a[h]$ is called the Jacobi operator of the geodesic Γ. Assumption: J_a is invertible.

We assume that if $h(\tilde{s})$, $\tilde{s} \in (0,l)$ is such that $h(0)=h(l)$, $\dot{h}(0)=\dot{h}(l)$ and $J_a[h]=0$ then $h\equiv 0$. $Ker(J_a)=\{0\}$, in the space of $l-$periodic C^2 functions. This implies (exercise) that the problem

$$J_a[h]=g, g\in C(0,l), g(0)=g(l), h(0)=h(l), \dot{h}(0)=\dot{h}(l)$$

has a unique solution ϕ. Moreover $\|\phi\|_{C^{2,\alpha}(0,l)} \le C\|g\|_{C^\alpha(0,l)}$.

Remember that the equation in coordinates (s,z) is

$$E(v)=\frac{1}{(1-\varepsilon k(\varepsilon s)z)}\left(\frac{1}{(1-\varepsilon k(\varepsilon s)z)}v_s\right)_s+v_{zz}-\frac{\varepsilon k(\varepsilon s)}{(1-\varepsilon k(\varepsilon s)z)}v_z+$$

$$\varepsilon\frac{a_{\tilde{s}}}{a}\frac{1}{(1-\varepsilon k(\varepsilon s)z)^2}v_s+\varepsilon\frac{a_{\tilde{z}}}{a}v_z+f(v)=0$$

Change of variables: Fix a function $h\in C^{2,\alpha}(0,l)$ with $\|h\|\le 1$ and do the change of variables $z-h(\varepsilon s)=t$ and take as first approximation $v_0\equiv w(t)$. Let us see that $v_0(s,z)=w(z-h(\varepsilon s))$ so

$$E(v_0)=\frac{1}{1-\varepsilon kz}\left(\frac{1}{1-\varepsilon kz}w'(-\dot{h}(\varepsilon s,\varepsilon z)\right)_s+w''+f(w)$$

$$+\varepsilon\left(\frac{a_{\tilde{z}}}{a}(\varepsilon s,\varepsilon z)-\frac{k(\varepsilon s)}{1-k(\varepsilon s)\varepsilon z}\right)w'-\varepsilon\dot{h}\frac{\varepsilon}{(1-\varepsilon kz)^2}\frac{a_{\tilde{s}}}{a}w'$$

Error in terms of coordinates (s,t) $z=t+h(\varepsilon s)$:

$$E(v_0)(s,t)=\varepsilon w'(t)\left[\frac{a_{\tilde{z}}}{a}(\varepsilon s,\varepsilon(t+h))-\frac{k(\varepsilon s)}{1-k(\varepsilon s)(t+h)\varepsilon}\right]-\frac{\varepsilon^2 w'}{(1-k\varepsilon(t+h))^2}h''$$

$$+\frac{1}{(1-k\varepsilon(t+h))^2}w''\dot{h}^2\varepsilon^2-\frac{1}{(1-\varepsilon k(t+h))^3}\varepsilon^2\dot{k}(t+h)\dot{h}w'(t)-\varepsilon\dot{h}\frac{\varepsilon}{(1-\varepsilon kz)^2}\frac{a_{\tilde{s}}}{a}w'$$

In fact

$$|E(v_0)(t,s)|\le C\varepsilon^2 e^{-\sigma|t|}$$

$\sigma<1$, and

$$\|e^{\sigma|t|}E(v_0)\|_{C^{0,\alpha}(|t|<\frac{\delta}{\varepsilon})}\le C\varepsilon^2$$

Formal computation: We would like $\int_{-\delta/\varepsilon}^{\delta/\varepsilon}E(v_0)(s,y)w'(t)dt\approx 0$. Observe that

$$-\varepsilon^2 h''(\varepsilon s)\int_{|t|<\delta/\varepsilon}\frac{w'^2}{(1-k\varepsilon(t+h))}=-\varepsilon^2 h''\int_{\mathbb{R}}w'^2dt+O(\varepsilon^3)$$

Also

$$\dot{h}^2\varepsilon^2\int\frac{1}{1-\varepsilon k(t+h)}w''w'dt=0+O(\varepsilon^3).$$

$$\varepsilon^2\dot{h}\int\frac{a_s}{a}(\varepsilon s,\varepsilon(t+h))w'^2/(1+k\varepsilon(t+h))^2=\varepsilon^2\dot{h}\frac{a_{\tilde{s}}}{a}(\varepsilon s,0)\int w'^2+O(\varepsilon^3)$$

and finally

$$\varepsilon\int_{|t|<\delta/\varepsilon} w'^2(\frac{a_{\tilde z}}{a}(\varepsilon s,\varepsilon(t+h))-\frac{k(\varepsilon s)}{1-k(\varepsilon s)(t+h)\varepsilon}) = \varepsilon^2\int_{\mathbb{R}} w'(t)^2(\varepsilon^2)((\frac{a_{\tilde z}}{a})(\varepsilon s,0)-k^2)h(\varepsilon s)+O(\varepsilon$$

Then

$$\frac{-\int Ew'dt}{\varepsilon^2\int w'^2} = h''+h'\frac{a_{\tilde s}}{a}-(\left(\frac{a_{\tilde z}}{a}\right)_{\tilde z}(\varepsilon s,0)-k^2)h+O(\varepsilon)$$

we call $\tilde s=\varepsilon s$, and we conclude that the right hand side of the above equality is equal to

$$\frac{1}{a(\tilde s,0)}((a(\tilde s,0))h'(\tilde s)'+(2k^2a(\tilde s,0)-a_{\tilde z\tilde z}(\tilde s,0))h)+O(\varepsilon)$$

and this is equal to

$$\frac{1}{a(\tilde s,0)}(J_a[h]+O(\varepsilon))$$

We need the equation for $v(s,z)=\tilde v(s,z-h(\varepsilon s))$. We have

$$\frac{\partial v}{\partial s}=\frac{\partial \tilde v}{\partial s}-\frac{\partial \tilde v}{\partial t}\dot h\varepsilon$$

We write $z=t+h$, so we have

$$S(\tilde v)=\frac{1}{(1-\varepsilon kz)}(\frac{\partial}{\partial s}-\varepsilon\dot h\frac{\partial}{\partial t})[\frac{1}{1-\varepsilon k(t+h)}(\frac{\partial}{\partial s}-\varepsilon\dot h\frac{\partial}{\partial t})]\tilde v+\tilde v_{tt}$$

$$\varepsilon[-\frac{k}{1-\varepsilon kz}+\frac{a_{\tilde z}}{a}]\tilde v_t+\varepsilon\frac{a_{\tilde s}}{a}\frac{1}{1-k\varepsilon z}^2[\tilde v_s-\varepsilon\dot h\tilde v_t]+f(\tilde v)=0$$

The first term of this equation is equal to

$$\frac{1}{1-\varepsilon kz}\{\frac{\varepsilon(\varepsilon\dot k(t+h)+\varepsilon k\dot h)}{(1-\varepsilon k(t+h))^2}(\tilde v_s-\varepsilon\dot h v_t)+\frac{1}{1-k\varepsilon(t+h)}(-\varepsilon^2h''v_t-2\varepsilon\dot h\tilde v_t s)+\frac{1}{1+\varepsilon k(t+h)}\tilde v_s s\}$$

$$-\varepsilon\dot h\{\frac{\varepsilon k}{(1-\varepsilon k(t+h))^2}(\tilde v_s-\varepsilon\dot h\tilde v_t)+\frac{1}{1-\varepsilon k(t+h)}(-\varepsilon\dot h\tilde v_{tt})\}+f(\tilde v)=0$$

Let us observe that for $|t|<\delta/\varepsilon$, $\delta\ll 1$

$$S[\tilde v](s,t)=\tilde v_{ss}+\tilde v_{tt}+O(\varepsilon)\partial_{ts}\tilde v+O(\tilde\varepsilon)\partial_{tt}\tilde v+O(\varepsilon k(|t|+1))\partial_{ss}\tilde v+O(\varepsilon)\partial_t\tilde v+O(\varepsilon)\partial_s\tilde v+f(v)=0$$

We will call the operator that appears in the equation $B[\tilde v]$. We look for a solution of the form $\tilde v(s,t)=w(t)+\phi(s,t)$. The equation for ϕ is

$$\phi_{ss}+\phi_{tt}+f'(w(t))\phi+E+B(\phi)+N(\phi)=0,\quad |t|<\delta/\varepsilon$$

where $E=S(w(t))=O(\varepsilon^2e^{-\sigma t})$, $N(\phi)=f(w+\phi)-f(w)-f'(w)\phi$, $s\in(0,l/\varepsilon)$. We use the notation $L(\phi)=\phi_{ss}+\phi_{tt}+f'(w(t))\phi$. We also need the boundary condition $\phi(0,t)=\phi(l/\varepsilon,t)$ and $\phi_s(0,t)=\phi_s(l/\varepsilon,t)$.

It is natural to study the linear operator in $\mathbb{R}^2$ and the linear projected problem

$$\phi_{ss} + \phi_{tt} + f'(w(t))\phi + g(t,s) = c(s)w'(t)$$

where $c(s) = \frac{\int_{\mathbb{R}} g(t,s)w'(t)dt}{\int_{\mathbb{R}} w'(t)^2 dt}$ and under the orthogonally condition

$$\int_{-\infty}^{\infty} \phi(s,t)w'(t)dt = 0, \quad \forall s \in \mathbb{R}$$

Basic ingredient: (Even more general) Consider the problem in $\mathbb{R}^m \times \mathbb{R}$, with variables (y,t):

$$\Delta_y \phi + \phi_{tt} + f'(w(t))\phi = 0, \quad \phi \in L^\infty(\mathbb{R}^m \times \mathbb{R})$$

If ϕ is a solution of the above problem, then $\phi(y,t) = \alpha w'(t)$ some $\alpha \in \mathbb{R}$. Ingredient: $\exists \gamma > 0 : \quad \int_{\mathbb{R}} p'(t)^2 - f'(w(t))p(t)^2 \geq \gamma \int_{\mathbb{R}} p^2(t)dt$ for all $p \in H^1$ with $\int_{\mathbb{R}} pw' = 0$. $\psi(y) = \int_{\mathbb{R}} \phi^2(y,t)dt$. This is well defined (as we will see) Indeed: It turns out that $|\phi(y,t)| \leq Ce^{-\sigma t}$, $\sigma < \sqrt{2}$, thanks to the fact that $\phi \in L^\infty$. We use $x = (y,t)$ and we obtain

$$\Delta_x \phi - (2 - 3(1 - w(t)^2))\phi = 0$$

Observe that $1 - w(t)^2$ is small if $|t| \gg 1$. Fix $0 < \sigma < \sqrt{2}$, for $|t| > R_0$ we have $2 - 3(1 - w^2(t)) > \sigma^2$. Let

$$\bar{\phi}_\rho(y,t) = \rho \sum_{i=1}^{n} \cosh(\sigma y_i) + \rho \cosh(\sigma t) + \|\phi\|_\infty e^{\sigma R_0} e^{-\sigma|t|}.$$

We have that

$$\phi(y,t) \leq \bar{\phi}_\rho(y,t), \quad \text{for } |t| = R_0$$

also true that for $|t| + |y| > R_\rho \gg 1$, $\phi(y,t) \leq \bar{\phi}_\rho$.

$$-\Delta_x \phi + (2 - 3(1 - w(t)^2))\bar{\phi} = (2 - \sigma^2 - 3(1 - w(t)^2)\bar{\phi}_\rho) > 0$$

for $|t| > R_0$. So is a supersolution of the operator

$$-\Delta_x \phi + (2 - 3(1 - w(t)^2))\phi$$

in D_ρ, which implies that $\phi \leq \bar{\phi}_\rho$ for $|t| > R_0$. This implies that $|\phi(x)| \leq C\bar{\phi}_\rho$ for all x, and we conclude the assertion taking $\rho \to 0$. If ϕ solves $-\Delta\phi + (1 - 3w^2)\phi = 0$, then $\|\phi\|_{C^{2,\alpha}}(B_1(x_0)) \leq C\|\phi\|_{L^\infty(B_2(x_0))}$. This implies that also

$$|\phi_y| + |\phi_{yy}| \leq Ce^{-\sigma t}.$$

Let $\phi(\tilde{y,t}) = \phi(y,t) - \frac{\int \phi(y,\tau)w'(\tau)d\tau}{\int w'^2} w'$. We call $\beta(y) = \frac{\int \phi(y,\tau)w'(\tau)d\tau}{\int w'^2}$

$$\Delta\tilde{\phi} + f'(w)\tilde{\phi} = \Delta\phi + f'(w)\phi + (\Delta_y \beta)w' + \beta(\Delta w' + f'(w))w' = 0$$

because $\Delta_y\beta = 0$ by integration by parts. Let $\psi(y) = \int_{\mathbb{R}} \tilde{\phi}^2 dt$.

$$\Delta_y\psi = \int_{\mathbb{R}} \nabla_y(2\tilde{\phi}\nabla_y\tilde{\phi})dt = 2\int |\nabla_y\tilde{\phi}|^2 dt + 2\int \tilde{\phi}\Delta_y\tilde{\phi} = 2\int |\nabla_y\tilde{\phi}|^2 - 2\int \tilde{\phi}[\tilde{\phi}_{tt} + f'(w)\tilde{\phi}]dt$$

Using $2\int |\nabla_y\tilde{\phi}|^2 dt + 2\int(\tilde{\phi}_t^2 - f'(w)\tilde{\phi}^2)$ This implies that $\Delta\psi \geq 2\gamma\psi$ which implies $-\Delta\psi + 2\gamma\psi \leq 0$, $0 \leq \psi \leq c$.

We obtain that $\psi \equiv 0$ and this implies $\tilde{\phi} = 0$. This implies that $\phi(t) = (\int \phi w')w' = \beta(y)w'$ and $\Delta\beta = 0$, $\beta \in L^\infty$. Liouville implies that $\beta = constant$ so $\phi = constant w'$.

Lemma: L^∞ a priori estimates for the linear projected problem: $\exists C : \|\phi\|_\infty \leq C\|g\|_\infty$.

Proof: If not exists $\|g_n\|_\infty \to 0$ and $\|\phi_n\|_\infty = 1$.

$$L[\phi_n] = -g_n + c_n(t)w'(t) = h_n(t)$$

and $h_n \to 0$ in L^∞. $\|\phi_n\| = 1$ which implies that $\exists (y_n, t_n)$: $|\phi(y_n, t_n)| \geq \gamma > 0$. Assume that $|t_n| \leq C$ and define $\tilde{\phi}(y,t) = \phi_n(y_n + y, t)$. Then

$$\Delta\tilde{\phi}_n + f'(w(t))\tilde{\phi}_n = \tilde{h}_n$$

but $f'(w(t))\tilde{\phi}_n$ is uniformly bounded and the right hand side goes to 0. This implies that $\|\phi\|_{C^1(\mathbb{R}^{m+|})} \leq C$ This implies that $\tilde{\phi}_n \to \tilde{\phi}$ passing to subsequence, and the convergence is uniformly on compacts, where $\Delta\tilde{\phi} + f'(w)\tilde{\phi} = 0$, $\tilde{\phi} \in L^\infty$. We conclude after a classic argument that $\tilde{\phi} = 0$. We have also that $\|e^{\sigma|t|}\phi\|_\infty \leq C\|e^{\sigma|t|}g\|_\infty$, $0 < \sigma < \sqrt{2}$. Elliptic regularity implies that $\|e^{\sigma|t|}\phi\|_{C^{2,\sigma}} \leq \|e^{\sigma|t|}g\|_{C^{0,\sigma}}$.

Existence: Assume g has compact support and take the weak formulation: Find $\phi \in H$ such that $\int_{\mathbb{R}^{m+1}} \nabla\phi\nabla\psi - f'(w)\phi\psi = \int gy$, for all $\psi \in H$, where $H = \{f \in H^1(\mathbb{R}^{m+1}) | \int_{\mathbb{R}} \psi w' dt = 0, \forall y \in \mathbb{R}^m\}$. Let us see that $a(\psi,\psi) = \int |\nabla\psi|^2 - f'(w)\psi^2 \geq \gamma\int \psi^2 + \psi^2$. So $a(\psi,\psi) \geq C\|\psi\|^2_{H^1(\mathbb{R}^{m+1})}$ This implies the unique existence solution. Observe that

$$\int (\Delta\phi + f'(w)\phi + g)\psi = 0$$

for all $\psi \in H$. Let $\psi \in H^1$ and $\psi = \tilde{\psi} - \frac{\int \tilde{\psi} w' dt}{\int w'^2} w' = \Pi(\tilde{\psi})$. We have that

$$\int dy \int g\Pi(\tilde{\psi})dt = \int \Pi(g)\psi$$

which implies that $\Pi(\Delta\phi + f'(w)\phi + g) = 0$ if and only if $\Delta\phi + f'(w) + \phi + g = \frac{\int(\Delta\phi + f'(w) + g)}{\int w'^2} w'$ Regularity implies that $\phi \in L^\infty$ and $\|\phi\|_\infty \leq C\|g\|_\infty$. Approximating $g \in L^\infty$ by $g_R \in C_c^\infty(\mathbb{R}^N)$ locally over compacts. This implies existence result.

We can bound ϕ in other norms. For example if $0<\sigma<\sqrt{2}$, then

$$\|e^{\sigma|t|}\phi\|_\infty \le C\|e^{\sigma|t|}g\|_\infty.$$

Indeed, $f'(w) < -\sigma^2-\eta$ if $|t|>R$, with $\eta=(2-\sigma^2)/2$. We set

$$\bar\phi = Me^{-\sigma|t|} + \rho\sum_{i=1}^n \cosh(\sigma y_i) + \rho\cosh(\sigma t).$$

Therefore

$$-\Delta\bar\phi + (-f'(w))\bar\phi \ge -\delta\bar\phi + (\sigma^2+\eta)\bar\phi = \eta\bar\phi > \tilde g = -g + c(y)w'(t)$$

if $M \ge \frac{A}{\eta}\|e^{\sigma|t|}g\|_\infty$. In addition we have $\bar\phi \ge \phi$ on $|t|=R$ if $M \ge \|\phi\|_\infty e^{\sigma R}$. By an standard argument based on maximum principle, we conclude that $\phi \le \bar\phi$. This means, letting $\rho\to 0$, $\phi \le Me^{-\sigma|t|}$, where $M \ge C\max\{\|\phi\|_\infty, \|ge^{\sigma|t|}\|_\infty\}$. Since $\|\phi\|_\infty \le C\|g\|_\infty \le C\|ge^{\sigma|t|}\|_\infty$, we can take $M = C\|ge^{\sigma|t|}\|_\infty$. Finally, we conclude $\|\phi e^{\sigma|t|}\|_\infty \le \|ge^{\sigma|t|}\|_\infty$.

Reminder: If $\Delta\phi = p$ implies that

$$\|\nabla\phi\|_{L^\infty(B_1(0))} \le C[\|\phi\|_{L^\infty B_2(0)} + \|p\|_{L^\infty(B_1(0))}].$$

Remember that

$$\|p\|_{C^{0,\alpha}(A)} = \|p\|_\infty + [\phi]_{0,\alpha,A}$$

where $[\phi]_{0,\alpha,A} = \sup_{x_1,x_2\in A, x_1\ne x_2}\frac{|p(x_1)-p(x_2)|}{|x_1-x_2|^\alpha}$. Also we have the following interior Schauder estimate: for $0<\alpha<1$

$$\|\phi\|_{C^{2,\sigma}(B_1)} \le C[\|\phi\|_{L^\infty(B_2(0))} + \|p\|_{C^{0,\alpha}(B_2(0))}].$$

Conclusion: If ϕ solves the equation in $\mathbb{R}^{n+1}$ then

$$\|\phi\|_{C^{2,\alpha}(\mathbb{R}^{n+1})} \le C\|g\|_{C^{0,\alpha}(\mathbb{R}^{n+1})}.$$

Sketch of the proof of this fact: Fix $x_0\in\mathbb{R}^{n+1}$, then

$$C[\phi]_{0,\alpha,B_1(x_0)} \le \|\nabla\phi\|_{L^\infty(B_1(x_0))} \le C[\|\phi\|_\infty + \|g\|_\infty] \le C\|g\|_\infty$$

This implies that $\|\phi\|_{C^{0,\alpha}(B_1(x_0))} \le C\|g\|_\infty$, which implies $\|\phi\|_{C^{0,\alpha}(\mathbb{R}^n)} \le C\|g\|_\infty$. Clearly $\|p\|_{C^{0,\alpha}(B_2(x_0))} \le C\|g\|_\infty$, so $\|\phi\|_{C^{0,\alpha}(B_1(x_0))} \le C\|g\|_{C^{0,\alpha}(\mathbb{R}^{n+1})}$, from where we deduce the estimate.

We also get

$$\|e^{\sigma|t|}\phi\|_{C^{2,\alpha}(\mathbb{R}^{n+1})} \le C\|e^{\sigma|t|}g\|_{C^{0,\alpha}(\mathbb{R}^{n+1})}.$$

The proof of this fact is very similar to the previous one (use that $g \le e^{-\sigma|t_0|\|ge^{\sigma|t|}\|}$, for $|t_0|\gg 1$).

Another result is the following

$$\|(1+|y|^2)^{\mu/2}\phi\|_\infty \le C\|(1+|y|^2)^{\mu/2}g\|_\infty$$

In order to prove this result we define $\rho(y) = (1+|y|^\mu)$ and we consider $\tilde\phi = \rho(\delta y)\phi$. Observe that

$\Delta\phi = \rho^{-1}\Delta\tilde{\phi} - 2\delta\nabla\tilde{\phi}\nabla(\rho^{-1}(\delta y)) + \tilde{\phi}\delta^2\Delta(\rho^{-1})(\delta y) = f'(w)\phi + g - cw'$

We get $L[\tilde{\phi}] + O(\delta^2)\tilde{\phi} + O(\delta)\nabla\tilde{\phi} = \rho(g - cw')$. We get

$$\|\nabla\tilde{\phi}\|_\infty + \|\tilde{\phi}\|_\infty \leq C[\delta^2\|\tilde{\phi}\|_\infty + \delta\|\nabla\tilde{\phi}\|_\infty + \|\rho g\|_\infty].$$

If δ is small we conclude that

$$\|\tilde{\phi}\|_\infty + \|\nabla\tilde{\phi}\|_\infty \leq C\|\rho g\|_\infty$$

and we obtain

$$\|\rho\phi\| C \leq \|\rho g\|.$$

Our setting:

$$\varepsilon^2[\delta u + \frac{\nabla a}{a} \cdot \nabla u] + f(u) = 0 \tag{1.3}$$

We want a solution to (1.3) $u_\varepsilon(x) \approx W(z/\varepsilon)$. Writing $x = y + z\gamma(y)$, $|z| < \delta$, we have

$$\Delta v + \nabla a(\varepsilon x)/a \cdot \nabla v + f(v) = 0,$$

in $\Gamma_\varepsilon = \frac{1}{\varepsilon}\Gamma$: $x = y + z\nu(\varepsilon y)$, which means $x = \frac{1}{\varepsilon}\gamma(\varepsilon s) + z\nu(\varepsilon s)$. Remember that $|\dot{\gamma}(\tilde{s})| = 1$ which implies $\dot{\nu}(\tilde{s}) = -k(\tilde{s})\dot{\gamma}(\tilde{s})$. We also set $z = h(\varepsilon s) + t$. $x = \frac{1}{\varepsilon}\gamma(\varepsilon s) + (t + h(\varepsilon s))\nu(\varepsilon s)$. We assume $\|h\|_{\alpha,(0,l)} \leq 1$, for $0 < \alpha < 1$. We wrote Δ_x in terms of this coordinates (t, s) and the equations $S(v) = 0$ is rewritten taking as first approximation $w(t)$. We evaluated $S(w(t))$ and got that $S(w(t)) = 0$.

From the expression of Δ_x we get $(x = \frac{1}{\varepsilon}\gamma(\varepsilon s) + (t + h(\varepsilon s))\nu(\varepsilon s))$

$$\Delta_x v = \partial_{ss} + \partial_{tt} + \varepsilon[b_1^\varepsilon(t, s)\partial_{ss} + b_2^\varepsilon\partial_{tt} + b_3^\varepsilon\partial_{st} + b_4^\varepsilon\partial_t + b_5^\varepsilon\partial_s]$$

$|\varepsilon b_i| \leq C\delta$ in the region $|t| < \delta/\varepsilon$. The coefficients are periodic (same values at $s = 0$ and $s = l/\varepsilon$). Our equation reads

$$\partial_{ss}v + \partial_{tt}v + B_\varepsilon[v] + f(v) = 0, \quad \text{for } s \in (0, l/\varepsilon), |t| < \delta/\varepsilon.$$

This expression does not make sense globally. We consider $\delta \ll 1$. We define

$$H(x) = \begin{cases} -1 & \text{in } \Omega_-^\varepsilon \\ +1 & \text{in } \Omega_+^\varepsilon \end{cases}$$

where Ω_+^ε is a bounded component of $\mathbb{R}^2 \setminus \Gamma$, and Ω_-^ε the other. For the equation

$$\Delta v + \varepsilon\frac{\nabla a}{a} \cdot \nabla v + f(v) = 0$$

we take as first (global) approximation

$$v_0(x) = w(t)\eta_3 + (1 - \eta_4)H(x)$$

where

$$\eta_l(x) = \begin{cases} \eta\left(\frac{\varepsilon|t|}{l\delta}\right) & \text{if } |t| < 2\delta l/\varepsilon \\ 0 & \text{otherwise} \end{cases}$$

Look for a solution of the form $v = v_0 + \tilde{\phi}$, so

$$\Delta_x \tilde{\phi} + \varepsilon \frac{\nabla a}{a} \cdot \nabla \tilde{\phi} + f'(v_o)\tilde{\phi} + E + N(\tilde{\phi}) = 0$$

where $E = S(v_0)$ and $N(\tilde{\phi}) = f(v_0 + \tilde{\phi}) - f(v_0) - f'(v_0)\tilde{\phi}$.

We write $\tilde{\phi} = \eta_3\phi + \psi$. We require that ϕ and ψ solve the system

$$\Delta_x\psi - 2\psi + (2 + f'(v_0))(1-\eta_1)\psi + \varepsilon\frac{\nabla a}{a}\nabla\psi + (1-\eta_1)E + (1-\eta_1)N(\eta_3\phi+\psi) + \nabla\eta_3\nabla\phi + \nabla\eta_3\nabla\phi + \varepsilon\frac{\nabla}{}$$

$$\eta_3\left[\Delta_x\phi + f'(w(t))\phi + \eta_1(2 + f'(w(t)))\psi + \eta_1 E + \eta_1 N(\phi+\psi) + \varepsilon\frac{\nabla a}{a}\cdot\nabla\phi\right] = 0.$$

We need that the ϕ above satisfies the equation just for $|t| < 6\delta/\varepsilon$. We assume that $\phi(s,t)$ is defined for all s and t (and it is l/ε- periodic in s). We require that ϕ satisfies globally

$$\phi_{tt} + \phi_{ss} + \eta_6 B_\varepsilon[\phi] + f'(w(t))\phi + \eta_1 E + \eta_1 N(\phi+\psi) + \eta_1(2 + f'(w))\psi = 0$$

and $\phi \in L^\infty(\mathbb{R}n+1)$ and periodic in s. Notice that $\phi_{tt} + \phi_{ss} + \eta_6 B_\varepsilon[\phi] = \Delta_x\phi$ inside the support of η_3. Rather than solving this problem directly we solve the projected problem

(1.4)

$$\phi_{tt} + \phi_{ss} + \eta_6 B_\varepsilon[\phi] + f'(w(t))\phi + \eta_1 E + \eta_1 N(\phi+\psi) + \eta_1(2 + f'(w))\psi = c(s)w'(t)$$

and $\int_{\mathbb{R}} \phi w'(t)dt = 0$. We solve (1)-(1.4) first, then we find h such that $c(s) \equiv 0$. We consider ϕ with $\|\phi\|_\infty + \|\nabla\phi\|_\infty \le \varepsilon$. The operator $-\Delta\psi + 2\psi$ is invertible $L^\infty(\mathbb{R}^3) \to C^1(\mathbb{R}^2)$. We conclude that if $g \in L^\infty$ the exist a unique solution $\psi = T[g] \in C^1(\mathbb{R}^2)$ with $\|\phi\|_{C^1} \le C\|g\|_\infty$ of equation $-\Delta\psi + 2\psi = g$ in $\mathbb{R}^2$. Observe that (1) is equivalent to

$$\psi = T[(2 + f'(v_0))(1-\eta_1)\psi + \varepsilon\frac{\nabla a}{a}\nabla\psi + (1-\eta_1)E + (1-\eta_1)N(\eta_3\phi+\psi) + \nabla\eta_3\nabla\phi + \nabla\eta_3\nabla\phi + \varepsilon\frac{\nabla a}{a}\nabla$$

Using contraction mapping in C^1 on $\|\psi\|_{C^1} \le C\varepsilon$, we conclude that there exist a unique solution of the this problem $\psi = \psi(\phi, h)$ such that

$$\|\psi\| \le C[\varepsilon^2 + \varepsilon\|\phi\|_{C^1}].$$

Even more, $\|\psi(\phi_1, h) - \psi(\phi_2, h)\|_{C^1} \le C\varepsilon\|\phi_1 - \phi_2\|_{C^1}$.