NONDEGENERACY OF THE TRAVELING LUMP SOLUTION TO THE $2+1$ TODA LATTICE

YONG LIU AND JUNCHENG WEI

Abstract. We consider the $2+1$ Toda system

$$\frac{1}{4}\Delta q_n = e^{q_{n-1}-q_n} - e^{q_n-q_{n+1}} \text{ in } \mathbb{R}^2,\ n\in\mathbb{Z}.$$

It has a traveling wave type solution $\{Q_n\}$ satisfying $Q_{n+1}(x,y) = Q_n(x+\frac{1}{2\sqrt{2}}, y)$, and is explicitly given by

$$Q_n(x,y) = \ln\frac{\frac{1}{4}+\left(n-1+2\sqrt{2}x\right)^2+4y^2}{\frac{1}{4}+\left(n+2\sqrt{2}x\right)^2+4y^2}.$$

In this paper we prove that $\{Q_n\}$ is nondegenerate.

1. Introduction and statement of main results

Toda lattice equation is a classical integrable system appearing in various different areas of mathematics, mechanics and physics. In this paper, we are interested in the lump solution to the following $2+1$ Toda lattice equation:

$$\frac{1}{4}\Delta q_n = e^{q_{n-1}-q_n} - e^{q_n-q_{n+1}} \text{ in } \mathbb{R}^2, n\in\mathbb{Z}. \tag{1.1}$$

Equation (1.1) has been studied in [18–21], using the inverse scattering transform(IST). A family of lump solution to (1.1) has been found in [18] (see equation (3.6) there). Let us consider one of these lumps:

$$Q_n(x,y) = \ln\frac{\frac{1}{4}+\left(n-1+2\sqrt{2}x\right)^2+4y^2}{\frac{1}{4}+\left(n+2\sqrt{2}x\right)^2+4y^2}. \tag{1.2}$$

Then Q_n decays at the rate $O\left(r^{-1}\right)$, as $r^2 = x^2+y^2 \to +\infty$. We also point out that in [14], families of rational and N-breather solutions to (1.1), including Q_n, have been found using Hirota's direct method. It turns out that Q_n is actually an analogy of the classical lump solution to the KP-I equation. As a matter of fact, the KP-I equation can be regarded as a continuum limit of a family of generalized Toda lattice, with (1.1) being in this family. We refer to [15] for more details on this correspondence.

It is worth noting that the hyperbolic version of (1.1) :

$$\frac{1}{4}\left(\partial_x^2-\partial_y^2\right)q_n = e^{q_{n-1}-q_n} - e^{q_n-q_{n+1}}, (x,y)\in\mathbb{R}^2, n\in\mathbb{Z}, \tag{1.3}$$

has also been investigated in [18–21]. From the IST point of view, (1.3) is quite different from (1.1). More precisely, the associated Cauchy problem in the IST formulation is well posed in (1.3), but ill-posed in (1.1).

The system (1.1) is a generalization of following $1+1$ Toda lattice

$$\frac{1}{4}q_n'' = e^{q_{n-1}-q_n} - e^{q_n-q_{n+1}},\ n \in \mathbb{Z}. \tag{1.4}$$

This is a classical integrable system. Compared to the $2+1$ Toda lattice (1.1), the system (1.4) has been extensively studied in the literature. We refer to [4, 17] and the reference therein for more discussion on this equation and related topics.

It is worth mentioning that there is another class of Toda equation, which we call finite Toda system (It is also called Toda molecule equation in [5]):

$$\left(\partial_x^2 - \partial_y^2\right) q_n = 4e^{q_{n-1}-q_n} - 4e^{q_n-q_{n+1}}, (x,y) \in \mathbb{R}^2, n \in 1, ..., N, \tag{1.5}$$

with $q_0 = -\infty, q_{N+1} = +\infty$. This is also an integrable system. The elliptic version of (1.5):

$$\left(\partial_x^2 + \partial_y^2\right) q_n = 4e^{q_{n-1}-q_n} - 4e^{q_n-q_{n+1}}, (x,y) \in \mathbb{R}^2, n \in 1, ..., N, \tag{1.6}$$

has been studied in [9], where classification and nondegeneracy of solutions have been proved, by analyzing various explicit conserved quantities of this system.

One of the motivations of studying Toda system comes from the following unexpected connection: the solutions of (1.1), (1.4), (1.6), actually describe the interface motion of the solutions of the Allen-Cahn equation

$$-\Delta u = u - u^3.$$

The general principle is the following: For each "regular" enough solution of the Toda system in $\mathbb{R}^1$ or $\mathbb{R}^2$, one should be able to construct an entire solution to the Allen-Cahn equation in $\mathbb{R}^2$ or $\mathbb{R}^3$ whose nodal sets resemble the solutions to the Toda system. This type of results has been obtained in [1, 3, 8], using the method of infinite dimensional Lyapunov-Schmidt reduction. The bounded domain case is considered in [2]. A key element in these constructions is the nondegeneracy of solutions to the Toda system. See [3, 8].

In this paper, we prove that $\{Q_n\}$ is nondegenerate. Our main result is

Theorem 1. *Let $\{U_n\}$ be a solution of the linearized equation*

$$\Delta U_n = e^{Q_{n-1}-Q_n}\left(U_{n-1} - U_n\right) - e^{Q_n-Q_{n+1}}\left(U_n - U_{n+1}\right). \tag{1.7}$$

Suppose $U_{n+1}(x,y) = U_n\left(x + \frac{1}{2\sqrt{2}}, y\right)$ and

$$U_n(x,y) \to 0 \quad \text{as } x^2 + y^2 \to +\infty.$$

Then

$$U_n = c_1 \partial_x Q_n + c_2 \partial_y Q_n,$$

for some constants c_1, c_2.

Theorem 1, combined with gluing arguments similar to those in [1, 8], yields the following result for Allen-Cahn equation

Corollary 2. *The Allen-Cahn equation*

$$-\Delta u = u - u^3 \qquad \text{in } \mathbb{R}^3 \tag{1.8}$$

has a family of singly periodic solutions whose zero level set $\{u = 0\}$ is approximately given by $\cup_n\{z = Q_n(x,y)\}$.

The main idea of the proofs of Theorem 1 is to consider the Bäcklund transformation at the linearized level. More precisely we use the linearized Bäcklund transformation to transform a kernel U_n of (1.7) to a kernel of the linearized equation with respect to the trivial solution, which is an operator of constant coefficient. This type of arguments has been used in [11,13] for the analysis of spectral property of some soliton solutions to $1+1$ Toda lattice and KdV equation. In this respect, we also refer to [12], where the stability of line solitons of the KP-II equation has been proved using Miura transformation. In [10] similar idea is used to prove the nondegeneracy of the lump solution to the KP-I equation.

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2. Preliminaries on the Bäcklund transformation of the 2+1 Toda lattice

Bäcklund transformation has been used to study soliton solutions for many integrable systems. We refer to [5, 16] for a general introduction this topic.

In this paper, we use D to denote the bilinear derivative operator. That is,

$$D_s^m D_t^n f \cdot g = \left[\left(\partial_s - \partial_{s'} \right)^m \left(\partial_t - \partial_{t'} \right)^n \right] \left(f\left(s,t\right) g\left(s',t'\right) \right) |_{s'=s,t'=t}.$$

We already know that the lump Q_n can be obtained via the inverse scattering transform. It turns out that we can also find Q_n by Bäcklund transformation. Let us explain this in the sequel.

To use the form of the Bäcklund transformation as studied in [5], we introduce the complex variables $s = x+iy, t = x-iy$. Then $\Delta = 4\partial_s\partial_t$. Setting $r_n = q_{n-1}-q_n$, we transform (1.1) into

$$\partial_s\partial_t r_n = e^{r_{n+1}} + e^{r_{n-1}} - 2e^{r_n}, n \in \mathbb{Z}. \tag{2.1}$$

Let us define V_n by

$$1 + V_n = e^{r_n}.$$

Equation (2.1) then becomes

$$\partial_s\partial_t \ln\left(1+V_n\right) = V_{n+1} + V_{n-1} - 2V_n.$$

Introducing the so-called τ-function τ_n by $V_n = \partial_s\partial_t \ln \tau_n$, we get the following bilinear form for the Toda lattice (2.1):

$$D_s D_t \tau_n \cdot \tau_n = 2\left(\tau_{n+1}\tau_{n-1} - \tau_n^2\right), n \in \mathbb{N}. \tag{2.2}$$

For $n \in \mathbb{N}$, we define

$$\kappa_n = 1,$$

$$\omega_n = \sqrt{2}\left(s+t\right) + n + \left(s-t\right) + \frac{\sqrt{2}-1}{2},$$

and

$$\theta_n = \left(\sqrt{2}\left(s+t\right)+n\right)^2 - \left(s-t\right)^2 + \frac{1}{4}. \tag{2.3}$$

Then $\{\kappa_n\}, \{\omega_n\}, \{\theta_n\}$ are solutions of (2.2). The lump solution Q_n is corresponding to θ_n.

We are interested in the transformation from $\{q_n\}$ to $\{\tau_n\}$ at the linearized level. Let us define the linearized operator T_θ of (2.2):

$$(T_\theta\eta)_n := \partial_s\partial_t\eta_n\theta_n - \partial_s\eta_n\partial_t\theta_n - \partial_t\eta_n\partial_s\theta_n + \eta_n\partial_s\partial_t\theta_n - (\eta_{n+1}\theta_{n-1} + \theta_{n+1}\eta_{n-1} - 2\theta_n\eta_n).$$

Similarly, we have T_ω and T_κ. Note that

$$(T_\kappa\eta)_n = \frac{1}{4}\Delta\eta_n + 2\eta_n - \eta_{n+1} - \eta_{n-1}.$$

Lemma 4. *Suppose $\{U_n\}$ satisfies the linearized equation*

$$\frac{1}{4}\Delta U_n = e^{Q_{n-1}-Q_n}(U_{n-1} - U_n) - e^{Q_n - Q_{n+1}}(U_n - U_{n+1}), n \in \mathbb{Z}. \tag{2.4}$$

Assume

$$U_{n+1}(x,y) = U_n\left(x + \frac{1}{2\sqrt{2}}, y\right)$$

and

$$U_n(x,y) \to 0, \text{ as } x^2+y^2 \to +\infty.$$

Then the equation

$$\frac{1}{4}\Delta\tilde{\eta}_n = e^{Q_{n-1}-Q_n}(U_{n-1} - U_n), \tag{2.5}$$

has a solution $\{\tilde{\eta}_n\}$ with $\tilde{\eta}_{n+1}(x,y) = \tilde{\eta}_n\left(x + \frac{1}{2\sqrt{2}}, y\right)$ and

$$|\tilde{\eta}_0| \leq \frac{C}{\sqrt{1+x^2+y^2}}.$$

Moreover, $\eta_n := \theta_n\tilde{\eta}_n$ solves the linearized equation $T_\theta\eta = 0$.

Proof. Let us denote the function $U_{n-1} - U_n$ by v_n. We deduce from (2.4) that

$$\frac{1}{4}\Delta v_n = e^{Q_{n-2}-Q_{n-1}}v_{n-1} + e^{Q_n-Q_{n+1}}v_{n+1} - 2e^{Q_{n-1}-Q_n}v_n. \tag{2.6}$$

Setting $f_n = \left(e^{Q_{n-1}-Q_n} - 1\right)v_n$, we can write (2.6) as

$$\frac{1}{4}\Delta v_n - v_{n+1} - v_{n-1} + 2v_n = f_{n+1} + f_{n-1} - 2f_n. \tag{2.7}$$

We use $\mathcal{F}(f)$ to denote the Fourier transform(in $\mathbb{R}^2$) of the function f. Taking Fourier transform in (2.7), we obtain

$$\mathcal{F}(v_n) = \frac{-\left(2 - 2\cos\frac{\pi\xi_1}{\sqrt{2}}\right)\mathcal{F}(f_n)}{-\pi^2(\xi_1^2+\xi_2^2) + 2 - 2\cos\frac{\pi\xi_1}{\sqrt{2}}}. \tag{2.8}$$

Observe that

$$-\pi^2\left(\xi_1^2+\xi_2^2\right) + 2 - 2\cos\frac{\pi\xi_1}{\sqrt{2}} \leq 0,$$

and it equals zero if and only if $\xi_1 = \xi_2 = 0$. Then using (2.8) and the estimate

$$Q_n - Q_{n+1} = O\left(\left(1+x^2+y^2\right)^{-1}\right),$$

we can show that

$$v_n = O\left(\left(1+x^2+y^2\right)^{-1}\right).$$

This implies

$$f_n = O\left(\left(1+x^2+y^2\right)^{-2}\right).$$

The equation (2.5) has a solution $\tilde{\eta}_n$ with

$$\mathcal{F}\left(\tilde{\eta}_n\right) = -\frac{1}{\pi^2\left(\xi_1^2+\xi_2^2\right)}\mathcal{F}\left(e^{Q_{n-1}-Q_n}v_n\right) = -\frac{\mathcal{F}\left(f_n+v_n\right)}{\pi^2\left(\xi_1^2+\xi_2^2\right)}.$$

In view of (2.8) , we get

$$\mathcal{F}\left(\tilde{\eta}_n\right) = \frac{\mathcal{F}\left(f_n\right)}{-\pi^2\left(\xi_1^2+\xi_2^2\right)+2-2\cos\frac{\pi\xi_1}{\sqrt{2}}}. \tag{2.9}$$

On the other hand, using equation (2.4), we obtain

$$\int_{\mathbb{R}^2} f_n\left(x,y\right)dxdy = \int_{\mathbb{R}^2} v_n\left(x,y\right)dxdy = 0.$$

Hence by (2.9) ,

$$\left|\tilde{\eta}_0\right| \leq \frac{C}{\sqrt{1+x^2+y^2}}. \tag{2.10}$$

The derivatives of $\tilde{\eta}_0$ can also be estimated.

Under the transformation

$$r_n = q_{n-1}-q_n,\ V_n = e^{r_n}-1, \partial_s\partial_t \ln\tau_n = V_n,$$

the original Toda lattice (1.1) becomes

$$\partial_s\partial_t \ln\left(\frac{\partial_s\partial_t\tau_n\tau_n - \partial_s\tau_n\partial_t\tau_n + \tau_n^2}{\tau_n^2}\right) = \partial_s\partial_t \ln\frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}. \tag{2.11}$$

Linearizing these relations at θ_n, we find that the function $\eta_n = \tilde{\eta}_n\theta_n$ satisfies

$$\partial_s\partial_t\frac{\left(T_\theta\eta\right)_n}{\theta_{n+1}\theta_{n-1}} = 0.$$

This together with the estimate (2.10) tells us $T_\theta\eta = 0$. □

Let

$$\mathcal{P} = \left[D_tD_s\tau_n\cdot\tau_n - 2\tau_{n+1}\tau_{n-1} + 2\tau_n^2\right]\tau_n'^2 - \left[D_tD_s\tau_n'\cdot\tau_n' - 2\tau_{n+1}'\tau_{n-1}' + 2\tau_n'^2\right]\tau_n^2.$$

Then we have the identity(Page 179, [5])

$$\begin{aligned}\frac{1}{2}\mathcal{P} &= D_t\left[D_s\tau_n\cdot\tau_n' - \lambda\tau_{n+1}\cdot\tau_{n-1}' + \lambda\tau_n\tau_n'\right]\cdot\left(\tau_n'\tau_n\right)\\ &\quad + \lambda\left[D_t\tau_{n+1}\cdot\tau_n' + \lambda^{-1}\tau_n\tau_{n+1}' - \lambda^{-1}\tau_{n+1}\tau_n'\right]\tau_{n-1}'\tau_n\\ &\quad - \lambda\left[D_t\tau_n\cdot\tau_{n-1}' + \lambda^{-1}\tau_{n-1}\tau_n' - \lambda^{-1}\tau_n\tau_{n-1}'\right]\tau_n'\tau_{n+1}.\end{aligned}$$

Here λ is a free parameter. From this, we get the following Bäcklund transformation between $\{\tau_n\}$ and $\{\tau_n'\}$:

$$\begin{cases} D_s\tau_n\cdot\tau_n' - \lambda\tau_{n+1}\cdot\tau_{n-1}' + \lambda\tau_n\tau_n' = 0,\\ D_t\tau_{n+1}\cdot\tau_n' + \lambda^{-1}\tau_n\tau_{n+1}' - \lambda^{-1}\tau_{n+1}\tau_n' = 0.\end{cases} \tag{2.12}$$

If $\{\tau_n\}$ is a solution of (2.2) and $\{\tau_n\},\{\tau_n'\}$ satisfy (2.12) , then $\{\tau_n'\}$ is also a solution of (2.2) .

In the rest of the paper, we choose $\lambda=\sqrt{2}+1$. The Bäcklund transformation from κ_n to ω_n is given by

$$\begin{cases} D_s\kappa_n\cdot\omega_n=\lambda\left(\kappa_{n+1}\omega_{n-1}-\kappa_n\omega_n\right), \\ D_t\kappa_{n+1}\cdot\omega_n=-\lambda^{-1}\left(\kappa_n\omega_{n+1}-\kappa_{n+1}\omega_n\right). \end{cases} \tag{2.13}$$

The Bäcklund transformation from ω_n to θ_n is

$$\begin{cases} D_s\omega_n\cdot\theta_n=\lambda^{-1}\left(\omega_{n+1}\theta_{n-1}-\omega_n\theta_n\right), \\ D_t\omega_{n+1}\cdot\theta_n=-\lambda\left(\omega_n\theta_{n+1}-\omega_{n+1}\theta_n\right). \end{cases} \tag{2.14}$$

We refer to [16] for related results on the Bäcklund transformation of $1+1$ Toda lattice and other integrable systems.

3. The linearized Bäcklund transformation between ω and θ

In this section, we study the linearized Bäcklund transformation between ω and θ. The linearization of the system (2.14) is

$$\begin{cases} \partial_s\phi_n\theta_n-\phi_n\partial_s\theta_n-\lambda^{-1}\left(\phi_{n+1}\theta_{n-1}-\phi_n\theta_n\right) \\ =-\partial_s\omega_n\eta_n+\omega_n\partial_s\eta_n+\lambda^{-1}\left(\omega_{n+1}\eta_{n-1}-\omega_n\eta_n\right), \\ \partial_t\phi_n\theta_{n-1}-\phi_n\partial_t\theta_{n-1}+\lambda\left(\phi_{n-1}\theta_n-\phi_n\theta_{n-1}\right) \\ =-\partial_t\omega_n\eta_{n-1}+\omega_n\partial_t\eta_{n-1}-\lambda\left(\omega_{n-1}\eta_n-\omega_n\eta_{n-1}\right). \end{cases} \tag{3.1}$$

Dividing the first equation by θ_n and the second one by θ_{n-1}, (3.1) can be rewritten as

$$\begin{cases} \left(F_1\phi\right)_n=\left(G_1\eta\right)_n, \\ \left(M_1\phi\right)_n=\left(N_1\eta\right)_n, \end{cases} \tag{3.2}$$

where

$$\left(F_1\phi\right)_n=\partial_x\phi_n-\left(\frac{\partial_s\theta_n}{\theta_n}+\frac{\partial_t\theta_{n-1}}{\theta_{n-1}}+2\right)\phi_n-\lambda^{-1}\frac{\theta_{n-1}}{\theta_n}\phi_{n+1}+\lambda\frac{\theta_n}{\theta_{n-1}}\phi_{n-1},$$

$$\left(M_1\phi\right)_n=\frac{1}{i}\partial_y\phi_n-\left(\frac{\partial_s\theta_n}{\theta_n}-\frac{\partial_t\theta_{n-1}}{\theta_{n-1}}-2\sqrt{2}\right)\phi_n-\lambda^{-1}\phi_{n+1}\frac{\theta_{n-1}}{\theta_n}-\lambda\phi_{n-1}\frac{\theta_n}{\theta_{n-1}},$$

and

$$\begin{aligned}\left(G_1\eta\right)_n=&\frac{\omega_n}{\theta_n}\partial_s\eta_n+\frac{\omega_n}{\theta_{n-1}}\partial_t\eta_{n-1}+\left(-\frac{\partial_s\omega_n}{\theta_n}-\lambda^{-1}\frac{\omega_n}{\theta_n}-\lambda\frac{\omega_{n-1}}{\theta_{n-1}}\right)\eta_n\\ &+\left(\lambda\frac{\omega_{n+1}}{\theta_n}-\frac{\partial_t\omega_n}{\theta_{n-1}}+\frac{\lambda\omega_n}{\theta_{n-1}}\right)\eta_{n-1},\end{aligned}$$

$$\begin{aligned}\left(N_1\eta\right)_n=&\frac{\omega_n}{\theta_n}\partial_s\eta_n-\frac{\omega_n}{\theta_{n-1}}\partial_t\eta_{n-1}+\left(-\frac{\partial_s\omega_n}{\theta_n}-\lambda^{-1}\frac{\omega_n}{\theta_n}+\lambda\frac{\omega_{n-1}}{\theta_{n-1}}\right)\eta_n\\ &+\left(\lambda\frac{\omega_{n+1}}{\theta_n}+\frac{\partial_t\omega_n}{\theta_{n-1}}-\frac{\lambda\omega_n}{\theta_{n-1}}\right)\eta_{n-1}.\end{aligned}$$

In this section, we would like to prove the following

Proposition 5. *Let $\{\eta_n\}$ be given by Lemma 4. Then* (3.2) *has a solution $\{\phi_n\}$ with $\phi_{n+1}(x,y)=\phi_n\left(x+\frac{1}{2\sqrt{2}},y\right)$ and*

$$\left|\phi_0(x,y)\right|\leq C\left(1+x^2+y^2\right)^{\frac{5}{8}}. \tag{3.3}$$

We remark that the exponent $\frac{5}{8}$ is not optimal, but it is suffice for our use in the proof of Theorem 1.

To prove Proposition 5, we will use the Fourier transform. Let us use $\phi^{\hat{}}$ to denote the Fourier transform of a generalized function $\phi = \phi(x, y)$ with respect to the x variable. In particular, if ϕ is a regular function, then

$$\phi^{\hat{}}(\xi, y) = \int_{\mathbb{R}} e^{-2\pi i x\xi} \phi(x, y)\, dx.$$

The Heaviside step function will be denoted by u. That is,

$$u(x) := \begin{cases} 0, x < 0, \\ 1, x \geq 0. \end{cases}$$

We will frequently use the following formulas for the Fourier transform(See Appendix 2 of the book [7]).

Lemma 6. *Let $a_1 \in \mathbb{R}, a_2 > 0$. Then*

$$\left(\frac{1}{x + a_1 - a_2 i}\right)^{\hat{}} = 2\pi i e^{2\pi i(a_1 - a_2 i)\xi} u(-\xi),$$

and

$$\left(\frac{1}{x + a_1 + a_2 i}\right)^{\hat{}} = (-2\pi i)\, e^{2\pi i(a_1 + a_2 i)\xi} u(\xi).$$

Lemma 7. *Let $a_1 \in \mathbb{R}$. Then*

$$\left(\frac{1}{x + a_1}\right)^{\hat{}} = -\pi i e^{2\pi i a_1 \xi} Sgn\xi.$$

Remark 8. *From Lemma 6 and Lemma 7, we see that formally,*

$$\left(\frac{1}{x}\right)^{\hat{}} = \lim_{\varepsilon \to 0} \frac{1}{2}\left(\frac{1}{x + \varepsilon i} + \frac{1}{x - \varepsilon i}\right).$$

Here the distribution $\frac{1}{x}$ is defined to be the derivative of $\ln|x|$.

Lemma 9. *Suppose $a_1 \in \mathbb{R}, a_2 > 0$, and $a_3 \in \mathbb{C}$. Then*

$$\begin{aligned} \left(\frac{(x + a_1) + a_3}{(x + a_1)^2 + a_2^2}\right)^{\hat{}} &= \left(\frac{1}{2} - \frac{a_3}{2a_2 i}\right)(-2\pi i)\, e^{2\pi i(a_1 + a_2 i)\xi} u(\xi) \\ &+ \left(\frac{1}{2} + \frac{a_3}{2a_2 i}\right)(2\pi i)\, e^{2\pi i(a_1 - a_2 i)\xi} u(-\xi). \end{aligned}$$

Proof. We have

$$\begin{aligned} \frac{(x + a_1) + a_3}{(x + a_1)^2 + a_2^2} &= \frac{1}{2}\frac{1}{x + a_1 + a_2 i} + \frac{1}{2}\frac{1}{x + a_1 - a_2 i} \\ &+ \frac{a_3}{2a_2 i}\left(\frac{1}{x + a_1 - a_2 i} - \frac{1}{x + a_1 + a_2 i}\right) \\ &= \left(\frac{1}{2} - \frac{a_3}{2a_2 i}\right)\frac{1}{x + a_1 + a_2 i} \\ &+ \left(\frac{1}{2} + \frac{a_3}{2a_2 i}\right)\frac{1}{x + a_1 - a_2 i}. \end{aligned}$$

The result then follows from Lemma 6. □

We define

$$\alpha = \frac{n}{2\sqrt{2}}, \beta = \sqrt{\frac{y^2}{2} + \frac{1}{32}}, b = -\frac{y}{2},$$

and

$$c = \frac{1}{2\sqrt{2}}, \alpha_1 = \alpha - c.$$

Set

$$A_1 = \frac{1}{2} - \frac{b}{2\beta}, A_2 = -\frac{\lambda}{\sqrt{2}}\left(\frac{1}{2} - \frac{\sqrt{2}}{16\beta i}\right),$$

$$A_3 = -\left(\frac{1}{2} + \frac{b}{2\beta}\right), A_4 = \frac{\lambda}{\sqrt{2}}\left(\frac{1}{2} + \frac{\sqrt{2}}{16\beta i}\right).$$

Lemma 10. *Let* θ_n *be defined by* (2.3). *Then*

$$\left[\frac{\partial_s \theta_n}{\theta_n}\right]^{\hat{}} = -2\pi i A_1 e^{2\pi i(\alpha+\beta i)\xi} u(\xi) - 2\pi i A_3 e^{2\pi i(\alpha-\beta i)\xi} u(-\xi),$$

$$\left[\frac{\partial_t \theta_{n-1}}{\theta_{n-1}}\right]^{\hat{}} = 2\pi i A_3 e^{2\pi i(\alpha_1+\beta i)\xi} u(\xi) + 2\pi i A_1 e^{2\pi i(\alpha_1-\beta i)\xi} u(-\xi),$$

and

$$\left[\frac{\theta_{n-1}}{\theta_n}\right]^{\hat{}} = \delta + 2\pi i \frac{A_4}{\lambda} e^{2\pi i(\alpha+\beta i)\xi} u(\xi) + 2\pi i \frac{A_2}{\lambda} e^{2\pi i(\alpha-\beta i)\xi} u(-\xi),$$

and

$$\left[\frac{\theta_n}{\theta_{n-1}}\right]^{\hat{}} = \delta + 2\pi i \frac{A_2}{\lambda} e^{2\pi i(\alpha_1+\beta i)\xi} u(\xi) + 2\pi i \frac{A_4}{\lambda} e^{2\pi i(\alpha_1-\beta i)\xi} u(-\xi).$$

Proof. We compute

$$\begin{aligned}\frac{\partial_s \theta_n}{\theta_n} &= \frac{2\sqrt{2}\left(2\sqrt{2}x + n\right) - 4yi}{\left(2\sqrt{2}x + n\right)^2 + 4y^2 + \frac{1}{4}} \\ &= \frac{x + \frac{n}{2\sqrt{2}} - \frac{yi}{2}}{\left(x + \frac{n}{2\sqrt{2}}\right)^2 + \frac{y^2}{2} + \frac{1}{32}}.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial_t \theta_n}{\theta_n} &= \frac{2\sqrt{2}\left(2\sqrt{2}x + n\right) + 4yi}{\left(2\sqrt{2}x + n\right)^2 + 4y^2 + \frac{1}{4}} \\ &= \frac{x + \frac{n}{2\sqrt{2}} + \frac{yi}{2}}{\left(x + \frac{n}{2\sqrt{2}}\right)^2 + 2y^2 + \frac{1}{32}}.\end{aligned}$$

Moreover,

$$\begin{aligned}\frac{\theta_{n-1}}{\theta_n} &= \frac{\theta_n - \left[2\left(2\sqrt{2}x + n\right) - 1\right]}{\theta_n} = 1 - \frac{2\left(2\sqrt{2}x + n\right) - 1}{\left(2\sqrt{2}x + n\right)^2 + 4y^2 + \frac{1}{4}} \\ &= 1 - \frac{1}{\sqrt{2}} \frac{\left(x + \frac{n}{2\sqrt{2}}\right) - \frac{\sqrt{2}}{8}}{\left(x + \frac{n}{2\sqrt{2}}\right)^2 + \frac{y^2}{2} + \frac{1}{32}}.\end{aligned}$$

We also have

$$\begin{aligned}\frac{\theta_n}{\theta_{n-1}} &= 1 + \frac{2\left(2\sqrt{2}x+n\right)-1}{\tau'_{n-1}} \\ &= 1 + \frac{1}{\sqrt{2}} \frac{\left(x+\frac{n-1}{2\sqrt{2}}\right)+\frac{\sqrt{2}}{8}}{\left(x+\frac{n-1}{2\sqrt{2}}\right)^2+\frac{y^2}{2}+\frac{1}{32}}.\end{aligned}$$

The desired results then follow from direct application of Lemma 9. □

To proceed, we need to introduce some notations. We define

$$\gamma^* := \frac{A_2}{A_3\lambda^2} = \frac{A_1}{A_4},$$

and

$$\gamma := \frac{A_4}{A_1\lambda^2} = \frac{A_3}{A_2}.$$

Then $\gamma\gamma^* = \lambda^{-2}$. Note that $|\gamma| < 1$, and $|\gamma| \to 1$ as $y \to +\infty$.

We introduce the functions

$$2\pi i P(\xi) = 2\pi i\xi - 2 - \lambda^{-1} e^{\frac{\pi i \xi}{\sqrt{2}}} + \lambda e^{-\frac{\pi i \xi}{\sqrt{2}}},$$

$$Q(\xi) = \left(1-\gamma e^{\frac{\pi i\xi}{\sqrt{2}}}\right)\left(A_1 + A_2 e^{-\frac{\pi i\xi}{\sqrt{2}}}\right) - \left(1-\gamma^* e^{\frac{\pi i\xi}{\sqrt{2}}}\right)\left(A_3 + A_4 e^{-\frac{\pi i\xi}{\sqrt{2}}}\right),$$

and

$$J(\xi) = \left(\frac{1-\gamma^* e^{\frac{\pi i\xi}{\sqrt{2}}}}{1-\gamma e^{\frac{\pi i\xi}{\sqrt{2}}}} e^{-4\pi\beta\xi}\right)',$$

$$R(\xi) = \left(1-\gamma e^{\frac{\pi i\xi}{\sqrt{2}}}\right)\left(A_3 + A_4 e^{-\frac{\pi i\xi}{\sqrt{2}}}\right) e^{4\pi\beta\xi}.$$

We then define

$$h(\xi) = \int_{-\infty}^{\xi} \left(1-\gamma e^{\frac{\pi i s}{\sqrt{2}}}\right) e^{2\pi i(\alpha+\beta i)(-s)} \hat{\phi_n}(s)\, ds. \tag{3.4}$$

Let

$$g(\xi) = \int_{\xi}^{+\infty} h(s) J(s)\, ds. \tag{3.5}$$

Proposition 11.

$$\left[(F_1\phi)_n\right]^{\hat{}} = -\frac{2\pi i e^{2\pi i(\alpha+\beta i)\xi}}{J}\left(Pg'' + \left(Q - P\frac{J'}{J}\right)g' + RJg\right). \tag{3.6}$$

Proof. Using Lemma 10, we find that the Fourier transform of $(F_1\phi)_n$ is equal to

$$\begin{aligned}&2\pi i\xi\hat{\phi_n} - 2\hat{\phi_n} - \lambda^{-1}\hat{\phi_{n+1}} + \lambda\hat{\phi_{n-1}} \\ &- \hat{\phi_n} * \left[-2\pi i A_1 e^{2\pi i(\alpha+\beta i)\xi} u(\xi) - 2\pi i A_3 e^{2\pi i(\alpha-\beta i)\xi} u(-\xi)\right] \\ &- \hat{\phi_n} * \left[2\pi i A_3 e^{2\pi i(\alpha_1+\beta i)\xi} u(\xi) + 2\pi i A_1 e^{2\pi i(\alpha_1-\beta i)\xi} u(-\xi)\right] \\ &- \lambda^{-1}\hat{\phi_{n+1}} * \left[2\pi i\frac{A_4}{\lambda} e^{2\pi i(\alpha+\beta i)\xi} u(\xi) + 2\pi i\frac{A_2}{\lambda} e^{2\pi i(\alpha-\beta i)\xi} u(-\xi)\right] \\ &+ \lambda\hat{\phi_{n-1}} * \left[2\pi i\frac{A_2}{\lambda} e^{2\pi i(\alpha_1+\beta i)\xi} u(\xi) + 2\pi i\frac{A_4}{\lambda} e^{2\pi i(\alpha_1-\beta i)\xi} u(-\xi)\right].\end{aligned}$$

We calculate

$$\begin{aligned}&\hat{\phi_{n-1}} * \left[e^{2\pi i(\alpha_1+\beta i)\xi}u(\xi)\right]\\&=\int_{-\infty}^{\xi} e^{2\pi i(\alpha_1+\beta i)(\xi-s)}e^{-2\pi ics}\hat{\phi_n}(s)\,ds.\end{aligned}$$

Using the fact that $\alpha_1=\alpha-c$, we obtain

$$\hat{\phi_{n-1}} * \left[e^{2\pi i(\alpha_1+\beta i)\xi}u(\xi)\right]=e^{2\pi i(\alpha_1+\beta i)\xi}\int_{-\infty}^{\xi} e^{2\pi i(\alpha+\beta i)(-s)}\hat{\phi_n}(s)\,ds.$$

Similarly,

$$\begin{aligned}\hat{\phi_{n+1}} * \left[e^{2\pi i(\alpha+\beta i)\xi}u(\xi)\right]&=\int_{-\infty}^{\xi} e^{2\pi i(\alpha+\beta i)(\xi-s)}e^{2\pi ics}\hat{\phi_n}(s)\,ds\\&=e^{2\pi i(\alpha+\beta i)\xi}\int_{-\infty}^{\xi} e^{2\pi i(\alpha_1+\beta i)(-s)}\hat{\phi_n}(s)\,ds.\end{aligned}$$

It follows that $\frac{1}{2\pi i}\left[(F_1\phi)_n\right]\hat{}$ is equal to

$$\begin{aligned}&P(\xi)\hat{\phi_n}+\left(A_1+A_2e^{-2\pi ic\xi}\right)e^{2\pi i(\alpha+\beta i)\xi}h\\&+\left(A_3+A_4e^{-2\pi ic\xi}\right)e^{2\pi i(\alpha-\beta i)\xi}\int_{\xi}^{+\infty}\left[1-\gamma^*e^{2\pi ics}\right]e^{2\pi i(\alpha-\beta i)(-s)}\hat{\phi_n}(s)\,ds\\&=P(\xi)\frac{e^{2\pi i(\alpha+\beta i)\xi}}{1-\gamma e^{2\pi ic\xi}}h'+\left(A_1+A_2e^{-2\pi ic\xi}\right)e^{2\pi i(\alpha+\beta i)\xi}h\\&+\left(A_3+A_4e^{-2\pi ic\xi}\right)e^{2\pi i(\alpha-\beta i)\xi}\int_{\xi}^{+\infty}\left[1-\gamma^*e^{2\pi ics}\right]e^{2\pi i(\alpha-\beta i)(-s)}\hat{\phi_n}(s)\,ds.\end{aligned}\tag{3.7}$$

The last term is equal to

$$\begin{aligned}&\left(A_3+A_4e^{-2\pi ic\xi}\right)e^{2\pi i(\alpha-\beta i)\xi}\int_{\xi}^{+\infty}\left[1-\gamma^*e^{2\pi ics}\right]e^{2\pi i(\alpha-\beta i)(-s)}\hat{\phi_n}(s)\,ds\\&=\left(A_3+A_4e^{-2\pi ic\xi}\right)e^{2\pi i(\alpha-\beta i)\xi}\int_{\xi}^{+\infty}\left[1-\gamma^*e^{2\pi ics}\right]e^{2\pi i(\alpha-\beta i)(-s)}\frac{e^{2\pi i(\alpha+\beta i)s}}{1-\gamma e^{2\pi ics}}h'ds\\&=\left(A_3+A_4e^{-2\pi ic\xi}\right)e^{2\pi i(\alpha-\beta i)\xi}\int_{\xi}^{+\infty}\frac{1-\gamma^*e^{2\pi ics}}{1-\gamma e^{2\pi ics}}e^{4\pi i(\beta i)s}h'ds\\&=-\frac{1-\gamma^*e^{2\pi ic\xi}}{1-\gamma e^{2\pi ic\xi}}\left(A_3+A_4e^{-2\pi ic\xi}\right)e^{2\pi i(\alpha+\beta i)\xi}h(\xi)\\&-\left(A_3+A_4e^{-2\pi ic\xi}\right)e^{2\pi i(\alpha-\beta i)\xi}\int_{\xi}^{+\infty}h(s)J(s)\,ds,\end{aligned}$$

Insert this identity into (3.7), we find that

$$\begin{aligned}&\frac{1}{2\pi ie^{2\pi i(\alpha+\beta i)\xi}}\left[(F_1\phi)_n\right]\hat{}=\\&\frac{P(\xi)h'}{1-\gamma e^{2\pi ic\xi}}+\left(A_1+A_2e^{-2\pi ic\xi}\right)h-\left(A_3+A_4e^{-2\pi ic\xi}\right)\frac{1-\gamma^*e^{2\pi ic\xi}}{1-\gamma e^{2\pi ic\xi}}h\\&-\left(A_3+A_4e^{-2\pi ic\xi}\right)e^{4\pi\beta\xi}\int_{\xi}^{+\infty}h(s)J(\xi)\,ds.\end{aligned}$$

Inserting the relation $g'(\xi) = -h(\xi) J(\xi)$ into this equation, we get the identity (3.6). □

Since $\frac{1}{P}$ has a singularity around 0, it will be important to understand the behavior of Q and RJ around 0.

Lemma 12. *For all $y \in \mathbb{R}$, we have*

$$Q(0) = 0 \text{ and } Q'(0) = \pi i.$$

Proof. Direct computation using the definition of γ and γ^* shows that

$$\begin{aligned} Q(0) &= (A_1 + A_2)(1-\gamma) - (A_3 + A_4)(1-\gamma^*) \\ &= A_1 + A_2 - \frac{1}{\lambda^2} A_4 - A_3 - A_3 - A_4 + \frac{1}{\lambda^2} A_2 + A_1 \\ &= 0. \end{aligned}$$

On the othe hand, since A_i does not depend on ξ, we calucate

$$\begin{aligned} Q'(\xi) = {} & A_2(-2\pi ic) e^{-2\pi ic\xi} - A_1 \gamma (2\pi ic) e^{2\pi ic\xi} \\ & + A_3 \gamma^* (2\pi ic) e^{2\pi ic\xi} - A_4(-2\pi ic) e^{-2\pi ic\xi}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{Q'(0)}{2\pi ic} &= -A_2 - A_1\gamma + A_3\gamma^* + A_4 \\ &= \lambda \frac{1}{\sqrt{2}} \left(\frac{1}{2} - \frac{\sqrt{2}}{16\beta i} \right) - \lambda^{-1} \frac{1}{\sqrt{2}} \left(\frac{1}{2} + \frac{\sqrt{2}}{16\beta i} \right) \\ &\quad - \lambda^{-1} \frac{1}{\sqrt{2}} \left(\frac{1}{2} - \frac{\sqrt{2}}{16\beta i} \right) + \lambda \frac{1}{\sqrt{2}} \left(\frac{1}{2} + \frac{\sqrt{2}}{16\beta i} \right) \\ &= \sqrt{2}. \end{aligned}$$

The proof is completed. □

Lemma 13. *For all $y \in \mathbb{R}$, we have*

$$J(0) = 0.$$

Proof. Since

$$\left(\frac{1 - \gamma^* e^{2\pi ics}}{1 - \gamma e^{2\pi ics}} \right)' = \frac{\gamma - \gamma^*}{(1 - \gamma e^{2\pi ics})^2} 2\pi ic e^{2\pi ics},$$

we have

$$\begin{aligned} J(s) &= \frac{\gamma - \gamma^*}{(1 - \gamma e^{2\pi ics})^2} 2\pi ic e^{2\pi ics} e^{-4\pi\beta s} + \frac{1 - \gamma^* e^{2\pi ics}}{1 - \gamma e^{2\pi ics}} (-4\pi\beta) e^{-4\pi\beta s} \\ &= \frac{2\pi e^{-4\pi\beta s}}{(1 - \gamma e^{2\pi ics})^2} \left[(\gamma - \gamma^*) ic e^{2\pi ics} + \left(1 - \gamma e^{2\pi ics}\right)\left(1 - \gamma^* e^{2\pi ics}\right)(-2\beta) \right]. \end{aligned}$$

Letting $s = 0$, it follows that

$$\frac{(1-\gamma)^2}{2\pi} J(0) = \frac{(\gamma - \gamma^*) i}{2\sqrt{2}} - (1-\gamma)(1-\gamma^*) 2\beta.$$

Using software such as *Mathematica*, one can directly verify that $J(0) = 0$. □

Consider the equation

$$Pg'' + \left(Q - P\frac{J'}{J}\right) g' + RJg = 0.$$

Let us write it as

$$P_1 g'' + Q_1 g' + R_1 g = 0. \tag{3.8}$$

Here $P_1 = P$, $Q_1 = Q - P\frac{J'}{J}$, and $R_1 = RJ$. Note that under the transformation $\phi_n \to h \to g$, the function $\phi_n = \omega_n$ corresponding to $g = 0$.

We are interested in the asymptotic behavior of the solutions to this equation. At this point, it is worth pointing out that according to Lemma 13, the function Q_1 has singularities at $\xi = 2\sqrt{2}\pi j, j \in \mathbb{N}$.

Lemma 14. *The equation* (3.8) *has two solutions* g_1, g_2*, satisfying*

$$g_1(\xi) = 1 + O(\xi), \text{ as } \xi \to 0,$$

and

$$g_2(\xi) = \xi^{-2} + O\left(\xi^{-1}\right), \text{ as } \xi \to 0.$$

Moreover, g_1, g_2 *are smooth at* $\xi \neq 0$.

Proof. For ξ close to 0,

$$P(\xi) = \frac{\pi i}{4}\xi^2 + O\left(\xi^3\right), \ Q(\xi) = \pi i \xi + O\left(\xi^2\right).$$

Hence

$$Q_1(\xi) = \frac{3\pi i}{4}\xi + O\left(\xi^2\right).$$

The existence of g_1, g_2 then follows from a perturbation argument.

Near the points $\xi_j = 2\sqrt{2}\pi j$, $j \in \mathbb{N}\setminus\{0\}$, the equation (3.8) can be written as

$$g'' + \left((\xi - \xi_j)^{-1} + O(1)\right) g' + O(\xi - \xi_j) g = 0.$$

We then can deduce the existence of two smooth linearly independent solutions using perturbation arguments again. Indeed, we can also find the asymptotic behavior of these solutions. □

Taking Fourier transform for the equation $(F_1\phi)_n = (G_1\eta)_n$, using Proposition 11, we obtain

$$P_1 g'' + Q_1 g' + R_1 g = -\frac{J}{2\pi i e^{2\pi i(\alpha+\beta i)\xi}} \left[(G_1\eta)_n\right]^{\hat{}} := B(\xi, y). \tag{3.9}$$

Variation of parameter formula tells us that equation (3.9) has a solution of the form

$$g^*(\xi, y) = g_2(\xi, y) \int_{+\infty}^{\xi} \frac{g_1(s,y)}{W(s,y)} \frac{B(s,y)}{P_1(s,y)} ds - g_1(\xi, y) \int_{+\infty}^{\xi} \frac{g_2(s,y)}{W(s,y)} \frac{B(s,y)}{P_1(s,y)} ds.$$

Here $W(s, y)$ is the Wronskian of g_1 and g_2. Let

$$h^*(\xi, y) = -\frac{g'(\xi, y)}{J(\xi, y)},$$

and we define ϕ_n^* by

$$(\phi_n^*)^{\hat{}} = \frac{h^{*\prime}}{\left(1 - \gamma e^{2\pi i c\xi}\right) e^{2\pi i(\alpha+\beta i)(-\xi)}}.$$

With these preparation, now we seek a solution $\phi = \{\phi_n\}$ for the system

$$\begin{cases} (F_1\phi)_n = (G_1\eta)_n\,, \\ (M_1\phi)_n = (N_1\eta)_n\,. \end{cases}$$

in the form

$$\phi_n = \phi_n^* + \zeta(y)\,\omega_n. \tag{3.10}$$

Lemma 15. *There exists a solution ζ to the equation*

$$(M_1\phi)_n(0,y) = (N_1\eta)_n(0,y)\,,$$

with initial condition $\zeta(0) = 0$.

Proof. We need to solve

$$[M_1(\zeta\omega)]_n = (N_1\eta)_n - (M_1\phi^*)_n\,, \text{ for } x = 0. \tag{3.11}$$

Since ω_n satisfies $(F_1\omega)_n = (M_1\omega)_n = 0$, we find that the equation (3.11) has the form

$$\frac{1}{i}\zeta'(y)\,\omega_n = (N_1\eta)_n - (M_1\phi^*)_n\,,\ x = 0.$$

This is a first order ODE for ζ and has a unique solution with the initial condition $\zeta(0) = 0$.

We remark that due to the condition $\eta_{n+1}(x,y) = \eta_n\left(x + \frac{1}{2\sqrt{2}}, y\right)$, the above argument gives us same ζ for different $n \in \mathbb{N}$. □

Lemma 16. *Let $\{\phi_n\}$ be given by* (3.10)*. Define $\Phi_n = (M_1\phi)_n - (N_1\eta)_n$. Suppose $T_\theta\eta = 0$ and $(F_1\phi)_n = (G_1\eta)_n$. Then*

$$\partial_x\Phi_n = \lambda^{-1}\frac{\theta_{n-1}}{\theta_n}\Phi_{n+1} + \left[\left(\frac{\partial_s\theta_n}{\theta_n} + \frac{\partial_t\theta_{n-1}}{\theta_{n-1}}\right) + \lambda + \lambda^{-1}\right]\Phi_n - \lambda\frac{\theta_n}{\theta_{n-1}}\Phi_{n-1}.$$

Proof. We compute

$$\begin{aligned}
\partial_x(M_1\phi)_n &= -i\partial_y\partial_x\phi_n - \partial_x\left[\left(\frac{\partial_s\theta_n}{\theta_n} - \frac{\partial_t\theta_{n-1}}{\theta_{n-1}}\right)\phi_n\right] \\
&\quad - \lambda^{-1}\partial_x\left(\phi_{n+1}\frac{\theta_{n-1}}{\theta_n} - \phi_n\right) - \lambda\partial_x\left(\phi_{n-1}\frac{\theta_n}{\theta_{n-1}} - \phi_n\right) \\
&= -i\partial_y\left[\left(\frac{\partial_s\theta_n}{\theta_n} + \frac{\partial_t\theta_{n-1}}{\theta_{n-1}}\right)\phi_n + \lambda^{-1}\left(\phi_{n+1}\frac{\theta_{n-1}}{\theta_n} - \phi_n\right)\right] \\
&\quad - i\partial_y\left[-\lambda\left(\phi_{n-1}\frac{\theta_n}{\theta_{n-1}} - \phi_n\right) + (G_1\eta)_n\right] - \partial_x\left[\left(\frac{\partial_s\theta_n}{\theta_n} - \frac{\partial_t\theta_{n-1}}{\theta_{n-1}}\right)\phi_n\right] \\
&\quad - \lambda^{-1}\partial_x\left(\frac{\theta_{n-1}}{\theta_n}\right)\phi_{n+1} - \lambda^{-1}\frac{\theta_{n-1}}{\theta_n}\partial_x\phi_{n+1} + \lambda^{-1}\partial_x\phi_n \\
&\quad - \lambda\partial_x\left(\frac{\theta_n}{\theta_{n-1}}\right)\phi_{n-1} - \lambda\frac{\theta_n}{\theta_{n-1}}\partial_x\phi_{n-1} + \lambda\partial_x\phi_n.
\end{aligned}$$

Plugging the identity

$$\begin{aligned}
\partial_x\phi_n &= \left(\frac{\partial_s\theta_n}{\theta_n} + \frac{\partial_t\theta_{n-1}}{\theta_{n-1}}\right)\phi_n + \lambda^{-1}\left(\phi_{n+1}\frac{\theta_{n-1}}{\tau_n'} - \phi_n\right) - \lambda\left(\phi_{n-1}\frac{\theta_n}{\theta_{n-1}} - \phi_n\right) + (G_1\eta)_n\,, \\
-i\partial_y\phi_n &= \left(\frac{\partial_s\theta_n}{\theta_n} - \frac{\partial_t\theta_{n-1}}{\theta_{n-1}}\right)\phi_n + \lambda^{-1}\left(\phi_{n+1}\frac{\theta_{n-1}}{\theta_n} - \phi_n\right) + \lambda\left(\phi_{n-1}\frac{\theta_n}{\theta_{n-1}} - \phi_n\right) + (M_1\phi)_n
\end{aligned}$$

into $\partial_x (M_1\phi)_n$, we find that the coefficient before ϕ_{n+1} is

$$
\begin{aligned}
& - i\lambda^{-1}\partial_y\left(\frac{\theta_{n-1}}{\theta_n}\right) - \lambda^{-1}\partial_x\left(\frac{\theta_{n-1}}{\theta_n}\right) \\
& - i\lambda^{-1}\frac{\theta_{n-1}}{\theta_n}\left[i\left(\frac{\partial_s\theta_{n+1}}{\theta_{n+1}} - \frac{\partial_t\theta_n}{\theta_n}\right) - i\left(\lambda+\lambda^{-1}\right)\right] \\
& - \left(\frac{\partial_s\theta_n}{\theta_n} - \frac{\partial_t\theta_{n-1}}{\theta_{n-1}}\right)\lambda^{-1}\frac{\theta_{n-1}}{\theta_n} \\
& - i\left[\left(\frac{\partial_s\theta_n}{\theta_n} + \frac{\partial_t\theta_{n-1}}{\theta_{n-1}}\right) + \left(\lambda-\lambda^{-1}\right)\right] i\lambda^{-1}\frac{\theta_{n-1}}{\theta_n} \\
& - \lambda^{-1}\frac{\theta_{n-1}}{\theta_n}\left[\left(\frac{\partial_s\theta_{n+1}}{\theta_{n+1}} + \frac{\partial_t\theta_n}{\theta_n}\right) + \left(\lambda-\lambda^{-1}\right)\right] \\
& + \left(\lambda+\lambda^{-1}\right)\lambda^{-1}\frac{\theta_{n-1}}{\theta_n}.
\end{aligned}
$$

One could verify directly that this is equal to 0. Similarly the coefficient before ϕ_{n-1} and ϕ_n is equal to 0. Hence we get

$$
\begin{aligned}
\partial_x (M_1\phi)_n = {} & \lambda^{-1}\frac{\theta_{n-1}}{\theta_n}(M_1\phi)_{n+1} + \left[\left(\frac{\partial_s\theta_n}{\theta_n} + \frac{\partial_t\theta_{n-1}}{\theta_{n-1}}\right) + \lambda + \lambda^{-1}\right](M_1\phi)_n \\
& - \lambda\frac{\theta_n}{\theta_{n-1}}(M_1\phi)_{n-1} - i\partial_y (G_1\eta)_n - \lambda^{-1}\frac{\theta_{n-1}}{\theta_n}(G_1\eta)_{n+1} \\
& - \lambda\frac{\theta_n}{\theta_{n-1}}(G_1\eta)_{n-1} + \left(\lambda+\lambda^{-1}\right)(G_1\eta)_n.
\end{aligned}
$$

After some tedious computations(alternatively, we can use the software *Mathematica* to verify this), we find that

$$
\partial_x\Phi_n = \lambda^{-1}\frac{\theta_{n-1}}{\theta_n}\Phi_{n+1} + \left[\left(\frac{\partial_s\theta_n}{\theta_n} + \frac{\partial_t\theta_{n-1}}{\theta_{n-1}}\right) + \lambda + \lambda^{-1}\right]\Phi_n - \lambda\frac{\theta_n}{\theta_{n-1}}\Phi_{n-1}.
$$

□

For $y \in \mathbb{R}$, we define

$$
k(y) := \int_{+\infty}^{0} \frac{g_1(s,y)\,B(s,y)}{W(s,y)\,P_1(s,y)}ds,
$$

where B is defined in (3.9).

Lemma 17. *As $y \to +\infty$, $k(y) \to 0$.*

Proof. Recall that under the notation of Lemma 4, $\eta_n = \theta_n \tilde{\eta}_n$. We compute

$$\begin{aligned}
(G_1\eta)_n &= \frac{\omega_n}{\theta_n}\partial_s\eta_n + \frac{\omega_n}{\theta_{n-1}}\partial_t\eta_{n-1} + \left(-\frac{\partial_s\omega_n}{\theta_n} - \lambda^{-1}\frac{\omega_n}{\theta_n} - \lambda\frac{\omega_{n-1}}{\theta_{n-1}}\right)\eta_n \\
&+ \left(\lambda\frac{\omega_{n+1}}{\theta_n} - \frac{\partial_t\omega_n}{\theta_{n-1}} + \frac{\lambda\omega_n}{\theta_{n-1}}\right)\eta_{n-1} \\
&= \frac{\omega_n}{\theta_n}\left(\frac{1}{2}\partial_x\left(\theta_n\tilde{\eta}_n\right) + \frac{1}{2i}\partial_y\left(\theta_n\tilde{\eta}_n\right)\right) + \frac{\omega_n}{\theta_{n-1}}\left(\frac{1}{2}\partial_x\left(\theta_{n-1}\tilde{\eta}_{n-1}\right) - \frac{1}{2i}\partial_y\left(\theta_{n-1}\tilde{\eta}_{n-1}\right)\right) \\
&- \left(\frac{\lambda}{\theta_n} + \lambda^{-1}\frac{\omega_n}{\theta_n} + \lambda\frac{\omega_{n-1}}{\theta_{n-1}}\right)\theta_n\tilde{\eta}_n + \left(\lambda\frac{\omega_{n+1}}{\theta_n} - \frac{\lambda^{-1}}{\theta_{n-1}} + \frac{\lambda\omega_n}{\theta_{n-1}}\right)\theta_{n-1}\tilde{\eta}_{n-1} \\
&= \frac{\omega_n}{2}\partial_x\tilde{\eta}_n + \frac{1}{2}\omega_n\frac{\partial_x\theta_n}{\theta_n}\tilde{\eta}_n + \frac{1}{2i}\omega_n\partial_y\tilde{\eta}_n + \frac{1}{2i}\frac{\omega_n\partial_y\theta_n}{\theta_n}\tilde{\eta}_n \\
&+ \frac{\omega_n}{2}\partial_x\tilde{\eta}_{n-1} + \frac{1}{2}\frac{\omega_n\partial_x\theta_{n-1}}{\theta_{n-1}}\tilde{\eta}_{n-1} - \frac{1}{2i}\omega_n\partial_y\tilde{\eta}_{n-1} - \frac{1}{2i}\frac{\omega_n\partial_y\theta_{n-1}}{\theta_{n-1}}\tilde{\eta}_{n-1} \\
&- \left(\lambda + \lambda^{-1}\omega_n + \lambda\frac{\omega_{n-1}\theta_n}{\theta_{n-1}}\right)\tilde{\eta}_n + \left(\lambda\frac{\omega_{n+1}}{\theta_n}\theta_{n-1} - \lambda^{-1} + \lambda\omega_n\right)\tilde{\eta}_{n-1}.
\end{aligned}$$

On the other hand, since $\tilde{\eta}_n$ satisfies (2.9) and $\mathcal{F}(f_n)(0,0) = 0$, we infer that

$$\left|(G_1\eta)_n^{\hat{}}\right| \le C.$$

From this estimate and the fact that $W(s,y) = O\left(s^3\right)$, we get

$$\left|\frac{g_1(s,y)B(s,y)}{W(s,y)P_1(s,y)}\right| \le Cs^2, \text{ for } s \in (0,1), \tag{3.12}$$

and

$$\left|\frac{g_2(s,y)B(s,y)}{W(s,y)P_1(s,y)}\right| \le Ce^{-2\pi\beta s}, \text{ for all } s > 0. \tag{3.13}$$

It follows that

$$|k(y)| \le C\int_0^{\sqrt{\frac{1}{y}}} s^2 ds + C\int_{\sqrt{\frac{1}{y}}}^{+\infty} e^{-2\pi\beta s} ds \to 0, \text{ as } y \to +\infty.$$

□

Proof of Proposition 5. For each fixed y, using Lemma 16, $\Phi_n(0,y) = 0$, and the Gronwall inequality, we deduce $\Phi_n(x,y) = 0$ for all x. Hence $\{\phi_n\}$ solves the system (3.2). It remains to prove the growth estimate for ϕ_0.

Let us define

$$\chi(\xi,y) = -\frac{1}{\left(1-\gamma e^{2\pi i c\xi}\right)e^{2\pi i(\alpha+\beta i)(-\xi)}}\left(\frac{g_1'}{J}\right)'.$$

Note that by the asymptotic behavior of g_1,

$$\chi(\xi,y) = O\left(\xi^{-5}\right). \tag{3.14}$$

Near $\xi = 0$, we can write

$$\hat{\phi_n}(\xi,y) = k(y)\chi(\xi,y) + O\left(\xi^{-2}\right) + \zeta(y)\hat{\omega_n}.$$

Inserting this into the equation $[(M_1\phi)_n]\hat{} = [(N_1\eta)_n]\hat{}$, using the asymptotic behavior (3.14) of χ, we find that $k'(y) = 0$. Applying Lemma 17, we then deduce $k(y) = 0$. This in turn implies that

$$g^* = -g_2 \int_0^{\xi} \frac{g_1 B}{W P_1} ds - g_1 \int_{+\infty}^{\xi} \frac{g_2 B}{W P_1} ds.$$

This together with (3.12) and (3.13) in particular tells us that

$$\left|\hat{\phi_0}(\xi, y) - \zeta(y)\hat{\omega_0}\right| \leq C e^{2\pi\beta|\xi|}\left(1 + \xi^{-2}\right). \tag{3.15}$$

After some manipulation on the Fourier integral representation of ϕ_0, we obtain the desired estimate (3.3) for ϕ_0. Indeed, the exponent $\frac{5}{8}$ can be replaced by any number larger than $\frac{1}{2}$(But in general can not be $\frac{1}{2}$). □

4. The linearized Bäcklund transformation between κ and ω

Linearizing the Bäcklund transformation (2.13) at $\{\kappa_n\}, \{\omega_n\}$, we get

$$\begin{cases} \partial_s \sigma_n \omega_n - \sigma_n \partial_s \omega_n - \lambda\left(\sigma_{n+1}\omega_{n-1} - \sigma_n \omega_n\right) \\ = -\partial_s \kappa_n \phi_n + \kappa_n \partial_s \phi_n + \lambda\left(\kappa_{n+1}\phi_{n-1} - \kappa_n \phi_n\right), \\ \partial_t \sigma_{n+1}\omega_n - \sigma_{n+1}\partial_t \omega_n + \lambda^{-1}\left(\sigma_n \omega_{n+1} - \sigma_{n+1}\omega_n\right) \\ = -\partial_t \kappa_{n+1}\phi_n + \kappa_{n+1}\partial_t \phi_n - \lambda^{-1}\left(\kappa_n \phi_{n+1} - \kappa_{n+1}\phi_n\right). \end{cases} \tag{4.1}$$

We write this system as

$$\begin{cases} (F_0\sigma)_n = (G_0\phi)_n, \\ (M_0\sigma)_n = (N_0\phi)_n. \end{cases} \tag{4.2}$$

Here

$$(F_0\sigma)_n = \partial_x \sigma_n - \lambda \frac{\omega_{n-1}}{\omega_n}\left(\sigma_{n+1} - \sigma_n\right) - \lambda^{-1}\frac{\omega_n}{\omega_{n-1}}\left(\sigma_n - \sigma_{n-1}\right),$$

$$(M_0\sigma)_n = -i\partial_y \sigma_n - \lambda \frac{\omega_{n-1}}{\omega_n}\left(\sigma_{n+1} - \sigma_n\right) + \lambda^{-1}\frac{\omega_n}{\omega_{n-1}}\left(\sigma_n - \sigma_{n-1}\right),$$

and

$$\begin{aligned} (G_0\phi)_n = &\frac{-\partial_s \kappa_n \phi_n + \kappa_n \partial_s \phi_n + \lambda\left(\kappa_{n+1}\phi_{n-1} - \kappa_n \phi_n\right)}{\omega_n} \\ &+ \frac{-\partial_t \kappa_n \phi_{n-1} + \kappa_n \partial_t \phi_{n-1} - \lambda^{-1}\left(\kappa_{n-1}\phi_n - \kappa_n \phi_{n-1}\right)}{\omega_{n-1}}, \end{aligned}$$

$$\begin{aligned} (N_0\phi)_n = &\frac{-\partial_s \kappa_n \phi_n + \kappa_n \partial_s \phi_n + \lambda\left(\kappa_{n+1}\phi_{n-1} - \kappa_n \phi_n\right)}{\omega_n} \\ &- \frac{-\partial_t \kappa_n \phi_{n-1} + \kappa_n \partial_t \phi_{n-1} - \lambda^{-1}\left(\kappa_{n-1}\phi_n - \kappa_n \phi_{n-1}\right)}{\omega_{n-1}}. \end{aligned}$$

The main result of this section is the following

Proposition 18. *Let $\{\phi_n\}$ be given by Proposition 5. Then the system* (4.2) *has a solution $\{\sigma_n\}$ with $\sigma_{n+1}(x, y) = \sigma_n\left(x + \frac{1}{2\sqrt{2}}, y\right)$ and*

$$|\sigma_0(x, y)| \leq C\left(1 + x^2 + y^2\right)^{\frac{5}{8}}.$$

Let us define

$$P_0(\xi) = 2\pi i\xi - \lambda\left(e^{\frac{\pi i}{\sqrt{2}}\xi} - 1\right) - \lambda^{-1}\left(1 - e^{-\frac{\pi i}{\sqrt{2}}\xi}\right), \tag{4.3}$$

$$Q_0(\xi) = -\lambda\frac{\pi i}{\sqrt{2}}\left(e^{\frac{\pi i}{\sqrt{2}}\xi} - 1\right) + \lambda^{-1}\frac{\pi i}{\sqrt{2}}\left(1 - e^{-\frac{\pi i}{\sqrt{2}}\xi}\right).$$

We see from Lemma 6 that the Fourier transform of $\frac{1}{\omega_n}$ will depend on the sign of y. Define

$$\alpha_0 = \frac{n + \frac{\sqrt{2}-1}{2}}{2\sqrt{2}},\ \beta_0 = \frac{|y|}{\sqrt{2}},$$

and introduce a new function

$$h_1(\xi) = \int_{-\infty}^{\xi} e^{2\pi i(\alpha_0+\beta_0 i)(-s)}\left(e^{\frac{\pi i}{\sqrt{2}}s} - 1\right)\hat{\sigma_n}(s)\,ds.$$

Lemma 19. *Suppose $y > 0$. Then*

$$\left[(F_0\sigma)_n\right]\hat{} = \frac{e^{2\pi i(\alpha_0+\beta_0 i)\xi}}{e^{\frac{\pi i}{\sqrt{2}}\xi} - 1}\left(P_0(\xi)\,h_1'(\xi) + Q_0(\xi)\,h_1(\xi)\right). \tag{4.4}$$

Proof. Using Lemma 6, we get

$$\begin{aligned}\left(\frac{\omega_{n-1}}{\omega_n}\right)\hat{} &= 1\hat{} - \left[\frac{1}{2\sqrt{2}x + n + \frac{\sqrt{2}-1}{2} + 2yi}\right]\hat{}\\ &= \delta - \frac{1}{2\sqrt{2}}(-2\pi i)\,e^{2\pi i(\alpha_0+\beta_0 i)\xi}u(\xi).\end{aligned}$$

Then we compute

$$\begin{aligned}\left[\frac{\omega_{n-1}}{\omega_n}(\sigma_{n+1} - \sigma_n)\right]\hat{} &= \left(e^{\frac{\pi i}{\sqrt{2}}\xi} - 1\right)\hat{\sigma_n} + \frac{\pi i}{\sqrt{2}}\left[e^{2\pi i(\alpha_0+\beta_0 i)\xi}u(\xi)\right] * \left[\left(e^{\frac{\pi i}{\sqrt{2}}\xi} - 1\right)\hat{\sigma_n}\right]\\ &= \left(e^{\frac{\pi i}{\sqrt{2}}\xi} - 1\right)\hat{\sigma_n} + \frac{\pi i}{\sqrt{2}}\int_{-\infty}^{\xi} e^{2\pi i(\alpha_0+\beta_0 i)(\xi-s)}\left(e^{\frac{\pi i}{\sqrt{2}}s} - 1\right)\hat{\sigma_n}(s)\,ds\end{aligned}$$

Similarly, we have

$$\begin{aligned}\left[\frac{\omega_n}{\omega_{n-1}}(\sigma_n - \sigma_{n-1})\right]\hat{} &= \left(1 - e^{-\frac{\pi i}{\sqrt{2}}\xi}\right)\hat{\sigma_n}\\ &\quad + \frac{1}{2\sqrt{2}}(-2\pi i)\left[e^{2\pi i(\alpha_0+\beta_0 i)\xi}e^{-\frac{\pi i}{\sqrt{2}}\xi}u(\xi)\right] * \left[\left(1 - e^{-\frac{\pi i}{\sqrt{2}}\xi}\right)\hat{\sigma_n}\right]\\ &= \left(1 - e^{-\frac{\pi i}{\sqrt{2}}\xi}\right)\hat{\sigma_n}\\ &\quad - \frac{\pi i}{\sqrt{2}}\int_{-\infty}^{\xi} e^{2\pi i(\alpha_0+\beta_0 i)(\xi-s)}e^{-\frac{\pi i}{\sqrt{2}}(\xi-s)}\left(1 - e^{-\frac{\pi i}{\sqrt{2}}s}\right)\hat{\sigma_n}(s)\,ds.\end{aligned}$$

Consequently,

$$
\begin{aligned}
\left[(F_0\sigma)_n\right]\hat{} &= 2\pi i\xi\hat{\sigma_n} - \lambda\left(e^{\frac{\pi i}{\sqrt{2}}\xi}-1\right)\hat{\sigma_n} \\
&- \lambda\frac{\pi i}{\sqrt{2}}\int_{-\infty}^{\xi} e^{2\pi i(\alpha_0+\beta_0 i)(\xi-s)}\left(e^{\frac{\pi i}{\sqrt{2}}s}-1\right)\hat{\sigma_n}(s)\,ds \\
&- \lambda^{-1}\left(1-e^{-\frac{\pi i}{\sqrt{2}}\xi}\right)\hat{\sigma_n} \\
&+ \lambda^{-1}\frac{\pi i}{\sqrt{2}}\int_{-\infty}^{\xi} e^{2\pi i(\alpha_0+\beta_0 i)(\xi-s)}e^{-\frac{\pi i}{\sqrt{2}}(\xi-s)}\left(1-e^{-\frac{\pi i}{\sqrt{2}}s}\right)\hat{\sigma_n}(s)\,ds \\
&= 2\pi i\xi\frac{h_1'(\xi)\,e^{2\pi i(\alpha_0+\beta_0 i)\xi}}{\left(e^{\frac{\pi i}{\sqrt{2}}\xi}-1\right)} - \lambda h_1'(\xi)\,e^{2\pi i(\alpha_0+\beta_0 i)\xi} - \lambda\frac{\pi i}{\sqrt{2}}e^{2\pi i(\alpha_0+\beta_0 i)\xi}h_1 \\
&- \lambda^{-1}e^{-\frac{\pi i}{\sqrt{2}}\xi}h_1'(\xi)\,e^{2\pi i(\alpha_0+\beta_0 i)\xi} + \lambda^{-1}\frac{\pi i}{\sqrt{2}}e^{-\frac{\pi i}{\sqrt{2}}\xi}e^{2\pi i(\alpha_0+\beta_0 i)\xi}h_1.
\end{aligned}
$$

This is (4.4). □

Now let us define

$$
h_2(\xi) = \int_{\xi}^{+\infty} e^{2\pi i(\alpha_0-\beta_0 i)(-s)}\left(e^{\frac{\pi i}{\sqrt{2}}s}-1\right)\hat{\sigma_n}(s)\,ds.
$$

Lemma 20. *Suppose $y<0$. Then*

$$
\left[(F_0\sigma)_n\right]\hat{} = -\frac{e^{2\pi i(\alpha_0-\beta_0 i)\xi}}{e^{\frac{\pi i}{\sqrt{2}}\xi}-1}\left(P(\xi)\,h_2'(\xi)+Q(\xi)\,h_2(\xi)\right).
$$

Proof. The proof is similar to that in the previous lemma. By Lemma 7,

$$
\begin{aligned}
\left(\frac{\omega_{n-1}}{\omega_n}\right)\hat{} &= 1\hat{} - \left[\frac{1}{2\sqrt{2}x+n+\frac{\sqrt{2}-1}{2}-|y|\,i}\right]\hat{} \\
&= \delta - \frac{1}{2\sqrt{2}}(2\pi i)\,e^{2\pi i(\alpha_0-\beta_0 i)\xi}u(-\xi).
\end{aligned}
$$

Then we compute

$$
\begin{aligned}
\left[\frac{\omega_{n-1}}{\omega_n}(\sigma_{n+1}-\sigma_n)\right]\hat{} &= \left(e^{\frac{\pi i}{\sqrt{2}}\xi}-1\right)\hat{\sigma_n} - \frac{\pi i}{\sqrt{2}}\left[e^{2\pi i(\alpha_0-\beta_0 i)\xi}u(-\xi)\right]*\left[\left(e^{\frac{\pi i}{\sqrt{2}}\xi}-1\right)\hat{\sigma_n}\right] \\
&= \left(e^{\frac{\pi i}{\sqrt{2}}\xi}-1\right)\hat{\sigma_n} - \frac{\pi i}{\sqrt{2}}\int_{\xi}^{+\infty} e^{2\pi i(\alpha_0-\beta_0 i)(\xi-s)}\left(e^{\frac{\pi i}{\sqrt{2}}s}-1\right)\hat{\sigma_n}(s)\,ds.
\end{aligned}
$$

We also have

$$
\begin{aligned}
\left[\frac{\omega_n}{\omega_{n-1}}(\sigma_n-\sigma_{n-1})\right]\hat{} &= \left(1-e^{-\frac{\pi i}{\sqrt{2}}\xi}\right)\hat{\sigma_n} \\
&+ \frac{1}{2\sqrt{2}}(2\pi i)\left[e^{2\pi i(\alpha_0-\beta_0 i)\xi}e^{-\frac{\pi i}{\sqrt{2}}\xi}u(-\xi)\right]*\left[\left(1-e^{-\frac{\pi i}{\sqrt{2}}\xi}\right)\hat{\sigma_n}\right] \\
&= \left(1-e^{-\frac{\pi i}{\sqrt{2}}\xi}\right)\hat{\sigma_n} \\
&+ \frac{\pi i}{\sqrt{2}}\int_{\xi}^{+\infty} e^{2\pi i(\alpha_0-\beta_0 i)(\xi-s)}e^{-\frac{\pi i}{\sqrt{2}}(\xi-s)}\left(1-e^{-\frac{\pi i}{\sqrt{2}}s}\right)\hat{\sigma_n}(s)\,ds.
\end{aligned}
$$

It follows that

$$[(F_0\sigma)_n]\hat{} = 2\pi i\xi\hat{\sigma_n} - \lambda\left(e^{\frac{\pi i}{\sqrt{2}}\xi} - 1\right)\hat{\sigma_n}$$
$$+\lambda\frac{\pi i}{\sqrt{2}}\int_{-\infty}^{\xi} e^{2\pi i(\alpha_0+\beta_0 i)(\xi-s)}\left(e^{\frac{\pi i}{\sqrt{2}}s} - 1\right)\hat{\sigma_n}(s)\,ds - \lambda^{-1}\left(1 - e^{-\frac{\pi i}{\sqrt{2}}\xi}\right)\hat{\sigma_n}$$
$$-\lambda^{-1}\frac{\pi i}{\sqrt{2}}\int_{-\infty}^{\xi} e^{2\pi i(\alpha_0+\beta_0 i)(\xi-s)}e^{-\frac{\pi i}{\sqrt{2}}(\xi-s)}\left(1 - e^{-\frac{\pi i}{\sqrt{2}}s}\right)\hat{\sigma_n}(s)\,ds$$
$$= -2\pi i\xi\frac{h_2' e^{2\pi i(\alpha_0-\beta_0 i)\xi}}{\left(e^{\frac{\pi i}{\sqrt{2}}\xi} - 1\right)} + \lambda h_2' e^{2\pi i(\alpha_0-\beta_0 i)\xi}$$
$$+\lambda\frac{\pi i}{\sqrt{2}}e^{2\pi i(\alpha_0-\beta_0 i)\xi}h_2 + \lambda^{-1}e^{-\frac{\pi i}{\sqrt{2}}\xi}h_2' e^{2\pi i(\alpha_0-\beta_0 i)\xi} - \lambda^{-1}\frac{\pi i}{\sqrt{2}}e^{-\frac{\pi i}{\sqrt{2}}\xi}e^{2\pi i(\alpha_0-\beta_0 i)\xi}h_2.$$

We conclude that

$$e^{-2\pi i(\alpha_0-\beta_0 i)\xi}[(F_0\sigma)_n]\hat{} = \left[\frac{-2\pi i\xi}{\left(e^{\frac{\pi i}{\sqrt{2}}\xi} - 1\right)} + \lambda + \lambda^{-1}e^{-\frac{\pi i}{\sqrt{2}}\xi}\right]h_2'$$
$$+\left[\lambda\frac{\pi i}{\sqrt{2}} - \lambda^{-1}\frac{\pi i}{\sqrt{2}}e^{-\frac{\pi i}{\sqrt{2}}\xi}\right]h_2.$$

The proof is completed. □

Lemma 21. *The functions P_0 and Q_0 have the following asymptotic behavior as $\xi \to 0$:*

$$P_0(\xi) = \frac{\pi^2\xi^2}{2} + O\left(\xi^3\right),$$
$$Q_0(\xi) = \pi^2\xi + O\left(\xi^2\right).$$

Moreover,

$$P_0(\xi) \neq 0, \textit{for } \xi \neq 0.$$

Proof. This follows from direct computations. □

Lemma 22. *The equation*

$$P_0(\xi)h'(\xi) + Q_0(\xi)h(\xi) = 0$$

has a solution ρ such that

$$\rho(\xi) = \xi^{-2} + O\left(\xi^{-1}\right), \textit{ as } \xi \to 0.$$

Proof. This follows from the asymptotic behavior of P_0 and Q_0 near 0. □

Having understood the Fourier transform of the $(F_0\sigma)_n$, we proceed to solve the equation

$$(F_0\sigma)_n = (G_0\sigma)_n.$$

Taking Fourier transform, we get $[(F_0\sigma)_n]\hat{} = [(G_0\sigma)_n]\hat{}$. Using Lemma 19 and Lemma 20, we arrive at a first order ODE. Let

$$A(\xi, y) = \frac{e^{\frac{\pi i}{\sqrt{2}}\xi} - 1}{e^{2\pi i\left(\alpha_0 + \frac{yi}{\sqrt{2}}\right)\xi}P_0}[(G_0\phi)_n]\hat{}.$$

For $y>0$, we define

$$h^{*}(\xi,y)=\rho(\xi)\int_{-\infty}^{\xi}(\rho(s))^{-1}A(\xi,y)\,ds,$$

and

$$\hat{\sigma_n^{*}}(\xi,y)=\frac{e^{2\pi i\left(\alpha_0+\frac{yi}{\sqrt{2}}\right)\xi}h^{*\prime}}{e^{\frac{\pi i}{\sqrt{2}}\xi}-1}=-\frac{e^{2\pi i\left(\alpha_0+\frac{yi}{\sqrt{2}}\right)\xi}h^{*}Q_0}{\left(e^{\frac{\pi i}{\sqrt{2}}\xi}-1\right)P_0}.$$

Similarly, if $y<0$, we define

$$h^{*}(\xi,y)=\rho(\xi)\int_{+\infty}^{\xi}(\rho(s))^{-1}A(\xi,y)\,ds,$$

and

$$\hat{\sigma_n^{*}}(\xi,y)=\frac{e^{2\pi i\left(\alpha_0+\frac{yi}{\sqrt{2}}\right)\xi}h^{*\prime}}{e^{\frac{\pi i}{\sqrt{2}}\xi}-1}=-\frac{e^{2\pi i\left(\alpha_0+\frac{yi}{\sqrt{2}}\right)\xi}h^{*}Q_0}{\left(e^{\frac{\pi i}{\sqrt{2}}\xi}-1\right)P_0}.$$

We would like to solve the equation $(M_0\sigma)_n=(N_0\phi)_n$.

Lemma 23. *The equation*

$$(M_0\sigma)_n(0,y)=(N_0\phi)_n(0,y),\,y>0.$$

has a solution of the form $\sigma_n^{*}+\gamma(y)$, *with the initial condition* $\gamma(0)=0$.

Proof. This is a first order ODE for the function γ, of the form

$$\frac{1}{i}\gamma'(y)=(N_0\phi)_n(0,y)-(M_0\sigma^{*})_n(0,y).$$

Integrating this equation, we get the solution. □

Similarly, we have

Lemma 24. *The equation*

$$(M_0\sigma)_n(0,y)=(N_0\phi)_n(0,y),\,y<0.$$

has a solution σ_n *of the form* $\sigma_n^{*}+\gamma(y)$, *with the initial condition* $\gamma(0)=0$.

To proceed, slightly abuse the notation with the previous section, we define $\Phi_n=(M_0\sigma)_n-(N_0\phi)_n$.

Lemma 25. *Assume* $y\neq 0$. *Suppose* $T_\omega\phi=0$ *and* $(F_0\sigma)_n=(G_0\phi)_n$, $n\in\mathbb{N}$. *Then*

$$\partial_x\Phi_n=\lambda\frac{\omega_{n-1}}{\omega_n}\Phi_{n+1}+\left(\lambda^{-1}\frac{\omega_n}{\omega_{n-1}}-\lambda\frac{\omega_{n-1}}{\omega_n}\right)\Phi_n-\lambda^{-1}\frac{\omega_n}{\omega_{n-1}}\Phi_{n-1}. \tag{4.5}$$

Proof. We compute

$$\begin{aligned}\partial_x(M_0\sigma)_n&=-i\partial_y\partial_x\sigma_n-\lambda\partial_x\left[\frac{\omega_{n-1}}{\omega_n}(\sigma_{n+1}-\sigma_n)\right]+\lambda^{-1}\partial_x\left[\frac{\omega_n}{\omega_{n-1}}(\sigma_n-\sigma_{n-1})\right]\\&=-i\partial_y\left[\lambda\frac{\omega_{n-1}}{\omega_n}(\sigma_{n+1}-\sigma_n)+\lambda^{-1}\frac{\omega_n}{\omega_{n-1}}(\sigma_n-\sigma_{n-1})+(G_0\phi)_n\right]\\&\quad-\lambda\partial_x\left(\frac{\omega_{n-1}}{\omega_n}\right)(\sigma_{n+1}-\sigma_n)-\lambda\frac{\omega_{n-1}}{\omega_n}\partial_x(\sigma_{n+1}-\sigma_n)\\&\quad+\lambda^{-1}\partial_x\left(\frac{\omega_n}{\omega_{n-1}}\right)(\sigma_n-\sigma_{n-1})+\lambda^{-1}\frac{\omega_n}{\omega_{n-1}}\partial_x(\sigma_n-\sigma_{n-1}).\end{aligned}$$

Inserting the identity

$$\partial_x \sigma_n = \lambda \frac{\omega_{n-1}}{\omega_n} (\sigma_{n+1} - \sigma_n) + \lambda^{-1} \frac{\omega_n}{\omega_{n-1}} (\sigma_n - \sigma_{n-1}) + (G_0 \phi)_n ,$$

$$\partial_y \sigma_n = i \left[-\lambda \frac{\omega_{n-1}}{\omega_n} (\sigma_{n+1} - \sigma_n) + \lambda^{-1} \frac{\omega_n}{\omega_{n-1}} (\sigma_n - \sigma_{n-1}) \right] + i (M_0 \sigma)_n ,$$

into $\partial_x (M_0 \sigma)_n$, we find that

$$\begin{aligned} \partial_x (M_0 \sigma)_n = {} & \lambda \frac{\omega_{n-1}}{\omega_n} (M_0 \sigma)_{n+1} + \left(\lambda^{-1} \frac{\omega_n}{\omega_{n-1}} - \lambda \frac{\omega_{n-1}}{\omega_n} \right) (M_0 \sigma)_n \\ & - \lambda^{-1} \frac{\omega_n}{\omega_{n-1}} (M_0 \sigma)_{n-1} - i \partial_y (G_0 \phi)_n \\ & - \lambda \frac{\omega_{n-1}}{\omega_n} \left((G_0 \phi)_{n+1} - (G_0 \phi)_n \right) \\ & + \lambda^{-1} \frac{\omega_n}{\omega_{n-1}} \left((G_0 \phi)_n - (G_0 \phi)_{n-1} \right) . \end{aligned} \tag{4.6}$$

On the other hand, we compute

$$\begin{aligned} -i \partial_y (G_0 \phi)_n - \partial_x (N_0 \phi)_n = {} & -2 \partial_t \left[\frac{\partial_s \phi_n + \lambda (\phi_{n-1} - \phi_n)}{\omega_n} \right] \\ & + 2 \partial_s \left[\frac{\partial_t \phi_{n-1} - \lambda^{-1} (\phi_n - \phi_{n-1})}{\omega_{n-1}} \right] . \end{aligned}$$

Using the fact that $T_\omega \phi = 0$, we get

$$\begin{aligned} & - i \partial_y (G_0 \phi)_n - \partial_x (N_0 \phi)_n \\ & = -\lambda \frac{\omega_{n-1}}{\omega_n} (N_0 \phi)_{n+1} - \left(\lambda^{-1} \frac{\omega_n}{\omega_{n-1}} - \lambda \frac{\omega_{n-1}}{\omega_n} \right) (N_0 \phi)_n + \lambda^{-1} \frac{\omega_n}{\omega_{n-1}} (N_0 \phi)_{n-1} \\ & + \lambda \frac{\omega_{n-1}}{\omega_n} \left((G_0 \phi)_{n+1} - (G_0 \phi)_n \right) - \lambda^{-1} \frac{\omega_n}{\omega_{n-1}} \left((G_0 \phi)_n - (G_0 \phi)_{n-1} \right) . \end{aligned}$$

This identity together with (4.6) yield (4.5). □

Proof of Proposition 18. For each fixed $y \neq 0$, since Φ_n satisfies (4.5) and the initial condition $\Phi_n (0, y) = 0$, we deduce that $\Phi_n (x, y) = 0$, for all $x \in \mathbb{R}$. Observe that σ_n may have a jump across the x axis.

We would like to show that actually σ_n is continuous at $y = 0$, that is,

$$\lim_{y \to 0^+} \sigma_n (x, y) = \lim_{y \to 0^-} \sigma_n (x, y) .$$

To see this, according to the definition of σ_n^*, it will be suffice to prove

$$\int_{-\infty}^{0} \rho^{-1} (s) A (s, y) \, ds = 0, y > 0, \tag{4.7}$$

and

$$\int_{+\infty}^{0} \rho^{-1} (s) A (s, y) \, ds = 0, y < 0. \tag{4.8}$$

Let $k_1 (y) = \int_{-\infty}^{0} \rho^{-1} (s) A (s, y) \, ds, y > 0$. Using the estimate of the Fourier transform of ϕ_n(See (3.15)) and the asymptotic behavior of ρ, we can show that $k_1 (y) \to$

0, as $y \to +\infty$. On the other hand, letting

$$w(\xi, y) = -\frac{e^{2\pi i\left(\alpha_0 + \frac{yi}{\sqrt{2}}\right)\xi} \rho' Q_0}{\left(e^{\frac{\pi i}{\sqrt{2}}\xi} - 1\right) P_0},$$

we have $\hat{\sigma_n^*}(\xi, y) = k_1(y) w(\xi, y) + O\left(\xi^{-2}\right)$, for ξ close to 0. Inserting this into the second equation of (4.2) we conclude $k_1' = 0$. Hence $k_1(y) = 0$. Similarly, $\int_{+\infty}^0 \rho^{-1}(s) A(s, y)\, ds = 0$ for $y < 0$.

Once (4.7) and (4.8) have been proved, we get

$$\left|\hat{\sigma_0}(\xi, y) - \gamma(y)\delta\right| \le Ce^{2\pi\beta_0|\xi|}\xi^{-2}, \text{ for } \xi \text{ close to } 0.$$

Analysis of the Fourier integral of σ_0 then tells us that

$$|\sigma_0(x, y)| \le C\left(1 + x^2 + y^2\right)^{\frac{5}{8}}.$$

This finishes the proof. □

5. Proof of Theorem 1

We have analyzed the linear Bäcklund transformation in the previous sections. Based on this, we will prove our main theorem in this section. Let us define

$$(F_0^*\sigma)_n = \partial_s\sigma_n\omega_n - \sigma_n\partial_s\omega_n - \lambda\left(\sigma_{n+1}\omega_{n-1} - \sigma_n\omega_n\right)$$

and

$$(M_0^*\sigma)_n = \partial_t\sigma_{n+1}\omega_n - \sigma_{n+1}\partial_t\omega_n + \lambda^{-1}\left(\sigma_n\omega_{n+1} - \sigma_{n+1}\omega_n\right).$$

Lemma 26. *Suppose* $\{\sigma_n\}, \{\phi_n\}$ *satisfy the system* (4.1)*. Then*

$$\partial_x\phi_n - 2\phi_n + \lambda\phi_{n-1} - \lambda_{n+1}^{-1}\phi = (F_0^*\sigma)_n + (M_0^*\sigma)_n, \tag{5.1}$$

and

$$\frac{1}{i}\partial_y\phi_n - 2\sqrt{2}\phi_n + \lambda\phi_{n-1} + \lambda_{n+1}^{-1}\phi = (F_0^*\sigma)_n - (M_0^*\sigma)_n. \tag{5.2}$$

In particular, if $F_0^*\sigma = M_0^*\sigma = 0$*, and*

$$\phi_{n+1}(x, y) = \phi_n\left(x + \frac{1}{2\sqrt{2}}, y\right),$$

$$|\phi_0| \le C\left(1 + x^2 + y^2\right)^{\frac{5}{8}}, \tag{5.3}$$

then $\phi_n = c_1 + c_2\omega_n$ *for some constants* c_1, c_2*.*

Proof. The equations (5.1) and (5.2) are obtained from adding and subtracting the two equations in (4.1). If $F_0^*\sigma = M_0^*\sigma = 0$, then by (5.1), we have

$$\partial_x\phi_n - 2\phi_n + \lambda\phi_{n-1} - \lambda^{-1}\phi_{n+1} = 0.$$

Fourier transform tells us that $\phi_n = a_1 + a_2x + b(y)$. Inserting this into the equation

$$\frac{1}{i}\partial_y\phi_n - 2\sqrt{2}\phi_n + \lambda\phi_{n-1} + \lambda_{n+1}^{-1}\phi_{n-1} = 0,$$

using the estimate (5.3), we find that $\phi_n = c_1 + c_2\omega_n$ for some constants c_1, c_2. This finishes the proof. □

Lemma 27.

$$\begin{cases} F_0\left(2\sqrt{2}x+n\right)=G_0\left(\left(2\sqrt{2}x+n\right)^2+2yi\left(2\sqrt{2}x+n\right)+\left(1-\sqrt{2}\right)yi\right), \\ M_0\left(2\sqrt{2}x+n\right)=N_0\left(\left(2\sqrt{2}x+n\right)^2+2yi\left(2\sqrt{2}x+n\right)+\left(1-\sqrt{2}\right)yi\right). \end{cases}$$

$$\begin{cases} F_0\left(y\right)=G_0\left(\left(2\sqrt{2}x+n\right)y+\frac{\sqrt{2}-1}{2}y\right), \\ M_0\left(y\right)=N_0\left(\left(2\sqrt{2}x+n\right)y+\frac{\sqrt{2}-1}{2}y\right). \end{cases}$$

Proof. We compute

$$\left[F_0^*\left(2\sqrt{2}x+n\right)\right]_n=-2\sqrt{2}x-2iy-n+\frac{\sqrt{2}+3}{2},$$

$$\left[M_0^*\left(2\sqrt{2}x+n\right)\right]_n=n+2iy+2\sqrt{2}x-\frac{1}{2}\sqrt{2}+\frac{1}{2}.$$

Solving the equations

$$\begin{cases} \partial_x\phi_n-2\phi_n+\lambda\phi_{n-1}-\lambda_{n+1}^{-1}\phi_{n+1}=\left[F_0^*\left(2\sqrt{2}x+n\right)\right]_n+\left[M_0^*\left(2\sqrt{2}x+n\right)\right]_n, \\ \frac{1}{i}\partial_y\phi_n-2\sqrt{2}\phi_n+\lambda\phi_{n-1}+\lambda^{-1}\phi_{n+1}=\left[F_0^*\left(2\sqrt{2}x+n\right)\right]_n-\left[M_0^*\left(2\sqrt{2}x+n\right)\right]_n, \end{cases}$$

we get a solution

$$\left(2\sqrt{2}x+n\right)^2+2yi\left(2\sqrt{2}x+n\right)+\left(1-\sqrt{2}\right)yi.$$

Similarly,

$$F_0^*\left(y\right)=\frac{1}{2i}\left(2\sqrt{2}x+n+2yi+\frac{\sqrt{2}-1}{2}\right),$$

$$M_0^*\left(y\right)=-\frac{1}{2i}\left(2\sqrt{2}x+n+2yi+\frac{\sqrt{2}-1}{2}\right).$$

Solving the system

$$\begin{cases} \partial_x\phi_n-2\phi_n+\lambda\phi_{n-1}-\lambda_{n+1}^{-1}\phi_{n+1}=0, \\ \frac{1}{i}\partial_y\phi_n-2\sqrt{2}\phi_n+\lambda\phi_{n-1}+\lambda^{-1}\phi_{n+1}=-i\left(2\sqrt{2}x+n+2yi+\frac{\sqrt{2}-1}{2}\right), \end{cases}$$

we get a solution $\left(2\sqrt{2}x+n\right)y+\frac{\sqrt{2}-1}{2}y$. The proof is completed. □

We now define

$$\left(F_1^*\phi\right)_n=\partial_s\phi_n\theta_n-\phi_n\partial_s\theta_n-\lambda^{-1}\left(\phi_{n+1}\theta_{n-1}-\phi_n\theta_n\right),$$

$$\left(M_1^*\phi\right)_n=\partial_t\phi_{n+1}\theta_n-\phi_{n+1}\partial_t\theta_n+\lambda\left(\phi_n\theta_{n+1}-\phi_{n+1}\theta_n\right).$$

Lemma 28. *Suppose* $\{\phi_n\},\{\eta_n\}$ *satisfy* (3.1). *Then*

$$\partial_x\eta_n+\left(2-\frac{\lambda}{\omega_n}-\frac{\lambda^{-1}}{\omega_{n+1}}\right)\eta_n+\frac{\omega_{n+1}\lambda^{-1}}{\omega_n}\eta_{n-1}-\frac{\omega_n\lambda}{\omega_{n+1}}\eta_{n+1}=\frac{\left(F_1^*\phi\right)_n}{\omega_n}+\frac{\left(M_1^*\phi\right)_n}{\omega_{n+1}},$$

and

$$\frac{1}{i}\partial_y\eta_n+\left(-\frac{\lambda}{\omega_n}+\frac{\lambda^{-1}}{\omega_{n+1}}-2\sqrt{2}\right)\eta_n+\frac{\omega_{n+1}\lambda^{-1}}{\omega_n}\eta_{n-1}+\frac{\omega_n\lambda}{\omega_{n+1}}\eta_{n+1}=\frac{\left(F_1^*\phi\right)_n}{\omega_n}-\frac{\left(M_1^*\phi\right)_n}{\omega_{n+1}}.$$

In particular, if $\left(F_1^*\phi\right)_n=\left(M_1^*\phi\right)_n=0$,

$$\eta_{n+1}\left(x,y\right)=\eta_n\left(x+\frac{1}{2\sqrt{2}},y\right),$$

and

$$|\eta_n| \le C\sqrt{1+x^2+y^2},$$

then

$$\eta_n = c_1\left(2\sqrt{2}x+n+2yi\right).$$

Proof. If $(F_1^*\phi)_n = (M_1^*\phi)_n = 0$, then

$$\partial_x\eta_n + \left(2-\frac{\lambda}{\omega_n}-\frac{\lambda^{-1}}{\omega_{n+1}}\right)\eta_n + \frac{\omega_{n+1}\lambda^{-1}}{\omega_n}\eta_{n-1} - \frac{\omega_n\lambda}{\omega_{n+1}}\eta_{n+1} = 0.$$

Taking Fourier transform, we get

$$\begin{aligned}&\left(2\pi i\xi + 2 + \lambda^{-1}e^{-\frac{\pi i\xi}{\sqrt{2}}} - \lambda e^{\frac{\pi i\xi}{\sqrt{2}}}\right)\hat{\eta}_n \\ &+\left(1-e^{\frac{\pi i}{\sqrt{2}}\xi}\right)\frac{\pi i}{\sqrt{2}}e^{2\pi i(\alpha+\beta i)\xi}\int_{-\infty}^{\xi} e^{-2\pi i(\alpha+\beta i)s}\left(\lambda-\lambda^{-1}e^{-\frac{\pi i}{\sqrt{2}}s}\right)\hat{\eta}_n(s)\,ds \\ &=0,\ \text{ if } y>0,\end{aligned}$$

and

$$\begin{aligned}&\left(2\pi i\xi + 2 + \lambda^{-1}e^{-\frac{\pi i\xi}{\sqrt{2}}} - \lambda e^{\frac{\pi i\xi}{\sqrt{2}}}\right)\hat{\eta}_n \\ &+\left(1-e^{\frac{\pi i}{\sqrt{2}}\xi}\right)\frac{\pi i}{\sqrt{2}}e^{2\pi i(\alpha-\beta i)\xi}\int_{+\infty}^{\xi} e^{-2\pi i(\alpha-\beta i)s}\left(\lambda-\lambda^{-1}e^{-\frac{\pi i}{\sqrt{2}}s}\right)\hat{\eta}_n(s)\,ds \\ &=0,\ \text{ if } y<0.\end{aligned}$$

Then using the growth estimate of η_n, we find that $\eta_n = c_1\left(2\sqrt{2}x+n+2yi\right)$. □

We are in a position to prove our main theorem.

Proof of Theorem 1. Let $\{\eta_n\}$ be the solution given by Lemma 4. If $G_1\eta = N_1\eta = 0$, then by Lemma 28, $\eta_n = c_1\left(2\sqrt{2}x+n+2yi\right)$. This implies that $\eta_n = a_1\partial_x\theta_n + a_2\partial_y\theta_n$, for some constants a_1, a_2.

Now suppose $G_1\eta \ne 0$ or $N_1\eta \ne 0$. By Propositon 5, there exists $\{\phi_n\}$ such that $F_1\phi = G_1\eta$, $M_1\phi = N_1\eta$, and $|\phi_n| \le C\left(1+x^2+y^2\right)^{\frac{5}{8}}$. Moreover, $T_\omega\phi = 0$.

Case 1. $G_0\phi = N_0\phi = 0$.

In this case, by Lemma 26, $\phi_n = c_1 + c_2\omega_n$. From this, we deduce $\eta_n = a_1\partial_x\theta_n + a_2\partial_y\theta_n$.

Case 2. $G_0\phi \ne 0$, or $N_0\phi \ne 0$.

In this case, by Proposition 18, we can find $\{\sigma_n\}$, such that

$$F_0\sigma = G_0\phi,\ M_0\sigma = N_1\phi.$$

Moreover, $|\sigma_0| \le C\left(1+x^2+y^2\right)^{\frac{5}{8}}$, and $T_\kappa\sigma = 0$. Taking Fourier transform in the equation $T_\kappa\sigma = 0$, we conclude that

$$\sigma_n = c_1 + c_2\left(2\sqrt{2}x+n\right) + c_3 y,$$

for some constants c_1, c_2, c_3. However, in view of Lemma 27, after some computations, we find that η can not satisfy the growth control

$$|\partial_x\eta_0| + |\partial_y\eta_0| \le C\frac{1}{\sqrt{1+x^2+y^2}}.$$

Hence this case is also excluded. This finishes the proof.

□

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E-mail address: `liuyong@ncepu.edu.cn`
E-mail address: `jcwei@math.ubc.ca`

School of Mathematics and Physics, North China Electric Power University, Beijing, China

Department of Mathematics, University of British Columbia, Vancouver, B.C., Canada, V6T 1Z2