SHARP QUANTITATIVE STABILITY ESTIMATES FOR THE BREZIS-NIRENBERG PROBLEM

HAIXIA CHEN, SEUNGHYEOK KIM, AND JUNCHENG WEI

Abstract. We study the quantitative stability for the classical Brezis-Nirenberg problem associated with the critical Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$. To the best of our knowledge, this work presents the first quantitative stability result for the Sobolev inequality on bounded domains. A key discovery is the emergence of unexpected stability exponents in our estimates, which arise from the intricate interaction among the nonnegative solution u_0 and the linear term λu of the Brezis–Nirenberg equation, bubble formation, and the boundary effect of the domain Ω. One of the main challenges is to capture the boundary effect quantitatively, a feature that fundamentally distinguishes our setting from the Euclidean case treated in [21, 31, 23] and the smooth closed manifold case studied in [15]. Our proof refines and streamlines several arguments from the existing literature while also resolving new analytical difficulties specific to our setting.

1. Introduction

1.1. **Backgrounds.** The Brezis-Nirenberg problem is one of the most celebrated problems in nonlinear analysis. It is formulated as

$$\begin{cases} -\Delta u - \lambda u = u^p & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda \in \mathbb{R}$, $p := 2^* - 1 = \frac{n+2}{n-2}$, and $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is a smooth bounded domain.[1] Equation (1.1) was first introduced by Brezis and Nirenberg in their groundbreaking work [11], which is closely linked to the critical Sobolev embedding via the Rayleigh quotient

$$Q_\lambda(u) := \frac{\int_\Omega (|\nabla u|^2 - \lambda u^2) dx}{\|u\|^2_{L^{p+1}(\Omega)}}, \quad u \in H_0^1(\Omega) \setminus \{0\},$$

with associated energy threshold

$$S_\lambda := \inf_{u \in H_0^1(\Omega) \setminus \{0\}} Q_\lambda(u).$$

When $\lambda = 0$, the constant S_0 coincides with the best constant of the Sobolev inequality in $\mathbb{R}^n$

$$S_0 \left(\int_{\mathbb{R}^n} |u|^{p+1} dx \right)^{\frac{2}{p+1}} \leq \int_{\mathbb{R}^n} |\nabla u|^2 dx \quad \text{for all } u \in D^{1,2}(\mathbb{R}^n), \tag{1.2}$$

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[1] The Brezis-Nirenberg problem may also refer to finding (sign-changing) solutions to $-\Delta u - \lambda u = |u|^{p-1} u$ in Ω and $u = 0$ on $\partial\Omega$. This paper is primarily concerned with its non-negative solutions, that is, solutions to (1.1).

where $D^{1,2}(\mathbb{R}^n)$ is the closure of the space $C_c^\infty(\mathbb{R}^n)$ with respect to the norm $\|\nabla u\|_{L^2(\mathbb{R}^n)}$. It is well-known that S_0 is achieved if and only if u is a constant multiple of the Aubin-Talenti bubbles [3, 55] defined as

$$U_{\delta,\xi}(x) = a_n \left(\frac{\delta}{\delta^2 + |x-\xi|^2}\right)^{\frac{n-2}{2}}, \quad \xi \in \mathbb{R}^n,\ \delta > 0,\ a_n := (n(n-2))^{\frac{n-2}{4}}. \tag{1.3}$$

The constant $a_n > 0$ is chosen so that $U := U_{1,0}$ solves the associated Euler-Lagrange equation

$$-\Delta u = |u|^{p-1}u \quad \text{in } \mathbb{R}^n. \tag{1.4}$$

In view of the Sobolev inequality, all solutions to (1.4) are critical points of the energy functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^n} |u|^{p+1} dx \quad \text{for } u \in D^{1,2}(\mathbb{R}^n),$$

and all Aubin-Talenti bubbles share the same energy level: $J(U_{\delta,\xi}) = \frac{1}{n}S_0^{\frac{n}{2}}$.

A key role is played by the critical parameter

$$\lambda_* := \inf\{\lambda > 0 : S_\lambda < S_0\}. \tag{1.5}$$

In their seminal work [11], Brezis and Nirenberg demonstrated that if $n \ge 4$, then $\lambda_* = 0$ and positive least energy solutions exist for all $\lambda \in (0, \lambda_1)$, where λ_1 is the first Dirichlet eigenvalue of $-\Delta$ on Ω. The situation is drastically different when $n = 3$: They showed that $\lambda_* > 0$, computed $\lambda_* = \lambda_1/4$ for the unit ball $\Omega = B(0,1)$, and established the existence of positive least energy solutions for $\lambda \in (\lambda_*, \lambda_1)$. Later, Druet [27] proved that

$$\lambda_* = \sup\{\lambda > 0 : \min_\Omega \varphi_\lambda > 0\}, \tag{1.6}$$

where φ_λ is the Robin function of the operator $-\Delta - \lambda$ in Ω with Dirichlet boundary condition (defined by (2.8) below).

Nonexistence results emerge from various mechanisms: Testing the equation against the first eigenfunction eliminates the possibility of positive solutions when $\lambda \ge \lambda_1$, and Pohozaev's identity [51] prohibits nontrivial solutions for $\lambda \le 0$ in star-shaped domains. Conversely, Bahri and Coron [4] illustrated that certain topological features can allow for existence even at $\lambda = 0$.

Apart from these existence results, the Brezis-Nirenberg problem (1.1) serves as a fundamental model for understanding bubbling phenomena in nonlinear PDEs.

For $n \ge 4$ and $\lambda \to 0^+$, Han [35] and Rey [52] characterized single-bubble blow-up profiles. The existence of single- or multi-bubble solutions concentrating at distinct isolated points was studied by Rey [52] and Musso and Pistoia [45] for $n \ge 5$, and by Pistoia, Rago, and Vaira [50] for $n = 4$. Furthermore, Cao, Luo, and Peng [12] investigated the number of bubbling solutions for $n \ge 6$, and König and Laurain [40] carried out a refined analysis of bubbling phenomena for $n \ge 4$.

In the case $n = 3$, Druet [28] described single-bubble blow-up profiles as $\lambda \to \lambda_*$, del Pino, Dolbeault, and Musso [25] constructed single bubble solution, and Musso and Salazar [46] constructed multi-bubble solutions concentrating at distinct isolated points as λ tends to a certain value $\lambda_0 \in (0, \lambda_1)$. Moreover, Druet and Laurain [29] examined the Pohozaev obstruction, and König and Laurain [41] conducted a fine analysis in this setting.

It is worth noting that the existence or nonexistence of positive cluster or tower solutions for the Brezis-Nirenberg problem remains not fully understood, although Cerqueti [14] established the nonexistence of such solutions in symmetric domains for $n \ge 5$ as $\lambda \to 0$. For the results concerning sign-changing solutions, we refer interested readers to the recent papers [43, 54] and the references therein.

In this paper, we aim to investigate the *quantitative stability* of the Brezis-Nirenberg problem, a topic that has attracted considerable attention of researchers, with numerous generalizations and refinements in various directions.

One prominent line of research concerns the stability of functional inequalities. The study of sharp functional inequalities naturally proceeds through three stages: Identifying optimal constants, characterizing extremal functions, and understanding quantitative stability. Once extremal functions are established, a fundamental question arises: How does the *deficit*–the difference between the two sides of the inequality at the sharp constant–influence the distance to the set of extremals? This stability question was initially posed by Brezis and Lieb [10] and subsequently resolved for the critical Sobolev inequality (1.2) by Bianchi and Egnell [7], who provided a quantitative estimate regarding the distance to Aubin-Talenti bubbles in $D^{1,2}(\mathbb{R}^n)$. Extending the Bianchi-Egnell stability result to general L^p-Sobolev inequalities has required the development of novel techniques, with major contributions from Cianchi, Figalli, Fusco, Maggi, Neumayer, Pratell and Zhang [20, 32, 47, 33]. Related advances have been developed for a variety of Sobolev-type inequalities [26, 56, 58], and so on. Furthermore, a recent progress has also been achieved in geometric contexts, including product spaces [34] and general Riemannian manifolds [30, 49, 48, 1, 8]. Notably, König's recent breakthroughs [37, 39, 38] on the attainability of the sharp Bianchi-Egnell constant represent a significant milestone in the pursuit of optimal stability constants.

Another significant direction focuses on stability through the viewpoint of the Euler–Lagrange equation induced by a sharp inequality. This perspective refines the classical concentration–compactness principle (refer to Theorem A) by providing explicit convergence rates. In a seminal work [21], Ciraolo, Figalli, and Maggi established the sharp stability result near a single-bubble for the Sobolev inequality in dimensions $n \geq 3$, with extensions to multiple-bubble configurations by Figalli and Glaudo [31] and Deng, Sun, and Wei [23]. Specifically, suppose that $\nu \in \mathbb{N}$ and u is a nonnegative element in $D^{1,2}(\mathbb{R}^n)$ with $(\nu - \frac{1}{2})S_0^{n/2} \leq \|u\|^2_{D^{1,2}(\mathbb{R}^n)} \leq (\nu + \frac{1}{2})S_0^{n/2}$ and sufficiently small $\Gamma(u) := \|\Delta u + u^{\frac{n+2}{n-2}}\|_{(D^{1,2}(\mathbb{R}^n))^*}$, where $(D^{1,2}(\mathbb{R}^n))^*$ is the dual space of $D^{1,2}(\mathbb{R}^n)$. Then there is a constant $C > 0$ depending only on n and ν such that

$$\left\| u - \sum_{i=1}^{\nu} U_i \right\|_{D^{1,2}(\mathbb{R}^n)} \leq C \begin{cases} \Gamma(u) & \text{if } n \geq 3,\ \nu = 1 \text{ (by Ciraolo, Figalli and Maggi [21])}, \\ \Gamma(u) & \text{if } 3 \leq n \leq 5,\ \nu \geq 2 \text{ (by Figalli and Glaudo [31])}, \\ \Gamma(u)|\log \Gamma(u)|^{\frac{1}{2}} & \text{if } n = 6,\ \nu \geq 2 \text{ (by Deng, Sun, and Wei [23])}, \\ \Gamma(u)^{\frac{n+2}{2(n-2)}} & \text{if } n \geq 7,\ \nu \geq 2 \text{ (by Deng, Sun, and Wei [23])} \end{cases} \tag{1.7}$$

for some bubbles $U_1, \ldots, U_\nu$ and this estimate is optimal. These results have been further generalized to a broad range of inequalities, including the fractional Sobolev inequality [2, 24, 16], the Caffarelli-Kohn-Nirenberg inequality [56, 59], the logarithmic Sobolev inequality [57], Sobolev inequalities involving p-Laplacian [22, 44], the subcritical case [18], as well as settings on the hyperbolic spaces [5, 6], general Riemannian manifolds [15, 17], the Heisenberg group [19], and so forth.

Beyond their intrinsic interest, quantitative stability estimates have powerful applications in nonlinear PDE dynamics, such as the asymptotic behavior of solutions to the Keller-Segel system [13] and the fast diffusion equation [21, 31, 24, 42].

Our present work is interested in the latter direction, devoted to the quantitative stability of almost solutions to the Euler-Lagrange equation associated with the inequality $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ in bounded domains Ω. We begin with a well-known global compactness result associated with the functional corresponding to (1.1), commonly referred to as Struwe's decomposition. This result was established in [53, Proposition 2.1], [9, Theorem 2] and [4, Proposition 4], which we restate below.

Theorem A. *Let Ω be a smooth open bounded domain in $\mathbb{R}^n$ with $n \geq 3$ and $\lambda_1 > 0$ be the first eigenvalue of $-\Delta$ with Dirichlet boundary condition in Ω. For $\lambda \in (0, \lambda_1)$, we endow the Sobolev space $H_0^1(\Omega)$ with the norm*

$$\|u\|_{H_0^1(\Omega)} := \left[\int_\Omega \left(|\nabla u|^2 - \lambda u^2\right) dx\right]^{\frac{1}{2}},$$

and denote by $(H_0^1(\Omega))^*$ its dual space.

Let $\{u_m\}_{m\in\mathbb{N}}$ be a sequence of nonnegative functions in $H_0^1(\Omega)$ such that

$$\|u_m\|_{H_0^1(\Omega)} \leq C_0 \quad \text{and} \quad \|\Delta u_m + \lambda u_m + u_m^p\|_{(H_0^1(\Omega))^*} \to 0 \text{ as } m \to \infty$$

for some constant $C_0 > 0$. Then, up to a subsequence, there exist a nonnegative function $u_0 \in C^\infty(\Omega)$, an integer $\nu \in \mathbb{N} \cup \{0\}$ satisfying $\nu \leq C_0^2 S_0^{-n/2}$, and a sequence of parameters $\{(\delta_{1,m}, \ldots, \delta_{\nu,m}, \xi_{1,m}, \ldots, \xi_{\nu,m})\}_{m\in\mathbb{N}} \subset (0,\infty)^\nu \times \Omega^\nu$ such that the followings hold:

- *u_0 is a smooth solution to (1.1). By the strong maximum principle, we have either $u_0 > 0$ or $u_0 = 0$ in Ω.*
- *For all $1 \leq i \neq j \leq \nu$, we have that $\delta_{i,m} \to 0$ and*

$$\frac{d(\xi_{i,m}, \partial\Omega)}{\delta_{i,m}} \to \infty, \quad \frac{\delta_{i,m}}{\delta_{j,m}} + \frac{\delta_{j,m}}{\delta_{i,m}} + \frac{|\xi_{i,m} - \xi_{j,m}|^2}{\delta_{i,m}\delta_{j,m}} \to \infty \quad \text{as } m \to \infty.$$

- *It holds that*

$$\left\|u_m - \left(u_0 + \sum_{i=1}^{\nu} U_{\delta_{i,m},\xi_{i,m}}\right)\right\|_{H_0^1(\Omega)} \to 0 \quad \text{as } m \to \infty.$$

1.2. **Main results.** Our objective is to derive a quantitative version of above decomposition. To this end, we consider the following two auxiliary equations:

$$\begin{cases} -\Delta u = U_{\delta,\xi}^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

and

$$\begin{cases} -\Delta u - \lambda u = U_{\delta,\xi}^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

Before presenting our main results, we introduce the following assumption:

Assumption B. Given any open bounded set $\Omega \subset \mathbb{R}^n$ with $n \geq 3$ and any $\lambda \in (0, \lambda_1)$. Suppose that a nonnegative function u in $H_0^1(\Omega)$ satisfies

$$\left\|u - \left(u_0 + \sum_{i=1}^{\nu} U_{\tilde{\delta}_i,\tilde{\xi}_i}\right)\right\|_{H_0^1(\Omega)} \leq \varepsilon_0 \tag{1.10}$$

for some small $\varepsilon_0 > 0$ and $\nu \in \mathbb{N}$. Here, u_0 is a solution of (1.1) and $(\tilde{\delta}_i, \tilde{\xi}_i) \in (0,\infty) \times \Omega$ satisfies

$$\max_{i=1,\ldots,\nu} \tilde{\delta}_i + \max_{i=1,\ldots,\nu} \frac{\tilde{\delta}_i}{d(\tilde{\xi}_i, \partial\Omega)} \leq \varepsilon_0$$

and

$$\max\left\{\left(\frac{\tilde{\delta}_i}{\tilde{\delta}_j} + \frac{\tilde{\delta}_j}{\tilde{\delta}_i} + \frac{|\tilde{\xi}_i - \tilde{\xi}_j|^2}{\tilde{\delta}_i\tilde{\delta}_j}\right)^{-\frac{n-2}{2}} : i, j = 1, \ldots, \nu,\ i \neq j\right\} \leq \varepsilon_0.$$

If $u_0 > 0$ in Ω, we further assume that u_0 is **non-degenerate** in the sense that the only $H_0^1(\Omega)$-solution to $\Delta\phi + \lambda\phi + pu_0^{p-1}\phi = 0$ in Ω is identically zero in Ω. For later use, we define $\Gamma(u) := \|\Delta u + \lambda u + u^p\|_{(H_0^1(\Omega))^*}$.

We note that the condition $\max_i \frac{\tilde{\delta}_i}{d(\tilde{\xi}_i,\partial\Omega)} \le \varepsilon_0$ admits two possibilities: Either $\tilde{\xi}_i$ is away from $\partial\Omega$ or close to $\partial\Omega$. Accordingly, we divide our main results into two theorems.

Our first theorem addresses the case where $\tilde{\xi}_i$ is away from the boundary of Ω, covering both single and multi-bubble cases.

Theorem 1.1. *Let $\lambda_* \ge 0$ be the number in (1.5) and $\varphi_\lambda^3(x) = H_\lambda^3(x,x)$ be the function defined by (2.4) below. Under the **Assumption B**, we further assume the followings:*

- *Each $\tilde{\xi}_i$ lies on a compact set of Ω for $i = 1, \ldots, \nu$.*
- *If $n = 3$ and $u_0 > 0$, then $\lambda \in (\lambda_*, \lambda_1)$, which ensures the existence of such u_0.*
- *If $n = 3$, $u_0 = 0$, and $\nu \ge 2$, then $\lambda \in (\lambda_*, \lambda_1)$ and $\varphi_\lambda^3(\tilde{\xi}_i) < 0$ for each $i = 1, \ldots, \nu$.*[2]

Then, by possibly reducing $\varepsilon_0 > 0$, one can find a large constant $C = C(n, \nu, \lambda, u_0, \Omega) > 0$ and functions $PU_1 := PU_{\delta_1,\xi_1}, \ldots, PU_\nu := PU_{\delta_\nu,\xi_\nu}$ satisfying (1.8) if either $[n = 3, 4$ and $u_0 > 0]$ or $n \ge 5$, and satisfying (1.9) if $n = 3, 4$ and $u_0 = 0$, such that

$$\left\| u - \left(u_0 + \sum_{i=1}^{\nu} PU_i \right) \right\|_{H_0^1(\Omega)} \le C\zeta(\Gamma(u)), \tag{1.11}$$

where $\zeta \in C^0([0,\infty))$ satisfies

$$\zeta(t) = \begin{cases} t & \text{if } [n = 3, 4,\ \nu \ge 1] \text{ or } [n = 5,\ \nu \ge 1,\ u_0 > 0] \text{ or } [n \ge 7,\ \nu = 1], \\ t^{\frac{3}{4}} & \text{if } [n = 5,\ \nu \ge 1,\ u_0 = 0], \\ t|\log t|^{\frac{1}{2}} & \text{if } [n = 6,\ \nu \ge 1], \\ t^{\frac{n+2}{2(n-2)}} & \text{if } [n \ge 7,\ \nu \ge 2] \end{cases} \tag{1.12}$$

for $t > 0$.

The estimate above is optimal in the sense that the function ζ cannot be improved.

Before we proceed further, we leave some remarks.

Remark 1.2.

(1) Compared to the Euclidean case summarized in (1.7), the new exponents appear when $[n = 5,\ u_0 = 0,\ \nu \ge 1]$ or $[n = 6,\ \nu = 1]$.

(2) Solutions to certain specific perturbation of the equation $\Delta u + \lambda u + u^p = 0$ in Ω cannot exhibit boundary blow-up, thereby fulfilling the first additional assumption in Theorem 1.1. Moreover, in some cases, only one of the conditions $u_0 = 0$ or $\nu = 0$ is permitted; refer to e.g. [41, 28].

(3) When $u_0 > 0$, the non-degeneracy assumption on u_0 is generic; see [36, Lemma 4.9]. In the case $u_0 = 0$ and $n = 3$ or 4, defining PU_i via solutions to (1.9) rather than (1.8) turns out to be more natural; see Subsection 1.3(2). Similar observation was made in constructing positive solutions to the Brezis-Nirenberg-type problem in low dimensions; see e.g. [25].

(4) For $n = 3$, $u_0 = 0$, and $\nu \ge 2$, we use the condition $\varphi_\lambda^3(\tilde{\xi}_i) < 0$ so that no sign competition occurs between the terms $\int_\Omega \mathcal{I}_2 PZ_j^0$ in Lemma 2.8 and $\int_\Omega \mathcal{I}_3 PZ_j^0$ in Lemma 2.7.

(5) If $n = 5$, $u_0 = 0$, and $\nu \ge 1$, the linear term λu is the dominant factor determining $\zeta(t) = t^{3/4}$ in (1.12). In this case, one may instead choose the projected bubble $PU_{\delta,\xi}$ as in (1.9) rather than (1.8). Since (1.9) already incorporates the effect of the linear term, it leads to the stability function $\zeta(t) = t$, as opposed to $t^{3/4}$, and this improved rate can again be shown to be sharp. Such a sensitive dependence on the choice of the test function is a distinctive characteristic of the

[2] Using (2.8)–(2.9), Druet's characterization (1.6) of the number λ_* in (1.5) can be rewritten as $\lambda_* = \sup\{\lambda > 0 : \min_\Omega \varphi_\lambda^3 > 0\}$.

Brezis-Nirenberg problem in Ω, and does not appear in the Euclidean setting or in the Yamabe problem.

Our second main result concerns the boundary effect when $\tilde{\xi}_i$ may approach $\partial\Omega$. We fully characterize the single-bubble case in this setting.

Theorem 1.3. *Under the* ***Assumption B****, we further assume that* $\nu = 1$ *and* $\tilde{\xi}_1 \in \Omega$*. If* $n = 3$ *and* $u_0 > 0$*, we also need* $\lambda \in (\lambda_*, \lambda_1)$*. Then, by possibly reducing* $\varepsilon_0 > 0$*, one can find a large constant* $C = C(n, \lambda, u_0, \Omega) > 0$ *and a function* $PU_1 := PU_{\delta_1,\xi_1}$ *satisfying* (1.8) *if either* $[n = 4, 5$ *and* $u_0 > 0]$ *or* $n \geq 6$*, and satisfying* (1.9) *if either* $n = 3$ *and* $[n = 4, 5$ *and* $u_0 = 0]$*, such that*

$$\|u - (u_0 + PU_1)\|_{H_0^1(\Omega)} \leq C\zeta(\Gamma(u)), \tag{1.13}$$

where $\zeta \in C^0([0, \infty))$ *satisfies*

$$\zeta(t) = \begin{cases} t & \text{if } n = 3 \text{ or } [n = 4,\ u_0 = 0], \\ t^{\frac{n-2}{n-1}} & \text{if } [n = 4,\ u_0 > 0] \text{ or } n = 5, \\ t|\log t|^{\frac{1}{2}} & \text{if } n = 6, \\ t^{\frac{n+2}{2(n-1)}} & \text{if } n \geq 7 \end{cases} \tag{1.14}$$

for $t > 0$*. The above estimate is also optimal.*

Remark 1.4.

(1) Even in single-bubble case, the surprising new exponents in (1.14) emerge due to the possibility that $d(\tilde{\xi}_1, \partial\Omega)$ is small. This phenomenon occurs exclusively in domains with nonempty boundary. The multi-bubble case remains an open problem due to a serious technical issue. See Subsection 1.3(6).

(2) Unlike in Theorem 1.1, we choose PU_1 to satisfy (1.9) for the cases $[n = 3,\ u_0 > 0]$ or $[n = 5,\ u_0 = 0]$ to avoid difficulties arising from the boundary effects. We believe that this choice is nearly unavoidable.

(3) Similar to Remark 1.2(4), when $[n = 4,\ u_0 > 0]$ or $[n = 5,\ u_0 > 0]$, choosing $PU_{\delta,\xi}$ as in (1.9) again yields the optimal stability function $\zeta(t) = t$. In both cases, the sharp stability function depends explicitly on the choice of the projected bubble $PU_{\delta,\xi}$ within the framework of this theorem.

As an application of Theorem 1.3 and Struwe's profile decomposition Theorem A, we obtain the following corollary.

Corollary 1.5. *Let* $S_0 > 0$ *be the sharp Sobolev constant in* (1.2)*. We assume that every positive solution to* (1.1) *is non-degenerate.*

If u *is a nonnegative function in* $H_0^1(\Omega)$ *with*

$$\|u\|_{H_0^1(\Omega)}^2 \leq \frac{3}{2} S_0^{\frac{n}{2}}, \tag{1.15}$$

then there exists a constant $C > 0$ *depending only on* n, λ, Ω *such that*

$$\inf \left\{ \left\| u - \left(u_0 + \sum_{i=1}^{\nu} PU_{\delta_i,\xi_i} \right) \right\|_{H_0^1(\Omega)} : u_0 \text{ solves } (1.1),\ PU_{\delta_i,\xi_i} \in \mathcal{B},\ \nu = 0, 1 \right\} \leq C\zeta(\Gamma(u)),$$

where $\zeta(t)$ *satisfies* (1.14) *for* $t \in [0, \infty)$ *and*

$$\mathcal{B} := \big\{ PU_{\delta,\xi} : PU_{\delta,\xi} \text{ satisfies } (1.8) \text{ for } n \geq 6 \text{ or } [n = 4, 5,\ u_0 > 0] \text{ and satisfies } (1.9) \text{ for } n = 3 \text{ or } [n = 4, 5,\ u_0 = 0],\ (\delta, \xi) \in (0, \infty) \times \Omega \big\}.$$

Here $\sum_{i=1}^{0} PU_{\delta_i,\xi_i} = 0$.

Remark 1.6. In this corollary, we modify the class of admissible functions u in (1.10) to those with uniformly bounded energy as in (1.15). This necessitates assuming the non-degeneracy for all positive solutions to (1.1), since u_0 cannot be determined a priori. The proof proceeds by contradiction, following an argument similar to that in [15, Section 6], and is therefore omitted.

1.3. **Comments on the proof.** Our proof is primarily inspired by the approaches developed in [21, 31, 23, 15, 16]. To clarify the new technical challenges involved in our setting, we begin by outlining a general strategy for proving quantitative stability of sharp inequalities in the critical point setting:

(i) The starting point is that the infimum $\inf \|u-(u_0+\sum_{i=1}^{\nu}\mathcal{V}_{\tilde{\delta}_i,\tilde{\xi}_i})\|_{H^1}$ can be achieved by $\mathcal{V}_{\delta_i,\xi_i}$ where $\mathcal{V}_{\delta,\xi}$ is an appropriate bubble-like function, then $\rho := u-(u_0+\sum_{i=1}^{\nu}\mathcal{V}_{\delta_i,\xi_i})$ satisfies an auxiliary equation (e.g. (2.1)) along with certain orthogonality conditions, at least in a Hilbert space framework.

(ii) By testing the equation of ρ with ρ itself, one can derive a estimate $\|\rho\|_{H^1} \lesssim \|f\|_{(H^1)^*} + \|\mathcal{I}\|_{(H^1)^*} := \|f\|_{(H^1)^*} + \mathcal{J}_1$ where $f := -\Delta u - \lambda u - u^p$, $\mathcal{I}$ is an error term (in our setting, $\mathcal{I} := \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3$ given by (2.2)), and $\mathcal{J}_1$ is a small quantity.

(iii) By choosing suitable test functions originated from bubbles (see Subsection 1.3(7) below), one can find another small quantity $\mathcal{J}_2$ such that $\mathcal{J}_2 \lesssim \|f\|_{(H^1)^*}$. If one can determine a function $\tilde{\zeta}$ such that $\mathcal{J}_1 \lesssim \tilde{\zeta}(\mathcal{J}_2)$, the final stability function will be determined as $\zeta(t) := \max\{t, \tilde{\zeta}(t)\}$ for small $t > 0$.

(iv) Once one finds special parameters (δ_i, ξ_i) and functions ρ and f satisfying $\|\rho\|_{H^1} \gtrsim \zeta(\|f\|_{(H^1)^*})$, then the nonnegative function $u_* = (u_0 + \sum_{i=1}^{\nu}\mathcal{V}_{\delta_i,\xi_i} + \rho)_+$ usually provides an optimal example.

Although our proof could follow the procedures outlined above, several non-trivial and novel challenges arise in our specific setting. We now present the new strategies devised to overcome or mitigate these difficulties.

(1) Due to the presence of u_0 and the linear term λu, more precise computations are needed for the interactions among bubbles with various powers, as well as those between a bubble and u_0, for all dimensions $n \geq 3$.

(2) The selection of bubble-like functions is subtle. For our problem, depending on the dimension n and the solution u_0 of (1.1), we make appropriate use of two different projected bubbles given by (1.8) and (1.9).

Let us explain why we must define PU_i via solutions of (1.9) in deducing Theorem 1.1 for $n = 3$ or 4 and $u_0 = 0$:

If $n = 3$ and $u_0 = 0$, then the function PU_i defined via (1.8) fails to produce any quantitative estimates even in the single-bubble case due to the excessive size of $\|U_{\delta,\xi}\|_{L^{6/5}(\Omega)}$.

If $n = 4$, $u_0 = 0$, and $\nu = 1$, then such a definition yields a valid but a non-sharp estimate.

If $n = 4$, $u_0 = 0$, and $\nu \geq 2$, then the use of the above-defined PU_i fails completely, because the interaction terms $\int U_iU_j$ are not negligible compared to the presumably dominant term $\max_i \int U_i^2$.

In Lemmas 2.1 and B.1, we rigorously analyze the behavior of the function $PU_{\delta,\xi}$ defined via (1.9), which may be independent of interests.

As previously noted, there seems be no results on positive cluster or tower solutions for the low-dimensional Brezis-Nirenberg problem. Our calculations take into account all possible bubbles and may be helpful for constructing such solutions, should they exist.

(3) In [23, 16], the authors derive a pointwise estimate for the main part ρ_0 of ρ across all bubble configurations in all dimensions $n \geq 6$. Instead, our proofs of stability estimates (1.11) and its

optimality make full use of the definition of $\|\cdot\|_{(H_0^1(\Omega))^*}$-norm, which can simplify the argument of [23, 16]. Part of the idea is influenced by [19, Lemma 5.4]. See Subsections 2.3, 3.1 and 3.2.

(4) In Step (ii), many seminal works in the critical regime (see [21, 31, 23] and their generalizations, e.g., [6, 15, 16]) devote substantial effort to deriving appropriate coercivity inequalities. In [23, Section 6], such inequalities play a crucial role in deducing a Sobolev norm estimate for the term $\rho_1 := \rho - \rho_0$. In contrast, our approach provides a direct derivation of the Sobolev norm estimate for ρ_1 based solely on blow-up analysis (refer to Subsection 3.1). As a result, the proof avoids coercivity inequalities entirely, greatly shortening the argument again.

(5) Regarding the sharpness of our results, we conduct a comprehensive analysis of all admissible forms of the function ζ, dealing with the linear ($\zeta(t) = t$) and sublinear ($\zeta(t) \gg t$) regimes through two distinct strategies. In the linear case, sharpness is verified by constructing a smooth perturbation of $u_0 + \sum_{i=1}^{\nu} PU_i$. For the sublinear case, a more delicate analysis is required for the multiple bubble scenario whose idea differs from that in $\mathbb{R}^n$, and it is important to identify which of the dominant factors–interactions among u_0, the boundary effect, the bubbles, and the linear term λu–govern the exponent of ζ.

(6) In the proof of Theorem 1.3, the scenario in which $d(\xi, \partial\Omega)$ is small introduces a crucial challenge: The projection of $\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3$ in the direction of the dilation derivative $\delta_i \partial_{\delta_i} \mathcal{V}_i$ of the bubble-like function $\mathcal{V}_i$ has a negative leading term of the form $-\delta^{n-2}/d(\xi, \partial\Omega)^{n-2}$; see (4.4). In the single-bubble case, we address this projection by carefully analyzing all possible scenarios, as detailed in Section 4. The reason that one primarily uses $\delta_i \partial_{\delta_i} \mathcal{V}_i$ as a test function in both Euclidean and manifold settings–instead of using a spatial derivative $\delta_i \partial_{\xi_i^k} \mathcal{V}_i$–is that the latter generally lead to weaker estimates. However, in our setting, it is sometimes necessary to consider the projections of $\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3$ in the direction $\delta_i \partial_{\xi_i^k} \mathcal{V}_i$, since the dilation projection may suffer from sign cancellations among its leading-order terms, weakening their overall effect. As such, precise term-by-term estimates like (4.4) and (4.5) are indispensable.

In the multi-bubble case $\nu \geq 2$, these challenges become significantly more difficult. We currently lack a clear strategy to effectively handle the competition between the negative term involving $d(\xi_i, \partial\Omega)$ and the interaction between different bubbles. Additionally, integrals such as $\int_\Omega [(PU_i)^p - U_i^p] U_j$ for $i \neq j$ and $\int_\Omega [(\sum_{i=1}^{\nu} PU_i)^p - \sum_{i=1}^{\nu} (PU_i)^p] PZ_j^0$ (when $n \geq 3$) pose formidable analytical obstacles.

Our structure of this paper is described as follows: In Section 2, we present some necessary estimates for proving our main theorems. In Sections 3 and 4, we provide the detailed proofs of Theorem 1.1 and Theorem 1.3, respectively. In Appendix A, we include several elementary estimates that are frequently used throughout the main text. In Appendix B, we give a proof for an important lemma used in Section 4.

Notations. Here, we list some notations that will be frequently used later.

- $\mathbb{N}$ denotes the set of positive integers.

- Let (A) be a condition. We set $\mathbf{1}_{(\mathrm{A})} = 1$ if (A) holds and 0 otherwise.

- For $x \in \Omega$ and $r > 0$, we write $B(x,r) = \{\omega \in \Omega : |\omega - x| < r\}$ and $B(x,r)^c = \{\omega \in \Omega : |\omega - x| \geq r\}$.

- We use the Japanese bracket notation $\langle x \rangle = \sqrt{1 + |x|^2}$ for $x \in \mathbb{R}^n$.

- Unless otherwise stated, $C > 0$ is a universal constant that may vary from line to line and even in the same line. We write $a_1 \lesssim a_2$ if $a_1 \leq Ca_2$, $a_1 \gtrsim a_2$ if $a_1 \geq Ca_2$, and $a_1 \simeq a_2$ if $a_1 \lesssim a_2$ and $a_1 \gtrsim a_2$.

2. Setting and analysis of bubbles

2.1. **Problem setting.** By (1.10), there exist parameters $(\delta_1,\dots,\delta_\nu,\xi_1,\dots,\xi_\nu)\subset(0,\infty)^\nu\times\Omega^\nu$ and $\varepsilon_1>0$ small such that $\varepsilon_1\to 0$ as $\varepsilon_0\to 0$,

$$\left\|u-\left(u_0+\sum_{i=1}^{\nu}PU_i\right)\right\|_{H_0^1(\Omega)}=\inf\left\{\left\|u-\left(u_0+\sum_{i=1}^{\nu}PU_{\tilde\delta_i,\tilde\xi_i}\right)\right\|_{H_0^1(\Omega)}:\left(\tilde\delta_i,\tilde\xi_i\right)\in(0,\infty)\times\Omega,\ i=1,\dots,\nu\right\}\le\varepsilon_1,$$

where $PU_i=PU_{\delta_i,\xi_i}$, and

$$\max_i\delta_i+\max_i\frac{\delta_i}{d(\xi_i,\partial\Omega)}\le\varepsilon_1,$$

as well as

$$\max\left\{\left(\frac{\delta_i}{\delta_j}+\frac{\delta_j}{\delta_i}+\frac{|\xi_i-\xi_j|^2}{\delta_i\delta_j}\right)^{-\frac{n-2}{2}}:i,j=1,\dots,\nu\right\}\le\varepsilon_1.$$

Throughout the paper, we write $\kappa_i:=\frac{\delta_i}{d(\xi_i,\partial\Omega)}$.

Setting $\sigma:=\sum_{i=1}^{\nu}PU_i$, $\rho:=u-(u_0+\sigma)$, and $f:=-\Delta u-\lambda u-u^p$, we have

$$\begin{cases}-\Delta\rho-\lambda\rho-p(u_0+\sigma)^{p-1}\rho=f+\mathcal{I}_0[\rho]+\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3\quad\text{in }\Omega,\\ \rho=0\quad\text{on }\partial\Omega,\\ \left\langle\rho,PZ_i^k\right\rangle_{H_0^1(\Omega)}:=\displaystyle\int_\Omega\nabla\rho\cdot\nabla PZ_i^k-\lambda\rho PZ_i^k=0\quad\text{for }i=1,\dots,\nu\text{ and }k=0,\dots,n,\end{cases}\tag{2.1}$$

where

$$\begin{aligned}&PZ_i^0:=\delta_i\frac{\partial PU_i}{\partial\delta_i},\quad PZ_i^k:=\delta_i\frac{\partial PU_i}{\partial\xi_i^k}\quad\text{for }k=1,\dots,n,\\ &\mathcal{I}_0[\rho]:=|u_0+\sigma+\rho|^{p-1}(u_0+\sigma+\rho)-(u_0+\sigma)^p-p(u_0+\sigma)^{p-1}\rho,\\ &\mathcal{I}_1:=(u_0+\sigma)^p-u_0^p-\sigma^p,\\ &\mathcal{I}_2:=\sigma^p-\sum_{i=1}^{\nu}(PU_i)^p,\quad\text{and}\quad\mathcal{I}_3:=\sum_{i=1}^{\nu}\left[\Delta PU_i+\lambda PU_i+(PU_i)^p\right].\end{aligned}\tag{2.2}$$

We recall a well-known non-degeneracy result: Given any $\delta>0$ and $\xi=(\xi^1,\dots,\xi^n)\in\mathbb{R}^n$, the solution space of the linear problem

$$-\Delta v=pU_{\delta,\xi}^{p-1}v\quad\text{in }\mathbb{R}^n,\quad v\in D^{1,2}(\mathbb{R}^n)$$

is spanned by the functions

$$Z_{\delta,\xi}^0:=\delta\frac{\partial U_{\delta,\xi}}{\partial\delta}\quad\text{and}\quad Z_{\delta,\xi}^k:=\delta\frac{\partial U_{\delta,\xi}}{\partial\xi^k}\quad\text{for }k=1,\dots,n.$$

We rewrite $U_i:=U_{\delta_i,\xi_i}$, $Z^k:=Z_{1,0}^k$, and $Z_i^k:=Z_{\delta_i,\xi_i}^k$ for $i=1,\dots,\nu$ and $k=0,\dots,n$.

Let us define four quantities

$$\begin{cases}q_{ij}:=\left[\dfrac{\delta_i}{\delta_j}+\dfrac{\delta_j}{\delta_i}+\dfrac{|\xi_i-\xi_j|^2}{\delta_i\delta_j}\right]^{-\frac{n-2}{2}},\quad Q:=\max\{q_{ij}:i,j=1,\dots,\nu\}\le\varepsilon_1,\\ \mathscr{R}_{ij}:=\max\left\{\sqrt{\dfrac{\delta_i}{\delta_j}},\sqrt{\dfrac{\delta_j}{\delta_i}},\dfrac{|\xi_i-\xi_j|}{\sqrt{\delta_i\delta_j}}\right\}\simeq q_{ij}^{-\frac{1}{n-2}},\quad\mathscr{R}:=\dfrac{1}{2}\min\mathscr{R}_{ij}.\end{cases}\tag{2.3}$$

2.2. **Expansions of $PU_{\delta,\xi}$.** Given the projected bubbles $PU_{\delta,\xi}$ via either (1.8) or (1.9), we expand them.

Lemma 2.1. *Suppose that $x, \xi \in \Omega$ and $\delta > 0$ is small. Then, $0 < PU_{\delta,\xi} \le U_{\delta,\xi}$ in Ω, and for any $\tau \in (0,1)$, the following holds:*

$$PU_{\delta,\xi}(x) = U_{\delta,\xi}(x) - a_n\delta^{\frac{n-2}{2}}H(x,\xi) + O\left(\delta^{\frac{n+2}{2}}d(\xi,\partial\Omega)^{-n}\right)$$

provided $n \ge 3$ and $PU_{\delta,\xi}$ is given by equation (1.8), and

$$PU_{\delta,\xi}(x) = U_{\delta,\xi}(x) + \frac{\lambda}{2}a_n\delta^{\frac{n-2}{2}}\begin{Bmatrix} -|x-\xi| & \text{if } n=3 \\ -\log|x-\xi| & \text{if } n=4 \\ \frac{1}{|x-\xi|} - 4\lambda|x-\xi| & \text{if } n=5 \end{Bmatrix} - \delta^{\frac{n-2}{2}}a_nH_\lambda^n(x,\xi) + \delta^{2-\frac{n-2}{2}}\mathcal{D}_n\left(\frac{x-\xi}{\delta}\right)$$
$$+ \begin{Bmatrix} O(\delta^{\frac{5}{2}-\tau}) & \text{if } n=3,5 \\ O(\delta^{3-\tau}) & \text{if } n=4 \end{Bmatrix} + O\left(\delta^{\frac{n+2}{2}}\left[d(\xi,\partial\Omega)^{-(n-2)}\left|\log\frac{d(\xi,\partial\Omega)}{\delta}\right| + d(\xi,\partial\Omega)^{-n}\right]\right)$$

provided $n = 3,4,5$ and $PU_{\delta,\xi}$ is given by equation (1.9). Here, $a_n = (n(n-2))^{\frac{n-2}{4}}$ (see (1.3)), the function $H(x,y)$ satisfies

$$\begin{cases} -\Delta_x H(x,y) = 0 & \text{in } \Omega, \\ H(x,y) = \frac{1}{|x-y|^{n-2}} & \text{on } \partial\Omega, \end{cases}$$

the function $H_\lambda^3(x,y)$ satisfies

$$\begin{cases} \Delta_x H_\lambda^3(x,y) + \lambda H_\lambda^3(x,y) = -\frac{\lambda^2}{2}|x-y| & \text{in } \Omega, \\ H_\lambda^3(x,y) = \frac{1}{|x-y|} - \frac{\lambda}{2}|x-y| & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

the function $H_\lambda^4(x,y)$ satisfies

$$\begin{cases} \Delta_x H_\lambda^4(x,y) + \lambda H_\lambda^4(x,y) = \lambda\log|x-y| & \text{in } \Omega, \\ H_\lambda^4(x,y) = \frac{1}{|x-y|^2} - \frac{\lambda}{2}\log|x-y| & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

and the function $H_\lambda^5(x,y)$ satisfies

$$\begin{cases} \Delta_x H_\lambda^5(x,y) + \lambda H_\lambda^5(x,y) = -\frac{\lambda^2}{2}|x-y| & \text{in } \Omega, \\ H_\lambda^5(x,y) = \frac{1}{|x-y|^3} + \frac{\lambda}{2}\frac{1}{|x-y|} - \frac{\lambda^2}{2}|x-y| & \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

for each fixed $y \in \Omega$. Besides, the function $\mathcal{D}_n = \mathcal{D}_n(z)$ satisfies

$$\begin{cases} -\Delta\mathcal{D}_n = \lambda a_n\left[\frac{1}{(1+|z|^2)^{\frac{n-2}{2}}} - \frac{1}{|z|^{n-2}}\right] & \text{in } \mathbb{R}^n, \\ \mathcal{D}_n \approx |z|^{-(n-2)}|\log|z|| & \text{as } |z| \to \infty. \end{cases}$$

Proof. The inequality $0 < PU_{\delta,\xi} \le U_{\delta,\xi}$ in Ω holds by the maximum principle.

The proof for the case where $PU_{\delta,\xi}$ satisfies (1.8), or it satisfies (1.9) with $n = 3$, can be found in [52, Proposition 1] or [25, Lemma 2.2], respectively. Here, we provide a proof for $PU_{\delta,\xi}$ satisfying (1.9) that applies to $n = 3,4,5$ simultaneously.

Let $G_\lambda(x,y)$ be the Green's function of $-\Delta - \lambda$ in $\Omega \subset \mathbb{R}^n$ with Dirichlet boundary condition, which solves

$$\begin{cases} -\Delta_x G_\lambda(x,y) - \lambda G_\lambda(x,y) = \delta_y & \text{in } \Omega, \\ G_\lambda(x,y) = 0 & \text{on } \partial\Omega \end{cases} \tag{2.7}$$

in the sense of distributions. The function $G_\lambda(x,y)$ is symmetric with respect to the two variables x and y. Also, one can write

$$G_\lambda(x,y) = \frac{1}{(n-2)|\mathbb{S}^{n-1}|}\left[\frac{1}{|x-y|^{n-2}} - H_\lambda(x,y)\right],$$

where $|\mathbb{S}^{n-1}|$ is the surface area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ and H_λ solves

$$\begin{cases} \Delta_x H_\lambda(x,y) + \lambda H_\lambda(x,y) = \lambda \frac{1}{|x-y|^{n-2}} & \text{in } \Omega, \\ H_\lambda(x,y) = \frac{1}{|x-y|^{n-2}} & \text{on } \partial\Omega. \end{cases} \tag{2.8}$$

We decompose $H_\lambda(x,y)$ as

$$H_\lambda(x,y) = \left\{ \begin{matrix} \frac{\lambda}{2}|x-y| & \text{if } n=3 \\ \frac{\lambda}{2}\log|x-y| & \text{if } n=4 \\ -\frac{\lambda}{2}\frac{1}{|x-y|} + 2\lambda^2|x-y| & \text{if } n=5 \end{matrix} \right\} + H_\lambda^n(x,y) \tag{2.9}$$

and apply elliptic regularity theory to ensure that $H_\lambda^n(x,y) \in C^{1,\alpha}(\Omega\times\Omega)$ for any $\alpha \in (0,1)$.

Next, we define

$$\mathcal{S}_{\delta,\xi}(x) = PU_{\delta,\xi} - U_{\delta,\xi} + a_n\delta^{\frac{n-2}{2}} H_\lambda(x,\xi) - \widetilde{\mathcal{D}}_n(x).$$

Here, $\widetilde{\mathcal{D}}_n(x) := \delta^{2-\frac{n-2}{2}}\mathcal{D}_n(\frac{x-\xi}{\delta})$ so that

$$\begin{cases} -\Delta\widetilde{\mathcal{D}}_n = \lambda a_n \left[\left(\frac{\delta}{\delta^2+|x-\xi|^2}\right)^{\frac{n-2}{2}} - \frac{\delta^{\frac{n-2}{2}}}{|x-\xi|^{n-2}} \right] & \text{in } \Omega, \\ \widetilde{\mathcal{D}}_n \approx \frac{\delta^{2+\frac{n-2}{2}}}{|x-\xi|^{n-2}} \left| \log \frac{|x-\xi|}{\delta} \right| & \text{on } \partial\Omega. \end{cases}$$

By observing that

$$\mathcal{S}_{\delta,\xi}(x) = -a_n \left[\left(\frac{\delta}{\delta^2+|x-\xi|^2} \right)^{\frac{n-2}{2}} - \frac{\delta^{\frac{n-2}{2}}}{|x-\xi|^{n-2}} \right] - \widetilde{\mathcal{D}}_n(x) \quad \text{for } x \in \partial\Omega,$$

we obtain

$$\begin{cases} \Delta\mathcal{S}_{\delta,\xi} + \lambda\mathcal{S}_{\delta,\xi} = \lambda\widetilde{\mathcal{D}}_n & \text{in } \Omega, \\ \mathcal{S}_{\delta,\xi} = O\left(\delta^{2+\frac{n-2}{2}} \left[d(\xi,\partial\Omega)^{-(n-2)} \left| \log \frac{d(\xi,\partial\Omega)}{\delta} \right| + d(\xi,\partial\Omega)^{-n} \right] \right) & \text{on } \partial\Omega. \end{cases}$$

We notice that

$$|\mathcal{D}_n(z)| \simeq \begin{cases} |z| & \text{if } n=3, \\ |\log|z|| & \text{if } n=4, \\ |z|^{-1} & \text{if } n=5 \end{cases} \quad \text{as } |z| \to 0.$$

Thus, elliptic estimates yield that $\|\widetilde{\mathcal{D}}_3\|_{L^t} \lesssim \delta^{\frac{3}{2}+\frac{3}{t}}$ for any $t>3$, $\|\widetilde{\mathcal{D}}_4\|_{L^t} \lesssim \delta^{1+\frac{4}{t}}$ for any $t>2$, and $\|\widetilde{\mathcal{D}}_5\|_{L^t} \lesssim \delta^{\frac{1}{2}+\frac{5}{t}}$ for any $t \in (\frac{5}{2},5)$. Thus, we conclude for any $\tau \in (0,1)$,

$$\|\mathcal{S}_{\delta,\xi}\|_{L^\infty(\Omega)} = O\left(\left\{ \begin{matrix} \delta^{\frac{5}{2}-\tau} & \text{if } n=3,5 \\ \delta^{3-\tau} & \text{if } n=4 \end{matrix} \right\} + \delta^{2+\frac{n-2}{2}} \left[d(\xi,\partial\Omega)^{-(n-2)} \left| \log \frac{d(\xi,\partial\Omega)}{\delta} \right| + d(\xi,\partial\Omega)^{-n} \right] \right),$$

which completes the proof. □

Remark 2.2.

(1) To construct solutions to the Brezis-Nirenberg problem via a perturbative approach, additional information about $H_\lambda^n(x,y)$ might be necessary. However, since the coefficient λ is fixed in this paper, the $C^{1,\alpha}$ regularity suffices for our purpose.

(2) Define $\varphi_\lambda^n(x) := H_\lambda^n(x,x)$ for $n=3,4,5$ and $\varphi(x) := H(x,x)$ for $n \ge 3$. Indeed, it is not difficult to realize that $\varphi_\lambda^n \in C^\infty(\Omega)$ for $n=3,4,5$ and $\varphi \in C^\infty(\Omega)$ for $n \ge 3$. When $d(x,\partial\Omega)$ is

small enough, the following estimates hold:

$$\begin{aligned}
&\begin{Bmatrix} \varphi_\lambda^n(x) & \text{if } n=3,4,5 \\ \varphi(x) & \text{if } n\ge 3 \end{Bmatrix} = \frac{1}{(2d(x,\partial\Omega))^{n-2}}[1+O(d(x,\partial\Omega))], \\
&\begin{Bmatrix} |\nabla\varphi_\lambda^n(x)| & \text{if } n=3,4,5 \\ |\nabla\varphi(x)| & \text{if } n\ge 3 \end{Bmatrix} = \frac{2(n-2)}{(2d(x,\partial\Omega))^{n-1}}[1+O(d(x,\partial\Omega))].
\end{aligned} \tag{2.10}$$

We postpone their proofs to Appendix B.

Corollary 2.3. *Suppose that $x,\,\xi\in\Omega$ and $\delta>0$ is small. For any $\tau\in(0,1)$, it holds that*

$$PZ^0_{\delta,\xi}(x) = Z^0_{\delta,\xi}(x) - \frac{n-2}{2}a_n\delta^{\frac{n-2}{2}}H(x,\xi) + O\left(\delta^{\frac{n+2}{2}}d(\xi,\partial\Omega)^{-n}\right)$$

provided $n\ge 3$ and $PU_{\delta,\xi}$ is given by equation (1.8), and

$$\begin{aligned}
PZ^0_{\delta,\xi}(x) =&\ Z^0_{\delta,\xi}(x) + \frac{n-2}{4}\lambda a_n\delta^{\frac{n-2}{2}}\begin{Bmatrix} -|x-\xi| & \textit{if } n=3 \\ -\log|x-\xi| & \textit{if } n=4 \\ \frac{1}{|x-\xi|}-4\lambda|x-\xi| & \textit{if } n=5 \end{Bmatrix} - \frac{n-2}{2}a_n\delta^{\frac{n-2}{2}}H_\lambda^n(x,\xi) \\
&+\delta\partial_\delta\left[\delta^{2-\frac{n-2}{2}}\mathcal{D}_n\Big(\frac{x-\xi}{\delta}\Big)\right] + \begin{Bmatrix} O(\delta^{\frac{5}{2}-\tau}) & \textit{if } n=3,5 \\ O(\delta^{3-\tau}) & \textit{if } n=4 \end{Bmatrix} \\
&+O\left(\delta^{\frac{n+2}{2}}\left[d(\xi,\partial\Omega)^{-(n-2)}\left|\log\frac{d(\xi,\partial\Omega)}{\delta}\right| + d(\xi,\partial\Omega)^{-n}\right]\right)
\end{aligned}$$

provided $n=3,4,5$ and $PU_{\delta,\xi}$ is given by equation (1.9).

Proof. We can argue as in the proof of Lemma 2.1. We omit the details. □

Corollary 2.4. *Suppose that $x,\,\xi\in\Omega$, $\delta>0$ is small, and $k=1,\dots,n$. For any $\tau\in(0,1)$, it holds that*

$$PZ^k_{\delta,\xi}(x) = Z^k_{\delta,\xi}(x) - a_n\delta^{\frac{n}{2}}\partial_{\xi^k}H(x,\xi) + O\left(\delta^{\frac{n+2}{2}}d(\xi,\partial\Omega)^{-n}\right)$$

provided $n\ge 3$ and $PU_{\delta,\xi}$ is given by equation (1.8), and

$$\begin{aligned}
PZ^k_{\delta,\xi}(x) =&\ Z^k_{\delta,\xi}(x) + a_n\delta^{\frac{n}{2}}\begin{Bmatrix} \dfrac{\lambda}{2}\dfrac{(x-\xi)^k}{|x-\xi|} & \textit{if } n=3 \\ \dfrac{\lambda}{2}\dfrac{(x-\xi)^k}{|x-\xi|^2} & \textit{if } n=4 \end{Bmatrix} - \delta^{\frac{n}{2}}a_n\partial_{\xi^k}H_\lambda^n(x,\xi) + \delta\partial_{\xi^k}\left[\delta^{2-\frac{n-2}{2}}\mathcal{D}_n\left(\frac{x-\xi}{\delta}\right)\right] \\
&+\begin{Bmatrix} O(\delta^{\frac{5}{2}-\tau}) + O\left(\delta^{\frac{n+2}{2}}\left[d(\xi,\partial\Omega)^{-(n-2)}\left|\log\frac{d(\xi,\partial\Omega)}{\delta}\right| + d(\xi,\partial\Omega)^{-n}\right]\right) & \textit{if } n=3 \\ O(\delta^{3-\tau}) + O\left(\delta^{\frac{n+2}{2}}\left[d(\xi,\partial\Omega)^{-(n-2)}\left|\log\frac{d(\xi,\partial\Omega)}{\delta}\right| + d(\xi,\partial\Omega)^{-n}\right]\right) & \textit{if } n=4 \end{Bmatrix}
\end{aligned}$$

provided $n=3,4$ and $PU_{\delta,\xi}$ is given by equation (1.9). Furthermore, if $n=5$ and $PU_{\delta,\xi}$ is given by equation (1.9), then

$$\begin{aligned}
PZ^k_{\delta,\xi}(x) =&\ Z^k_{\delta,\xi}(x) + a_n\delta^{\frac{n}{2}}\left[\frac{\lambda}{2}\frac{(x-\xi)^k}{|x-\xi|^3} + 2\lambda^2\frac{(x-\xi)^k}{|x-\xi|}\right] - \delta^{\frac{n}{2}}a_n\partial_{\xi^k}H_\lambda^n(x,\xi) \\
&+\delta\partial_{\xi^k}\left[\delta^{2-\frac{n-2}{2}}\mathcal{D}_n\left(\frac{x-\xi}{\delta}\right)\right] + \delta\partial_{\xi^k}\mathcal{S}_{\delta,\xi}(x),
\end{aligned}$$

where the function $\mathcal{S}_{\delta,\xi}$ satisfies

$$\|\delta\partial_{\xi^k}\mathcal{S}_{\delta,\xi}\|_{L^t(\Omega)} \lesssim \delta^{\frac{1}{2}+\frac{5}{t}} + O\left(\delta^{\frac{n+2}{2}}\left[d(\xi,\partial\Omega)^{-(n-2)} + \delta d(\xi,\partial\Omega)^{-(n+1)}\right]\right)$$

for any $t\in(\frac{5}{3},\frac{5}{2})$.[3]

[3]We have not deduced a pointwise estimate of $|\delta\partial_{\xi^k}\mathcal{S}_{\delta,\xi}|$ for this case. Its L^t-estimate is sufficient for later use.

Proof. We notice that

$$|\nabla \mathcal{D}_n(z)| \simeq \begin{cases} |\log|z|| & \text{if } n=3, \\ |z|^{-1} & \text{if } n=4, \\ |z|^{-2} & \text{if } n=5 \end{cases} \quad \text{as } |z|\to 0, \quad \text{and} \quad |\nabla \mathcal{D}_n(z)| \simeq |z|^{-(n-2)} \quad \text{as } |z|\to\infty.$$

Thus, elliptic estimates yield that $\|\delta\partial_{\xi^k}\widetilde{\mathcal{D}}_3\|_{L^t} \lesssim \delta^{\frac{3}{2}+\frac{3}{t}}$ for any $t>3$, $\|\delta\partial_{\xi^k}\widetilde{\mathcal{D}}_4\|_{L^t} \lesssim \delta^{1+\frac{4}{t}}$ for any $t\in(2,4)$, and $\|\delta\partial_{\xi^k}\widetilde{\mathcal{D}}_5\|_{L^t} \lesssim \delta^{\frac{1}{2}+\frac{5}{t}}$ for any $t\in(\frac{5}{3},\frac{5}{2})$. Using these results, we employ the same strategy in the proof of Lemma 2.1. □

2.3. $(H_0^1(\Omega))^*$**-norm estimates for the terms $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$.** We recall the quantities $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$ from (2.2). We estimate their $(H_0^1(\Omega))^*$-norms.

Lemma 2.5. *For each $i\in\{1,\dots,\nu\}$, we assume that $PU_i = PU_{\delta_i,\xi_i}$ satisfies (1.8) if $n\ge 5$ or $[n=3,4$ and $u_0>0]$, and satisfies (1.9) if $n=3,4$ and $u_0=0$. Then it holds that*

$$\|\mathcal{I}_1\|_{(H_0^1(\Omega))^*} + \|\mathcal{I}_2\|_{(H_0^1(\Omega))^*} + \|\mathcal{I}_3\|_{(H_0^1(\Omega))^*} \lesssim \left\{\begin{matrix} 0 & \text{if } n=3, u_0=0 \\ \max_i \delta_i^2|\log\delta_i| & \text{if } n=4, u_0=0 \\ \max_i \delta_i^{\frac{n-2}{2}} & \text{if } [n=3,4 \text{ and } u_0>0] \text{ or } n=5 \\ \max_i \delta_i^2|\log\delta_i|^{\frac{1}{2}} & \text{if } n=6 \\ \max_i \delta_i^2, & \text{if } n\ge 7 \end{matrix}\right\}$$
$$+ \left\{\begin{matrix} \max_i \kappa_i^{n-2} & \text{if } n=3,4,5 \\ \max_i \kappa_i^4|\log\kappa_i|^{\frac{1}{2}} & \text{if } n=6 \\ \max_i \kappa_i^{\frac{n+2}{2}} & \text{if } n\ge 7 \end{matrix}\right\} + \left\{\begin{matrix} Q & \text{if } n=3,4,5 \\ Q|\log Q|^{\frac{1}{2}} & \text{if } n=6 \\ Q^{\frac{n+2}{2(n-2)}} & \text{if } n\ge 7 \end{matrix}\right\} \mathbf{1}_{\{\nu\ge 2\}}$$

provided $\epsilon_1>0$ is small.

Proof. We begin by introducing an elementary inequality: For fixed $m\in\mathbb{N}$, $s>1$, and any $a_1,\dots,a_m\ge 0$, it holds that

$$0 \le \left(\sum_{i=1}^m a_i\right)^s - \sum_{i=1}^m a_i^s \lesssim \sum_{i\ne j}\left[(a_i+a_j)^s - a_i^s - a_j^s\right] \lesssim \begin{cases} \sum_{i\ne j} a_i^{s-1}a_j & \text{if } s>2, \\ \sum_{i\ne j} \min\{a_i^{s-1}a_j, a_i a_j^{s-1}\} & \text{if } s\le 2. \end{cases}$$

From this, we derive that

$$0 \le \mathcal{I}_1 + \mathcal{I}_2 \lesssim \sum_{i=1}^{\nu}(U_i^{p-1}+U_j) + \sum_{i\ne j} U_i^{p-1}U_j \quad \text{for } n=3,4,5. \tag{2.11}$$

We next consider when $n\ge 6$. Fixing any $i\in\{1,\dots,\nu\}$, we decompose $\mathcal{I}_1$ into three parts:

$$\mathcal{I}_1 = \mathcal{I}_{11} + \mathcal{I}_{12} + \mathcal{I}_{13},$$

where

$$\begin{aligned} \mathcal{I}_{11} &:= (u_0+PU_i)^p - u_0^p - (PU_i)^p, \\ \mathcal{I}_{12} &:= (u_0+\sigma)^p - (u_0+PU_i)^p - \sum_{j\ne i}(PU_j)^p, \\ \mathcal{I}_{13} &:= (PU_i)^p + \sum_{j\ne i}(PU_j)^p - \sigma^p. \end{aligned}$$

Considering the relationship between u_0 and U_i in different regions, i.e., $u_0 \lesssim U_i$ when $|x-\xi_i| \le \sqrt{\delta_i}$ and $u_0 \gtrsim U_i$ when $|x-\xi_i| \ge \sqrt{\delta_i}$, we obtain that

$$|\mathcal{I}_{11}| \lesssim \min\{u_0(PU_i)^{p-1}, u_0^{p-1}PU_i\} \lesssim \min\{U_i^{p-1}, U_i\} \simeq U_i^{p-1}\mathbf{1}_{|x-\xi_i|\le\sqrt{\delta_i}} + U_i\mathbf{1}_{|x-\xi_i|\ge\sqrt{\delta_i}}.$$

Similarly, we have

$$\begin{aligned}|\mathcal{I}_{12}| &\lesssim \sum_{j\neq i} \min\left\{(u_0+PU_i)^{p-1}PU_j, (u_0+PU_i)(PU_j)^{p-1}\right\} \\ &\lesssim \sum_{j\neq i}\left[\min\{U_i^{p-1}U_j, U_j^{p-1}U_i\}\mathbf{1}_{|x-\xi_i|\le\sqrt{\delta_i}} + \min\{U_j, U_j^{p-1}\}\mathbf{1}_{|x-\xi_i|\ge\sqrt{\delta_i}}\right].\end{aligned}$$

In addition,

$$|\mathcal{I}_{13}| + \mathcal{I}_2 \lesssim \sum_{j\neq i} \min\{U_i^{p-1}U_j, U_j^{p-1}U_i\}.$$

By introducing the rescaled variable $x_i := \delta_i^{-1}(x-\xi_i)$ and using [23, Proposition 3.4], we deduce a pointwise estimate for $\mathcal{I}_1+\mathcal{I}_2$:

$$\begin{aligned}&\mathcal{I}_1+\mathcal{I}_2 \\ &\lesssim \sum_{i=1}^{\nu} \min\{U_i, U_i^{p-1}\}\mathbf{1}_{\{u_0>0\}} + \sum_{j\neq i}\min\{U_i^{p-1}U_j, U_j^{p-1}U_i\}\mathbf{1}_{\{\nu\ge2\}} \\ &\lesssim \sum_{i=1}^{\nu}\left[\frac{\delta_i^{-2}}{\langle x_i\rangle^4}\mathbf{1}_{\{|x_i|\le\delta_i{}^{-1/2}\}} + \frac{\delta_i^{-\frac{n-2}{2}}}{\langle x_i\rangle^{n-2}}\mathbf{1}_{\{|x_i|\ge\delta_i{}^{-1/2}\}}\right]\mathbf{1}_{\{u_0>0\}} \\ &+\begin{cases}\displaystyle\sum_{i=1}^{\nu}\left[\delta_i^{-4}\frac{\mathscr{R}^{-4}}{\langle x_i\rangle^4}\mathbf{1}_{\{|x_i|<\mathscr{R}^2\}}(x) + \delta_i^{-4}\frac{\mathscr{R}^{-2}}{|x_i|^5}\mathbf{1}_{\{|x_i|\ge\mathscr{R}^2\}}(x)\right] & \text{if } n=6 \\ \displaystyle\sum_{i=1}^{\nu}\left[\delta_i^{-\frac{n+2}{2}}\frac{\mathscr{R}^{2-n}}{\langle x_i\rangle^4}\mathbf{1}_{\{|x_i|<\mathscr{R}\}}(x) + \delta_i^{-\frac{n+2}{2}}\frac{\mathscr{R}^{-4}}{|x_i|^{n-2}}\mathbf{1}_{\{|x_i|\ge\mathscr{R}\}}(x)\right] & \text{if } n\ge7\end{cases}\mathbf{1}_{\{\nu\ge2\}}.\end{aligned} \tag{2.12}$$

Consequently, direct computations using (2.11)–(2.12) and Lemmas A.2–A.3 give the $(H_0^1(\Omega))^*$-norm estimates for $\mathcal{I}_1$ and $\mathcal{I}_2$ provided $n=3,4,5$ or $n\ge7$:

$$\begin{aligned}&\|\mathcal{I}_1\|_{(H_0^1(\Omega))^*} + \|\mathcal{I}_2\|_{(H_0^1(\Omega))^*} \lesssim \|\mathcal{I}_1\|_{L^{\frac{p+1}{p}}(\Omega)} + \|\mathcal{I}_2\|_{L^{\frac{p+1}{p}}(\Omega)} \\ &\qquad\lesssim \begin{cases}\max_i \delta_i^{\frac{n-2}{2}} & \text{if } n=3,4,5 \\ \max_i \delta_i^{\frac{n+2}{4}} & \text{if } n\ge7\end{cases}\mathbf{1}_{\{u_0>0\}} + \begin{cases}Q & \text{if } n=3,4,5 \\ Q^{\frac{n+2}{2(n-2)}} & \text{if } n\ge7\end{cases}\mathbf{1}_{\{\nu\ge2\}}.\end{aligned} \tag{2.13}$$

Furthermore, by applying Lemma 2.1, we see that

$$\begin{aligned}&\|(PU_i-U_i)U_i^{p-1}\|_{L^{\frac{p+1}{p}}(\Omega)} \\ &\lesssim \|(PU_i-U_i)U_i^{p-1}\|_{L^{\frac{p+1}{p}}(B(\xi_i,d(\xi_i,\partial\Omega)))} + \|U_i^p\|_{L^{\frac{p+1}{p}}(B(\xi_i,d(\xi_i,\partial\Omega))^c)} \\ &\lesssim \begin{cases}\max_i \kappa_i^{n-2} & \text{if } n=3 \text{ or } [n=4,\ \text{each } PU_i \text{ satisfies (1.8)}] \text{ or } n=5, \\ \max_i\left(\delta_i^2|\log\delta_i| + \kappa_i^2\right) & \text{if } n=4 \text{ and each } PU_i \text{ satisfies (1.9)}, \\ \max_i \kappa_i^{\frac{n+2}{2}} & \text{if } n\ge7\end{cases} \\ &=: J_1.\end{aligned} \tag{2.14}$$

Using estimate (A.1) and Lemma A.2, we derive the $(H_0^1(\Omega))^*$-norm estimate for $\mathcal{I}_3$ provided $n = 3, 4, 5$ or $n \geq 7$:

$$\begin{aligned}
\|\mathcal{I}_3\|_{(H_0^1(\Omega))^*} &\lesssim \|\mathcal{I}_3\|_{L^{\frac{p+1}{p}}(\Omega)} \\
&\lesssim \max_i \|(PU_i - U_i)U_i^{p-1}\|_{L^{\frac{p+1}{p}}(\Omega)} + \max_i \|(PU_i - U_i)^{p-1}U_i\|_{L^{\frac{p+1}{p}}(\Omega)} \\
&\quad + \lambda \max_i \|U_i\|_{L^{\frac{p+1}{p}}(\Omega)} \mathbf{1}_{\{\text{each } PU_i \text{ satisfies } (1.8)\}} \\
&\lesssim J_1 + \begin{Bmatrix} \max_i \delta_i^{\frac{n-2}{2}} & \text{if } n = 3,4,5 \\ \max_i \delta_i^2 & \text{if } n \geq 7 \end{Bmatrix} \mathbf{1}_{\{\text{each } PU_i \text{ satisfies } (1.8)\}}.
\end{aligned} \tag{2.15}$$

For the case $n = 6$, we fully exploit the definition of the dual norm $\|\mathcal{I}_i\|_{(H_0^1(\Omega))^*}$ rather than relying on estimates of $\|\mathcal{I}_i\|_{L^{(p+1)/p}(\Omega)}$ as above. Recall the dual norm is defined as

$$\|\mathcal{I}_i\|_{(H_0^1(\Omega))^*} = \sup_{\chi \in H_0^1(\Omega)\setminus\{0\}} \frac{\left|\int_\Omega \mathcal{I}_i \chi\right|}{\|\chi\|_{H_0^1(\Omega)}} \quad \text{for } i = 1,2,3.$$

Consider the boundary value problem

$$\begin{cases} (-\Delta - \lambda)v = \mathcal{I}_2 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.16}$$

By observing that

$$\left|\int_\Omega \mathcal{I}_2 \chi\right| = \left|\int_\Omega \nabla v \cdot \nabla \chi - \lambda v \chi\right| \lesssim \|v\|_{H_0^1(\Omega)} \|\chi\|_{H_0^1(\Omega)} \quad \text{for any } \chi \in H_0^1(\Omega),$$

we deduce

$$\|\mathcal{I}_2\|_{(H_0^1(\Omega))^*} \lesssim \|v\|_{H_0^1(\Omega)}.$$

Testing (2.16) with v itself and using (2.7) and (2.12), we obtain

$$\begin{aligned}
\|v\|_{H_0^1(\Omega)}^2 &= \int_\Omega (\mathcal{I}_2 v)(x)dx = \int_\Omega \mathcal{I}_2(x) \int_\Omega G_\lambda(x,\omega)\mathcal{I}_2(\omega)d\omega dx \\
&\lesssim \int_\Omega \mathcal{I}_2(x) \int_\Omega \frac{1}{|x-\omega|^4} \mathcal{I}_2(\omega) d\omega dx \\
&\lesssim \int_\Omega \sum_{i=1}^{\nu} \left[\delta_i^{-4} \frac{\mathscr{R}^{-4}}{\langle x_i \rangle^4} \mathbf{1}_{\{|x_i|<\mathscr{R}^2\}}(x) + \delta_i^{-4} \frac{\mathscr{R}^{-2}}{|x_i|^5} \mathbf{1}_{\{|x_i|\geq\mathscr{R}^2\}}(x)\right] \\
&\qquad \times \sum_{j=1}^{\nu} \left[\delta_j^{-2} \frac{\mathscr{R}^{-4}}{\langle x_j \rangle^2} \mathbf{1}_{\{|x_j|<\mathscr{R}^2\}}(x) + \delta_j^{-2} \frac{\mathscr{R}^{-2}}{|x_j|^3} \mathbf{1}_{\{|x_j|\geq\mathscr{R}^2\}}(x)\right] dx \\
&\lesssim \left\|\sum_{i=1}^{\nu} \left[\delta_i^{-4} \frac{\mathscr{R}^{-4}}{\langle x_i \rangle^4} \mathbf{1}_{\{|x_i|<\mathscr{R}^2\}}(x) + \delta_i^{-4} \frac{\mathscr{R}^{-2}}{|x_i|^5} \mathbf{1}_{\{|x_i|\geq\mathscr{R}^2\}}(x)\right]\right\|_{L^{\frac{3}{2}}(\Omega)} \\
&\qquad \times \left\|\sum_{j=1}^{\nu} \left[\delta_j^{-2} \frac{\mathscr{R}^{-4}}{\langle x_j \rangle^2} \mathbf{1}_{\{|x_j|<\mathscr{R}^2\}}(x) + \delta_j^{-2} \frac{\mathscr{R}^{-2}}{|x_j|^3} \mathbf{1}_{\{|x_j|\geq\mathscr{R}^2\}}(x)\right]\right\|_{L^3(\Omega)} \\
&\lesssim \mathscr{R}^{-8} |\log \mathscr{R}| \simeq Q^2 |\log Q|,
\end{aligned}$$

which yields

$$\|\mathcal{I}_2\|_{(H_0^1(\Omega))^*} \lesssim Q |\log Q|^{\frac{1}{2}}. \tag{2.17}$$

Similarly, we derive

$$\begin{aligned}\|\mathcal{I}_1\|_{(H_0^1(\Omega))^*} &\lesssim \sum_{i=1}^{\nu}\|U_i\|_{(H_0^1(\Omega))^*} \lesssim \sum_{i=1}^{\nu}\left[\int_\Omega U_i(x)\int_\Omega \frac{1}{|x-\omega|^4}U_i(\omega)d\omega dx\right]^{\frac12}\\ &\lesssim \sum_{i=1}^{\nu}\left[\int_\Omega \frac{\delta_i^4}{(\delta_i^2+|x-\xi_i|^2)^3}dx\right]^{\frac12} \lesssim \max_i \delta_i^2|\log\delta_i|^{\frac12}.\end{aligned} \tag{2.18}$$

By Lemma 2.1, we further estimate

$$\begin{aligned}\|\mathcal{I}_3\|_{(H_0^1(\Omega))^*} &\lesssim \sum_{i=1}^{\nu}\left[\|(PU_i-U_i)U_i\|_{(H_0^1(\Omega))^*}+\|U_i\|_{(H_0^1(\Omega))^*}\right]\\ &\lesssim \sum_{i=1}^{\nu}\left[\left|\int_\Omega[(PU_i-U_i)U_i](x)\int_\Omega\frac{1}{|x-\omega|^4}[(PU_i-U_i)U_i](\omega)d\omega dx\right|^{\frac12}+\delta_i^2|\log\delta_i|^{\frac12}\right]\\ &\lesssim \max_i \kappa_i^4|\log\kappa_i|^{\frac12}+\max_i\delta_i^2|\log\delta_i|^{\frac12} =: J_1.\end{aligned} \tag{2.19}$$

To derive the third inequality in (2.19), we considered that for $d(\xi_i,\partial\Omega)\le c$ with $c>0$ small,

$$\begin{aligned}&\int_\Omega\frac{1}{|x-\omega|^4}|(PU_i-U_i)U_i|(\omega)d\omega\\ &\lesssim \int_\Omega\frac{1}{|x-\omega|^4}\left[\delta_i^2(|\varphi(\xi_i)|+|H(\omega,\xi_i)-\varphi(\xi_i)|+\delta_i^2d(\xi_i,\partial\Omega)^{-4})U_i\mathbf{1}_{B(\xi_i,d(\xi_i,\partial\Omega))}(\omega)+U_i^2\mathbf{1}_{\Omega\setminus B(\xi_i,d(\xi_i,\partial\Omega))}\right]d\omega\\ &\lesssim \int_\Omega\frac{1}{|x-\omega|^4}\Bigg[\delta_i^2\left(|H(\omega,\xi_i)-\varphi(\xi_i)|+\delta_i^2d(\xi_i,\partial\Omega)^{-4}\right)U_i\mathbf{1}_{B(\xi_i,d(\xi_i,\partial\Omega))}(\omega)\\ &\qquad\qquad+\left\{\frac{\delta_i^{-4}\kappa_i^4}{\langle\omega_i\rangle^4}\mathbf{1}_{\{|\omega_i|\le\kappa_i^{-1}\}}+\frac{\delta_i^{-4}\kappa_i^3}{|\omega_i|^5}\mathbf{1}_{\{|\omega_i|\ge\kappa_i^{-1}\}}\right\}\Bigg]d\omega\\ &\lesssim \int_\Omega\frac{1}{|x-\omega|^4}\delta_i^2\left(|H(\omega,\xi_i)-\varphi(\xi_i)|+\delta_i^2d(\xi_i,\partial\Omega)^{-4}\right)U_i\mathbf{1}_{B(\xi_i,d(\xi_i,\partial\Omega))}(\omega)d\omega\\ &\qquad\qquad+\frac{\delta_i^{-2}\kappa_i^4}{\langle x_i\rangle^2}\mathbf{1}_{\{|x_i|\le\kappa_i^{-1}\}}+\frac{\delta_i^{-2}\kappa_i^3}{|x_i|^3}\mathbf{1}_{\{|x_i|\ge\kappa_i^{-1}\}},\end{aligned}$$

where $\omega_i := \delta_i^{-1}(\omega-\xi_i)$, and subsequently,

$$\begin{aligned}&\int_\Omega[(PU_i-U_i)U_i](x)\Bigg[\int_\Omega\frac{1}{|x-\omega|^4}\delta_i^2(|H(\omega,\xi_i)-\varphi(\xi_i)|+\delta_i^2d(\xi_i,\partial\Omega)^{-4})U_i\mathbf{1}_{B(\xi_i,d(\xi_i,\partial\Omega))}(\omega)d\omega\\ &\qquad\qquad+\frac{\delta_i^{-2}\kappa_i^4}{\langle x_i\rangle^2}\mathbf{1}_{\{|x_i|\le\kappa_i^{-1}\}}+\frac{\delta_i^{-2}\kappa_i^3}{|x_i|^3}\mathbf{1}_{\{|x_i|\ge\kappa_i^{-1}\}}\Bigg]dx\\ &\lesssim \|(PU_i-U_i)U_i\|_{L^{\frac32}(\Omega)}\times\left\|\delta_i^2\left(|H(\cdot,\xi_i)-\varphi(\xi_i)|+\delta_i^2d(\xi_i,\partial\Omega)^{-4}\right)U_i\mathbf{1}_{B(\xi_i,d(\xi_i,\partial\Omega))}\right\|_{L^{\frac32}(\Omega)}\\ &+\Bigg[\left\|\frac{\delta_i^{-4}\kappa_i^4}{\langle x_i\rangle^4}\mathbf{1}_{\{|x_i|\le\kappa_i^{-1}\}}+\frac{\delta_i^{-4}\kappa_i^3}{|x_i|^5}\mathbf{1}_{\{|x_i|\ge\kappa_i^{-1}\}}\right\|_{L^{\frac32}(\Omega)}\\ &\qquad+\left\|\delta_i^2\left(|H(\cdot,\xi_i)-\varphi(\xi_i)|+\delta_i^2d(\xi_i,\partial\Omega)^{-4}\right)U_i\mathbf{1}_{B(\xi_i,d(\xi_i,\partial\Omega))}\right\|_{L^{\frac32}(\Omega)}\Bigg]\\ &\qquad\times\left\|\frac{\delta_i^{-2}\kappa_i^4}{\langle x_i\rangle^2}\mathbf{1}_{\{|x_i|\le\kappa_i^{-1}\}}+\frac{\delta_i^{-2}\kappa_i^3}{|x_i|^3}\mathbf{1}_{\{|x_i|\ge\kappa_i^{-1}\}}\right\|_{L^3(\Omega)}\lesssim\kappa_i^8|\log\kappa_i|.\end{aligned}$$

Also, we applied the Hardy-Littlewood-Sobolev inequality to treat when $d(\xi_i,\partial\Omega)\ge c$ as follows:

$$\left|\int_\Omega [(PU_i - U_i)U_i](x) \int_\Omega \frac{1}{|x-\omega|^4}[(PU_i - U_i)U_i](\omega) d\omega dx\right|$$
$$\lesssim \|(PU_i - U_i)U_i\|^2_{L^{\frac{3}{2}}(\Omega)} \lesssim \delta_i^8 |\log \delta_i|^{\frac{4}{3}} \lesssim \delta_i^4 |\log \delta_i|.$$

Collecting estimates (2.13), (2.15), and (2.17)–(2.19), we conclude the proof. □

2.4. **Projections of $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$ onto the PZ_j^0-direction.** Given $j = 1, \dots, \nu$, we evaluate the integrals $\int_\Omega \mathcal{I}_1 PZ_j^0$, $\int_\Omega \mathcal{I}_2 PZ_j^0$, and $\int_\Omega \mathcal{I}_3 PZ_j^0$, which correspond to the projections of $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$ onto the directions of PZ_j^0, respectively.

Lemma 2.6. *Assume that $u_0 > 0$. Moreover, when $n = 3$, each PU_i satisfies (1.8) or (1.9), and when $n \geq 4$, each PU_i satisfies (1.8). For any $j \in \{1, \dots, \nu\}$, it holds that*

$$\int_\Omega \mathcal{I}_1 PZ_j^0 = \mathfrak{a}_n u_0(\xi_j) \delta_j^{\frac{n-2}{2}} + o(Q) + \begin{Bmatrix} O(\max_i \delta_i) & \text{if } n = 3 \\ O(\max_i \delta_i^2 |\log \delta_i|) & \text{if } n = 4 \\ O(\max_i \delta_i^2) & \text{if } n = 5 \end{Bmatrix} \mathbf{1}_{\{p>2\}}$$
$$+ O\left(\max_i \delta_i^{\frac{n}{2}} + \max_i \kappa_i^n\right), \tag{2.20}$$

where $\mathfrak{a}_n := p \int_{\mathbb{R}^n} U^{p-1} Z^0 > 0$.

Proof. By (A.3), there exists a constant $\eta > 0$ such that

$$\begin{aligned} \mathcal{I}_1 = &\left[p u_0 \sigma^{p-1} + O(u_0^2 \sigma^{p-2}) \mathbf{1}_{\{p>2\}} + O\left(u_0^p\right)\right] \mathbf{1}_{\cup_{i=1}^\nu B(\xi_i, \eta\sqrt{\delta_i})} \\ &+ \left[p u_0^{p-1} \sigma + O(u_0^{p-2} \sigma^2) \mathbf{1}_{\{p>2\}} + O(\sigma^p)\right] \mathbf{1}_{\cap_{i=1}^\nu B(\xi_i, \eta\sqrt{\delta_i})^c}. \end{aligned} \tag{2.21}$$

The remainder of the proof is split into two steps.

STEP 1. It follows from $|PZ_j^0| \lesssim U_j$, Lemma 2.1, Corollary 2.3, and Young's inequality that

$$\begin{aligned} &\left|\int_{B(\xi_j, d(\xi_j, \partial\Omega))} \left[(PU_j)^{p-1} PZ_j^0 - U_j^{p-1} Z_j^0\right]\right| \\ &\lesssim \int_{B(\xi_j, d(\xi_j, \partial\Omega))} (|PU_j - U_j| + |PZ_j^0 - Z_j^0|) U_j^{p-1} + \int_{B(\xi_j, d(\xi_j, \partial\Omega))} |PU_j - U_j|^{p-1} U_j \\ &\lesssim \delta_j^{\frac{n-2}{2}} \kappa_j^2 \lesssim \delta_j^{\frac{n}{2}} + \kappa_j^n. \end{aligned}$$

Therefore,

$$\begin{aligned} &p \int_{\cup_{i=1}^\nu B(\xi_i, \eta\sqrt{\delta_i})} u_0 (PU_j)^{p-1} PZ_j^0 \\ &= p \int_{B(\xi_j, d(\xi_j, \partial\Omega))} u_0 (PU_j)^{p-1} PZ_j^0 + O\bigg(\int_{B(\xi_j, d(\xi_j, \partial\Omega))^c} U_j^p\bigg) \\ &= p \delta_j^{\frac{n-2}{2}} \left[u_0(\xi_j) \int_{\mathbb{R}^n} U^{p-1} Z^0 + O\bigg(\int_{B(0, \kappa_j^{-1})} |\delta_j y|^2 U^p(y) dy\bigg)\right] + O\left(\delta_j^{\frac{n}{2}} + \kappa_j^n\right) \\ &= \mathfrak{a}_n \delta_j^{\frac{n-2}{2}} u_0(\xi_j) + O\left(\delta_j^{\frac{n}{2}} + \kappa_j^n\right). \end{aligned} \tag{2.22}$$

We claim that

$$\left|\int_\Omega u_0 \left[\sigma^{p-1} - (PU_j)^{p-1}\right] PZ_j^0\right| \lesssim \sum_{i \neq l} \int_\Omega U_i^{p-1} U_l = o(Q) + O\left(\max_i \delta_i^{\frac{n}{2}}\right). \tag{2.23}$$

The inequality immediately follows from (A.2). To analyze the equality, we set $z_{ij} := \delta_i^{-1}(\xi_j - \xi_i)$ and $d_{ij} := |\xi_i - \xi_j|$. We distinguish three cases based on the value of $\mathscr{R}_{ij}$.

Case 1: Suppose that $\mathscr{R}_{ij} = \frac{d_{ij}}{\sqrt{\delta_i\delta_j}}$. Then, it holds that $d_{ij} \ge \delta_i$ and $(\sqrt{\delta_i\delta_j}/d_{ij})^{n-2} \simeq q_{ij} \le Q$. In view of Lemma A.4, we confirm that

$$\int_\Omega U_i^{p-1}U_j \lesssim \begin{Bmatrix} \delta_i\delta_j^{\frac{1}{2}}d_{ij}^{-1} & \text{if } n=3 \\ \delta_i^2\delta_j d_{ij}^{-2}\log\left(2+d_{ij}\delta_i^{-1}\right) & \text{if } n=4 \\ \delta_i^2\delta_j^{\frac{n-2}{2}}d_{ij}^{-2} & \text{if } n\ge 5 \end{Bmatrix} = O\left(\max_i \delta_i^{\frac{n}{2}}\right) + o(Q).$$

Case 2: Suppose that $\mathscr{R}_{ij} = \sqrt{\frac{\delta_i}{\delta_j}}$. Then, it holds that $d_{ij} \le \delta_i$, i.e., $|z_{ij}| \le 1$, and $(\frac{\delta_j}{\delta_i})^{\frac{n-2}{2}} \simeq q_{ij} \le Q$. Therefore,

$$\begin{aligned}\int_\Omega U_i^{p-1}U_j &\lesssim \int_\Omega \left(\frac{\delta_i}{\delta_i^2+|x-\xi_i|^2}\right)^2\left(\frac{\delta_j}{\delta_j^2+|x-\xi_j|^2}\right)^{\frac{n-2}{2}}dx \\ &\lesssim \delta_j^{\frac{n-2}{2}}\int_{B(0,C\delta_i^{-1})}\frac{1}{(1+|y|^2)^2}\frac{dy}{[(\frac{\delta_j}{\delta_i})^2+|y-z_{ij}|^2]^{\frac{n-2}{2}}} \\ &\lesssim \delta_j^{\frac{n-2}{2}}\left(1+\int_2^{C\delta_i^{-1}}t^{-3}dt\right) \simeq \delta_j^{\frac{n-2}{2}} = o(Q).\end{aligned}$$

Case 3: Suppose that $\mathscr{R}_{ij} = \sqrt{\frac{\delta_j}{\delta_i}}$. Then, it holds that $d_{ij} \le \delta_j$ and $(\frac{\delta_i}{\delta_j})^{\frac{n-2}{2}} \simeq q_{ij} \le Q$. We divide the domain Ω into $B(\xi_i,\sqrt{\delta_i})$ and $(B(\xi_i,\sqrt{\delta_i}))^c$, and compute

$$\begin{aligned}\int_{B(\xi_i,\sqrt{\delta_i})} U_i^{p-1}U_j &\lesssim \frac{\delta_i^{n-2}}{\delta_j^{\frac{n-2}{2}}}\int_{B(0,\delta_i^{-1/2})}\frac{1}{(1+|y|^2)^2}\frac{dy}{[1+(\frac{\delta_i}{\delta_j}|y-z_{ij}|)^2]^{\frac{n-2}{2}}} \\ &\lesssim \frac{\delta_i^{n-2}}{\delta_j^{\frac{n-2}{2}}}\left(1+\int_1^{\delta_i^{-1/2}}t^{n-5}dt\right) = o(Q)\end{aligned}$$

and

$$\int_{B(\xi_i,\sqrt{\delta_i})^c} U_i^{p-1}U_j \lesssim \delta_i^2\delta_j^{\frac{n-2}{2}}\int_{B(0,\sqrt{\delta_i})^c}\frac{1}{|y|^4}\frac{1}{|y-(\xi_j-\xi_i)|^{n-2}}dy \lesssim \delta_i\delta_j^{\frac{n-2}{2}} = O\left(\max_i \delta_i^{\frac{n}{2}}\right).$$

These estimates justify (2.23).

STEP 2. Applying $|PZ_j^0| \lesssim \sum_{i=1}^\nu U_i$, we observe

$$\int_\Omega u_0^2\sigma^{p-2}\left|PZ_j^0\right|\mathbf{1}_{\{p>2\}} \lesssim \sum_{i=1}^\nu\int_\Omega U_i^{p-1}\mathbf{1}_{\{p>2\}} \lesssim \begin{cases}\max_i \delta_i & \text{if } n=3, \\ \max_i \delta_i^2|\log\delta_i| & \text{if } n=4, \\ \max_i \delta_i^2 & \text{if } n=5.\end{cases} \tag{2.24}$$

Furthermore, since $u_0(x) \lesssim U_i(x)$ for $x \in B(\xi_i,\eta\sqrt{\delta_i})$, we infer from (2.23) that

$$\begin{aligned}\int_{\cup_{i=1}^\nu B(\xi_i,\eta\sqrt{\delta_i})} u_0^p|PZ_j^0| &\lesssim \int_{B(\xi_j,\eta\sqrt{\delta_j})}U_j + \sum_{i\ne j}\int_{B(\xi_i,\eta\sqrt{\delta_i})}U_i^{p-1}U_j \\ &= O\left(\max_i \delta_i^{\frac{n}{2}}\right) + o(Q).\end{aligned} \tag{2.25}$$

We also estimate the integrals over the exterior region:

$$\begin{aligned}\int_{\cap_{i=1}^{\nu} B(\xi_i,\eta\sqrt{\delta_i})^c} u_0^{p-2}\sigma^2|PZ_j^0|\mathbf{1}_{\{p>2\}} &\lesssim \sum_{i=1}^{\nu}\int_{B(\xi_i,\eta\sqrt{\delta_i})^c} U_i^3\mathbf{1}_{\{p>2\}} \\ &\lesssim \begin{cases}\max_i \delta_i^{\frac{3}{2}}|\log\delta_i| & \text{if } n=3,\\ \max_i \delta_i^{\frac{n}{2}} & \text{if } n=4,5\end{cases}\end{aligned} \tag{2.26}$$

and

$$\int_{\cap_{i=1}^{\nu} B(\xi_i,\eta\sqrt{\delta_i})^c} (u_0^{p-1}\sigma+\sigma^p)|PZ_j^0| \lesssim \sum_{i=1}^{\nu}\int_{B(\xi_i,\eta\sqrt{\delta_i})^c}(U_i^2+U_i^p) \lesssim \max_i \delta_i^{\frac{n}{2}}. \tag{2.27}$$

Combining estimates (2.22)–(2.27), we conclude the proof of (2.20). □

Lemma 2.7. *For any $j\in\{1,\dots,\nu\}$, it holds that*

$$\begin{aligned}\int_\Omega \mathcal{I}_3 PZ_j^0 &= \begin{Bmatrix*}[l] -\delta_j\mathfrak{c}_n\varphi(\xi_j)+O(\kappa_j^3)+O(\delta_j) & \text{if } n=3\\ -\delta_j^2\mathfrak{c}_n\varphi(\xi_j)+O(\kappa_j^4)+O(\delta_j^2|\log\delta_j|) & \text{if } n=4\\ \lambda\mathfrak{b}_n\delta_j^2-\delta_j^{n-2}\mathfrak{c}_n\varphi(\xi_j)+O\left(\delta_j^2\kappa_j^{n-4}\right)+O(\kappa_j^n) & \text{if } n\geq 5\end{Bmatrix*} \\ &+\left[\begin{Bmatrix*}[l] O(\max_i\delta_i) & \text{if } n=3\\ O(\max_i\delta_i^2|\log\delta_i|) & \text{if } n=4\\ o(\max_i\delta_i^2) & \text{if } n\geq 5\end{Bmatrix*}+o(Q)\right]\mathbf{1}_{\{\nu\geq 2,\text{ each }\xi_i\text{ is in a compact set of }\Omega\}}\end{aligned} \tag{2.28}$$

provided $n\geq 3$ and each PU_i satisfies (1.8), *and*

$$\begin{aligned}\int_\Omega \mathcal{I}_3 PZ_j^0 &= \begin{Bmatrix*}[l] -\mathfrak{c}_3\varphi_\lambda^3(\xi_j)\delta_j+O(\delta_j^2)+O(\kappa_j^3) & \text{if } n=3\\ \mathfrak{b}_4\lambda\delta_j^2|\log\delta_j|-\mathfrak{c}_4\delta_j^2\varphi_\lambda^4(\xi_j)-96|\mathbb{S}^3|\lambda\delta_j^2+O(\delta_j^3)+O(\kappa_j^4) & \text{if } n=4\end{Bmatrix*} \\ &+\left[\begin{Bmatrix*}[l] O(\max_i\delta_i^2) & \text{if } n=3\\ O(\max_i\delta_i^3|\log\delta_i|) & \text{if } n=4\end{Bmatrix*}+o(Q)\right]\mathbf{1}_{\{\nu\geq 2,\text{ each }\xi_i\text{ is in a compact set of }\Omega\}}\end{aligned} \tag{2.29}$$

provided $n=3,4$ and each PU_i satisfies (1.9). *Here,* $\mathfrak{b}_4 := 3\sqrt{2}\int_{\mathbb{R}^4}U^{p-1}Z^0>0$, $\mathfrak{b}_n:=\int_{\mathbb{R}^n}UZ^0>0$ *for $n\geq 5$, and* $\mathfrak{c}_n := a_np\int_{\mathbb{R}^n}U^{p-1}Z^0>0$.

Proof. We present the proof by dividing it into two steps.

STEP 1. Assuming that each PU_i satisfies (1.8), we assert that

$$\int_\Omega\sum_{i=1}^{\nu}\lambda PU_iPZ_j^0 = \begin{cases} O(\max_i\delta_i)+o(Q)\mathbf{1}_{\{\nu\geq 2\}} & \text{if } n=3,\\ O(\max_i\delta_i^2|\log\delta_i|)+o(Q)\mathbf{1}_{\{\nu\geq 2\}} & \text{if } n=4,\\ \lambda\mathfrak{b}_n\delta_j^2+O\left(\delta_j^2\kappa_j^{n-4}\right)+o(Q+\max_i\delta_i^2)\mathbf{1}_{\{\nu\geq 2\}} & \text{if } n\geq 5.\end{cases} \tag{2.30}$$

To verify (2.30), we first estimate

$$\int_\Omega PU_jPZ_j^0 = \begin{cases} O(\delta_j) & \text{if } n=3,\\ O(\delta_j^2|\log\delta_j|) & \text{if } n=4,\\ \mathfrak{b}_n\delta_j^2+O\left(\delta_j^2\kappa_j^{n-4}\right) & \text{if } n\geq 5.\end{cases}$$

Indeed, for the case $n \ge 5$, we have

$$\left| \int_{B(\xi_j, d(\xi_j,\partial\Omega))^c} PU_j PZ_j^0 \right| \lesssim \delta_j^2 \kappa_j^{n-4}$$

and

$$\begin{aligned} \int_{B(\xi_j, d(\xi_j,\partial\Omega))} PU_j PZ_j^0 &= \int_{B(\xi_j, d(\xi_j,\partial\Omega))} U_j Z_j^0 + O\left(\frac{\delta_j^{\frac{n-2}{2}}}{d(\xi_j,\partial\Omega)^{n-2}} \cdot \delta_j^{\frac{n+2}{2}} \int_{B(0,\kappa_j^{-1})} U \right) \\ &= \mathfrak{b}_n \delta_j^2 + O\left(\delta_j^2 \kappa_j^{n-4}\right). \end{aligned} \tag{2.31}$$

It remains to estimate the interaction terms $\int_\Omega U_i U_j$ for $1 \le i \ne j \le \nu$ provided $\nu \ge 2$. As in (2.23), we separate the analysis into three cases.

Case 1: Suppose that $\mathscr{R}_{ij} = \frac{d_{ij}}{\sqrt{\delta_i \delta_j}}$. We verify that

$$\begin{aligned} \int_\Omega U_i U_j &\lesssim \delta_i^{\frac{n-2}{2}} \delta_j^{\frac{n-2}{2}} \times \begin{cases} 1 & \text{if } n = 3, \\ 1 + |\log d_{ij}| & \text{if } n = 4, \\ d_{ij}^{-(n-4)} & \text{if } n \ge 5 \end{cases} \\ &\simeq \begin{cases} O(\max_i \delta_i) & \text{if } n = 3, \\ O(\max_i \delta_i^2 |\log \delta_i|) & \text{if } n = 4, \\ \max_i \delta_i^2 Q^{\frac{n-4}{n-2}} = o(\max_i \delta_i^2) & \text{if } n \ge 5. \end{cases} \end{aligned}$$

Case 2: Suppose that $\mathscr{R}_{ij} = \sqrt{\frac{\delta_j}{\delta_i}}$. We evaluate

$$\begin{aligned} \int_\Omega U_i U_j &\lesssim \int_\Omega \left(\frac{\delta_i}{\delta_i^2 + |x-\xi_i|^2} \right)^{\frac{n-2}{2}} \left(\frac{\delta_j}{\delta_j^2 + |x-\xi_j|^2} \right)^{\frac{n-2}{2}} dx \\ &\lesssim \delta_j^{\frac{n+2}{2}-(n-2)} \delta_i^{\frac{n-2}{2}} \int_{B(0,C\delta_j^{-1})} \frac{1}{(1+|y|^2)^{\frac{n-2}{2}}} \frac{dy}{[(\frac{\delta_i}{\delta_j})^2 + |y - \frac{\xi_i - \xi_j}{\delta_j}|^2]^{\frac{n-2}{2}}} \\ &\lesssim \delta_j^2 \frac{\delta_i^{\frac{n-2}{2}}}{\delta_j^{\frac{n-2}{2}}} \left(1 + \int_2^{C\delta_j^{-1}} t^{-(n-3)} dt \right) = o(Q). \end{aligned} \tag{2.32}$$

Here, we used $|\xi_i - \xi_j| \le \delta_j$.

Case 3: Suppose that $\mathscr{R}_{ij} = \sqrt{\frac{\delta_i}{\delta_j}}$. We can similarly estimate as above to deduce

$$\int_\Omega U_i U_j = o(Q). \tag{2.33}$$

This concludes the proof of (2.30).

Step 2. We claim that

$$\begin{aligned} \int_\Omega \sum_{i=1}^{\nu} [(PU_i)^p - U_i^p] PZ_j^0 &= -\delta_j^{n-2} \mathfrak{c}_n \varphi(\xi_j) + O(\kappa_j^n) \\ &\quad + \left[O(\max_i \delta_i^{n-1}) + o(Q) \right] \mathbf{1}_{\{\nu \ge 2,\ \text{each } \xi_i \text{ is in a compact set of } \Omega\}} \end{aligned}$$

provided $n \ge 3$ and PU_i satisfies (1.8) for each $i = 1, \dots, \nu$, and

$$\begin{aligned}&\int_\Omega \sum_{i=1}^{\nu} [(PU_i)^p - U_i^p] PZ_j^0 \\ &= \begin{Bmatrix} -\mathfrak{c}_3 \varphi_\lambda^3(\xi_j)\delta_j + O(\delta_j^2) + O(\kappa_j^3) & \text{if } n = 3 \\ \mathfrak{b}_4 \lambda \delta_j^2 |\log \delta_j| - \mathfrak{c}_4 \delta_j^2 \varphi_\lambda^4(\xi_j) - 96|\mathbb{S}^3|\lambda\delta_j^2 + O(\delta_j^3) + O(\kappa_j^4) & \text{if } n = 4 \end{Bmatrix} \\ &+ \left[\begin{Bmatrix} O(\max_i \delta_i^2) & \text{if } n = 3 \\ O(\max_i \delta_i^3 |\log \delta_i|) & \text{if } n = 4 \end{Bmatrix} + o(Q) \right] \mathbf{1}_{\{\nu \ge 2,\ \text{each } \xi_i \text{ is in a compact set of } \Omega\}}\end{aligned}$$

provided $n = 3, 4$ and PU_i satisfies (1.9) for each $i = 1, \dots, \nu$.

To prove this, we decompose the domain by $\Omega = B(\xi_j, d(\xi_j, \partial\Omega)) \cup [\Omega \setminus B(\xi_j, d(\xi_j, \partial\Omega))]$. First, we observe that

$$\left| \int_{\Omega \setminus B(\xi_j, d(\xi_j, \partial\Omega))} [(PU_j)^p - U_j^p] PZ_j^0 \right| \lesssim \int_{B(0, \kappa_j^{-1})^c} U^{p+1} \lesssim \kappa_j^n. \tag{2.34}$$

Suppose that PU_i satisfies (1.8) for each $i = 1, \dots, \nu$. By Lemma 2.1, Corollary 2.3, and (A.3), we obtain

$$\begin{aligned}&\int_{B(\xi_j, d(\xi_j, \partial\Omega))} [(PU_j)^p - U_j^p] PZ_j^0 \\ &= p \int_{B(\xi_j, d(\xi_j, \partial\Omega))} (PU_j - U_j) U_j^{p-1} PZ_j^0 + O\left(\int_{B(\xi_j, d(\xi_j, \partial\Omega))} (PU_j - U_j)^2 U_j^{p-2} |PZ_j^0| \right) \mathbf{1}_{\{p>2\}} \\ &+ O\left(\int_{B(\xi_j, d(\xi_j, \partial\Omega))} |PU_j - U_j|^p |PZ_j^0| \right) \\ &= -\delta_j^{n-2} \mathfrak{c}_n \varphi(\xi_j) + O(\kappa_j^n).\end{aligned} \tag{2.35}$$

Suppose next that $n = 3, 4$ and PU_i satisfies (1.9) for each $i = 1, \dots, \nu$. Noticing that

$$\begin{aligned}p \int_{B(\xi_j, d(\xi_j, \partial\Omega))} \delta_j^{2 - \frac{n-2}{2}} \mathcal{D}_n \left(\frac{\cdot - \xi_j}{\delta_j} \right) U_j^{p-1} Z_j^0 &= \delta_j^2 \int_{B(0, \kappa_j^{-1})} (-\Delta \mathcal{D}_n) Z^0 \\ &+ \delta_j^2 O\left(\int_{\partial B(0, \kappa_j^{-1})} \frac{\partial \mathcal{D}_n}{\partial \nu} |Z^0| dS + \int_{\partial B(0, \kappa_j^{-1})} \left| \frac{\partial Z^0}{\partial \nu} \mathcal{D}_n \right| dS \right) \\ &= \delta_j^2 \int_{B(0, \kappa_j^{-1})} (-\Delta \mathcal{D}_n) Z^0 + O\left(\delta_j^n + \kappa_j^n\right),\end{aligned} \tag{2.36}$$

where $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative and dS is the surface measure, we deduce

$$\begin{aligned}&\int_{B(\xi_j, d(\xi_j, \partial\Omega))} [(PU_j)^p - U_j^p] PZ_j^0 \\ &= -\delta_j^{\frac{1}{2}} a_3 p H_\lambda^3(\xi_j, \xi_j) \int_{B(\xi_j, d(\xi_j, \partial\Omega))} U_j^{p-1} Z_j^0 - \frac{\lambda}{2} a_n \delta_j^{\frac{1}{2}} \int_{B(\xi_j, d(\xi_j, \partial\Omega))} |x - \xi_j| (U_j^{p-1} Z_j^0)(x) dx \\ &+ \lambda a_3^2 p \delta_j^2 \int_{B(0, \kappa_j^{-1})} \left[\frac{1}{\sqrt{1 + |z|^2}} - \frac{1}{|z|} \right] \frac{|z|^2 - 1}{(1 + |z|^2)^{\frac{3}{2}}} dz + O(\delta_j^{3-\tau}) + O(\kappa_j^3) \\ &= -\mathfrak{c}_3 \delta_j \varphi_\lambda^3(\xi_j) + O(\delta_j^2) + O(\kappa_j^3) \text{ for } n = 3,\end{aligned} \tag{2.37}$$

and

$$\begin{aligned}
&\int_{B(\xi_j,d(\xi_j,\partial\Omega))}[(PU_j)^p-U_j^p]PZ_j^0\\
&=\frac{\lambda}{2}a_4\delta_j|\log\delta_j|p\int_{B(\xi_j,d(\xi_j,\partial\Omega))}U_j^{p-1}Z_j^0-\frac{\lambda}{2}a_4p\delta_j^2\int_{B(0,\kappa_j^{-1})}\log|x|(U^2Z^0)(x)dx \qquad (2.38)\\
&+\lambda a_4^2p\delta_j^2\int_{B(0,\kappa_j^{-1})}\left[\frac{1}{1+|z|^2}-\frac{1}{|z|^2}\right]\frac{|z|^2-1}{(1+|z|^2)^2}dz-\delta_ja_4pH_\lambda^4(\xi_j,\xi_j)\int_{B(\xi_j,d(\xi_j,\partial\Omega))}U_j^{p-1}Z_j^0\\
&+O(\delta_j^3)+O(\kappa_j^4)\\
&=\mathfrak{b}_4\lambda\delta_j^2|\log\delta_j|-\mathfrak{c}_4\delta_j^2\varphi_\lambda^4(\xi_j)-96|\mathbb{S}^3|\lambda\delta_j^2+O(\delta_j^3)+O(\kappa_j^4)\text{ for }n=4.
\end{aligned}$$

Here, we used

$$\int_{\mathbb{R}^4}\log|z|\frac{|z|^2-1}{(1+|z|^2)^4}dz=\frac{|\mathbb{S}^3|}{8}\quad\text{and}\quad\int_{\mathbb{R}^4}\left[\frac{1}{1+|z|^2}-\frac{1}{|z|^2}\right]\frac{|z|^2-1}{(1+|z|^2)^2}dz=0.$$

Finally, we assume that $\nu\ge 2$ and each $\xi_1,\dots,\xi_\nu$ is in a compact set of Ω. Given $1\le i\ne j\le\nu$, we infer from (2.23) that

$$\left|\int_\Omega[(PU_i)^p-U_i^p]PZ_j^0\right|\lesssim\delta_i^{\frac{n-2}{2}}\int_{B(\xi_i,d(\xi_i,\partial\Omega))}U_i^{p-1}U_j=O(\max_i\delta_i^{n-1})+o(Q) \qquad (2.39)$$

provided $n\ge 3$ and each PU_i satisfies (1.8), and

$$\begin{aligned}
\left|\int_\Omega[(PU_i)^p-U_i^p]PZ_j^0\right|&\lesssim\begin{Bmatrix}O(\delta_i^{\frac12}) & \text{if } n=3\\ O(\delta_i|\log\delta_i|) & \text{if } n=4\end{Bmatrix}\times\int_{B(\xi_i,d(\xi_i,\partial\Omega))}U_i^{p-1}U_j\\
&=\begin{Bmatrix}O(\max_i\delta_i^2) & \text{if } n=3\\ O(\max_i\delta_i^3|\log\delta_i|) & \text{if } n=4\end{Bmatrix}+o(Q)
\end{aligned} \qquad (2.40)$$

provided $n=3,4$ and each PU_i satisfies (1.9). Here, we used

$$\int_\Omega\left|\log\left|\frac{x-\xi_i}{\delta_i}\right|\right|(U_i^{p-1}U_j)(x)dx=o(Q+\max_i\delta_i^2)\text{ for }n=4,$$

which can be argued as (2.23), and

$$\int_\Omega\left|\delta_i^{2-\frac{n-2}{2}}\mathcal{D}_n\left(\frac{\cdot-\xi_i}{\delta_i}\right)\right|U_i^{p-1}U_j\lesssim\|U_i^{p-1}U_j\|_{L^{\frac{p+1}{p}}(\Omega)}\left\|\delta_i^{2-\frac{n-2}{2}}\mathcal{D}_n\left(\frac{\cdot-\xi_i}{\delta_i}\right)\right\|_{L^{p+1}(\Omega)}\lesssim\delta_i^2Q$$

for $n=3,4$.

This completes the proof of the claim. □

Lemma 2.8. *Assume that $\nu\ge 2$ and each of the $\xi_1,\dots,\xi_\nu$ lies on a compact set of Ω. For any $j\in\{1,\dots,\nu\}$, it holds that*

$$\int_\Omega\mathcal{I}_2PZ_j^0=\sum_{i\ne j}\mathfrak{c}_n\left(q_{ij}^{-\frac{2}{n-2}}-2\frac{\delta_j}{\delta_i}\right)q_{ij}^{\frac{n}{n-2}}+o(Q)$$

$$+\begin{cases}O(\max_i\delta_i^{n-2}) & \textit{if } n\ge 3,\ \textit{each } PU_i \textit{ satisfies (1.8)},\\ \sum_{i\ne j}\left[-\frac{\mathfrak{c}_3}{2}\lambda|\xi_j-\xi_i|-\mathfrak{c}_3H_\lambda^3(\xi_i,\xi_j)\right]\delta_i^{\frac12}\delta_j^{\frac12}\mathbf{1}_{\left\{\mathscr{R}_{ij}=\frac{|\xi_i-\xi_j|}{\sqrt{\delta_i\delta_j}}\right\}}+o(\max_i\delta_i) & \textit{if } n=3,\ \textit{each } PU_i \textit{ satisfies (1.9)},\\ \sum_{i\ne j}\left[-\frac{\mathfrak{c}_4}{2}\lambda\log|\xi_j-\xi_i|-\mathfrak{c}_4H_\lambda^4(\xi_i,\xi_j)\right]\delta_i\delta_j\mathbf{1}_{\left\{\mathscr{R}_{ij}=\frac{|\xi_i-\xi_j|}{\sqrt{\delta_i\delta_j}}\right\}}+o(\max_i\delta_i^2|\log\delta_i|) & \textit{if } n=4,\ \textit{each } PU_i \textit{ satisfies (1.9)},\end{cases}$$

provided q_{ij} *in* (2.3) *is small.*

Proof. Adapting the proof of [23, Lemma 2.1], and employing Lemma 2.1, Corollary 2.3, (2.39)–(2.40), and [23, Lemma A.2], we discover

$$\begin{aligned}
&\int_\Omega \mathcal{I}_2 PZ_j^0 \\
&= \sum_{i\neq j} p\int_\Omega (PU_j)^{p-1}PU_iPZ_j^0\mathbf{1}_{\{\nu\geq 2\}} + o(Q) \\
&= \sum_{i\neq j}\int_{\mathbb{R}^n} U_i^p\delta_j\frac{\partial U_j}{\partial\delta_j} + p\sum_{i\neq j}\int_\Omega (PU_j)^{p-1}(PU_i-U_i)PZ_j^0 + O\left(\sum_{i\neq j}\int_\Omega (PU_j)^{p-1}PU_i|PZ_j^0-Z_j^0|\right) + o(Q) \\
&= \sum_{i\neq j}\mathfrak{c}_n\left(q_{ij}^{-\frac{2}{n-2}} - 2\frac{\delta_j}{\delta_i}\right)q_{ij}^{\frac{n}{n-2}} + o(Q) \qquad (2.41)\\
&+\begin{cases} O(\max_i \delta_i^{n-2}) & \text{if } n\geq 3, \text{ each } PU_i \text{ satisfies (1.8)},\\ p\sum_{i\neq j}\int_\Omega (PU_i-U_i)U_j^{p-1}Z_j^0 + o(\max_i\delta_i) & \text{if } n=3, \text{ each } PU_i \text{ satisfies (1.9)},\\ p\sum_{i\neq j}\int_\Omega (PU_i-U_i)U_j^{p-1}Z_j^0 + o(\max_i\delta_i^2|\log\delta_i|) & \text{if } n=4, \text{ each } PU_i \text{ satisfies (1.9)}.\end{cases}
\end{aligned}$$

Next, we only need to estimate $p\int_\Omega (PU_i-U_i)U_j^{p-1}Z_j^0$ if $i\neq j$ when $n=3,4$ and each PU_i satisfies (1.9).

If $\mathscr{R}_{ij}=\sqrt{\frac{\delta_i}{\delta_j}}$ or $\sqrt{\frac{\delta_j}{\delta_i}}$, then integrating by part and (2.32)–(2.33) imply

$$\begin{aligned}
p\int_\Omega (PU_i-U_i)U_j^{p-1}Z_j^0 &= \int_\Omega (PU_i-U_i)(-\Delta PZ_j^0-\lambda PZ_j^0)\\
&= \int_\Omega \lambda U_iPZ_j^0 + O\left(\delta_i^{\frac{n-2}{2}}\delta_j^{\frac{n-2}{2}}\right) = o(Q).
\end{aligned}$$

If $\mathscr{R}_{ij}=\frac{|\xi_i-\xi_j|}{\sqrt{\delta_i\delta_j}}$, then Taylor's expansion yields

$$\begin{aligned}
&p\int_\Omega (PU_i-U_i)U_j^{p-1}Z_j^0\\
&=\begin{cases}\left[-\frac{\mathfrak{c}_3}{2}\lambda|\xi_j-\xi_i| - \mathfrak{c}_3H_\lambda^3(\xi_i,\xi_j)\right]\delta_i^{\frac{1}{2}}\delta_j^{\frac{1}{2}} + o(Q+\max_i\delta_i) & \text{if } n=3,\\ \left[-\frac{\mathfrak{c}_4}{2}\lambda\log|\xi_j-\xi_i| - \mathfrak{c}_4H_\lambda^4(\xi_i,\xi_j)\right]\delta_i\delta_j + o(Q+\max_i\delta_i^2|\log\delta_i|) & \text{if } n=4.\end{cases}
\end{aligned}$$

Here, we used

$$\begin{aligned}
&\delta_i^{\frac{1}{2}}\int_{B(\xi_j,c)}||x-\xi_i|-|\xi_j-\xi_i||U_j^p(x)dx = o(Q+\max_i\delta_i) \quad \text{for } n=3,\\
&\delta_i\int_{B(\xi_j,c)}|\log|x-\xi_i|-\log|\xi_j-\xi_i||U_j^p(x)dx = o(Q+\max_i\delta_i^2|\log\delta_i|) \quad \text{for } n=4,\\
&\int_\Omega\left|\delta_i\mathcal{D}_4\left(\frac{\cdot-\xi_i}{\delta_i}\right)\right|U_j^p \lesssim \left\|\delta_i\mathcal{D}_4\left(\frac{\cdot-\xi_i}{\delta_i}\right)\right\|_{L^4(\Omega)} \lesssim \delta_i^2 \quad \text{for } n=4
\end{aligned}$$

to achieve the last equality, where $c>0$ is a small constant independent of δ_j for $j=1,\ldots,\nu$. This finishes the proof. □

3. Proof of Theorem 1.1

The proof of Theorem 1.1 is divided into two parts: In Subsection 3.1, we prove that (1.11) holds. In Subsection 3.2, we show that this estimate is optimal.

3.1. Proof of estimate (1.11).

Firstly, we establish the $H_0^1(\Omega)$-norm estimate of ρ.

Proposition 3.1. *Assume that $\epsilon_1 > 0$ is small enough. There exists a constant $C > 0$ depending only on n, ν, λ, u_0, and Ω that*

$$\|\rho\|_{H_0^1(\Omega)} \le C\left[\|f\|_{(H_0^1(\Omega))^*} + \left(\|\mathcal{I}_1\|_{(H_0^1(\Omega))^*} + \|\mathcal{I}_2\|_{(H_0^1(\Omega))^*} + \|\mathcal{I}_3\|_{(H_0^1(\Omega))^*}\right)\right]. \tag{3.1}$$

To deduce analogous $H_0^1(\Omega)$-norm estimates for ρ, the authors of [23, 15, 16] decomposed ρ into the main-order term ρ_0 and the remainder ρ_1, and further split ρ_1 into smaller pieces by introducing auxiliary parameters[4]. Besides, their analyses rely on coercivity inequalities. Refer to Subsection 1.3(4) for a prior discussion. Our argument in this paper is direct. We first derive an $H_0^1(\Omega)$-norm estimate for the solution to the associated linear problem, whose proof is based on a blow-up argument.

Lemma 3.2. *Let $\lambda \in (0, \lambda_1)$ and $\Pi^\perp : H_0^1(\Omega) \to \operatorname{span}\{PZ_i^k : i = 1, \dots, \nu \text{ and } k = 0, \dots, n\}^\perp \subset H_0^1(\Omega)$ be the projection operator. For any functions $\varrho \in H_0^1(\Omega)$ and $h \in (H_0^1(\Omega))^*$ satisfying*

$$\begin{cases} \varrho - \Pi^\perp[(-\Delta - \lambda)^{-1}(p(u_0+\sigma)^{p-1})] = \Pi^\perp[(-\Delta - \lambda)^{-1}(h)] & \text{in } \Omega, \\ \varrho = 0 & \text{on } \partial\Omega, \\ \left\langle \varrho, PZ_i^k \right\rangle = 0 \quad \text{for } i = 1, \dots, \nu \text{ and } k = 0, \dots, n, \end{cases}$$

it holds that

$$\|\varrho\|_{H_0^1(\Omega)} \lesssim \|h\|_{(H_0^1(\Omega))^*}. \tag{3.2}$$

Proof. We proceed by contradiction. Suppose that there exist sequences of parameters $\{(\delta_{i,m}, \xi_{i,m})\}_{m\in\mathbb{N}}$, and functions $\{\varrho_m\}_{m\in\mathbb{N}}$ and $\{h_m\}_{m\in\mathbb{N}}$ such that

$$\begin{cases} \max_i \delta_{i,m} + \max_i \kappa_{i,m} + \|h_m\|_{(H_0^1(\Omega))^*} \to 0 & \text{as } m \to \infty, \\ \|\varrho_m\|_{H_0^1(\Omega)} = 1 & \text{for all } m \in \mathbb{N}, \end{cases} \tag{3.3}$$

and

$$\begin{cases} \varrho_m - (-\Delta - \lambda)^{-1}[p(u_0+\sigma_m)^{p-1}\varrho_m] = \Pi^\perp[(-\Delta - \lambda)^{-1}h_m] + \displaystyle\sum_{i=1}^{\nu}\sum_{k=0}^{n} \mu_{i,m}^k PZ_{i,m}^k & \text{in } \Omega, \\ \varrho_m = 0 & \text{on } \partial\Omega, \\ \left\langle \varrho_m, PZ_{i,m}^k \right\rangle_{H_0^1(\Omega)} = 0 \quad \text{for } i = 1, \dots, \nu \text{ and } k = 0, 1, \dots, n. \end{cases} \tag{3.4}$$

Here, $PU_{i,m} := PU_{\delta_{i,m},\xi_{i,m}}$, $PZ_{i,m}^0 := \delta_{i,m}\frac{\partial PU_{i,m}}{\partial \delta_{i,m}}$, and $PZ_{i,m}^k := \delta_{i,m}\frac{\partial PU_{i,m}}{\partial \xi_{i,m}^k}$. Besides, $\mu_{i,m}^k \in \mathbb{R}$ denote Lagrange multipliers.

[4] Similar auxiliary parameters were used also in [31].

First, we observe that

$$\begin{aligned}\|\Pi^\perp[(-\Delta-\lambda)^{-1}h_m]\|_{H_0^1(\Omega)} &\lesssim \left\|(-\Delta-\lambda)^{-1}h_m + \sum_{i=1}^{\nu}\sum_{k=0}^{n} \frac{\int_\Omega h_m PZ_{i,m}^k}{\|PZ_{i,m}^k\|_{H_0^1(\Omega)}} \cdot PZ_{i,m}^k\right\|_{H_0^1(\Omega)} \\ &\lesssim \|h_m\|_{(H_0^1(\Omega))^*} + \sum_{i=1}^{\nu}\sum_{k=0}^{n}\left|\int_\Omega h_m PZ_{i,m}^k\right| \\ &\lesssim \|h_m\|_{(H_0^1(\Omega))^*}.\end{aligned} \tag{3.5}$$

Second, we verify that

$$\sum_{i=1}^{\nu}\sum_{k=0}^{n}|\mu_{i,m}^k| = o_m(1) \tag{3.6}$$

where $o_m(1)\to 0$ as $m\to\infty$.

For this aim, we test (3.4) with $PZ_{j,m}^q$ for each $j\in\{1,\dots,\nu\}$ and $q\in\{0,1,\dots,n\}$. We only need to focus on

$$\begin{aligned}&\left|\int_\Omega \left[(-\Delta-\lambda)\varrho_m - p(u_0+\sigma_m)^{p-1}\varrho_m\right] PZ_{j,m}^q\right| \\ &\lesssim \left|\int_\Omega \left[(-\Delta-\lambda)PZ_{j,m}^q - p(PU_{j,m})^{p-1}PZ_{j,m}^q\right]\varrho_m\right| \\ &+ \int_\Omega \left[\sigma_m^{p-1} - (PU_{j,m})^{p-1}\right]|\varrho_m||PZ_{j,m}^q|\mathbf{1}_{\{\nu\ge 2\}} + \int_\Omega \left[(u_0+\sigma_m)^{p-1}-\sigma_m^{p-1}\right]|\varrho_m|U_{j,m}.\end{aligned} \tag{3.7}$$

We now estimate each of the integrals on the right-hand side of (3.7).

It holds that

$$\begin{aligned}&\left|\int_\Omega \left[(-\Delta-\lambda)PZ_{j,m}^q - p(PU_{j,m})^{p-1}PZ_{j,m}^q\right]\varrho_m\right| \lesssim \|\varrho_m\|_{H_0^1(\Omega)} \\ &\quad\times\left[\left\|(PU_{j,m})^{p-1}PZ_{j,m}^q - U_{j,m}^{p-1}Z_{j,m}^q\right\|_{L^{\frac{p+1}{p}}(\Omega)} + \|U_{j,m}\|_{L^{\frac{p+1}{p}}(\Omega)}\mathbf{1}_{\{\text{each } PU_{j,m} \text{ satisfies (1.8)}\}}\right].\end{aligned}$$

Arguing as in (2.14) and (2.15), we deduce

$$\left\|\left[(PU_{j,m})^{p-1} - U_{j,m}^{p-1}\right]PZ_{j,m}^q\right\|_{L^{\frac{p+1}{p}}(\Omega)} + \left\|U_{j,m}^{p-1}(PZ_{j,m}^q - Z_{j,m}^q)\right\|_{L^{\frac{p+1}{p}}(\Omega)} \lesssim J_{1,m},$$

where $J_{1,m}$ is the quantity J_1 in (2.14) for $n\ne 6$ and (2.19) for $n=6$ with $(\delta_i,\delta_j,\xi_i,\xi_j)$ replaced by $(\delta_{i,m},\delta_{j,m},\xi_{i,m},\xi_{j,m})$.

Also, by applying the inequality $|PZ_j^q|\lesssim PU_j$ (which directly comes from the maximum principle), (A.1), (A.2), and Hölder's inequality, we obtain

$$\begin{aligned}&\int_\Omega \left[\sigma^{p-1} - (PU_{j,m})^{p-1}\right]|\varrho_m||PZ_{j,m}^q| \\ &\lesssim \int_\Omega \left[\sigma^{p-1}PU_{j,m} - (PU_{j,m})^p\right]|\varrho_m| \lesssim \int_\Omega \left[\sigma^p - \sum_{i=1}^{\nu}(PU_{i,m})^p\right]|\varrho_m| \\ &\lesssim \begin{cases}\displaystyle\sum_{i\ne j}\|U_{i,m}^{p-1}U_{j,m}\|_{L^{\frac{p+1}{p}}(\Omega)}\|\varrho_m\|_{H_0^1(\Omega)} & \text{if } n=3,4,5,\\ \displaystyle\sum_{i\ne j}\left\|\min\{U_{i,m}^{p-1}U_{j,m}, U_{j,m}^{p-1}U_{i,m}\}\right\|_{L^{\frac{p+1}{p}}(\Omega)}\|\varrho_m\|_{H_0^1(\Omega)} & \text{if } n\ge 6.\end{cases}\end{aligned}$$

On the other hand, using (A.2), we have

$$\begin{aligned}\int_\Omega \left[(u_0+\sigma_m)^{p-1}-\sigma_m^{p-1}\right]|\varrho_m|U_{j,m} &\lesssim \int_\Omega \left[(u_0\sigma_m^{p-2}\mathbf{1}_{\{u_0>0,p>2\}}+u_0^{p-1}\mathbf{1}_{\{u_0>0\}}\right]|\varrho_m|U_{j,m}\\ &\lesssim \|U_{j,m}\|_{L^{\frac{p+1}{p}}(\Omega)}\mathbf{1}_{\{u_0>0\}}+\sum_{i=1}^{\nu}\|U_{i,m}^{p-1}\|_{L^{\frac{p+1}{p}}(\Omega)}\mathbf{1}_{\{u_0>0,p>2\}}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\left|\int_\Omega \left[(-\Delta-\lambda)\varrho_m - p(u_0+\sigma_m)^{p-1}\varrho_m\right]PZ_{j,m}^q\right|\\ &\lesssim \|\varrho_m\|_{H_0^1(\Omega)}\Bigg[\|U_{j,m}\|_{L^{\frac{p+1}{p}}(\Omega)}\mathbf{1}_{\{u_0>0\}\cup\{\text{each } PU_{j,m} \text{ satisfies (1.8)}\}}+\sum_{i=1}^{\nu}\|U_{i,m}^{p-1}\|_{L^{\frac{p+1}{p}}(\Omega)}\mathbf{1}_{\{u_0>0,p>2\}}\\ &\qquad + \begin{cases}\sum_{i\neq j}\|U_{i,m}^{p-1}U_{j,m}\|_{L^{\frac{p+1}{p}}(\Omega)} & \text{if } n=3,4,5\\ \sum_{i\neq j}\left\|\min\{U_{i,m}^{p-1}U_{j,m},U_{j,m}^{p-1}U_{i,m}\}\right\|_{L^{\frac{p+1}{p}}(\Omega)} & \text{if } n\geq 6\end{cases}\Bigg\}\mathbf{1}_{\{\nu\geq 2\}}+J_{1,m}\Bigg]\\ &= o_m(1), \end{aligned} \tag{3.8}$$

where the last equality follows from Lemmas A.2 and A.3, (2.12), (2.13), and $\|\varrho_m\|_{H_0^1(\Omega)}=1$.

Third, we assert that

$$\begin{cases}\varrho_m \rightharpoonup 0 & \text{weakly in } H_0^1(\Omega),\\ \varrho_m\to 0 & \text{strongly in } L^s(\Omega) \text{ for } s\in(1,2^*)\end{cases}\quad \text{as } m\to\infty.$$

Since $\|\varrho_m\|_{H_0^1(\Omega)}=1$, there exists $\varrho_\infty\in H_0^1(\Omega)$ such that

$$\begin{cases}\varrho_m \rightharpoonup \varrho_\infty & \text{weakly in } H_0^1(\Omega),\\ \varrho_m\to \varrho_\infty & \text{strongly in } L^s(\Omega) \text{ for } s\in(1,2^*)\end{cases}\quad \text{as } m\to\infty,$$

along a subsequence. Given any $\chi\in C_c^\infty(\Omega)$, we test (3.4) with χ and passing to the limit $m\to\infty$. We can derive from (A.2) and Lemma A.5 that

$$\begin{aligned}\left|\int_\Omega \left[(u_0+\sigma_m)^{p-1}-u_0^{p-1}\right]\varrho_m\chi\right| &\lesssim \int_\Omega\left[\sigma_m^{p-1}+u_0^{p-2}\sigma_m\mathbf{1}_{\{p>2\}}\right]|\varrho_m\chi|\\ &\lesssim \|\sigma_m^{p-1}\|_{L^{\frac{p+1}{p}}(\Omega)}+\|\sigma_m\|_{L^{\frac{p+1}{p}}(\Omega)}\mathbf{1}_{\{p>2\}}=o_m(1).\end{aligned}$$

This fact and (3.3)–(3.6) imply that

$$\begin{cases}(-\Delta-\lambda)\varrho_\infty = pu_0^{p-1}\varrho_\infty & \text{in } \Omega,\\ \varrho_\infty = 0 & \text{on } \partial\Omega,\end{cases}$$

which together with the non-degeneracy of u_0 yields $\varrho_\infty=0$ in Ω.

Let us now fix an index $j\in\{1,\dots,\nu\}$, and define the rescaled function

$$\tilde\varrho_{j,m}(y) := \delta_{j,m}^{\frac{n-2}{2}}\varrho_m\big(\delta_{j,m}y+\xi_{j,m}\big)\quad \text{for any } y\in\frac{\Omega-\xi_{j,m}}{\delta_{j,m}}$$

for all sufficiently large $m\in\mathbb{N}$. We extend $\tilde\varrho_{j,m}(y)$ to $\mathbb{R}^n$ by setting it to zero outside its original domain. We will show that

$$\begin{cases}\tilde\varrho_{j,m} \rightharpoonup 0 & \text{weakly in } D^{1,2}(\mathbb{R}^n),\\ \tilde\varrho_{j,m}\to 0 & \text{strongly in } L^s_{\mathrm{loc}}(\mathbb{R}^n) \text{ for } s\in(1,2^*)\end{cases}\quad \text{as } m\to\infty. \tag{3.9}$$

Because $\|\varrho_m\|_{H_0^1(\Omega)} = 1$, the sequence $\{\tilde{\varrho}_{j,m}\}_{n\in\mathbb{N}}$ is uniformly bounded in $D^{1,2}(\mathbb{R}^n)$, and so there exists $\tilde{\varrho}_{j,\infty} \in D^{1,2}(\mathbb{R}^N)$ such that

$$\begin{cases} \tilde{\varrho}_{j,m} \rightharpoonup \tilde{\varrho}_{j,\infty} & \text{weakly in } D^{1,2}(\mathbb{R}^n), \\ \tilde{\varrho}_{j,m} \to \tilde{\varrho}_{j,\infty} & \text{strongly in } L^s_{\text{loc}}(\mathbb{R}^n) \text{ for } s \in (1, 2^*) \end{cases} \quad \text{as } m \to \infty,$$

up to a subsequence. Given a function $\chi \in C_c^\infty(\mathbb{R}^n)$, we set

$$\tilde{\chi}_{j,m}(x) = \delta_{j,m}^{\frac{2-n}{2}} \chi\left(\delta_{j,m}^{-1}(x - \xi_{j,m})\right) \quad \text{for } x \in \Omega.$$

After testing (3.4) with $\tilde{\chi}_{j,m}$, the only technical point we encounter is to derive

$$\int_\Omega (u_0 + \sigma_m)^{p-1} \varrho_m \tilde{\chi}_{j,m} = \int_{\mathbb{R}^n} U^{p-1} \tilde{\varrho}_{j,\infty} \chi + o_m(1) \tag{3.10}$$

as $m \to \infty$.

Indeed, direct calculations give us that

$$\begin{aligned} \int_\Omega (PU_{j,m})^{p-1} \varrho_m \tilde{\chi}_{j,m} &= \int_{\frac{\Omega - \xi_{j,m}}{\delta_{j,m}}} U^{p-1} \tilde{\varrho}_{j,m} \chi + O\left(\kappa_{j,m}^{\frac{n-2}{n}}\right) \\ &= \int_{\mathbb{R}^n} U^{p-1} \tilde{\varrho}_{j,\infty} \chi + o_m(1), \end{aligned}$$

because

$$\begin{aligned} \int_\Omega \left| (PU_{j,m})^{p-1} - U_{j,m}^{p-1} \right|^{\frac{p+1}{p-1}} &\lesssim \left\| |PU_{j,m} - U_{j,m}| U_{j,m}^{p-2} \mathbf{1}_{\{p>2\}} \right\|_{L^{\frac{p+1}{p-1}}(B(\xi_{j,m}, d(\xi_{j,m}, \partial\Omega)))}^{\frac{p+1}{p-1}} \\ &\quad + \| |PU_{j,m} - U_{j,m}|^{p+1} \|_{L^1(B(\xi_{j,m}, d(\xi_{j,m}, \partial\Omega)))} \\ &\quad + \int_{B(\xi_{j,m}, d(\xi_{j,m}, \partial\Omega))^c} U_{j,m}^{p+1} \lesssim \kappa_{j,m}^{\frac{n-2}{2}}, \end{aligned} \tag{3.11}$$

while we know

$$\int_\Omega u_0^{p-1} \varrho_m \tilde{\chi}_{j,m} \simeq \delta_{j,m}^2 \int_{\text{supp}(\chi)} u_0^{p-1}(\xi_{j,m} + \delta_{j,m} y)(\tilde{\varrho}_{j,m} \chi)(y) dy = o_m(1)$$

thanks to the boundedness of u_0. Furthermore, for $1 \le i \ne j \le \nu$,

$$\left| \int_\Omega (PU_{i,m})^{p-1} \varrho_m \tilde{\chi}_{j,m} \right| \lesssim \left\| \left[\delta_{j,m}^{\frac{n-2}{2}} U_{i,m}\left(\xi_{j,m} + \delta_{j,m} \cdot\right) \right]^{p-1} \right\|_{L^{\frac{p+1}{p}}(\text{supp}(\chi))} = o_m(1),$$

since

$$\begin{aligned} &\left(\frac{\delta_{j,m}}{\delta_{i,m}}\right)^{\frac{4n}{n+2}} \int_{\text{supp}(\chi)} \frac{dy}{\left(1 + \left(\frac{\delta_{j,m}}{\delta_{i,m}} |y - z_{ij.m}|\right)^2\right)^{\frac{4n}{n+2}}} \\ &\lesssim \begin{cases} \left(\dfrac{\delta_{j,m}}{\delta_{i,m}}\right)^{-\frac{4n}{n+2}} |z_{ij,m}|^{-\frac{8n}{n+2}} & \text{if } |z_{ij,m}| \to \infty, \\ \left(\dfrac{\delta_{j,m}}{\delta_{i,m}}\right)^{\frac{4n}{n+2} - n} \displaystyle\int_{|z| \le \frac{\delta_{j,m}}{\delta_{i,m}}} \frac{1}{(1 + |z|)^{\frac{8n}{n+2}}} dz & \text{if } |z_{ij,m}| \text{ is bounded}, \delta_{i,m} \ll \delta_{j,m}, \\ \left(\dfrac{\delta_{j,m}}{\delta_{i,m}}\right)^{\frac{4n}{n+2}} & \text{if } |z_{ij,m}| \text{ is bounded}, \delta_{i,m} \gg \delta_{j,m} \end{cases} \end{aligned}$$

$$\lesssim \begin{cases} \mathscr{R}_{ij,m}^{-\frac{8n}{n+2}} & \text{if } |z_{ij,m}| \to \infty, \\ \left(\frac{\delta_{j,m}}{\delta_{i,m}}\right)^{\frac{4n}{n+2}-n} \left[\mathbf{1}_{\{p>2\}} + \left|\log \frac{\delta_{j,m}}{\delta_{i,m}}\right| \mathbf{1}_{\{p=2\}} + \left(\frac{\delta_{j,m}}{\delta_{i,m}}\right)^{n-\frac{8n}{n+2}} \mathbf{1}_{\{p<2\}}\right] & \text{if } |z_{ij,m}| \text{ is bounded}, \delta_{i,m} \ll \delta_{j,m}, \\ \left(\frac{\delta_{j,m}}{\delta_{i,m}}\right)^{\frac{4n}{n+2}} & \text{if } |z_{ij,m}| \text{ is bounded}, \delta_{i,m} \gg \delta_{j,m}, \end{cases}$$

where $\mathscr{R}_{ij,m}$ is the quantity introduced in (2.3) with $(\xi_i, \xi_j, \delta_i, \delta_j)$ replaced by $(\xi_{i,m}, \xi_{j,m}, \delta_{i,m}, \delta_{j,m})$. By (A.1), and Lemmas A.2 and A.3, we also have that

$$\begin{aligned} &\int_\Omega (PU_{j,m})^{p-2}\Big(u_0 + \sum_{i\neq j} PU_{i,m}\Big)|\varrho_m \tilde{\chi}_{j,m}|\mathbf{1}_{\{p>2\}} \\ &\lesssim \Big[\sum_{i\neq j} \|U_{i,m}U_{j,m}^{p-2}\|_{L^{\frac{p+1}{p-1}}(\Omega)} + \max_i \|U_{i,m}^{p-2}\|_{L^{\frac{p+1}{p-1}}(\Omega)}\Big]\mathbf{1}_{\{p>2\}} = o_m(1). \end{aligned}$$

Combining the above calculations, we derive (3.10).

Taking $m \to \infty$, we observe from (3.4) that

$$\begin{cases} -\Delta \tilde{\varrho}_{j,\infty} = pU^{p-1}\tilde{\varrho}_{j,\infty} \quad \text{in } \mathbb{R}^n, \quad \tilde{\varrho}_{j,\infty} \in D^{1,2}(\mathbb{R}^n), \\ \displaystyle\int_{\mathbb{R}^n} \nabla \tilde{\varrho}_{j,\infty} \cdot \nabla Z^k = 0 \quad \text{for all } k = 0, \dots, n. \end{cases}$$

The nondegeneracy of U implies that $\tilde{\varrho}_{j,\infty} = 0$, yielding (3.9).

Finally, we will prove

$$\lim_{m\to\infty} \|\varrho_m\|_{H_0^1(\Omega)} = 0. \tag{3.12}$$

Since (3.12) contradicts (3.3), we will be able to conclude that (3.2) must hold.

To deduce (3.12), we test (3.4) with ϱ_m. Then, we only have to consider

$$\begin{aligned} \int_\Omega (u_0+\sigma_m)^{p-1}\varrho_m^2 &\lesssim \int_\Omega u_0^{p-1}\varrho_m^2 + \sum_{i=1}^{\nu}\int_\Omega (PU_{i,m})^{p-1}\varrho_m^2 \\ &\lesssim o_m(1) + \int_{\mathbb{R}^n} U^{p-1}\tilde{\varrho}_{i,m}^2 + O\left(\max_i \kappa_{i,m}^{\frac{n-2}{n}}\right)\|\varrho_m\|_{H_0^1(\Omega)}^2 \\ &= o_m(1). \end{aligned}$$

Here, we employed (3.11) and the facts that $\varrho_m \to 0$ strongly in $L^2(\Omega)$ and $\tilde{\varrho}_{i,m}^2 \rightharpoonup 0$ weakly in $L^{\frac{n}{n-2}}(\mathbb{R}^n)$. We are done. □

Proof of Proposition 3.1. We set

$$h := f + \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_0[\rho].$$

From (2.1), we have

$$\begin{cases} \rho - \Pi^\perp[(-\Delta-\lambda)^{-1}(p(u_0+\sigma)^{p-1}\rho)] = \Pi^\perp[(-\Delta-\lambda)^{-1}h] \quad \text{in } \Omega, \\ \rho = 0 \quad \text{on } \partial\Omega, \\ \langle \rho, PZ_i^k\rangle_{H_0^1(\Omega)} = 0 \quad \text{for } i = 1,\dots,\nu \text{ and } k = 0,\dots,n. \end{cases}$$

By making use of (3.2), (A.2), (A.3) and Hölder's inequality

$$\begin{aligned} \|\rho\|_{H_0^1(\Omega)} &\lesssim \|f\|_{(H_0^1(\Omega))^*} + \|\rho\|_{H_0^1(\Omega)}^2\mathbf{1}_{\{p>2\}} + \|\rho\|_{H_0^1(\Omega)}^p \\ &\quad + \|\mathcal{I}_1\|_{(H_0^1(\Omega))^*} + \|\mathcal{I}_2\|_{(H_0^1(\Omega))^*} + \|\mathcal{I}_3\|_{(H_0^1(\Omega))^*}. \end{aligned}$$

Since $p > 1$ and $\|\rho\|_{H_0^1(\Omega)} = o_{\epsilon_1}(1)$, we immediately deduce (3.1). □

Corollary 3.3. *For each $i = 1, \dots, \nu$, we assume that PU_i satisfies (1.8) if $n \ge 5$ or $[n = 3, 4$ and $u_0 > 0]$, and satisfies (1.9) if $n = 3, 4$ and $u_0 = 0$. We define*

$$\mathcal{J}_{11}(\delta_1, \dots, \delta_\nu) := \begin{cases} \max_i \delta_i & \text{if } n = 3 \text{ and } u_0 = 0, \\ \max_i \delta_i^2 |\log \delta_i| & \text{if } n = 4 \text{ and } u_0 = 0, \\ \max_i \delta_i^{\frac{n-2}{2}} & \text{if } [n = 3, 4 \text{ and } u_0 > 0] \text{ or } n = 5, \\ \max_i \delta_i^2 |\log \delta_i|^{\frac{1}{2}} & \text{if } n = 6, \\ \max_i \delta_i^2 & \text{if } n \ge 7, \end{cases}$$

$$\mathcal{J}_{12}(\kappa_1, \dots, \kappa_\nu) := \begin{cases} \max_i \kappa_i^{n-2} & \text{if } n = 3, 4, 5, \\ \max_i \kappa_i^4 |\log \kappa_i|^{\frac{1}{2}} & \text{if } n = 6, \\ \max_i \kappa_i^{\frac{n+2}{2}} & \text{if } n \ge 7, \end{cases}$$

$$\mathcal{J}_{13}(Q) := \begin{Bmatrix} Q & \text{if } n = 3, 4, 5 \\ Q |\log Q|^{\frac{1}{2}} & \text{if } n = 6 \\ Q^{\frac{n+2}{2(n-2)}} & \text{if } n \ge 7 \end{Bmatrix} \mathbf{1}_{\{\nu \ge 2\}}.$$

Then

$$\|\rho\|_{H_0^1(\Omega)} \lesssim \|f\|_{(H_0^1(\Omega))^*} + \mathcal{J}_{11}(\delta_1, \dots, \delta_\nu) + \mathcal{J}_{12}(\kappa_1, \dots, \kappa_\nu) + \mathcal{J}_{13}(Q). \tag{3.13}$$

Proof. The result is a consequence of Lemma 2.5 and Proposition 3.1. □

Proposition 3.4. *For each $i = 1, \dots, \nu$, we assume that PU_i satisfies (1.8) if $n \ge 5$ or $[n = 3, 4$ and $u_0 > 0]$, and satisfies (1.9) if $n = 3, 4$ and $u_0 = 0$. We set*

$$\mathcal{J}_{21}(\delta_1, \dots, \delta_\nu) := \begin{cases} \max_i \delta_i & \text{if } n = 3 \text{ and } u_0 = 0, \\ \max_i \delta_i^2 |\log \delta_i| & \text{if } n = 4 \text{ and } u_0 = 0, \\ \max_i \delta_i^{\frac{n-2}{2}} & \text{if } n = 3, 4, 5 \text{ and } u_0 > 0, \\ \max_i \delta_i^2 & \text{if } [n = 5 \text{ and } u_0 = 0] \text{ or } n \ge 6, \end{cases}$$

and $\mathcal{J}_{23}(Q) := Q \mathbf{1}_{\{\nu \ge 2\}}$. If each $\xi_1, \dots, \xi_\nu$ lies on a compact set of Ω, then it holds that

$$\mathcal{J}_{21}(\delta_1, \dots, \delta_\nu) + \mathcal{J}_{23}(Q) \lesssim \|f\|_{(H_0^1(\Omega))^*}. \tag{3.14}$$

Proof. Let $j \in \{1, \dots, \nu\}$ be fixed. By testing (2.1) with PZ_j^0, we obtain

$$\begin{aligned} \int_\Omega \mathcal{I}_1 PZ_j^0 + \int_\Omega \mathcal{I}_2 PZ_j^0 + \int_\Omega \mathcal{I}_3 PZ_j^0 &= - \int_\Omega f PZ_j^0 - \int_\Omega \mathcal{I}_0[\rho] PZ_j^0 \\ &\quad + \int_\Omega \left[(-\Delta - \lambda)\rho - p(u_0 + \sigma)^{p-1} \rho \right] PZ_j^0. \end{aligned}$$

As in (3.8), we apply Lemmas A.2–A.3 and (3.13), and the assumption that ξ_i lies on a compact set of Ω for $i = 1, \dots, \nu$ to deduce

$$\begin{aligned} &\left| \int_\Omega \left[(-\Delta - \lambda)\rho - p(u_0 + \sigma)^{p-1} \rho \right] PZ_j^0 \right| \\ &\lesssim \|\rho\|_{H_0^1(\Omega)} \Bigg[\|U_j\|_{L^{\frac{p+1}{p}}(\Omega)} \mathbf{1}_{\{u_0 > 0\} \cup \{PU_j \text{ satisfies } (1.8)\}} + \sum_{i=1}^{\nu} \|U_i^{p-1}\|_{L^{\frac{p+1}{p}}(\Omega)} \mathbf{1}_{\{u_0 > 0, p > 2\}} \end{aligned}$$

$$+\left.\begin{cases}\sum_{i\neq j}\|U_i^{p-1}U_j\|_{L^{\frac{p+1}{p}}(\Omega)} & \text{if } n=3,4,5\\ \sum_{i\neq j}\left\|\min\{U_i^{p-1}U_j,U_j^{p-1}U_i\}\right\|_{L^{\frac{p+1}{p}}(\Omega)} & \text{if } n\geq 6\end{cases}\right\}\mathbf{1}_{\{\nu\geq 2\}}\Bigg]$$
$$= o(\mathcal{J}_{21}(\delta_1,\dots,\delta_\nu)+\mathcal{J}_{23}(Q)).$$

Using (A.5) and the fact that $|PZ_j^0|\leq \sum_{i=1}^{\nu}U_i$, we also know that

$$\begin{aligned}\left|\int_\Omega \mathcal{I}_0[\rho]PZ_j^0\right| &\lesssim \begin{cases}\int_\Omega \min\{\sigma^{p-2}\rho^2,|\rho|^p\}|PZ_j^0| & \text{if } 1<p<2,\\ \int_\Omega \left(\sigma^{p-2}\rho^2+|\rho|^p\right)|PZ_j^0| & \text{if } p\geq 2\end{cases}\\ &\lesssim \int_\Omega \sum_{i=1}^{\nu}U_i^{p-1}|\rho|^2+\|\rho\|_{H_0^1(\Omega)}^p\mathbf{1}_{\{p>2\}}\lesssim \|\rho\|_{H_0^1(\Omega)}^2.\end{aligned} \tag{3.15}$$

Without loss of generality, one may assume that $\delta_1\geq\delta_2\geq\cdots\geq\delta_\nu$. By employing Lemmas 2.6–2.8 together with $\mathfrak{a}_n,\mathfrak{b}_n>0$, $d(\xi_i,\partial\Omega)\gtrsim 1$, $-\varphi_\lambda^3(\xi_i)>0$ provided $n=3$, $u_0=0$, and $\nu\geq 2$, and

$$\mathfrak{c}_nq_{ij}+\left.\begin{cases}\sum_{i\neq j}\left[-\frac{\mathfrak{c}_3}{2}\lambda|\xi_j-\xi_i|-\mathfrak{c}_3H_\lambda^3(\xi_i,\xi_j)\right]\delta_i^{\frac12}\delta_j^{\frac12} & \text{if } n=3\\ \sum_{i\neq j}\left[-\frac{\mathfrak{c}_4}{2}\lambda\log|\xi_j-\xi_i|-\mathfrak{c}_4H_\lambda^4(\xi_i,\xi_j)\right]\delta_i\delta_j & \text{if } n=4\end{cases}\right\}\mathbf{1}_{\left\{n=3,4,\ u_0=0,\text{ and } q_{ij}=\left(\frac{|\xi_i-\xi_j|}{\sqrt{\delta_i\delta_j}}\right)^{2-n}\right\}}\simeq q_{ij},$$

we adopt the same reasoning as in [23, Lemma 2.3] (which is based on mathematical induction) to achieve

$$\mathcal{J}_{23}(Q)\lesssim \|f\|_{(H_0^1(\Omega))^*}+o(\mathcal{J}_{21}(\delta_1,\dots,\delta_\nu)). \tag{3.16}$$

Then, one may take the test function PZ_1^0, where $\delta_1=\max_i\delta_i$, to prove

$$\begin{aligned}\mathcal{J}_{21}(\delta_1,\dots,\delta_\nu)&\lesssim \|f\|_{(H_0^1(\Omega))^*}+\left|\int_\Omega \mathcal{I}_2PZ_1^0\right|+o(\mathcal{J}_{21}(\delta_1,\dots,\delta_\nu)+\mathcal{J}_{23}(Q))\\ &\lesssim \|f\|_{(H_0^1(\Omega))^*}+o(\mathcal{J}_{21}(\delta_1,\dots,\delta_\nu)+\mathcal{J}_{23}(Q)).\end{aligned} \tag{3.17}$$

Here, we used $\left|\int_\Omega \mathcal{I}_2PZ_1^0\right|\lesssim Q$, which comes from (2.41) and Lemma A.3.

Putting (3.16) and (3.17), we establish (3.14), concluding the proof. □

We are now in a position to establish estimate (1.11).

Proof of Estimate (1.11). Since $d(\xi_i,\partial\Omega)\gtrsim 1$, we have

$$\mathcal{J}_{12}(\kappa_1,\dots,\kappa_\nu)\lesssim \mathcal{J}_{11}(\delta_1,\dots,\delta_\nu).$$

From (3.13) and (3.14), one can identify two optimal functions $\tilde\zeta_1(t)$ and $\tilde\zeta_3(t)$ of the form $t^a|\log t|^b$, with $a>0$ and $b\geq 0$ ($b=0$ unless $n=6$), such that

$$\mathcal{J}_{11}(\delta_1,\dots,\delta_\nu)\lesssim\tilde\zeta_1(\mathcal{J}_{21}(\delta_1,\dots,\delta_\nu))\quad\text{and}\quad \mathcal{J}_{13}(Q)\lesssim\tilde\zeta_3(\mathcal{J}_{23}(Q)).$$

Recognizing that $\tilde\zeta_1(t)$ and $\tilde\zeta_3(t)$ are non-decreasing for $t>0$, we obtain

$$\|\rho\|_{H_0^1(\Omega)}\lesssim\max\left\{\|f\|_{(H_0^1(\Omega))^*},\ \tilde\zeta_1(\|f\|_{(H_0^1(\Omega))^*}),\ \tilde\zeta_3(\|f\|_{(H_0^1(\Omega))^*})\right\}=\zeta(\|f\|_{(H_0^1(\Omega))^*}),$$

where $\zeta(t)$ is the function introduced in (1.12). □

3.2. **Sharpness of estimate (1.11).** Let us divide it into two cases.

Case 1: We prove the optimality of (1.11) when $[n = 3, 4,\ \nu \geq 1]$, or $[n = 5,\ \nu \geq 1,\ u_0 > 0]$ or $[n \geq 7,\ \nu = 1]$. In this case, we have that $\zeta(t) = t$.

We select numbers $\delta = \delta_i \in (0,1)$ for each $i \in \{1, \dots, \nu\}$ and points $\xi_i \in \Omega$ such that $d(\xi_i, \partial\Omega) \gtrsim 1$ and $|\xi_i - \xi_j| \gtrsim 1$ for all distinct indices $1 \leq i \neq j \leq \nu$. Under these conditions, it holds that $Q \simeq \delta^{n-2} \cdot \mathbf{1}_{\nu \geq 2}$.

Taking

$$\epsilon \simeq \begin{cases} \delta & \text{if } n = 3 \text{ and } u_0 = 0, \\ \delta^2 |\log \delta| & \text{if } n = 4 \text{ and } u_0 = 0, \\ \delta^{\frac{n-2}{2}} & \text{if } n = 3, 4, 5 \text{ and } u_0 > 0, \\ \delta^2 & \text{if } n \geq 7 \text{ and } \nu = 1, \end{cases}$$

and using $|PZ_i^k| \leq CPU_i$ in Ω, we construct a nonnegative function of the form

$$\phi_\delta = \sum_{i=1}^{\nu} PU_i + \sum_{i=1}^{\nu} \sum_{k=0}^{n} \beta_i^k PZ_i^k,$$

where $\beta_i^k = o_\delta(1)$, $\langle \phi_\delta, PZ_i^k \rangle = 0$ for each $i = 1, \dots, \nu$ and $k = 0, 1, \dots, n$, and $\|\phi_\delta\|_{H_0^1(\Omega)} \simeq 1$.

Letting $\rho := \epsilon \phi_\delta$, we define $u_* := u_0 + \sum_{i=1}^{\nu} PU_i + \rho$ so that $u_* = 0$ on $\partial\Omega$. Then we set

$$f := -\Delta u_* - \lambda u_* - u_*^{p-1} = -\Delta\rho - \lambda\rho - p\left(u_0 + \sum_{i=1}^{\nu} PU_i\right)^{p-1} \rho + \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_0[\rho]$$

where $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$, and $\mathcal{I}_0[\rho]$ are defined as in (2.2) with parameters (δ_i, ξ_i) satisfying the above conditions. By Lemmas 2.1 and 2.5, we have that $\|\rho\|_{H_0^1(\Omega)} \simeq \epsilon$ and

$$\begin{aligned} \|f\|_{(H_0^1(\Omega))^*} &\lesssim \|\rho\|_{H_0^1(\Omega)} + \|\rho\|_{H_0^1(\Omega)}^{\min\{2,p\}} + \|\mathcal{I}_1\|_{L^{\frac{p+1}{p}}(\Omega)} \mathbf{1}_{\{u_0 \neq 0\}} + \|\mathcal{I}_2\|_{L^{\frac{p+1}{p}}(\Omega)} \mathbf{1}_{\{\nu \geq 2\}} + \|\mathcal{I}_3\|_{L^{\frac{p+1}{p}}(\Omega)} \\ &\simeq \epsilon \simeq \|\rho\|_{H_0^1(\Omega)}. \end{aligned}$$

Proceeding as in Step 2 of [15, Subsection 5.1], we deduce that

$$\inf_{\substack{(\tilde{\delta}_i, \tilde{\xi}_i) \in (0,1) \times \Omega, \\ i = 1, \dots, \nu}} \left\| u_* - \left(u_0 + \sum_{i=1}^{\nu} PU_{\tilde{\delta}_i, \tilde{\xi}_i}\right) \right\|_{H_0^1(\Omega)} \gtrsim \|\rho\|_{H_0^1(\Omega)},$$

thereby establishing the optimality of (1.11).

Case 2: We prove the optimality of (1.11) when $[n = 5,\ \nu \geq 1,\ u_0 = 0]$ or $[n = 6,\ \nu \geq 1]$ or $[n \geq 7,\ \nu \geq 2]$. In this case, we have that $\zeta(t) \gg t$. The proof is split into three steps.

STEP 1. We select $\delta = \delta_i \in (0,1)$ and $\xi_i \in \Omega$ such that $d(\xi_i, \partial\Omega) \gtrsim 1$ and $|\xi_i - \xi_j| \simeq \delta^b$ for each $i \neq j$, where $i, j \in \{1, \dots, \nu\}$ and $b \in [0, 1)$. This choice ensures that $Q \simeq \delta^{(1-b)(n-2)}$. We impose a further restriction $b \in (\frac{n-4}{n-2}, 1)$ for $n \geq 7$, and set $b = 0$ in dimensions $n = 5, 6$.

We now consider the function ρ solving the boundary value problem

$$\begin{cases} -\Delta\rho - \lambda\rho - p(u_0 + \sigma)^{p-1}\rho = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_0[\rho] + \displaystyle\sum_{i=1}^{\nu} \sum_{k=0}^{n} c_i^k (-\Delta - \lambda) PZ_i^k & \text{in } \Omega, \\ \rho = 0 \quad \text{on } \partial\Omega, \quad c_i^k \in \mathbb{R} \quad \text{for } i = 1, \dots, \nu \text{ and } k = 0, \dots, n, \\ \langle \rho, PZ_i^k \rangle_{H_0^1(\Omega)} = 0 \quad \text{for } i = 1, \dots, \nu \text{ and } k = 0, \dots, n, \end{cases} \tag{3.18}$$

where PZ_i^k, $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$, and $\mathcal{I}_0[\rho]$ are defined as in (2.2) with parameters (δ_i, ξ_i) satisfying the above conditions. We set $f := \sum_{k=0}^n \sum_{i=1}^{\nu} c_i^k(-\Delta - \lambda)PZ_i^k$. Then

$$\begin{aligned} \|f\|_{(H_0^1(\Omega))^*} &\lesssim \sum_{k=0}^{n}\sum_{i=1}^{\nu} |c_i^k| \\ &\lesssim \varsigma_1(\delta) := \begin{cases} \delta^2 & \text{if } [n=5,\ u_0=0,\ \nu \ge 1] \text{ or } [n=6,\ \nu\ge 1], \\ \delta^{(1-b)(n-2)} & \text{if } n \ge 7 \text{ and } \nu \ge 2. \end{cases} \end{aligned} \tag{3.19}$$

By applying Lemmas 3.2, 2.5 and (3.19), we see that

$$\begin{aligned} \|\rho\|_{H_0^1(\Omega)} &\lesssim \|f\|_{(H_0^1(\Omega))^*} + \|\mathcal{I}_1\|_{(H_0^1(\Omega))^*} + \|\mathcal{I}_2\|_{(H_0^1(\Omega))^*} + \|\mathcal{I}_3\|_{(H_0^1(\Omega))^*} \\ &\lesssim \varsigma_2(\delta) := \begin{cases} \delta^{\frac{3}{2}} & \text{if } n=5, u_0=0, \text{ and } \nu \ge 1, \\ \delta^2|\log\delta|^{\frac12} & \text{if } n=6 \text{ and } \nu \ge 1, \\ \delta^{\frac{(1-b)(n+2)}{2}} & \text{if } n\ge 7 \text{ and } \nu\ge 2. \end{cases} \end{aligned} \tag{3.20}$$

STEP 2. We now establish the lower bound

$$\|\rho\|_{H_0^1(\Omega)} \gtrsim \varsigma_2(\delta), \tag{3.21}$$

which in turn implies

$$\|\rho\|_{H_0^1(\Omega)} \gtrsim \zeta(\|f\|_{(H_0^1(\Omega))^*}).$$

Testing equation (3.18) against any $\chi \in H_0^1(\Omega)$ and applying Holder's inequality yield

$$\begin{aligned} &\left|\int_\Omega \left(\mathcal{I}_1 \mathbf{1}_{\{n\ge 6,\ u_0>0\}} + \mathcal{I}_2 \mathbf{1}_{\{n\ge 6,\ \nu\ge 2\}} + \sum_{i=1}^{\nu} \lambda PU_i \mathbf{1}_{\{n=5,6\}}\right)\chi\right| \\ &\lesssim \|\chi\|_{H_0^1(\Omega)} \Bigg[\|\rho\|_{H_0^1(\Omega)} + \|\rho\|_{H_0^1(\Omega)}^{\min\{2,p\}} + \|\mathcal{I}_1+\mathcal{I}_2\|_{L^{\frac{2n}{n+2}}(\Omega)} \mathbf{1}_{\{n=5,u_0=0,\nu\ge 1\}} \\ &\qquad + \left\|\mathcal{I}_3 - \sum_{i=1}^{\nu}\lambda PU_i \mathbf{1}_{\{n=5,6\}}\right\|_{L^{\frac{2n}{n+2}}(\Omega)} + \sum_{i=1}^{\nu}\sum_{k=0}^{n}|c_i^k|\Bigg], \end{aligned}$$

and so

$$\begin{aligned} \|\rho\|_{H_0^1(\Omega)} &\gtrsim \left\|\mathcal{I}_1 \mathbf{1}_{\{n\ge 6,\ u_0>0\}} + \mathcal{I}_2 \mathbf{1}_{\{n\ge 6,\ \nu\ge 2\}} + \sum_{i=1}^{\nu} \lambda PU_i \mathbf{1}_{\{n=5,6\}}\right\|_{(H_0^1(\Omega))^*} \\ &\quad - C\left[\left\|\mathcal{I}_3 - \sum_{i=1}^{\nu}\lambda PU_i \mathbf{1}_{\{n=5,6\}}\right\|_{L^{\frac{2n}{n+2}}(\Omega)} + \|\mathcal{I}_2\|_{L^{\frac{2n}{n+2}}(\Omega)}\mathbf{1}_{\{n=5\}} + \sum_{i=1}^{\nu}\sum_{k=0}^{n}|c_i^k|\right] \\ &=: J_2 + o(\varsigma_2(\delta)) \end{aligned}$$

where we have invoked (2.13), (2.15), (3.19), and (3.20).

Let G_λ be defined as in (2.7) for $n \ge 3$. We recall the lower bound estimate of G_λ:

$$G_\lambda(x,y) \gtrsim \frac{1}{|x-y|^{n-2}}.$$

Drawing on the idea in the proof of Lemma 2.5 for $n = 6$, together with the non-negativity of the functions $\mathcal{I}_1$, $\mathcal{I}_2$ and λPU_i, we observe that

$$J_2 \gtrsim J_3^{\frac12}, \tag{3.22}$$

where the quantity J_3 is defined as

$$J_3 := \begin{cases} \int_\Omega \sum_{i=1}^{\nu} \lambda PU_i(x) \int_\Omega G_\lambda(x,\omega) \sum_{j=1}^{\nu} \lambda PU_j(\omega) dx d\omega & \text{if } n = 5,6, \\ \int_\Omega \mathcal{I}_2(x) \int_\Omega G_\lambda(x,\omega) \mathcal{I}_2(\omega) dx d\omega & \text{if } n \geq 7. \end{cases}$$

As a result, we only need to estimate J_3. Assume that $n = 5, 6$. A direct computation shows

$$\begin{aligned} &\int_\Omega \sum_{i=1}^{\nu} PU_i(x) \int_\Omega G_\lambda(x,\omega) \sum_{j=1}^{\nu} PU_j(\omega) dx d\omega \\ &\gtrsim \int_\Omega \sum_{i,j=1}^{\nu} \left(\frac{\delta}{\delta^2 + |x-\xi_i|^2} \right)^{\frac{n-2}{2}} \frac{\delta^{\frac{n-2}{2}}}{(\delta^2 + |x-\xi_j|^2)^{\frac{n-4}{2}}} \simeq \begin{cases} \delta^3 & \text{if } n = 5, \\ \delta^4 |\log \delta| & \text{if } n = 6. \end{cases} \end{aligned} \tag{3.23}$$

Assume that $n \geq 7$ and $\nu \geq 2$. If $|x_1| \lesssim \frac{1}{2}\delta^{b-1}$, then $|x_2| \leq |x_1| + \frac{|\xi_1 - \xi_2|}{\delta} \lesssim \delta^{b-1}$. From this, we derive

$$\mathcal{I}_2 = \sigma^p - \sum_{i=1}^{\nu} (PU_i)^p \gtrsim (PU_1)^{p-1} PU_2$$

and

$$U_1^{p-1} U_2 \gtrsim \frac{\delta^{-2}}{\langle x_1 \rangle^4} \frac{\delta^{-\frac{n-2}{2}}}{\langle x_2 \rangle^{n-2}} \gtrsim \frac{\delta^{-2}}{\langle x_1 \rangle^4} \delta^{-\frac{n-2}{2}} \delta^{(1-b)(n-2)} \gtrsim \frac{\delta^{(\frac{1}{2}-b)(n-2)-2}}{\langle x_1 \rangle^4}.$$

As a consequence, we have

$$\begin{aligned} J_3 &= \int_\Omega \mathcal{I}_2(x) \int_\Omega G_\lambda(x,\omega) \mathcal{I}_2(\omega) dx d\omega \\ &\gtrsim \delta^{(2-2b)(n-2)} \int_{\{|x_1| \lesssim \frac{1}{2}\delta^{b-1}\}} \int_{\{|\omega_1| < \frac{1}{2}|x_1|\}} \frac{1}{\langle x_1 \rangle^4} \frac{1}{|x_1 - \omega_1|^{n-2}} \frac{1}{\langle \omega_1 \rangle^4} dx_1 d\omega_1 + o(\delta^{(1-b)(n+2)}) \\ &\gtrsim \delta^{(2-2b)(n-2)} \int_{\{|x_1| \lesssim \frac{1}{2}\delta^{b-1}\}} \frac{1}{\langle x_1 \rangle^6} dx_1 + o(\delta^{(1-b)(n+2)}) \gtrsim \delta^{(1-b)(n+2)}, \end{aligned} \tag{3.24}$$

where $\omega_1 := \delta_1^{-1}(\omega - \xi_1)$.

Putting (3.22)-(3.24) together implies the validity of (3.21).

STEP 3. Let $u_\sharp := u_0 + \sum_{i=1}^{\nu} PU_i + \rho$, $(u_\sharp)_\pm := \max\{\pm u_\sharp, 0\}$, and $u_* := (u_\sharp)_+$.

Observe that

$$\begin{cases} (-\Delta - \lambda) u_\sharp = |u_\sharp|^{p-1} u_\sharp + \sum_{k=0}^{n} \sum_{i=1}^{\nu} c_i^k (-\Delta - \lambda) PZ_i^k & \text{in } \Omega, \\ u_\sharp = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.25}$$

Assuming that $\tilde{\xi}_i$ satisfies the assumption of Theorem 1.1, we introduce

$$d_*(u) := \inf \left\{ \left\| u - \left(u_0 + \sum_{i=1}^{\nu} PU_{\tilde{\delta}_i, \tilde{\xi}_i} \right) \right\|_{H_0^1(\Omega)} : \left(\tilde{\delta}_i, \tilde{\xi}_i \right) \in (0,\infty) \times \Omega, \, i = 1, \ldots, \nu \right\}.$$

Arguing as in Case 1, we can verify

$$d_*(u_\sharp) \gtrsim \|\rho\|_{H_0^1(\Omega)} \simeq \varsigma_2(\delta). \tag{3.26}$$

Testing (3.25) with $(u_\sharp)_-$ gives

$$\|(u_\sharp)_-\|_{H_0^1(\Omega)}^2 = \|(u_\sharp)_-\|_{L^{p+1}(\Omega)}^{p+1} + \int_\Omega \sum_{k=0}^{n} \sum_{i=1}^{\nu} c_i^k (-\Delta - \lambda) PZ_i^k (u_\sharp)_-$$

$$\lesssim \|(u_\sharp)_-\|_{L^{p+1}(\Omega)}^{p+1} + \sum_{k=0}^{n}\sum_{i=1}^{\nu} |c_i^k| \|(u_\sharp)_-\|_{L^{p+1}(\Omega)}.$$

Using the estimate $0 \le (u_\sharp)_- \lesssim |\rho|$, we get

$$\|(u_\sharp)_-\|_{H_0^1(\Omega)}^2 \lesssim \|\rho\|_{L^{p+1}(\Omega)}^{p+1} + \sum_{k=0}^{n}\sum_{i=1}^{\nu} |c_i^k| \|\rho\|_{L^{p+1}(\Omega)} = o(1),$$

and so we obtain

$$\|(u_\sharp)_-\|_{H_0^1(\Omega)} \lesssim \sum_{k=0}^{n}\sum_{i=1}^{\nu} |c_i^k| \lesssim \varsigma_1(\delta). \tag{3.27}$$

Therefore, by combining estimates (3.19), (3.26) and (3.27), we infer

$$d_*(u_*) \gtrsim d_*(u_\sharp) - \|(u_\sharp)_-\|_{H_0^1(\Omega)} \gtrsim \|\rho\|_{H_0^1(\Omega)} \simeq \varsigma_2(\delta).$$

Moreover,

$$\Gamma(u_*) \lesssim \tilde{\Gamma}(u_\sharp) + \|(u_\sharp)_-\|_{H_0^1(\Omega)} \lesssim \varsigma_1(\delta),$$

where $\tilde{\Gamma}(u_\sharp) := \|\Delta u_\sharp + \lambda u_\sharp + |u_\sharp|^{p-1} u_\sharp\|_{(H_0^1(\Omega))^*} \lesssim \varsigma_1(\delta)$. In conclusion, we obtain a function $u_* \ge 0$ satisfying

$$d_*(u_*) \gtrsim \zeta(\Gamma(u_*)),$$

thereby establishing the optimality of (1.11).

4. Proof of Theorem 1.3

In this section, we investigate the single-bubble case ($\nu = 1$), allowing the distance between ξ_1 and $\partial\Omega$ to be arbitrarily small, and prove Theorem 1.3. We assume that the function PU_1 satisfies (1.9) when $n = 3$ or $[n = 4, 5,\ u_0 = 0]$, and satisfies (1.8) when $[n = 4, 5,\ u_0 > 0]$ or $n \ge 6$; see Remark 1.4(2).

We first examine the case when $n = 5$ and PU_1 satisfies (1.9). By Lemma 2.1, Corollary 2.3, and (2.36), we have

$$\begin{aligned}
\|\mathcal{I}_3\|_{L^{\frac{p+1}{p}}(\Omega)} &\lesssim \|(PU_1 - U_1)U_1^{p-1}\|_{L^{\frac{p+1}{p}}(\Omega)} \\
&\lesssim \delta_1^{\frac{3}{2}} \left(\left\| \frac{1}{|\cdot - \xi_1|} U_1^{p-1} \right\|_{L^{\frac{p+1}{p}}(\Omega)} + |\varphi_\lambda^5(\xi_1)| \|U_1^{p-1}\|_{L^{\frac{p+1}{p}}(\Omega)} \right) \\
&\quad + \left\| \delta_1^{1/2} \mathcal{D}_5 \left(\frac{\cdot - \xi_1}{\delta_1} \right) U_1^{p-1} \right\|_{L^{\frac{p+1}{p}}(\Omega)} \\
&\lesssim \delta_1^2 + \kappa_1^3
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
\int_{B(\xi_1, d(\xi_1,\partial\Omega))} [(PU_1)^p - U_1^p]\, PZ_1^0 &= \frac{\lambda}{2} a_5 \delta_1^{\frac{3}{2}} p \int_{B(\xi_1, d(\xi_1,\partial\Omega))} \frac{1}{|x-\xi_1|} (U_1^{p-1} Z_1^0)(x) dx \\
&\quad + \lambda a_5^2 p \delta_1^2 \int_{B(0,\kappa_1^{-1})} \left[\frac{1}{(1+|z|^2)^{\frac{3}{2}}} - \frac{1}{|z|^3} \right] \frac{|z|^2 - 1}{(1+|z|^2)^{\frac{5}{2}}} dz \\
&\quad - \delta_1^{\frac{3}{2}} a_5 p H_\lambda^5(\xi_1, \xi_1) \int_{B(\xi_1, d(\xi_1,\partial\Omega))} U_1^{p-1} Z_1^0 dx + O(\delta_1^3) + O(\kappa_1^5) \\
&= \bar{\mathfrak{b}}_5 \lambda \delta_1^2 - \mathfrak{c}_5 \delta_1^3 \varphi_\lambda^5(\xi_1) + O(\delta_1^3) + O(\kappa_1^5).
\end{aligned} \tag{4.2}$$

Here,

$$\bar{\mathfrak{b}}_5 := \frac{a_5}{2}\int_{\mathbb{R}^5}\frac{1}{|z|}(U^{p-1}Z^0)(z)dz + a_5^2 p\int_{\mathbb{R}^5}\left[\frac{1}{(1+|z|^2)^{\frac{3}{2}}} - \frac{1}{|z|^3}\right]\frac{|z|^2-1}{(1+|z|^2)^{\frac{5}{2}}}dz > 0$$

and $\mathfrak{c}_5 := a_5 p\int_{\mathbb{R}^5} U^{p-1}Z^0 > 0$.

Combining (4.1) with (3.13), we obtain

$$\begin{aligned}\|\rho\|_{H_0^1(\Omega)} \lesssim \|f\|_{(H_0^1(\Omega))^*} &+ \begin{cases}\delta_1 & \text{if } n=3 \text{ and } u_0=0\\ \delta_1^2|\log\delta_1| & \text{if } n=4 \text{ and } u_0=0\\ \delta_1^{\frac{n-2}{2}} & \text{if } n=3,4,5 \text{ and } u_0>0\\ \delta_1^2|\log\delta_1|^{\frac{1}{2}} & \text{if } n=6\\ \delta_1^2 & \text{if } [n=5,\ u_0=0] \text{ or } n\ge 7\end{cases}\\ &+ \begin{cases}\kappa_1^{n-2} & \text{if } n=3,4,5\\ \kappa_1^4|\log\kappa_1|^{\frac{1}{2}} & \text{if } n=6\\ \kappa_1^{\frac{n+2}{2}} & \text{if } n\ge 7\end{cases}.\end{aligned} \tag{4.3}$$

Also, applying (2.20), (2.28)–(2.29), and (4.2), we deduce

$$\begin{aligned}&\int_\Omega(\mathcal{I}_1+\mathcal{I}_3)PZ_1^0\\ &= \left[\mathfrak{a}_n u_0(\xi_1)\delta_1^{\frac{n-2}{2}} + \begin{Bmatrix}O(\delta_1) & \text{if } n=3\\ O(\delta_1^2|\log\delta_1|) & \text{if } n=4\\ O(\delta_1^2) & \text{if } n=5\end{Bmatrix}\mathbf{1}_{\{p>2\}} + O\left(\delta_1^{\frac{n}{2}}+\kappa_1^n\right)\right]\mathbf{1}_{\{u_0>0\}}\\ &+\begin{cases}-\mathfrak{c}_3\varphi_\lambda^3(\xi_1)\delta_1 + O(\delta_1^2)+O(\kappa_1^3) & \text{if } n=3,\\ \mathfrak{b}_4\lambda\delta_1^2|\log\delta_1| - \mathfrak{c}_4\delta_1^2\varphi_\lambda^4(\xi_1) - 96|\mathbb{S}^3|\lambda\delta_1^2 + O(\delta_1^3)+O(\kappa_1^4) & \text{if } n=4 \text{ and } u_0=0,\\ -\delta_1^2\mathfrak{c}_4\varphi(\xi_1)+O(\delta_1^2|\log\delta_1|)+O(\kappa_1^4) & \text{if } n=4 \text{ and } u_0>0,\\ \bar{\mathfrak{b}}_5\lambda\delta_1^2 - \mathfrak{c}_5\delta_1^3\varphi_\lambda^5(\xi_1)+O(\delta_1^3)+O(\kappa_1^5) & \text{if } n=5 \text{ and } u_0=0,\\ \lambda\mathfrak{b}_n\delta_1^2 - \delta_1^{n-2}\mathfrak{c}_n\varphi(\xi_1) + O\left(\delta_1^2\kappa_1^{n-4}\right)+O(\kappa_1^n) & \text{if } [n=5,\ u_0>0] \text{ or } n\ge 6.\end{cases}\end{aligned} \tag{4.4}$$

As mentioned earlier, certain cancellations between terms with opposite signs may occur in (4.4). To handle this issue, we establish an estimate for the projection of the term $\mathcal{I}_1+\mathcal{I}_3$ onto the direction of spatial derivatives of PU_1, as stated in the following lemma.

Lemma 4.1. *For any $k\in\{1,\dots,n\}$, there exists a constant $\mathfrak{e}_n>0$ such that*

$$\begin{aligned}\left|\int_\Omega(\mathcal{I}_1+\mathcal{I}_3)PZ_1^k\right| = (1+o(1))\mathfrak{e}_n\delta_1^{n-1}\times &\begin{Bmatrix}\left|\dfrac{\partial\varphi_\lambda^n}{\partial\xi_1^k}(\xi_1)\right| & \text{if } n=3 \text{ or } [n=4,5,\ u_0=0]\\ \left|\dfrac{\partial\varphi}{\partial\xi_1^k}(\xi_1)\right| & \text{if } [n=4,5,\ u_0>0] \text{ or } n\ge 6\end{Bmatrix}\\ &+\begin{cases}O(\delta_1^3) & \text{if } n=3,4,5 \text{ and } u_0=0,\\ O(\delta_1^{\frac{3}{2}}|\log\delta_1|) & \text{if } n=3 \text{ and } u_0>0,\\ O(\delta_1^2|\log\delta_1|) & \text{if } n=4 \text{ and } u_0>0,\\ O(\delta_1^2 d(\xi_1,\partial\Omega)+\delta_1^{\frac{n}{2}}) & \text{if } n=5 \text{ and } u_0>0,\\ O(\delta_1^{\frac{n}{2}}) & \text{if } n\ge 6.\end{cases}\end{aligned} \tag{4.5}$$

Proof. By using Corollary 2.4, we obtain

$$\begin{aligned}\int_{B(\xi_1,d(\xi_1,\partial\Omega))} u_0(PU_1)^{p-1}PZ_1^k &= \frac{\partial u_0}{\partial \xi_1^k}(\xi_1)\int_{B(\xi_1,d(\xi_1,\partial\Omega))}(x-\xi_1)^k(U_1^{p-1}Z_1^k)(x)dx + O\left(\delta_1^{\frac{n}{2}}+\kappa_1^n\right)\\ &= O\left(\delta_1^{\frac{n}{2}}+\kappa_1^n\right)\end{aligned}$$

for $n \ge 3$ (cf. (2.22)).

Let us refine estimate (2.24) for the cases $n = 3, 5$ and $u_0 > 0$. If $n = 5$ and $u_0 > 0$, then we have

$$\begin{aligned}\int_{B(\xi_1,\eta\sqrt{\delta_1})} u_0^2(PU_1)^{p-2}|PZ_1^k| &\lesssim \int_{B(\xi_1,d(\xi_1,\partial\Omega))} U_1^{p-1} + \int_{B(\xi_1,d(\xi_1,\partial\Omega))^c} U_1^{p+1}\\ &\lesssim \delta_1^2 d(\xi_1,\partial\Omega)+\kappa_1^n.\end{aligned}$$

Suppose that $n = 3$ and $u_0 > 0$. Applying (A.4), we expand $\mathcal{I}_1$ by

$$\begin{aligned}\mathcal{I}_1 = &\left[pu_0(PU_1)^{p-1} + \frac{p(p-1)}{2}u_0^2(PU_1)^{p-2} + O(u_0^3(PU_1)^{p-3}) + O\left(u_0^p\right)\right]\mathbf{1}_{B(\xi_1,\eta\sqrt{\delta_1})}\\ &+ \left[pu_0^{p-1}PU_1 + O(u_0^{p-2}(PU_1)^2)\mathbf{1}_{\{p>2\}} + O((PU_1)^p)\right]\mathbf{1}_{B(\xi_1,\eta\sqrt{\delta_1})^c}.\end{aligned}$$

Also, we have

$$\begin{aligned}\int_{B(\xi_1,\eta\sqrt{\delta_1})} u_0^2(PU_1)^{p-2}PZ_1^k &= 2(1+o(1))u_0(\xi_1)\frac{\partial u_0}{\partial \xi_1^k}(\xi_1)\int_{B(\xi_1,d(\xi_1,\partial\Omega))}(x-\xi_1)^k(U_1^{p-2}Z_1^k)(x)dx\\ &\quad + O\left(\frac{\delta_1^{\frac12}}{d(\xi_1,\partial\Omega)}\int_{B(\xi_1,d(\xi_1,\partial\Omega))}U_1^{p-2}\right) + \int_{B(\xi_1,d(\xi_1,\partial\Omega))^c}U_1^{p+1}\\ &\lesssim \delta_1^2|\log\delta_1| + \delta_1\kappa_1|\log\kappa_1| + \kappa_1^n\end{aligned}$$

and

$$\int_{B(\xi_1,\eta\sqrt{\delta_1})} u_0^3(PU_1)^{p-3}|PZ_1^k| \lesssim \int_{B(\xi_1,\eta\sqrt{\delta_1})} U_1^{p-2} \lesssim \delta_1^{\frac32}|\log\delta_1|.$$

By combining the above estimates with (2.21) and (2.25)–(2.27), we conclude

$$\int_\Omega \mathcal{I}_1 PZ_1^k = \left[\begin{Bmatrix} O(\delta_1^{\frac32}|\log\delta_1|) & \text{if } n=3\\ O(\delta_1^2|\log\delta_1|) & \text{if } n=4\\ O(\delta_1^2 d(\xi_1,\partial\Omega)) & \text{if } n=5\end{Bmatrix}\mathbf{1}_{\{p>2\}} + O\left(\delta_1^{\frac{n}{2}}+\kappa_1^n\right)\right]\mathbf{1}_{\{u_0>0\}}. \tag{4.6}$$

On the other hand, arguing as in (2.34)–(2.38) and (4.2), we can find a constant $\mathfrak{c}_n > 0$ such that

$$\begin{aligned}&\int_\Omega [(PU_1)^p - U_1^p]PZ_1^k\\ &= \begin{cases} -\delta_1^{\frac{n-2}{2}}a_n p\dfrac{\partial \varphi_\lambda^n}{\partial \xi_1^k}(\xi_1)\displaystyle\int_{B(\xi_1,d(\xi_1,\partial\Omega))}(x-\xi_1)^k(U_1^{p-1}Z_1^k)(x)dx + O(\delta_1^3) + O(\kappa_1^n)\\ \qquad\qquad\qquad\qquad \text{if } n=3 \text{ or } [n=4,5,\ u_0=0],\\ -\delta_1^{\frac{n-2}{2}}a_n p\dfrac{\partial \varphi}{\partial \xi_1^k}(\xi_1)\displaystyle\int_{B(\xi_1,d(\xi_1,\partial\Omega))}(x-\xi_1)^k(U_1^{p-1}Z_1^k)(x)dx + O(\kappa_1^n)\\ \qquad\qquad\qquad\qquad \text{if } [n=4,5,\ u_0>0] \text{ or } n\ge 6\end{cases}\end{aligned}$$

$$= \begin{cases} -(1+o(1))\mathfrak{c}_n\delta_1^{n-1}\dfrac{\partial\varphi_\lambda^n}{\partial\xi_1^k}(\xi_1)+O(\delta_1^3)+O(\kappa_1^n) & \text{if } n=3 \text{ or } [n=4,5,\ u_0=0], \\ -(1+o(1))\mathfrak{c}_n\delta_1^{n-1}\dfrac{\partial\varphi}{\partial\xi_1^k}(\xi_1)+O(\kappa_1^n) & \text{if } [n=4,5,\ u_0>0] \text{ or } n\geq 6. \end{cases} \tag{4.7}$$

Here, we also used Corollary 2.4.

Moreover, we see from (2.30) and (2.31) that

$$\int_\Omega \lambda PU_1PZ_1^k\mathbf{1}_{\{PU_1 \text{ satisfies } (1.8)\}} = \begin{cases} O(\delta_1^2|\log\delta_1|) & \text{if } n=4 \text{ and } u_0>0, \\ O\left(\delta_1^{\frac{n}{2}}+\kappa_1^n\right) & \text{if } [n=5,\ u_0>0] \text{ or } n\geq 6. \end{cases} \tag{4.8}$$

Consequently, (4.5) follows immediately from (4.6)–(4.8). □

Now we are in a position to prove Theorem 1.3.

Proof of Theorem 1.3. Throughout the proof, we keep in mind (2.10).

STEP 1. Let us prove estimate (1.13).

By testing (2.1) with PZ_1^k for $k\in\{0,1,\dots,n\}$, arguing as in (3.8), and using (3.15), we obtain

$$\begin{aligned} &\left|\int_\Omega(\mathcal{I}_1+\mathcal{I}_3)PZ_1^k\right| \\ &= \left|-\int_\Omega fPZ_1^k-\int_\Omega\mathcal{I}_0[\rho]PZ_1^k+\int_\Omega\left[(-\Delta-\lambda)\rho-p(u_0+PU_1)^{p-1}\rho\right]PZ_1^k\right| \\ &\lesssim \|f\|_{(H_0^1(\Omega))^*}+\|\rho\|_{H_0^1(\Omega)}^2+\|\rho\|_{H_0^1(\Omega)}\Big[\|[(PU_1)^{p-1}-U_1^{p-1}]PZ_1^k\|_{L^{\frac{p+1}{p}}(\Omega)} \\ &\qquad +\|U_1^{p-1}(PZ_1^k-Z_1^k)\|_{L^{\frac{p+1}{p}}(\Omega)}+\|U_1\|_{L^{\frac{p+1}{p}}(\Omega)}+\|U_1^{p-1}\|_{L^{\frac{p+1}{p}}(\Omega)}\mathbf{1}_{\{u_0>0,\ p>2\}}\Big]. \end{aligned} \tag{4.9}$$

Having (4.3)-(4.5) in mind, we proceed by distinguishing several cases according to the dimension n and the function u_0.

Case 1: Assume that $n\geq 7$.

We consider the following subcases:

- If $\mathfrak{b}_n\lambda\delta_1^2>\mathfrak{c}_n\varphi(\xi_1)\delta_1^{n-2}$, we have that $\delta_1^2\lesssim\|f\|_{(H_0^1(\Omega))^*}$.
  - When $\delta_1^2\gtrsim\kappa_1^{\frac{n+2}{2}}$, it follows that $\|\rho\|_{H_0^1(\Omega)}\lesssim\|f\|_{(H_0^1(\Omega))^*}+\delta_1^2$. Hence, $\|\rho\|_{H_0^1(\Omega)}\lesssim\|f\|_{(H_0^1(\Omega))^*}$.
  - When $\delta_1^2\lesssim\kappa_1^{\frac{n+2}{2}}$, it follows that $\|\rho\|_{H_0^1(\Omega)}\lesssim\|f\|_{(H_0^1(\Omega))^*}+\kappa_1^{\frac{n+2}{2}}$. Hence, $\|\rho\|_{H_0^1(\Omega)}\lesssim\|f\|_{(H_0^1(\Omega))^*}^{\frac{n+2}{2(n-2)}}$.
- If $\mathfrak{b}_n\lambda\delta_1^2<\mathfrak{c}_n\varphi(\xi_1)\delta_1^{n-2}$, we have that $\kappa_1^{n-2}\lesssim\|f\|_{(H_0^1(\Omega))^*}$ and $\|\rho\|_{H_0^1(\Omega)}\lesssim\|f\|_{(H_0^1(\Omega))^*}+\kappa_1^{\frac{n+2}{2}}$. Thus, $\|\rho\|_{H_0^1(\Omega)}\lesssim\|f\|_{(H_0^1(\Omega))^*}^{\frac{n+2}{2(n-2)}}$.
- If $\mathfrak{b}_n\lambda\delta_1^2=\mathfrak{c}_n\varphi(\xi_1)\delta_1^{n-2}$, we have that $\|\rho\|_{H_0^1(\Omega)}\lesssim\|f\|_{(H_0^1(\Omega))^*}+\kappa_1^{\frac{n+2}{2}}$. It follows from (4.5) and (4.9) that $\kappa_1^{n-1}\lesssim\|f\|_{(H_0^1(\Omega))^*}+\delta_1^{\frac{n+2}{n-2}+2}$. Consequently, $\|\rho\|_{H_0^1(\Omega)}\lesssim\|f\|_{(H_0^1(\Omega))^*}^{\frac{n+2}{2(n-1)}}$.

Case 2: Assume that $n=6$.

- If $\mathfrak{a}_nu_0(\xi_1)\delta_1^2+\mathfrak{b}_6\lambda\delta_1^2\neq\mathfrak{c}_6\varphi(\xi_1)\delta_1^4$, we have that

$$\|\rho\|_{H_0^1(\Omega)}\lesssim\|f\|_{(H_0^1(\Omega))^*}|\log\|f\|_{(H_0^1(\Omega))^*}|^{\frac{1}{2}}.$$

- If $\mathfrak{a}_6 u_0(\xi_1)\delta_1^2 + \mathfrak{b}_6 \lambda \delta_1^2 = \mathfrak{c}_6 \varphi(\xi_1)\delta_1^4$, a cancellation happens in (4.4), which leads to
$$\mathcal{I}_1 + \mathcal{I}_3 = 2(u_0(x) - u_0(\xi_1))PU_1 + 2(PU_1 - U_1)U_1 + 2a_6\varphi(\xi_1)\delta_1^2 PU_1.$$
Therefore,
$$\|\rho\|_{H_0^1(\Omega)} \lesssim \|f\|_{(H_0^1(\Omega))^*} + \|\mathcal{I}_1 + \mathcal{I}_3\|_{L^{\frac{p+1}{p}}(\Omega)} \lesssim \|f\|_{(H_0^1(\Omega))^*} + \delta_1^3 + \kappa_1^5.$$
Applying (4.5) and (4.9), we find that $\kappa_1^5 \simeq \delta_1^{\frac{5}{2}} \lesssim \|f\|_{(H_0^1(\Omega))^*} + \delta_1^4|\log \delta_1|$, and so $\|\rho\|_{H_0^1(\Omega)} \lesssim \|f\|_{(H_0^1(\Omega))^*}$.

Case 3: Assume that $n = 3, 4, 5$ and $u_0 = 0$.

- If $n = 3$ and $u_0 = 0$, we have that $\|\rho\|_{H_0^1(\Omega)} \lesssim \|f\|_{(H_0^1(\Omega))^*}$.
- Assume that $n = 4$ and $u_0 = 0$.
  - If $\mathfrak{b}_4 \lambda \delta_1^2 |\log \delta_1| \neq \mathfrak{c}_4 \varphi_\lambda^4(\xi_1)\delta_1^2$, we have that $\|\rho\|_{H_0^1(\Omega)} \lesssim \|f\|_{(H_0^1(\Omega))^*}$.
  - If $\mathfrak{b}_4 \lambda \delta_1^2 |\log \delta_1| = \mathfrak{c}_4 \varphi_\lambda^4(\xi_1)\delta_1^2$, we have that
  $$\mathcal{I}_3 = (PU_1)^p - U_1^p - p\lambda\delta_1|\log \delta_1| U_1^{p-1} + pa_4\delta_1\varphi_\lambda^4(\xi_1)U_1^{p-1}.$$
  Therefore,
  $$\|\rho\|_{H_0^1(\Omega)} \lesssim \|f\|_{(H_0^1(\Omega))^*} + \|\mathcal{I}_1 + \mathcal{I}_3\|_{L^{\frac{p+1}{p}}(\Omega)} \lesssim \|f\|_{(H_0^1(\Omega))^*} + \delta_1^2.$$
  Applying (4.4) and (4.9), we find that $|\int_\Omega \mathcal{I}_3 PZ_1^0| \simeq \delta_1^2 \lesssim \|f\|_{(H_0^1(\Omega))^*} + \delta_1^3$, and so $\|\rho\|_{H_0^1(\Omega)} \lesssim \|f\|_{(H_0^1(\Omega))^*}$.
- Assume that $n = 5$ and $u_0 = 0$.
  - If $\bar{\mathfrak{b}}_5 \lambda \delta_1^2 \neq \mathfrak{c}_5 \varphi_\lambda^5(\xi_1)\delta_1^3$, we have the same estimate as above.
  - If $\bar{\mathfrak{b}}_5 \lambda \delta_1^2 = \mathfrak{c}_5 \varphi_\lambda^5(\xi_1)\delta_1^3$, we have that $\|\rho\|_{H_0^1(\Omega)} \lesssim \|f\|_{(H_0^1(\Omega))^*} + \delta_1^2$. By Corollary 2.4, the inequality
  $$\|U_1^{p-1}\delta_1\partial_{\xi_1^k}\mathcal{S}_{\delta_1,\xi_1}\|_{L^{\frac{p+1}{p}}(\Omega)} \lesssim \|U_1^{p-1}\|_{L^5(\Omega)}\|\delta_1\partial_{\xi_1^k}\mathcal{S}_{\delta_1,\xi_1}\|_{L^2(\Omega)} \lesssim \delta_1^2,$$
  (4.5), and (4.9), one derives that $\kappa_1^4 \simeq \delta_1^{\frac{8}{3}} \lesssim \|f\|_{(H_0^1(\Omega))^*} + \delta_1^{\frac{7}{2}}$, which gives $\|\rho\|_{H_0^1(\Omega)} \lesssim \|f\|_{(H_0^1(\Omega))^*}^{\frac{3}{4}}$.

Case 4: Assume that $n = 3, 4, 5$ and $u_0 > 0$.

- If $\mathfrak{a}_n u_0(\xi_1)\delta_1^{\frac{n-2}{2}} \neq \mathfrak{c}_n \delta_1^{n-2} \begin{cases} \varphi_\lambda^3(\xi_1) & \text{if } n = 3 \\ \varphi(\xi_1) & \text{if } n = 4, 5 \end{cases}$, we obtain that $\|\rho\|_{H_0^1(\Omega)} \lesssim \|f\|_{(H_0^1(\Omega))^*}$.
- If $\mathfrak{a}_n u_0(\xi_1)\delta_1^{\frac{n-2}{2}} = \mathfrak{c}_n \delta_1^{n-2} \begin{cases} \varphi_\lambda^3(\xi_1) & \text{if } n = 3 \\ \varphi(\xi_1) & \text{if } n = 4, 5 \end{cases}$, the expansion
$$\begin{aligned}\mathcal{I}_1 + \mathcal{I}_3 &= (u_0 + PU_1)^p - u_0^p - (PU_1)^p - pu_0(\xi_1)(PU_1)^{p-1} + (PU_1)^p - U_1^p \\ &\quad + pa_n\delta_1^{\frac{n-2}{2}} \begin{Bmatrix} \varphi_\lambda^3(\xi_1) & \text{if } n = 3 \\ \varphi(\xi_1) & \text{if } n = 4, 5 \end{Bmatrix} (PU_1)^{p-1} + \lambda PU_1 \mathbf{1}_{\{n=4,5\}}\end{aligned}$$
gives
$$\|\rho\|_{H_0^1(\Omega)} \lesssim \|f\|_{(H_0^1(\Omega))^*} + \|\mathcal{I}_1 + \mathcal{I}_3\|_{L^{\frac{p+1}{p}}(\Omega)} \lesssim \|f\|_{(H_0^1(\Omega))^*} + \begin{cases} \delta_1 & \text{if } n = 3, \\ \delta_1^{\frac{n-2}{2}} & \text{if } n = 4, 5. \end{cases}$$

Besides, making use of (4.5) and (4.9), we know that

$$\kappa_1^{n-1} \simeq \delta_1^{\frac{n-1}{2}} \lesssim \|f\|_{(H_0^1(\Omega))^*} + \begin{cases} \delta_1^{\frac{3}{2}} & \text{if } n=3, \\ \delta_1^{n-2} & \text{if } n=4,5. \end{cases}$$

We conclude

$$\|\rho\|_{H_0^1(\Omega)} \lesssim \begin{cases} \|f\|_{(H_0^1(\Omega))^*} & \text{if } n=3 \text{ and } u_0>0, \\ \|f\|_{(H_0^1(\Omega))^*}^{\frac{n-2}{n-1}} & \text{if } n=4,5 \text{ and } u_0>0. \end{cases}$$

This completes the derivation of (1.13).

STEP 2. We prove the optimality of (1.13).

Case 1: Assume that $\zeta(t)=t$.

One can treat this case as in Case 1 of Subsection 3.2, by choosing a point $\xi_1 \in \Omega$ and setting

$$\epsilon \simeq \left.\begin{cases} \delta_1 & \text{if } n=3 \\ \delta_1^2 |\log \delta_1| & \text{if } n=4 \text{ and } u_0=0 \end{cases}\right\} + \kappa_1^{n-2}.$$

Case 2: Assume that $\zeta(t) \gg t$.

Let us choose $\delta_1>0$ and $\xi_1 \in \Omega$ satisfying the following conditions

$$\begin{cases} \mathfrak{a}_n u_0(\xi_1)\delta_1^{\frac{n-2}{2}} = \mathfrak{c}_n \varphi(\xi_1)\delta_1^{n-2} & \text{if } n=4,5 \text{ and } u_0>0, \\ \bar{\mathfrak{b}}_5 \lambda \delta_1^2 = \mathfrak{c}_5 \varphi_\lambda^5(\xi_1)\delta_1^3 & \text{if } n=5 \text{ and } u_0=0, \\ \mathfrak{a}_6 u_0(\xi_1)\delta_1^2 + \mathfrak{b}_6\lambda\delta_1^2 > \mathfrak{c}_n\varphi(\xi_1)\delta_1^4 & \text{if } n=6, \\ \mathfrak{b}_n\lambda\delta_1^2 = \mathfrak{c}_n\varphi(\xi_1)\delta_1^{n-2} & \text{if } n\geq 7. \end{cases} \tag{4.10}$$

We now consider a function ρ solving the following linearized problem

$$\begin{cases} -\Delta\rho - \lambda\rho - p(u_0+PU_1)^{p-1}\rho = \mathcal{I}_1 + \mathcal{I}_3 + \mathcal{I}_0[\rho] + \displaystyle\sum_{k=0}^n \tilde{c}_1^k(-\Delta-\lambda)PZ_1^k & \text{in } \Omega, \\ \rho = 0 \quad \text{on } \partial\Omega, \quad \tilde{c}_1^k \in \mathbb{R} \quad \text{for } k=0,\dots,n, \\ \left\langle \rho, PZ_1^k \right\rangle_{H_0^1(\Omega)} = 0 \quad \text{for } k=0,\dots,n. \end{cases}$$

where $\mathcal{I}_1$, $\mathcal{I}_3$, and $\mathcal{I}_0[\rho]$ are defined as in (2.2) with (δ_1,ξ_1) satisfying (4.10).

Denote $f := \sum_{k=0}^n \tilde{c}_1^k(-\Delta-\lambda)PZ_1^k$. Using (4.4) and (4.5), we obtain

$$\begin{aligned} \|f\|_{(H_0^1(\Omega))^*} &\lesssim |\tilde{c}_1^0| + \max_{k\in\{1,\dots,n\}} |\tilde{c}_1^k| \\ &\lesssim \left| \int_\Omega (\mathcal{I}_1+\mathcal{I}_3)PZ_1^0 \right| + \max_{k\in\{1,\dots,n\}} \left| \int_\Omega (\mathcal{I}_1+\mathcal{I}_3)PZ_1^k \right| \\ &\lesssim \begin{cases} \kappa_1^{n-1} & \text{if } [n=4,\ u_0>0], \text{ or } n=5, \text{ or } n\geq 7, \\ \delta_1^2 & \text{if } n=6. \end{cases} \end{aligned}$$

It follows that

$$\|\rho\|_{H_0^1(\Omega)} \lesssim \varsigma_3(\delta) := \begin{cases} \delta_1^{\frac{n-2}{2}} & \text{if } n=4,5 \text{ and } u_0>0, \\ \delta_1^2 & \text{if } n=5 \text{ and } u_0=0, \\ \delta_1^2|\log\delta_1|^{\frac{1}{2}} & \text{if } n=6, \\ \kappa_1^{\frac{n+2}{2}} & \text{if } n\geq 7. \end{cases}$$

On the other hand, similarly to (2.19), we can deduce a lower bound estimate

$$\begin{aligned}
&\|\rho\|_{H_0^1(\Omega)}^2 \\
&\gtrsim \begin{cases} \displaystyle\int_\Omega\int_\Omega (\lambda PU_1)(x)\frac{1}{|x-\omega|^{n-2}}(\lambda PU_1)(\omega)dxd\omega & \text{if } [n=4,5,\ u_0>0] \text{ or } n=6, \\ \displaystyle\int_\Omega\int_\Omega [(PU_1)^p-U_1^p](x)\frac{1}{|x-\omega|^{n-2}}[(PU_1)^p-U_1^p](\omega)dxd\omega & \text{if } [n=5,\ u_0=0] \text{ or } n\ge 7 \end{cases} \\
&\gtrsim (\varsigma_3(\delta))^2.
\end{aligned}$$

We set $u_* := (u_0+PU_1+\rho)_+$. Then, by proceeding as in Case 2 of Subsection 3.2, we finish the proof. □

Remark 4.2. Assume that $\nu \ge 2$. Arguing as above, one can find a nonnegative function $u_* \in H_0^1(\Omega)$ with $\delta_i=\delta_j$ and $|\xi_i-\xi_j| \gtrsim 1$ for $1\le i\ne j\le \nu$ such that

$$\inf\left\{\left\|u_*-\left(u_0+\sum_{i=1}^{\nu} PU_{\tilde\delta_i,\tilde\xi_i}\right)\right\|_{H_0^1(\Omega)} : \left(\tilde\delta_i,\tilde\xi_i\right)\in(0,\infty)\times\Omega,\ i=1,\dots,\nu\right\} \gtrsim \zeta(u_*),$$

where ζ is given by (1.14), except for the cases $[n=3,\ u_0>0]$ and $[n=4,\ u_0=0]$. In these exceptional cases, additional technical difficulties arise.

Appendix A. Some useful estimates

Lemma A.1. *Let $a,b>0$. Then the following estimates hold:*

$$|(a+b)^s-a^s-b^s| \lesssim \begin{cases} \min\{a^{s-1}b, ab^{s-1}\} & \text{if } 1\le s\le 2, \\ a^{s-1}b+ab^{s-1} & \text{if } s>2. \end{cases} \tag{A.1}$$

Moreover, we have the following asymptotic expansions:

$$(a+b)^s-a^s = O(a^{s-1}b)\mathbf{1}_{s>1}+O(b^s) \quad \text{for } s>0, \tag{A.2}$$

$$(a+b)^s = a^s+sa^{s-1}b+O(a^{s-2}b^2)\mathbf{1}_{s>2}+O(b^s) \quad \text{for } s>1, \tag{A.3}$$

$$(a+b)^s = a^s+sa^{s-1}b+\frac{p(p-1)}{2}a^{s-2}b^2+O(a^{s-3}b^3)\mathbf{1}_{s>3}+O(b^s) \quad \text{for } s>2. \tag{A.4}$$

For any $a>0$, $b\in\mathbb{R}$ such that $a+b\ge 0$ and $1<s<2$, it holds that

$$\left|(a+b)^s-a^s-sa^{s-1}b\right| \lesssim \min\left\{a^{s-2}|b|^2, |b|^s\right\}. \tag{A.5}$$

Lemma A.2. *Let $s>0$ and $U_{\delta,\xi}$ be the bubble defined in (1.3). Then*

$$\int_\Omega U_{\delta,\xi}^s \lesssim \begin{cases} \delta^{\frac{n-2}{2}s} & \text{if } 0<s<\frac{n}{n-2}, \\ \delta^{\frac{n}{2}}|\log\delta| & \text{if } s=\frac{n}{n-2}, \\ \delta^{n-\frac{n-2}{2}s} & \text{if } s>\frac{n}{n-2}. \end{cases}$$

Lemma A.3. *Let U_{δ_i,ξ_i} and U_{δ_j,ξ_j} be the bubbles for $1\le i\ne j\le\nu$. If $s,t\ge 0$ satisfy $s+t=2^*$, then for any fixed $\tau>0$, we have*

$$\int_{\mathbb{R}^n} U_{\delta_i,\xi_i}^s U_{\delta_j,\xi_j}^t \lesssim \begin{cases} q_{ij}^{\min\{s,t\}} & \text{if } |s-t|\ge\tau, \\ q_{ij}^{\frac{n}{n-2}}|\log q_{ij}| & \text{if } s=t, \end{cases}$$

provided q_{ij} in (2.3) is sufficiently small.

Proof. See [15, Lemma A.3]. □

Lemma A.4. *Suppose* $\alpha > 0$. *Then*

$$\int_\Omega \frac{1}{|x-z|^{n-2}} \left(\frac{\delta}{\delta^2+|z-\xi|^2}\right)^{\frac{\alpha}{2}} dz \lesssim \begin{cases} \delta^{\frac{\alpha}{2}} & \text{if } 0<\alpha<2,\\ \delta(1+|\log|x-\xi||) & \text{if } \alpha=2,\\ \delta^{\frac{\alpha}{2}}(\delta^2+|x-\xi|^2)^{-\frac{\alpha-2}{2}} & \text{if } 2<\alpha<n,\\ \delta^{\frac{n}{2}}(\delta^2+|x-\xi|^2)^{-\frac{n-2}{2}}\log(2+|x-\xi|\delta^{-1}) & \text{if } \alpha=n,\\ \delta^{n-\frac{\alpha}{2}}(\delta^2+|x-\xi|^2)^{-\frac{n-2}{2}} & \text{if } \alpha>n. \end{cases}$$

Proof. It follows from direct computations. □

Appendix B. Proof of (2.10)

Lemma B.1. *Let* $\varphi_\lambda^n(x) := H_\lambda^n(x,x)$ *for* $n = 3,4,5$, *where* $H_\lambda^n(x,y)$ *satisfies equations* (2.4)–(2.6). *If* $d(x,\partial\Omega)$ *is small, then we have*

$$\begin{cases} \varphi_\lambda^n(x) = \dfrac{1}{(2d(x,\partial\Omega))^{n-2}}\left(1+O(d(x,\partial\Omega))\right),\\ |\nabla\varphi_\lambda^n(x)| = \dfrac{2(n-2)}{(2d(x,\partial\Omega))^{n-1}}\left(1+O(d(x,\partial\Omega))\right). \end{cases}$$

Proof. Since Ω is a smooth domain, there exists $d_0 > 0$ such that for every $x \in \Omega$ with $d(x,\partial\Omega) < d_0$, there exists a unique $x' \in \partial\Omega$ such that $d(x,\partial\Omega) = |x-x'|$. By an appropriate translation and rotation, we may assume without loss of generality that $x = (0,d)$, $x' = 0$, and the boundary near the origin is locally given by a C^2 function ϕ with $\phi(0) = 0$, $\nabla\phi(0) = 0$. Specifically,

$$\partial\Omega \cap B(0,\tau) = \{y = (y', y^n) \in \mathbb{R}^n : y^n = \phi(y')\} \cap B(0,\tau),$$
$$\Omega \cap B(0,\tau) = \{y \in \mathbb{R}^n : y^n > \phi(y')\} \cap B(0,\tau)$$

for some small $\tau > 0$. Let $x'' = (0,-d)$ be the reflection of x across the boundary. For sufficiently small d, $x'' \notin \Omega$, and the function $\frac{1}{|y-x''|^{n-2}}$ is harmonic in Ω. Define

$$F_\lambda^n(y) := H_\lambda^n(y,x) - \begin{cases} \dfrac{1}{|y-x''|^{n-2}} - \frac{\lambda}{2}|y-x''| & \text{if } n=3,\\ \dfrac{1}{|y-x''|^{n-2}} - \dfrac{\lambda}{2}\log|x''-y| & \text{if } n=4,\\ \dfrac{1}{|y-x''|^{n-2}} + \dfrac{\lambda}{2}\dfrac{1}{|x''-y|} - 2\lambda^2|y-x''| & \text{if } n=5. \end{cases}$$

Then F_λ^n satisfies

$$\begin{cases} \Delta_y F_\lambda^n + \lambda F_\lambda^n = f_\lambda^n & \text{in } \Omega,\\ F_\lambda^n = g_\lambda^n & \text{on } \partial\Omega, \end{cases}$$

where

$$f_\lambda^n(y) := \begin{cases} -\frac{\lambda^2}{2}\left(|y-x| - |y-x''|\right) & \text{if } n=3,\\ -\lambda\log|x-y| - \dfrac{\lambda^2}{2}\log|x''-y| & \text{if } n=4,\\ -2\lambda^2|y-x| + 2\lambda^3|x''-y| & \text{if } n=5, \end{cases}$$

and

$g_\lambda^n(y)$

$$:= \begin{cases} \dfrac{1}{|y-x|^{n-2}} - \dfrac{1}{|y-x''|^{n-2}} - \dfrac{\lambda}{2}(|y-x| - |y-x''|) & \text{if } n=3, \\ \dfrac{1}{|y-x|^{n-2}} - \dfrac{1}{|y-x''|^{n-2}} - \dfrac{\lambda}{2}(\log|x-y| - \log|x''-y|) & \text{if } n=4, \\ \dfrac{1}{|y-x|^{n-2}} - \dfrac{1}{|y-x''|^{n-2}} + \dfrac{\lambda}{2}\left(\dfrac{1}{|x-y|} - \dfrac{1}{|x''-y|}\right) - 2\lambda^2(|y-x| - |y-x''|) & \text{if } n=5. \end{cases}$$

For $y \in \partial\Omega \cap B(0,\tau)$, we have the Taylor expansions

$$|y-x| = \sqrt{|y|^2 + d^2 - 2dy^n} = \sqrt{|y|^2+d^2}\left(1 + O\left(\frac{dy^n}{|y|^2+d^2}\right)\right),$$
$$\left|y-x''\right| = \sqrt{|y|^2 + d^2 + 2dy^n} = \sqrt{|y|^2+d^2}\left(1 + O\left(\frac{dy^n}{|y|^2+d^2}\right)\right),$$

where we used the smoothness of ϕ. Since $|y^n| = |\phi(y')| = O(|y'|^2)$, we observe

$$\begin{aligned} \frac{1}{|y-x|^{n-2}} - \frac{1}{|y-x''|^{n-2}} &= (|y|^2+d^2)^{-\frac{n-2}{2}} O\left(\frac{dy^n}{|y|^2+d^2}\right) \\ &= (|y|^2+d^2)^{-\frac{n-2}{2}} O(d) = O(d^{-n+3}). \end{aligned}$$

Similarly,

$$\begin{cases} |y-x| - |y-x''| = O(1) & \text{for } n=3,5, \\ \log|y-x| - \log|y-x''| = O(1) & \text{for } n=4, \\ \frac{1}{|y-x|} - \frac{1}{|y-x''|} = O(1) & \text{for } n=5. \end{cases}$$

For $y \in \partial\Omega \cap (\mathbb{R}^n \backslash B(0,\tau))$, the above differences are also uniformly bounded. In other words,

$$\|g_\lambda^n\|_{L^\infty(\partial\Omega)} = O(d^{-n+3}).$$

In particular, $\|f_\lambda^n\|_{L^t(\Omega)} \lesssim 1$ for any $t > n$. By standard elliptic estimates, we obtain

$$\|F_\lambda^n\|_{L^\infty(\Omega)} = O(d^{-n+3}).$$

Hence, evaluating at x, we get

$$\begin{aligned} \varphi_\lambda^n(x) = H_\lambda^n(x,x) &= \left.\begin{cases} \frac{1}{|y-x''|^{n-2}} - \frac{\lambda}{2}|y-x''| & \text{if } n=3 \\ \frac{1}{|y-x''|^{n-2}} - \frac{\lambda}{2}\log|x''-y| & \text{if } n=4 \\ \frac{1}{|y-x''|^{n-2}} + \frac{\lambda}{2}\frac{1}{|x''-y|} - 2\lambda^2|y-x''| & \text{if } n=5 \end{cases}\right\} + O(d^{-n+3}) \\ &= \frac{1}{(2d(x,\partial\Omega))^{n-2}}(1 + O(d(x,\partial\Omega))). \end{aligned}$$

The estimate for $|\nabla\varphi_\lambda^n(x)|$ follows analogously by applying interior gradient estimates under the same reflections. ☐

Remark B.2. The estimate for φ in (2.10) follows with slight modifications to the above proof.

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(Haixia Chen) Department of Mathematics and Research Institute for Natural Sciences, College of Natural Sciences, Hanyang University, 222 Wangsimni-ro Seongdong-gu, Seoul 04763, Republic of Korea

Email address: hxchen29@hanyang.ac.kr chenhaixia157@gmail.com

(Seunghyeok Kim) Department of Mathematics and Research Institute for Natural Sciences, College of Natural Sciences, Hanyang University, 222 Wangsimni-ro Seongdong-gu, Seoul 04763, Republic of Korea

Email address: shkim0401@hanyang.ac.kr shkim0401@gmail.com

(Juncheng Wei) Department of Mathematics, Chinese University of Hong Kong, Shatin, NT, Hong Kong

Email address: wei@math.cuhk.edu.hk