A LIOUVILLE THEOREM FOR SUPERLINEAR PARABOLIC EQUATIONS ON THE HEISENBERG GROUP

JUNCHENG WEI AND KE WU

Abstract. We consider a parabolic nonlinear equation on the Heisenberg group. Applying the Gidas-Spruck type estimates, we prove that under suitable conditions, the equation does not have positive solutions. As an application of the nonexistence result, we provide optimal universal estimates for positive solutions.

Dedicated to Joel Spruck with admirations

1. Introduction

In this paper, we consider the eternal solutions to the superlinear parabolic equation

$$\partial_s u = \Delta_{\mathbb{H}} u + u^p, u > 0 \quad \text{in} \quad \mathbb{H}^n \times (-\infty, +\infty), \tag{1.1}$$

where $\mathbb{H} = \mathbb{H}^n$ is the Heisenberg group. We recall that $\mathbb{H}^n$ is the Lie group $(\mathbb{C}^n \times \mathbb{R}, \circ)$ equipped with the group action

$$(z,t) \circ (\xi, t') = (z + \xi, t + t' + 2\text{Im} \sum_{i=1}^{n} z^i \bar{\xi}^i).$$

Let $Q = 2n + 2$ be the homogeneous dimension of $\mathbb{H}^n$ and $|(z,t)|_{\mathbb{H}}$ be the intrinsic metric, where

$$|(z,t)|_{\mathbb{H}} = (|z|^4 + t^2)^{\frac{1}{4}}.$$

The CR structure of $\mathbb{H}^n$ is spanned by the left-invariant vector fields

$$Z_i = \frac{\partial}{\partial Z^i} + \sqrt{-1}\bar{z}^i \frac{\partial}{\partial t}, \quad i = 1, \cdots, n.$$

For a smooth function u on $\mathbb{H}^n$, we denote

$$u_i = Z_i u, \quad u_{\bar{i}} = Z_{\bar{i}} u, \quad u_0 = \frac{\partial u}{\partial t},$$

where

$$Z_{\bar{i}} = \frac{\partial}{\partial \bar{Z}^i} - \sqrt{-1} z^i \frac{\partial}{\partial t}, \quad i = 1, \cdots, n.$$

Then

$$|\nabla u|^2 = \sum_{i=1}^{n} u_i u_{\bar{i}}$$

and $\Delta_{\mathbb{H}}$ is the subelliptic Laplacian defined by

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{n} (Z_i Z_{\bar{i}} + Z_{\bar{i}} Z_i).$$

In the Euclidean space, the analogous equation corresponding to (1.1) is

$$\partial_s u = \Delta u + u^p \quad \text{in} \quad \mathbb{R}^n \times (-\infty, +\infty), \tag{1.2}$$

If u is a stationary solution of the equation (1.2), then u satisfies the equation

$$\Delta u + u^p = 0 \quad \text{in} \quad \mathbb{R}^n. \tag{1.3}$$

For the equation (1.3), Gidas and Spruck proved in their seminal paper [7] that if $n > 2, 1 < p < (n+2)/(n-2)$ and if u is a non-negative C^2 solution of the equation (1.3), then $u \equiv 0$. This result is obtained through the use of a special vector field constructed from the solution. The basic form of the vector field used in [7] appeared previously in a geometric result of Obata (see [16]) concerning conformal deformations of the usual metric on S^n. In [6], Chen-Li give a new proof of the nonexistence result in [7]. The proof in [6] is based on moving plane method. If $p = (n+2)/(n-2)$, then (1.3) is a special case of the Yamabe problem in conformal geometry. In [5], using the asymptotic symmetry technique, Caffarelli, Gidas and Spruck were able to classify all the positive solutions of (1.3) for $n \geq 3$. They showed that any positive solutions of (1.3) can be written in the form

$$u_{a,\lambda}(x) = \left(\frac{\lambda\sqrt{n(n-2)}}{\lambda^2 + |x-a|^2}\right)^{\frac{n-2}{2}},$$

where $\lambda > 0$ and a is some point in $\mathbb{R}^n$.

For the parabolic equation (1.3), the so-called Fujita exponent $p_F = 1 + 2/n$ plays the role of the first critical exponent for the corresponding Cauchy problem. It is known that if $1 < p \leq p_F$, then the equation (1.2) does not admit any nontrivial non-negative distributional solution. However, in the range $p_F < p < (n+2)/(n-2)$, it is more difficult to classify the positive classical solutions of the equation (1.2). If we assume that u is radially symmetric, then it is proved in [19] that the nonexistence of positive bounded solutions hold for $1 < p < p_S$, where

$$p_S = \begin{cases} +\infty & \text{if} \quad 1 \leq n \leq 2, \\ \frac{n+2}{n-2} & \text{if} \quad n \geq 3. \end{cases}$$

Without the radial symmetry assumption, it is proved in [1] that if $1 < p < p_B$, where

$$p_B = \begin{cases} +\infty & \text{if} \quad n = 1, \\ \frac{n(n+2)}{(n-1)^2} & \text{if} \quad n \geq 2, \end{cases}$$

then equation (1.2) has no positive classical solution. The proof in [1] is based on a modification of the technique of local, integral gradient estimates developed in [7] for elliptic problems (see also [2]). By using the monotonicity formula and the energy estimates for the rescaled problems, Quittner proved in [22] that the nonexistence of positive solutions holds when $n > 2, 1 < p < (n+2)/(n-2)$. In a recent paper [23], Quittner proved the optimal Liouville theorem for the equation (1.2) by further improving the method used in [22].

Because of these important works, the structure of positive solutions of (1.2) has been well understood. Compared with the equation (1.2), the results concerning positive solutions of (1.1) are less known. For stationary equation of (1.1)

$$\Delta_{\mathbb{H}^n} u + u^p = 0, u > 0 \quad \text{in} \quad \mathbb{H}^n \tag{1.4}$$

it is proved in [3] that if $1 < p \leq Q/(Q-2)$ and if u is a nonnegative stationary solution of (1.4), then $u \equiv 0$. By applying the moving plane method, Birindelli and

Prajapat proved in [4] that if $1 < p < (Q-2)/(Q+2)$, and if u is a nonnegative stationary solution of (1.4) such that $u(z,t) = u(|z|,t)$, then $u \equiv 0$. In [14], Xu improved the Liouville type result in [3] to the range $n > 1, 1 < p < (Q(Q+2))/(Q-1)^2$. In a interesting paper [13], Ma and Ou gives a complete classification of nonnegative stationary solutions to the equation (1.4) when $1 < p < (Q-2)/(Q+2)$. The proof in [13] is based on a generalized formula of that found by Jerison and Lee ([11]).

In this paper, we are interested in positive solutions of (1.1) which are not necessary to be stationary solutions. The main result in this paper is the following.

Theorem 1.1. *If $n > 1, 1 < p < 1 + 4(Q^2 - 3Q - 1)/((Q-2)(Q-1)^2)$, then the equation (1.1) does not have positive solution.*

Remark 1.2. *If $n \geq 1, 1 < p < 1 + 2/Q$, then Theorem 1.1 is a special case of the results in [17] and [18]. Indeed, some Fujita-type results for the Cauchy problem are obtained in these two papers.*

Remark 1.3. *It is easy to check that*

$$1 < 1 + \frac{2}{Q} < 1 + \frac{4(Q^2 - 3Q - 1)}{(Q-2)(Q-1)^2}.$$

Remark 1.4. *Inspired by the equation (1.2), it is natural to conjecture that if $n \geq 1, 1 < p < (Q+2)/(Q-2)$, then the equation (1.1) does not possess positive classical solutions.*

In 2007, Poláčik, Quittner and Souplet proved in their seminar paper [21] that a Liouville theorem for scaling invariant superlinear parabolic problems would always imply optimal universal estimates for solutions of related initial value problems, including estimates of their singularities and decay. As a consequence of the main result in [21] and Theorem 1.1, we have the following result.

Proposition 1.5. *Let $n > 1, 1 < p < 1 + 4(Q^2 - 3Q - 1)/((Q-2)(Q-1)^2)$ and let u be a positive solution of the equation*

$$\partial_s u = \Delta_{\mathbb{H}} u + u^p \quad in \quad \mathbb{H}^n \times (-\infty, 0), \tag{1.5}$$

then there exists a positive constant c independent of u, n and p such that

$$u(z,t,s) \leq c(-s)^{-\frac{1}{p-1}} \quad in \quad \mathbb{H}^n \times (-\infty, 0). \tag{1.6}$$

Remark 1.6. *In [8], [9] and [10], Giga and Kohn studied the asymptotic behavior of positive classical solutions to the equation*

$$\partial_s u = \Delta u + u^p, \quad in \quad \mathbb{R}^n \times (-\infty, 0) \tag{1.7}$$

when $1 < p < p_S$. By combining the results in [8–10], we know that if u is a positive classical solution of (1.7), then either

$$\lim_{t \to 0} (-t)^{\frac{1}{p-1}} u(0,t) = (\frac{1}{p-1})^{\frac{1}{p-1}}$$

or 0 is not a blow up point. It will be an interesting problem to know whether similar result holds for the equation (1.1). The main difficulty is that we do not know how to establish a monotonicity formula for the equation (1.1) so far.

The content of this paper will be organized as follows. In Section 2, we derive a family of integral estimates relating any smooth function with its gradient and its sublaplacian. In section 3, we give the proof of main results by using the estimates established in Section 2.

2. An integral estimate based on the Bochner formula

In this section, we provide a family of integral estimates for smooth solutions. The proof relies on the Bochner identity.

Lemma 2.1. *Let u be a real smooth function, then*

$$\sum_{j=1}^{n}(|\nabla u|^2)_{j\bar{j}}=\sum_{i,j=1}^{n}(u_{ij\bar{j}}u_{\bar{i}}+u_iu_{\bar{i}j\bar{j}}+u_{ij}u_{\bar{i}\bar{j}}+u_{i\bar{j}}u_{\bar{i}j}). \tag{2.1}$$

Proof. The proof is straightforward, so we will omit the details. □

Since u is real, we know from Lemma 2.1 that

$$\sum_{j=1}^{n}(|\nabla u|^2)_{\bar{j}j}=\sum_{i,j=1}^{n}(u_{\bar{i}\bar{j}j}u_{i}+u_{\bar{i}}u_{i\bar{j}j}+u_{ij}u_{\bar{i}\bar{j}}+u_{i\bar{j}}u_{\bar{i}j}). \tag{2.2}$$

By combining (2.1) and (2.2), we have the following lemma.

Lemma 2.2. *Let u be a real smooth function, then*

$$\begin{aligned}\frac{1}{2}\Delta_{\mathbb{H}}(|\nabla u|^2)=&\frac{1}{2}\sum_{i,j=1}^{n}(u_{ij\bar{j}}u_{\bar{i}}+u_iu_{\bar{i}j\bar{j}}+u_{\bar{i}\bar{j}j}u_i+u_{\bar{i}}u_{i\bar{j}j})\\&+\sum_{i,j=1}^{n}(u_{ij}u_{\bar{i}\bar{j}}+u_{i\bar{j}}u_{\bar{i}j}).\end{aligned} \tag{2.3}$$

Remark 2.3. *The Bochner formula is used in [12] to derive an analogue of the Lichnerowicz estimate for the sublaplacian on a pseudo-hermitian manifold.*

In order to continue the computation, we define

$$E^u_{i\bar{j}}=:u_{i\bar{j}}-\frac{1}{n}\sum_{\alpha=1}^{n}u_{\alpha\bar{\alpha}}\delta_{i\bar{j}}, \tag{2.4}$$

then

$$\begin{aligned}\sum_{i,j=1}^{n}|E^u_{i\bar{j}}|^2&=\sum_{i,j=1}^{n}(u_{i\bar{j}}-\frac{1}{n}\sum_{\alpha=1}^{n}u_{\alpha\bar{\alpha}}\delta_{i\bar{j}})(u_{\bar{i}j}-\frac{1}{n}\sum_{\alpha=1}^{n}u_{\bar{\alpha}\alpha}\delta_{\bar{i}j})\\&=\sum_{i,j=1}^{n}u_{i\bar{j}}u_{\bar{i}j}-\frac{1}{n}(\sum_{\alpha=1}^{n}u_{\alpha\bar{\alpha}})(\sum_{\alpha=1}^{n}u_{\bar{\alpha}\alpha})\\&=\sum_{i,j=1}^{n}u_{i\bar{j}}u_{\bar{i}j}-\frac{1}{n}(n\sqrt{-1}u_0+\frac{1}{2}\Delta_{\mathbb{H}}u)(-n\sqrt{-1}u_0+\frac{1}{2}\Delta_{\mathbb{H}}u)\\&=\sum_{i,j=1}^{n}u_{i\bar{j}}u_{\bar{i}j}-\frac{1}{n}(n^2u_0^2+\frac{1}{4}(\Delta_{\mathbb{H}}u)^2).\end{aligned} \tag{2.5}$$

By taking (2.5) into (2.3), we have the following lemma.

Lemma 2.4. *Let u be a real smooth function, then*

$$\begin{aligned}\frac{1}{2}\Delta_{\mathbb{H}}(|\nabla u|^2) =&\frac{1}{2}\sum_{i,j=1}^{n}(u_{ij\bar{j}}u_{\bar{i}} + u_i u_{\bar{i}j\bar{j}} + u_{\bar{i}\bar{j}j}u_i + u_{\bar{i}}u_{i\bar{j}j}) \\ &+ \sum_{i,j=1}^{n}(u_{ij}u_{\bar{i}\bar{j}} + |E^u_{i\bar{j}}|^2) + \frac{1}{n}(n^2u_0^2 + \frac{1}{4}(\Delta_{\mathbb{H}}u)^2).\end{aligned} \tag{2.6}$$

Let $\phi \in C_0^\infty(\mathbb{H}^n), 0 \le \phi \le 1$. Multiplying the both sides of (2.6) by $\phi^q v^{r+2k+2}$, where $v = u^{-\frac{1}{k}}$ and q, k, r are to be determined. It follows from (2.6) that

$$0 = -I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$\begin{aligned}I_1 =&\frac{1}{2}\int \Delta_{\mathbb{H}}(|\nabla u|^2)\phi^q v^{r+2k+2},\\ I_2 =&\frac{1}{2}\sum_{i,j=1}^{n}\int(u_{\bar{i}j\bar{j}}u_i + u_{i\bar{j}j}u_{\bar{i}})\phi^q v^{r+2k+2},\\ I_3 =& \sum_{i,j=1}^{n}\int |E^u_{i\bar{j}}|^2\phi^q v^{r+2k+2},\\ I_4 =&\frac{1}{n}\int(n^2u_0^2 + \frac{1}{4}(\Delta_{\mathbb{H}}u)^2)\phi^q v^{r+2k+2},\\ I_5 =&\frac{1}{2}\sum_{i,j=1}^{n}\int(u_{ij}u_{\bar{i}\bar{j}} + u_i u_{\bar{i}\bar{j}j} + u_{ij}u_{\bar{i}\bar{j}} + u_{\bar{i}}u_{ij\bar{j}})\phi^q v^{r+2k+2}.\end{aligned}$$

We will compute the integrals terms by terms.

$$\begin{aligned}I_1 =&\frac{1}{2}\int \Delta_{\mathbb{H}}(|\nabla u|^2)\phi^q v^{r+2k+2}\\ =&\frac{1}{2}k^2\int v^r|\nabla v|^2\Delta_{\mathbb{H}}(\phi^q) - \frac{1}{2}k(r+2k+2)\int v^{r+k}\phi^q|\nabla v|^2\Delta_{\mathbb{H}}u\\ &+ k^2(r+2k+2)\int v^{r-1}|\nabla v|^2[(\phi^q)_i v_{\bar{i}} + (\phi^q)_{\bar{i}}v_i]\\ &+ k^2(r+2k+2)(r+3k+2)\int v^{r-2}\phi^q|\nabla v|^4.\end{aligned} \tag{2.7}$$

$$\begin{aligned}I_2 =&\frac{1}{2}\sum_{i,j=1}^{n}\int(u_{\bar{i}j\bar{j}}u_i + u_{i\bar{j}j}u_{\bar{i}})\phi^q v^{r+2k+2}\\ =&\frac{1}{2}\sum_{i,j=1}^{n}\int[(u_{\bar{j}j\bar{i}} + 2\sqrt{-1}\delta_{j\bar{j}}u_{0\bar{i}} - 2\sqrt{-1}\delta_{j\bar{i}}u_{0\bar{j}})]u_i\phi^q v^{r+2k+2}\\ &+ \frac{1}{2}\sum_{i,j=1}^{n}\int[(u_{j\bar{j}i} - 2\sqrt{-1}\delta_{j\bar{j}}u_{0i} + 2\sqrt{-1}\delta_{i\bar{j}}u_{0j})]u_{\bar{i}}\phi^q v^{r+2k+2}\\ =&(n-1)\sqrt{-1}\sum_{i=1}^{n}\int(u_{0\bar{i}}u_i - u_{0i}u_{\bar{i}})\phi^q v^{r+2k+2}\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\sum_{i,j=1}^{n}\int u_{\bar{j}j}(u_{i\bar{i}}\phi^q v^{r+2k+2}+u_i(\phi^q)_{\bar{i}}v^{r+2k+2}+\phi^q u_i(v^{r+2k+2})_{\bar{i}})\\
&-\frac{1}{2}\sum_{i,j=1}^{n}\int u_{j\bar{j}}(u_{\bar{i}i}\phi^q v^{r+2k+2}+u_{\bar{i}}(\phi^q)_i v^{r+2k+2}+\phi^q u_{\bar{i}}(v^{r+2k+2})_i)\\
=&2n(n-1)\int u_0^2\phi^q v^{r+2k+2}\\
&-(n-1)\sqrt{-1}\sum_{i=1}^{n}\int u_0[u_i(\phi^q)_{\bar{i}}-u_{\bar{i}}(\phi^q)_i]v^{r+2k+2}\\
&-\int(-n\sqrt{-1}u_0+\frac{1}{2}\Delta_{\mathbb{H}}u)(n\sqrt{-1}u_0+\frac{1}{2}\Delta_{\mathbb{H}}u)\phi^q v^{r+2k+2}\\
&-\frac{1}{2}\sum_{i=1}^{n}\int(-n\sqrt{-1}u_0+\frac{1}{2}\Delta_{\mathbb{H}}u)(u_i(\phi^q)_{\bar{i}}v^{r+2k+2})\\
&-\frac{1}{2}\sum_{i=1}^{n}\int(n\sqrt{-1}u_0+\frac{1}{2}\Delta_{\mathbb{H}}u)(u_{\bar{i}}(\phi^q)_i v^{r+2k+2})\\
&-\frac{1}{2}\sum_{i=1}^{n}\int(-n\sqrt{-1}u_0+\frac{1}{2}\Delta_{\mathbb{H}}u)(\phi^q u_i(v^{r+2k+2})_{\bar{i}}) \qquad (2.8)\\
&-\frac{1}{2}\sum_{i=1}^{n}\int(n\sqrt{-1}u_0+\frac{1}{2}\Delta_{\mathbb{H}}u)(\phi^q u_{\bar{i}}(v^{r+2k+2})_i)\\
=&n(n-2)k^2\int v_0^2\phi^q v^r-\frac{1}{4}\int(\Delta_{\mathbb{H}}u)^2\phi^q v^{r+2k+2}\\
&-\frac{1}{2}(n-2)k^2\sqrt{-1}\sum_{i=1}^{n}\int v_0(v_i(\phi^q)_{\bar{i}}-v_{\bar{i}}(\phi^q)_i)v^r\\
&+\frac{k}{4}\sum_{i=1}^{n}\int \Delta_{\mathbb{H}}u(v_i(\phi^q)_{\bar{i}}+v_{\bar{i}}(\phi^q)_i)v^{r+k+1}\\
&+\frac{k(r+2k+2)}{2}\int \Delta_{\mathbb{H}}uv^{r+k}\phi^q|\nabla v|^2.
\end{aligned}
$$

$$
\begin{aligned}
I_3=&\sum_{i,j=1}^{n}\int|E^u_{i\bar{j}}|^2\phi^q v^{r+2k+2}\\
=&\sum_{i,j=1}^{n}\int[k(k+1)v^{-2-k}(v_iv_{\bar{j}}-\frac{1}{n}|\nabla v|^2\delta_{i\bar{j}})-kv^{-1-k}E^v_{i\bar{j}}]\\
&\times[k(k+1)v^{-2-k}(v_{\bar{i}}v_j-\frac{1}{n}|\nabla v|^2\delta_{\bar{i}j})-kv^{-1-k}E^v_{\bar{i}j}]\phi^q v^{r+2k+2}\\
&-k^2(k+1)\sum_{i,j=1}^{n}\int v^{r-1}\phi^q(v_iv_{\bar{j}}E^v_{\bar{i}j}+v_{\bar{i}}v_jE^v_{i\bar{j}})\\
=&\sum_{i,j=1}^{n}k^2\int|E^v_{i\bar{j}}|^2\phi^q v^r+(1-\frac{1}{n})k^2(k+1)^2\int v^{r-2}\phi^q|\nabla v|^4
\end{aligned}
$$

$$
\begin{aligned}
&- k^2(k+1)\sum_{i,j=1}^{n}\int (v_{\bar i j}v_i v_{\bar j} + v_{i\bar j}v_{\bar i}v_j)\phi^q v^{r-1}\\
&+ \frac{k^2(k+1)}{n}\sum_{i,j=1}^{n}\int (\sum_{\alpha=1}^{n} v_{\bar\alpha\alpha}\delta_{\bar i j}v_i v_{\bar j} + \sum_{\alpha=1}^{n} v_{\alpha\bar\alpha}\delta_{i\bar j}v_{\bar i}v_j)\phi^q v^{r-1}\\
=&\sum_{i,j=1}^{n} k^2 \int |E^{v}_{i\bar j}|^2\phi^q v^r + (1-\frac{1}{n})k^2(k+1)^2\int v^{r-2}\phi^q|\nabla v|^4\\
&- k^2(k+1)\sum_{i,j=1}^{n}\int [|\nabla v|^2_j - v_{\bar i}v_{ij}]v_{\bar j}\phi^q v^{r-1}\\
&- k^2(k+1)\sum_{i,j=1}^{n}\int [|\nabla v|^2_{\bar j} - v_i v_{\bar i\bar j}]v_j\phi^q v^{r-1} \qquad (2.9)\\
&+ \frac{k^2(k+1)}{n}\int v^{r-1}\phi^q|\nabla v|^2\Delta_{\mathbb{H}}v\\
=&\sum_{i,j=1}^{n} k^2 \int |E^{v}_{i\bar j}|^2\phi^q v^r + (1-\frac{1}{n})k^2(k+1)^2\int v^{r-2}\phi^q|\nabla v|^4\\
&k^2(k+1)\sum_{j=1}^{n}\int |\nabla v|^2(v_{\bar j j}\phi^q v^{r-1} + v_{\bar j}(\phi^q)_j v^{r-1} + v_{\bar j}\phi^q(v^{r-1})_j)\\
&+ k^2(k+1)\sum_{j=1}^{n}\int |\nabla v|^2(v_{j\bar j}\phi^q v^{r-1} + v_j(\phi^q)_{\bar j}v^{r-1} + v_j\phi^q(v^{r-1})_{\bar j})\\
&+ \frac{k^2(k+1)}{n}\int v^{r-1}\phi^q|\nabla v|^2\Delta_{\mathbb{H}}v\\
&+ k^2(k+1)\sum_{i,j=1}^{n}\int v^{r-1}\phi^q(v_i v_j v_{\bar i\bar j} + v_{\bar i}v_{\bar j}v_{ij})\\
=&k^2\sum_{i,j=1}^{n}\int |E^{v}_{i\bar j}|^2\phi^q v^r - (1+\frac{1}{n})k(k+1)\int v^{r+k}\Delta_{\mathbb{H}}u\phi^q|\nabla v|^2\\
&+ k^2(k+1)[(k+1)(3+\frac{1}{n}) + 2(r-1)]\int v^{r-2}\phi^q|\nabla v|^4\\
&+ k^2(k+1)\sum_{j=1}^{n}\int v^{r-1}|\nabla v|^2[(\phi^q)_j v_{\bar j} + (\phi^q)_{\bar j}v_j]\\
&+ k^2(k+1)\sum_{i,j=1}^{n}\int v^{r-1}\phi^q(v_{\bar i}v_{\bar j}v_{ij} + v_j v_j v_{\bar i\bar j}).
\end{aligned}
$$

$$
\begin{aligned}
I_4 =&\frac{1}{n}\int (n^2u_0^2 + \frac{1}{4}(\Delta_{\mathbb{H}}u)^2)\phi^q v^{r+2k+2} \qquad (2.10)\\
=&nk^2\int v_0^2 v^r\phi^q + \frac{1}{4n}\int (\Delta_{\mathbb{H}}u)^2\phi^q v^{r+2k+2}.
\end{aligned}
$$

$$
I_5 = \frac{1}{2}\sum_{i,j=1}^{n}\int (u_{ij}u_{\bar i\bar j} + u_i u_{\bar i\bar j j} + u_{ij}u_{\bar i\bar j} + u_{\bar i}u_{ij\bar j})\phi^q v^{r+2k+2}
$$

$$
\begin{aligned}
&= \sum_{i,j=1}^{n} \int [-kv^{-1-k} v_{ij} + k(1+k)v^{-2-k} v_i v_i] \\
&\quad \times [-kv^{-1-k} v_{\bar{i}\bar{j}} + k(1+k)v^{-2-k} v_{\bar{i}} v_{\bar{j}}] \phi^q v^{r+2k+2} \\
&\quad + \frac{1}{2} \sum_{i,j=1}^{n} \int u_i [u_{\bar{j}j\bar{i}} - 2\sqrt{-1}\delta_{\bar{i}j} u_{\bar{j}0}] \phi^q v^{r+2k+2} \\
&\quad + \frac{1}{2} \sum_{i,j=1}^{n} \int u_{\bar{i}} [u_{j\bar{j}i} + 2\sqrt{-1}\delta_{i\bar{j}} u_{j0}] \phi^q v^{r+2k+2} \qquad (2.11) \\
&= k^2 \sum_{i,j=1}^{n} \int v^r \phi^q |v_{ij}|^2 + k^2(1+k)^2 \int v^{r-2} \phi^q |\nabla v|^4 \\
&\quad - k^2(1+k) \sum_{i,j=1}^{n} \int v^{r-1} \phi^q (v_{ij} v_{\bar{i}} v_{\bar{j}} + v_{\bar{i}\bar{j}} v_i v_j) \\
&\quad + \frac{1}{2} \sum_{i=1}^{n} \int u_i [(-n\sqrt{-1}u_0 + \frac{1}{2}\Delta_{\mathbb{H}} u)_{\bar{i}} - 2\sqrt{-1} u_{\bar{i}0}] \phi^q v^{r+2k+2} \\
&\quad + \frac{1}{2} \sum_{i=1}^{n} \int u_{\bar{i}} [(n\sqrt{-1}u_0 + \frac{1}{2}\Delta_{\mathbb{H}} u)_i + 2\sqrt{-1} u_{i0}] \phi^q v^{r+2k+2}.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\sum_{i=1}^{n} \int u_i [(-n\sqrt{-1}u_0 + \frac{1}{2}\Delta_{\mathbb{H}} u)_{\bar{i}} - 2\sqrt{-1} u_{\bar{i}0}] \phi^q v^{r+2k+2} \\
=& -(n+2)\sqrt{-1} \sum_{i=1}^{n} \int u_i u_{0\bar{i}} \phi^q v^{r+2k+2} + \frac{1}{2} \sum_{i=1}^{n} \int u_i (\Delta_{\mathbb{H}} u)_{\bar{i}} \phi^q v^{r+2k+2} \\
=& (n+2)\sqrt{-1} \int (n\sqrt{-1}u_0 + \frac{1}{2}\Delta_{\mathbb{H}} u) u_0 \phi^q v^{r+2k+2} \\
&+ (n+2)\sqrt{-1} \sum_{i=1}^{n} \int u_i u_0 (\phi^q)_{\bar{i}} v^{r+2k+2} \qquad (2.12) \\
&+ (n+2)\sqrt{-1} \sum_{i=1}^{n} \int u_i u_0 \phi^q (v^{r+2k+2})_{\bar{i}} \\
&- \frac{1}{2} \int (n\sqrt{-1}u_0 + \frac{1}{2}\Delta_{\mathbb{H}} u) \Delta_{\mathbb{H}} u \phi^q v^{r+2k+2} \\
&- \frac{1}{2} \int \Delta_{\mathbb{H}} u u_i (\phi^q)_{\bar{i}} v^{r+2k+2} - \frac{1}{2} \int \Delta_{\mathbb{H}} u u_i \phi^q (v^{r+2k+2})_{\bar{i}},
\end{aligned}
$$

we have

$$
\begin{aligned}
I_5 =& k^2 \sum_{i,j=1}^{n} \int v^r \phi^q |v_{ij}|^2 + k^2(1+k)^2 \int v^{r-2} \phi^q |\nabla v|^4 \\
&- k^2(1+k) \sum_{i,j=1}^{n} \int v^{r-1} \phi^q (v_{ij} v_{\bar{i}} v_{\bar{j}} + v_{\bar{i}\bar{j}} v_i v_j) \\
&- n(n+2)k^2 \int v^r \phi^q v_0^2 - \frac{1}{4} \int (\Delta_{\mathbb{H}} u)^2 \phi^q v^{r+2k+2} \qquad (2.13)
\end{aligned}
$$

$$+\frac{1}{2}k(r+2k+2)\int \Delta_{\mathbb{H}} u v^{r+k}\phi^q|\nabla v|^2$$
$$+\frac{k}{4}\int \Delta_{\mathbb{H}} u(v_i(\phi^q)_{\bar{i}}+v_{\bar{i}}(\phi^q)_i)v^{r+k+1}$$
$$+\frac{1}{2}(n+2)k^2\sqrt{-1}\int v_0(v_i(\phi^q)_{\bar{i}}-v_{\bar{i}}(\phi^q)_i)v^r.$$

On the other hand, using integration by parts, we have

$$\begin{aligned}
I_5=&\frac{1}{2}\sum_{i,j=1}^n\int(u_{ij}u_{\bar{i}\bar{j}}+u_iu_{\bar{i}\bar{j}j}+u_{ij}u_{\bar{i}\bar{j}}+u_{\bar{i}}u_{ij\bar{j}})\phi^q v^{r+2k+2}\\
=&-\frac{1}{2}\sum_{i,j=1}^n\int u_iu_{\bar{i}\bar{j}}\phi^q(v^{r+2k+2})_j-\frac{1}{2}\sum_{i,j=1}^n\int u_{\bar{i}}u_{ij}\phi^q(v^{r+2k+2})_{\bar{j}}\\
&-\frac{1}{2}\sum_{i,j=1}^n\int u_iu_{\bar{i}\bar{j}}(\phi^q)_jv^{r+2k+2}-\frac{1}{2}\sum_{i,j=1}^n\int u_{\bar{i}}u_{ij}(\phi^q)_{\bar{j}}v^{r+2k+2}\\
=&\frac{1}{2}\sum_{i,j=1}^n\int u_{i\bar{i}}u_{\bar{j}}(\phi^q)_jv^{r+2k+2}+\frac{1}{2}\sum_{i,j=1}^n\int u_iu_{\bar{j}}(\phi^q)_{j\bar{i}}v^{r+2k+2}\\
&+\frac{1}{2}\sum_{i,j=1}^n\int u_iu_{\bar{j}}(\phi^q)_j(v^{r+2k+2})_{\bar{i}}+\frac{1}{2}\sum_{i,j=1}^n\int u_{\bar{i}i}u_j(\phi^q)_{\bar{j}}v^{r+2k+2}\\
&+\frac{1}{2}\sum_{i,j=1}^n\int u_{\bar{i}}u_j(\phi^q)_{\bar{j}i}v^{r+2k+2}+\frac{1}{2}\sum_{i,j=1}^n\int u_{\bar{i}}u_j(\phi^q)_{\bar{j}}(v^{r+2k+2})_i\\
&-\frac{1}{2}\sum_{i,j=1}^n\int u_iu_{\bar{i}\bar{j}}\phi^q(v^{r+2k+2})_j-\frac{1}{2}\sum_{i,j=1}^n\int u_{\bar{i}}u_{ij}\phi^q(v^{r+2k+2})_{\bar{j}}\\
=&\frac{1}{2}\sum_{j=1}^n\int[-kn\sqrt{-1}v^{-1-k}v_0+\frac{1}{2}\Delta_{\mathbb{H}}u](-kv^{-1-k}v_{\bar{j}})(\phi^q)_jv^{r+2k+2}\\
&+\frac{1}{2}\sum_{j=1}^n\int[kn\sqrt{-1}v^{-1-k}v_0+\frac{1}{2}\Delta_{\mathbb{H}}u](-kv^{-1-k}v_j)(\phi^q)_{\bar{j}}v^{r+2k+2}\qquad(2.14)\\
&+\frac{1}{2}k^2\sum_{i,j=1}^n\int(v_iv_{\bar{j}}(\phi^q)_{j\bar{i}}+v_{\bar{i}}v_j(\phi^q)_{\bar{j}i})v^r\\
&+\frac{1}{2}k^2(r+2k+2)\sum_{i=1}^n\int|\nabla v|^2(v_{\bar{i}}(\phi^q)_i+v_i(\phi^q)_{\bar{i}})v^{r-1}\\
&\frac{1}{2}k(r+2k+2)\sum_{i,j=1}^n\int[-kv^{-1-k}v_{\bar{i}\bar{j}}+k(k+1)v^{-2-k}v_{\bar{i}}v_{\bar{j}}]\phi^qv^{r+k}v_iv_j\\
&\frac{1}{2}k(r+2k+2)\sum_{i,j=1}^n\int[-kv^{-1-k}v_{ij}+k(k+1)v^{-2-k}v_iv_j]\phi^qv^{r+k}v_{\bar{i}}v_{\bar{j}}\\
=&\frac{1}{2}k^2n\sqrt{-1}\sum_{i=1}^n\int v^r((\phi^q)_iv_{\bar{i}}-(\phi^q)_iv_{\bar{i}})v_0
\end{aligned}$$

$$
\begin{aligned}
&-\frac{k}{4}\sum_{i=1}^{n}\int v^{r+k+1}\Delta_{\mathbb{H}}u((\phi^q)_i v_{\bar{i}}+(\phi^q)_{\bar{i}}v_i)\\
&+\frac{k^2}{2}\sum_{i,j=1}^{n}\int v^r((\phi^q)_{j\bar{i}}v_i v_{\bar{j}}+(\phi^q)_{\bar{j}i}v_{\bar{i}}v_j)\\
&+\frac{k^2}{2}(r+2k+2)\sum_{i=1}^{n}\int v^{r-1}|\nabla v|^2(v_{\bar{i}}(\phi^q)_i+v_i(\phi^q)_{\bar{i}})\\
&+k^2(k+1)(r+2k+2)\int v^{r-2}\phi^q|\nabla v|^4\\
&-\frac{k^2}{2}(r+2k+2)\int v^{r-1}\phi^q(v_{\bar{i}\bar{j}}v_i v_j+v_{ij}v_{\bar{i}\bar{j}}).
\end{aligned}
$$

We write I_5 as η times (2.13) plus $(1-\eta)$ times (2.14), where $\eta\in(0,1)$ is a positive constant. From the above computations, we can get that

$$0=-I_1+I_2+I_3+I_4+\eta I_5+(1-\eta)I_5$$

is equivalent to

$$
\begin{aligned}
&k^2\sum_{i,j=1}^{n}\int|E^v_{i\bar{j}}|^2\phi^q v^r+\lambda_1\int v^{r-2}\phi^q|\nabla v|^4+\lambda_2\int(\Delta_{\mathbb{H}}u)^2\phi^q v^{r+2k+2}\\
&+\lambda_3\int v^r\phi^q v_0^2+\lambda_4\sum_{i,j=1}^{n}\int v^r\phi^q|v_{ij}|^2+\lambda_5\int(\Delta_{\mathbb{H}}u)\phi^q|\nabla v|^2 v^{r+k}\\
=&\lambda_6\sum_{i=1}^{n}\int v^{r-1}|\nabla v|^2[(\phi^q)_i v_{\bar{i}}+(\phi^q)_{\bar{i}}v_i]+\lambda_7\int v^r|\nabla v|^2\Delta_{\mathbb{H}}(\phi^q) \qquad (2.15)\\
&+\lambda_8\sum_{i,j=1}^{n}\int v^r[(\phi^q)_{j\bar{i}}v_i v_{\bar{j}}+(\phi^q)_{\bar{j}i}v_{\bar{i}}v_j]\\
&+\lambda_9\sum_{i=1}^{n}\int v^r[(\phi^q)_i v_{\bar{i}}-(\phi^q)_{\bar{i}}v_i]v_0\\
&+\lambda_{10}\sum_{i,j=1}^{n}\int v^{r-1}\phi^q(v_{\bar{i}}v_{\bar{j}}v_{ij}+v_i v_j v_{\bar{i}\bar{j}})\\
&+\lambda_{11}\sum_{i=1}^{n}\int[v_{\bar{i}}(\phi^q)_i+v_i(\phi^q)_{\bar{i}}]\Delta_{\mathbb{H}}u v^{r+1+k},
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda_1=&k^2[(\frac{1}{n}-1-\eta)(k+1)^2-(2+\eta)(k+1)r-r(r-1)],\\
\lambda_2=&-\frac{n-1}{4n}-\frac{\eta}{4},\\
\lambda_3=&n(n-1)k^2-\eta n(n+2)k^2,\\
\lambda_4=&\eta k^2,\\
\lambda_5=&-(1+\frac{1}{n})k(k+1)+k(r+2k+2)(1+\frac{\eta}{2}),
\end{aligned}
$$

$$\begin{aligned}\lambda_6 =&k^2[\frac{r+2k+2}{2}(1+\eta)-(k+1)],\\ \lambda_7 =&k^2/2,\\ \lambda_8 =&-\frac{k^2}{2}(1-\eta),\\ \lambda_9 =&k^2\sqrt{-1}[(n+1)\eta-(n-1)],\\ \lambda_{10} =&\frac{k^2 r}{2}(1-\eta),\\ \lambda_{11} =&\frac{k}{2}\eta.,\end{aligned} \tag{2.16}$$

3. The proof of the main results

In this section, we turn to the proof of the main results. The proof is the analogue of the one used in [24] (Theorem 21.1) to prove the Liouville-type theorem and the local estimates for the equation (1.2).

Proof of Theorem 1.1. Let u be a positive solution of the equation

$$\partial_s u = \Delta_{\mathbb{H}} u + u^p, \quad \text{in} \quad \mathbb{H}^n \times (-\infty, \infty) \tag{3.1}$$

and let ϕ be a smooth function such that

$$\phi(z,t,s) = \begin{cases} 1 \quad \text{in} \quad \{|(z,t)|_{\mathbb{H}} \le \frac{1}{2}\} \times (-\frac{1}{2}, \frac{1}{2}), \\ 0 \quad \text{in} \quad \mathbb{H}^n \times (-\infty,\infty)\backslash\{|(z,t)|_{\mathbb{H}} \le 1\} \times (-1,1), \\ 0 \le \phi \le 1 \quad \text{in} \quad \{|(z,t)|_{\mathbb{H}} \le 1\} \times (-1,1). \end{cases}$$

In the rest of this section, we use the notation $\iint = \int_{\infty}^{\infty}\int_{\mathbb{H}}$ for simplicity. It follows from (3.1) that

$$\begin{aligned}&\iint (\Delta_{\mathbb{H}} u)^2 \phi^q v^{r+2k+2}\\ =&k^2 \iint (\partial_s v)^2 \phi^q v^r + \iint \phi^q v^{r+2k+2-2pk} + 2k \iint \partial_s v \phi^q v^{r+k+1-pk}\\ =&k^2 \iint (\partial_s v)^2 \phi^q v^r + \iint \phi^q v^{r+2k+2-2pk}\\ &-\frac{2kq}{r+k+2-pk}\iint \phi^{q-1}\partial_s\phi v^{r+k+2-pk}.\end{aligned} \tag{3.2}$$

Since

$$\begin{aligned}&\iint \phi^q |\nabla v|^2 v^{r+k-pk}\\ =&\frac{1}{2(r+k+1-pk)}\sum_{i=1}^n \iint \phi^q v_i (v^{r+k+1-pk})_{\bar{i}}\\ &+\frac{1}{2(r+k+1-pk)}\sum_{i=1}^n \iint \phi^q v_{\bar{i}} (v^{r+k+1-pk})_i\\ =&-\frac{1}{2(r+k+1-pk)}\iint \phi^q v^{r+k+1-pk}\Delta_{\mathbb{H}} v\\ &-\frac{1}{2(r+k+1-pk)}\sum_{i=1}^n \iint v^{r+k+1-pk}(v_i(\phi^q)_{\bar{i}} + v_{\bar{i}}(\phi^q)_i)\end{aligned} \tag{3.3}$$

$$
\begin{aligned}
&= -\frac{1}{2(r+k+1-pk)}\iint \phi^q v^{r+k+1-pk}(\partial_s v + 2(1+k)\frac{|\nabla v|^2}{v} + \frac{1}{k}v^{1+k-pk})\\
&\quad -\frac{1}{2(r+k+1-pk)}\sum_{i=1}^{n}\iint v^{r+k+1-pk}(v_i(\phi^q)_{\bar{i}} + v_{\bar{i}}(\phi^q)_i)\\
&= -\frac{1}{2(r+k+1-pk)}\sum_{i=1}^{n}\iint v^{r+k+1-pk}(v_i(\phi^q)_{\bar{i}} + v_{\bar{i}}(\phi^q)_i)\\
&\quad -\frac{1}{2k(r+k+1-pk)}\iint \phi^q v^{r+2+2k-2pk}\\
&\quad -\frac{k+1}{r+k+1-pk}\iint \phi^q v^{r+k-pk}|\nabla v|^2\\
&\quad +\frac{q}{2(r+k+1-pk)(r+k+2-pk)}\iint \phi^{q-1}\partial_s\phi v^{r+2+k-pk},
\end{aligned}
$$

we conclude that

$$
\begin{aligned}
&\iint \phi^q|\nabla v|^2 v^{r+k-pk}\\
&= -\frac{1}{2(r+2k+2-pk)}\sum_{i=1}^{n}\iint v^{r+k+1-pk}(v_i(\phi^q)_{\bar{i}} + v_{\bar{i}}(\phi^q)_i)\\
&\quad -\frac{1}{2k(r+2k+2-pk)}\iint \phi^q v^{r+2+2k-2pk}\\
&\quad +\frac{q}{2(r+2k+2-pk)(r+k+2-pk)}\iint \phi^{q-1}\partial_s\phi v^{r+2+k-pk}.
\end{aligned}
\tag{3.4}
$$

Therefore,

$$
\begin{aligned}
&\iint \Delta_{\mathbb{H}} u\phi^q|\nabla v|^2 v^{r+k}\\
&= -k\iint \partial_s v\phi^q|\nabla v|^2 v^{r-1} - \iint \phi^q|\nabla v|^2 v^{r+k-pk}\\
&= -k\iint \partial_s v\phi^q|\nabla v|^2 v^{r-1}\\
&\quad +\frac{1}{2(r+2k+2-pk)}\sum_{i=1}^{n}\iint v^{r+k+1-pk}(v_i(\phi^q)_{\bar{i}} + v_{\bar{i}}(\phi^q)_i)\\
&\quad +\frac{1}{2k(r+2k+2-pk)}\iint \phi^q v^{r+2+2k-2pk}\\
&\quad -\frac{q}{2(r+2k+2-pk)(r+k+2-pk)}\iint \phi^{q-1}\partial_s\phi v^{r+2+k-pk}.
\end{aligned}
\tag{3.5}
$$

By (2.15), (2.16), (3.2) and (3.4), we have

$$
\begin{aligned}
&k^2\sum_{i,j=1}^{n}\iint |E^v_{i\bar{j}}|^2\phi^q v^r + \lambda_1\iint v^{r-2}\phi^q|\nabla v|^4 + \lambda_2'\iint \phi^q v^{r+2k+2-2pk}\\
&+\lambda_3\iint v^r\phi^q v_0^2 + \lambda_4\sum_{i,j=1}^{n}\iint v^r\phi^q|v_{ij}|^2
\end{aligned}
$$

$$
\begin{aligned}
=&\lambda_6 \sum_{i=1}^n \iint v^{r-1}|\nabla v|^2[(\phi^q)_i v_{\bar{i}} + (\phi^q)_{\bar{i}} v_i] + \lambda_7 \iint v^r |\nabla v|^2 \Delta_{\mathbb{H}}(\phi^q)\\
&+ \lambda_8 \sum_{i,j=1}^n \iint v^r[(\phi^q)_{j\bar{i}} v_i v_{\bar{j}} + (\phi^q)_{\bar{j}i} v_{\bar{i}} v_j]\\
&+ \lambda_9 \sum_{i=1}^n \iint v^r[(\phi^q)_i v_{\bar{i}} - (\phi^q)_{\bar{i}} v_i] v_0 \qquad (3.6)\\
&+ \lambda_{10} \sum_{i,j=1}^n \iint v^{r-1}\phi^q(v_{\bar{i}} v_{\bar{j}} v_{ij} + v_i v_j v_{\bar{i}\bar{j}})\\
&- \lambda_{11} k \sum_{i=1}^n \iint [v_{\bar{i}}(\phi^q)_i + v_i(\phi^q)_{\bar{i}}]\partial_s v v^r\\
&+ \lambda'_{11} \sum_{i=1}^n \iint v^{r+1+k-pk}((\phi^q)_{\bar{i}} v_i + (\phi^q)_i v_{\bar{i}})\\
&- \lambda_2 k^2 \iint (\partial_s v)^2 \phi^q v^r + \lambda_5 k \iint \partial_s v \phi^q |\nabla v|^2 v^{r-1}\\
&+ \lambda_{12} \iint \phi^{q-1}\partial_s \phi v^{r+k+2-pk},
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda'_2 &= \lambda_2 + \frac{1}{2k(r+2k+2-pk)}\lambda_5,\\
\lambda'_{11} &= \lambda_{11} - \frac{1}{2(r+2k+2-pk)}\lambda_5, \qquad (3.7)\\
\lambda_{12} &= \frac{1}{r+k+2-pk}[2\lambda_2 kq + \frac{\lambda_5 q}{2(r+2k+2-pk)}]. \qquad (3.8)
\end{aligned}
$$

We estimate the right hand side of (3.6). By using Young's inequality, we have

$$
\begin{aligned}
&\lambda_6 \sum_{i=1}^n \iint v^{r-1}|\nabla v|^2[(\phi^q)_i v_{\bar{i}} + (\phi^q)_{\bar{i}} v_i]\\
=&q\lambda_6 \sum_{i=1}^n \iint v^{r-1}|\nabla v|^2 \phi^{q-1}(\phi_i v_{\bar{i}} + \phi_{\bar{i}} v_i) \qquad (3.9)\\
\le&\epsilon \iint v^{r-2}\phi^q|\nabla v|^4 + c\iint v^{r+2}\phi^{q-4}|\nabla\phi|^4
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\lambda_7 \iint v^r|\nabla v|^2 \Delta_{\mathbb{H}}(\phi^q)\\
\le&\epsilon \iint v^{r-2}\phi^q|\nabla v|^4 + c\iint v^{r+2}\phi^{q-4}(|\Delta_{\mathbb{H}}\phi| + |\nabla\phi|^2)^2. \qquad (3.10)\\
&\lambda_8 \sum_{i,j=1}^n \iint v^r[(\phi^q)_{j\bar{i}} v_i v_{\bar{j}} + (\phi^q)_{\bar{j}i} v_{\bar{i}} v_j]\\
\le&\epsilon \iint v^{r-2}\phi^q|\nabla v|^4 + c\iint v^{r+2}\phi^{q-4}(|\nabla^2\phi| + |\nabla\phi|^2)^2, \qquad (3.11)
\end{aligned}
$$

$$
\begin{aligned}
&\lambda_9 \sum_{i=1}^n \iint v^r[(\phi^q)_i v_{\bar{i}} - (\phi^q)_{\bar{i}} v_i] v_0 \\
\leq &\epsilon \iint v^r v_0^2 \phi^q + \epsilon \iint v^{r-2}\phi^q |\nabla v|^4 + c \iint v^{r+2}\phi^{q-4}|\nabla \phi|^4, \qquad (3.12)\\
&\lambda_{10} \sum_{i,j=1}^n \iint v^{r-1}\phi^q (v_{\bar{i}} v_{\bar{j}} v_{ij} + v_i v_j v_{\bar{i}\bar{j}}) \\
\leq &k^2 \eta \sum_{i,j=1}^n \iint v^r \phi^q |v_{ij}|^2 + \frac{k^2 r^2 (1-\eta)^2}{4\eta} \iint v^{r-2}\phi^q |\nabla v|^4. \qquad (3.13)
\end{aligned}
$$

Next, we have

$$
\begin{aligned}
&-\lambda_{11} k \sum_{i=1}^n \iint [v_{\bar{i}}(\phi^q)_i + v_i(\phi^q)_{\bar{i}}] \partial_s v v^r \\
\leq &\epsilon \iint v^{r-2}\phi^q |\nabla v|^4 + c \iint v^{r+2}\phi^{q-4}|\nabla\phi|^4 + \epsilon \iint (\partial_s v)^2 \phi^q v^r, \qquad (3.14)\\
&\lambda_5 k \iint \partial_s v \phi^q |\nabla v|^2 v^{r-1} \\
\leq &\epsilon \iint v^{r-2}\phi^q |\nabla v|^4 + c \iint (\partial_s v)^2 \phi^q v^r. \qquad (3.15)\\
&\lambda_{12} \iint \phi^{q-1}\partial_s \phi v^{r+k+2-pk} \\
\leq &\epsilon \iint \phi^q v^{r+2k+2-2pk} + c \iint \phi^{q-2}(\partial_s\phi)^2 v^{r+2}, \qquad (3.16)\\
&\lambda_{11}' \sum_{i=1}^n \iint v^{r+1+k-pk}((\phi^q)_{\bar{i}} v_i + (\phi^q)_i v_{\bar{i}}) \\
\leq &\epsilon \iint \phi^q v^{r+2k+2-2pk} + c \iint v^{r+2}\phi^{q-4}(|\Delta_{\mathbb{H}}\phi| + |\nabla\phi|^2)^2 \qquad (3.17)
\end{aligned}
$$

Finally, we estimate $\iint (\partial_s v)^2 \phi^q v^r$. By combining (3.1) and integration by part, we have

$$
\begin{aligned}
&\iint (\partial_s v)^2 \phi^q v^r \\
= &\iint (\partial_s u)^2 \phi^q v^{r+2k+2} \\
= &\iint (\Delta_{\mathbb{H}} u + u^p)\phi^q u^{-\frac{r+2k+2}{k}} \partial_s u \\
= &\iint (\sum_{i=1}^n (u_{i\bar{i}} + u_{\bar{i}i}) + u^p)\phi^q u^{-\frac{r+2k+2}{k}} \partial_s u \\
= &-q \iint \phi^{q-1} v^r \nabla v \nabla \phi \partial_s v + q \iint \phi^{q-1} v^r |\nabla v|^2 \partial_s \phi \qquad (3.18)\\
&- \frac{q}{p+1-\frac{r+2k+2}{k}} \iint \phi^{q-1}\partial_s\phi v^{r+2+k-pk} \\
\leq &\epsilon \iint (\partial_s v)^2 v^r \phi^q + \epsilon \iint |\nabla v|^4 v^{r-2}\phi^q
\end{aligned}
$$

$$
\begin{aligned}
&+\epsilon \iint \phi^q v^{r+2k+2-2pk} + c\iint \phi^{q-2}(\partial_s\phi)^2 v^{r+2} \\
&+ c\iint \phi^{q-4}|\nabla\phi|^4 v^{r+2}.
\end{aligned}
\tag{3.19}
$$

By the above estimates, we conclude that

$$
\begin{aligned}
&(\lambda_1 - \frac{k^2r^2(1-\eta)^2}{4\eta} - \epsilon)\iint v^{r-2}\phi^q|\nabla v|^4 \\
&+(\lambda_2' - \epsilon)\iint \phi^q v^{r+2k+2-2pk} + (\lambda_3-\epsilon)\iint v^r\phi^q v_0^2 \\
\le& c\iint \phi^{q-4}v^{r+2}(|\nabla^2\phi| + |\Delta_{\mathbb{H}}\phi| + |\nabla\phi|^2 + |\partial_s\phi|)^2.
\end{aligned}
\tag{3.20}
$$

If we choose $k<0$, then we know from (3.20) that

$$
\begin{aligned}
&(\lambda_1 - \frac{k^2r^2(1-\eta)^2}{4\eta} - \epsilon)\iint v^{r-2}\phi^q|\nabla v|^4 \\
&+(\lambda_2' - \epsilon)\iint \phi^q v^{r+2k+2-2pk} + (\lambda_3-\epsilon)\iint v^r\phi^q v_0^2 \\
\le& c\iint \phi^{\mu_2}(|\nabla^2\phi| + |\Delta_{\mathbb{H}}\phi| + |\nabla\phi|^2 + |\partial_s\phi|)^{2s_2},
\end{aligned}
\tag{3.21}
$$

where

$$
\begin{aligned}
s_2 &= \frac{r+2k+2-2pk}{2(k-pk)}, \\
\mu_2 &= (q-4-\frac{q(r+2)}{r+2k+2-2pk})s_2.
\end{aligned}
$$

We choose $(r+2)/(k(1-p)) > 0$, then $s_2>0$. Moreover, $\mu_2>0$ provided that q is large enough. For every $R>1$, we take

$$
\phi_R(z,t,s) = \phi(\frac{z}{R}, \frac{t}{R^2}, \frac{s}{R^2})
$$

into (3.21), then

$$
\begin{aligned}
&(\lambda_1 - \frac{k^2r^2(1-\eta)^2}{4\eta} - \epsilon)\iint v^{r-2}\phi_R^q|\nabla v|^4 \\
&+(\lambda_2' - \epsilon)\iint \phi_R^q v^{r+2k+2-2pk} + (\lambda_3-\epsilon)\iint v^r\phi_R^q v_0^2 \\
\le& cR^{2n+4-4s_2}.
\end{aligned}
\tag{3.22}
$$

Let $0<\eta<(n-1)(n+2)$, then $\lambda_3>0$. In order to finish the proof, it is enough to choose $k<0, r>0, p>1$ such that

$$
\lambda_1 - \frac{k^2r^2(1-\eta)^2}{4\eta} > 0, \tag{3.23}
$$

$$
\lambda_2 + \frac{1}{2k(r+2k+2-pk)}\lambda_5 > 0, \tag{3.24}
$$

$$
2n+4-4s_2 < 0. \tag{3.25}
$$

We set $y = 1 + 1/k, \delta = -r/k$. Then (3.23), (3.24), (3.25) is equivalent to

$$(1+\eta-\frac{1}{n})y^2-(1+\eta)\delta y+[\frac{(1+\eta)^2}{4\eta}\delta^2-\delta]<0, \tag{3.26}$$

$$\frac{1}{-\delta+2y-p}[-\delta(1+\frac{1}{n})+(1+\eta-\frac{1}{n})p]>0, \tag{3.27}$$

$$n+2-\frac{-\delta+2y-2p}{1-p}<0. \tag{3.28}$$

Similar to [14], we need

$$\Delta_y=(1+\eta)^2\delta^2-4(1+\eta-\frac{1}{n})[\frac{(1+\eta)^2}{4\eta}\delta^2-\delta]>0 \tag{3.29}$$

or

$$0<\delta<\frac{4(n+n\eta-1)\eta}{(n-1)(1+\eta)^2}. \tag{3.30}$$

If (3.30) holds, we can choose y such that

$$y_1:=\frac{(1+\eta)\delta-\sqrt{\Delta_y}}{2(1+\eta-\frac{1}{n})}<y<\frac{(1+\eta)\delta+\sqrt{\Delta_y}}{2(1+\eta-\frac{1}{n})}:=y_2 \tag{3.31}$$

and

$$y_1<y<y_2<\frac{\delta+p}{2} \tag{3.32}$$

provided that $p > 1, n > 2$ and $0 < \eta < (n-1)(n+2)$ hold. In order that (3.27) and (3.28) hold, we need

$$1<p<\frac{n+1}{n+n\eta-1}\delta \tag{3.33}$$

and

$$1<p<\frac{n+2+\delta-2y}{n}. \tag{3.34}$$

We take $y = ((1+\eta)\delta)/(2(1+\eta-1/n))$, then (3.34) is equivalent to

$$1<p<1+\frac{2}{n}-\frac{\delta}{n(n+n\eta-1)}. \tag{3.35}$$

It is easy to check that

$$\frac{n+1}{n+n\eta-1}\delta=\frac{4(n+1)(n+2)}{(2n+1)^2},$$
$$1+\frac{2}{n}-\frac{\delta}{n(n+n\eta-1)}=\frac{(n+2)(4n^2+4n-3)}{n(2n+1)^2}$$

provided that

$$\eta=\frac{n-1}{n+2},\quad \delta=\frac{4(n+n\eta-1)\eta}{(n-1)(1+\eta)^2}.$$

Moreover, we can check that

$$\frac{(n+2)(4n^2+4n-3)}{n(2n+1)^2}<\frac{4(n+1)(n+2)}{(2n+1)^2}.$$

Since δ satisfies (3.30) and η satisfies $0 < \eta < (n-1)(n+2)$. For every

$$1 < p < \frac{(n+2)(4n^2+4n-3)}{n(2n+1)^2} = 1 + \frac{4(Q^2-3Q-1)}{(Q-2)(Q-1)^2},$$

there exist two small constants θ_1 and θ_2 such that (3.33) and (3.35) hold if we take

$$\eta = \frac{n-1}{n+2} - \theta_1, \quad \delta = \frac{8(n^2-1)}{(2n-1)^2} - \theta_2.$$

Moreover, if θ_1 and θ_2 are small enough, we have $y < 1$. Hence the proof of Theorem 1.1 is completed. □

The proof of Corollary 1.5. The proof is essentially based on the doubling lemma proved in [20]. Since the arguments are similar to the proof Theorem 4.1 in [19], so we will not give all the details. □

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Department of Mathematics, University of British Columbia, Vancouver, B.C., Canada, V6T 1Z2
E-mail address: jcwei@math.ubc.ca

School of Mathematics and Statistics, Wuhan University, Wuhan, Hubei, China, 430072
E-mail address: wukemail@whu.edu.cn