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LAPLACIAN VANISHING THEOREM FOR QUANTIZED SINGULAR LIOUVILLE EQUATION

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ABSTRACT. In this article we establish a vanishing theorem for singular Liouville equation with quantized singular source. If a blowup sequence tends to infinity near a quantized singular source and the blowup solutions violate the spherical Harnack inequality around the singular source (non-simple blow-ups), the Laplacian of a coefficient function must tend to zero. This seems to be the first second order estimates for Liouville equation with quantized sources and non-simple blow-ups. This result as well as the key ideas of the proof would be extremely useful for various applications.

1. INTRODUCTION

This is the third article in our series to study blowup solutions of

$$\Delta u+|x|^{2N}\mathrm{H}(x)e^{u}=0, \tag{1.1}$$

in a neighborhood of the origin in $\mathbb{R}^2$. Here H is a positive smooth function and $N\in\mathbb{N}$ is a positive integer. Since the analysis is local in nature we focus the discussion in a neighborhood of the origin: Let u_k be a sequence of solutions of

$$\Delta \mathrm{u}_k(x)+|x|^{2N}\mathrm{H}_k(x)e^{\mathrm{u}_k}=0, \quad \text{in} \quad B_\tau \tag{1.2}$$

for some $\tau>0$ independent of k. B_τ is the ball centered at the origin with radius τ. In addition we postulate the usual assumptions on u_k and H_k: For a positive constant C independent of k, the following holds:

$$\begin{cases} \|\mathrm{H}_k\|_{C^3(\bar{B}_\tau)}\le C, \quad \frac{1}{C}\le \mathrm{H}_k(x)\le C, \quad x\in \bar{B}_\tau,\\ \int_{B_\tau}\mathrm{H}_k e^{\mathrm{u}_k}\le C,\\ |\mathrm{u}_k(x)-\mathrm{u}_k(y)|\le C, \quad \forall x,y\in\partial B_\tau, \end{cases} \tag{1.3}$$

and since we study the asymptotic behavior of blowup solutions around the singular source, we assume that there is no blowup point except at the origin:

$$\max_{K\subset\subset B_\tau\setminus\{0\}} \mathrm{u}_k\le C(K). \tag{1.4}$$

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If a sequence of solutions $\{u^k\}_{k=1}^{\infty}$ of (1.1) satisfies

$$\lim_{k\to\infty} u^k(x_k) = \infty, \quad \text{for some } \bar{x} \in B_\tau \text{ and } x_k \to \bar{x},$$

we say $\{u^k\}$ is a sequence of bubbling solutions or blowup solutions, $\bar{x}$ is called a blowup point. The question we consider in this work is when 0 is the only blowup point in a neighborhood of the origin, what vanishing theorems will the coefficient functions H_k satisfy?

One indispensable assumption is that the blowup solutions violate the spherical Harnack inequality around the origin:

$$\max_{x\in B_\tau} \mathrm{u}_k(x) + 2(1+N)\log|x| \to \infty, \tag{1.5}$$

It is also mentioned in literature (see [21, 26]) that 0 is called an non-simple blowup point. The main result of this article is

Theorem 1.1. *Let $\{\mathrm{u}_k\}$ be a sequence of solutions of (1.2) such that (1.3),(1.4) hold and the spherical Harnack inequality is violated as in (1.5). Then along a sub-sequence*

$$\lim_{k\to\infty} \Delta(\log \mathrm{H}_k)(0) = 0.$$

Theorem 1.1 is a continuation of our previous result in [27]:

Theorem A: Let $\{\mathrm{u}_k\}$ be a sequence of solutions of (1.2) such that (1.3),(1.4) and (1.5) hold. Then along a subsequence

$$\lim_{k\to\infty} \nabla(\log \mathrm{H}_k + \phi_k)(0) = 0$$

where ϕ_k is defined as

$$\begin{cases} \Delta\phi_k(x) = 0, \quad in \quad B_\tau, \\ \\ \phi_k(x) = \mathrm{u}_k(x) - \frac{1}{2\pi\tau}\int_{\partial B_\tau} \mathrm{u}_k dS, \quad x \in \partial B_\tau. \end{cases} \tag{1.6}$$

The equation (1.1) comes from its equivalent form

$$\Delta v + \mathrm{H}e^v = 4\pi N\delta_0$$

by using a logarithmic function to eliminate the Dirac mass on the right hand side. Since the strength of the Dirac mass is a multiple of 4π, this type of singularity is called "quantized". An equation with a quantized singular source is ubiquitous in literature. In particular the following mean field equation defined on a Riemann surface (M,g):

$$\Delta_g u + \rho\Big(\frac{h(x)e^{u(x)}}{\int_M he^u} - \frac{1}{Vol_g(M)}\Big) = 4\pi\sum_j \alpha_j\Big(\delta_{p_j} - \frac{1}{Vol_g(M)}\Big), \tag{1.7}$$

represents a conformal metric with prescribed conic singularities (see [16, 24, 25]).If the singular source is quantized, the equation is profoundly linked to Algebraic geometry, integrable system, number theory and complex Monge-Ampere equations (see [13]). In Physics the main equation reveals key features of mean field limits of point vortices in the Euler flow [8, 9] and models in the Chern-Simons-Higgs theory [20] and in the electroweak theory [2], etc.

So far the non-simple bubbling situation has been observed in Liouville equation [21, 4], Liouville systems [18, 19, 28] and fourth order equations [1]. The main theorem in this article would impact the study of these equations as well as some well known open questions in Monge-Ampere equation [26].

When compared with Theorem A, Theorem 1.1 is clearly more challenging in analysis. As a matter of fact the proof of Theorem A is a special case of one step of the proof of Theorem 1.1. However, their major difference is on applications. Theorem 1.1 is significantly more influential for many reasons: First the main motivation to study equation (1.1) is for equations or systems defined on manifolds. Usually blowup analysis near a singular point needs to reflect the curvature at the blowup point. In this respect Theorem 1.1 is directly related to the Gauss curvature at the blowup point. Second, the harmonic function in Theorem A causes inconvenience in application since it is generally hard to identify what the harmonic function is. On the other hand Theorem 1.1 is only involved with the Laplace of the coefficient function. This may lead to substantial advances in applications: In many degree counting programs one major difficulty is bubble-coalition, which means bubbling disks may collide into one point. The formation of bubbling disks tending to one point is accurately represented by equation (1.1). Theorem 1.1 and its proof could be extremely useful to simplify blowup pictures. The Green's function of Laplace operator on manifold is generally not harmonic, the statement of Theorem 1.1 could imply that bubbling-coalition does not exist. Third, the proof of Theorem 1.1 is also important for proving uniqueness of bubbling solutions, and the results for Liouville equation with quantized singular sources is inspirational to many equations and systems with similar singular poles. As far as we know before our series of work most of the study of singular equations or systems focus on non-quantized singular situations. However it is the "quantized situations" that manifest profound connections to different fields of mathematics and Physics. Theorem 1.1 may be a starting point of multiple directions of exciting adventures.

The organization of the article is as follows. In section two we cite preliminary results related to the proof of the main theorem. Then in section three approximate the blowup solutions by a family of global solutions that agree with the blowup solutions at one local maximum point. This is crucial for our argument. Then we derive some intermediate estimates as preparation of more precise analysis. In section four we prove the first order estimates that cover the main result in [27]. This section proves stronger result than [27] and provides more detail. Finally in section five we take advantage of the result of the first order estimate and complete the proof of the main theorem.

Notation: We will use $B(x_0,r)$ to denote a ball centered at x_0 with radius r. If x_0 is the origin we use B_r. C represents a positive constant that may change from place to place.

2. Preliminary discussions

In the first stage of the proof of Theorem 1.1 we set up some notations and cite some preliminary results. For simple notation we set

$$u_k(x) = \mathrm{u}_k(x) - \phi_k(x), \quad \text{and} \tag{2.1}$$

$$h_k(x) = \mathrm{H}_k(x)e^{\phi_k(x)}. \tag{2.2}$$

to write the equation of u_k as

$$\Delta u_k(x) + |x|^{2N}h_k(x)e^{u_k} = 0, \quad \text{in} \quad B_\tau \tag{2.3}$$

Without loss of generality we assume

$$\lim_{k\to\infty} h_k(0) = 1. \tag{2.4}$$

Obviously (1.5) is equivalent to

$$\max_{x\in B_\tau} u_k(x) + 2(1+N)\log|x| \to \infty, \tag{2.5}$$

It is well known [21, 4] that u_k exhibits a non-simple blowup profile. It is established in [21, 4] that there are $N+1$ local maximum points of u_k: p_0^k,....,p_N^k and they are evenly distributed on $\mathbb{S}^1$ after scaling according to their magnitude: Suppose along a subsequence

$$\lim_{k\to\infty} p_0^k/|p_0^k| = e^{i\theta_0},$$

then

$$\lim_{k\to\infty} \frac{p_l^k}{|p_0^k|} = e^{i(\theta_0 + \frac{2\pi l}{N+1})}, \quad l = 1,...,N.$$

For many reasons it is convenient to denote $|p_0^k|$ as δ_k and define μ_k as follows:

$$\delta_k = |p_0^k| \quad \text{and} \quad \mu_k = u_k(p_0^k) + 2(1+N)\log\delta_k. \tag{2.6}$$

Also we use

$$\varepsilon_k = e^{-\frac{1}{2}\mu_k}$$

to be the scaling factor most of the time. Since p_l^k's are evenly distributed around ∂B_{δ_k}, standard results for Liouville equations around a regular blowup point can be applied to have $u_k(p_l^k) = u_k(p_0^k) + o(1)$. Also, (1.5) gives $\mu_k \to \infty$. The interested readers may look into [21, 4] for more detailed information.

Finally we shall use E to denote a frequently appearing error term of the size $O(\delta_k^2) + O(\mu_k e^{-\mu_k})$.

3. Approximating bubbling solutions by global solutions

We write p_0^k as $p_0^k = \delta_k e^{i\theta_k}$ and define v_k as

$$v_k(y) = u_k(\delta_k y e^{i\theta_k}) + 2(N+1)\log\delta_k, \quad |y| < \tau\delta_k^{-1}. \tag{3.1}$$

If we write out each component, (3.1) is

$$v_k(y_1,y_2) = u_k(\delta_k(y_1\cos\theta_k - y_2\sin\theta_k), \delta_k(y_1\sin\theta_k + y_2\cos\theta_k)) + 2(1+N)\log\delta_k.$$

Then it is standard to verify that v_k solves

(3.2) $$\Delta v_k(y)+|y|^{2N}\mathfrak{h}_k(\delta_k y)e^{v_k(y)}=0,\quad |y|<\tau/\delta_k,$$

where

(3.3) $$\mathfrak{h}_k(x)=h_k(xe^{i\theta_k}),\quad |x|<\tau.$$

Thus the image of p_0^k after scaling is $Q_1^k=e_1=(1,0)$. Let Q_1^k, Q_2^k,...,Q_N^k be the images of p_i^k $(i=1,...,N)$ after the scaling:

$$Q_l^k=\frac{p_l^k}{\delta_k}e^{-i\theta_k},\quad l=1,...,N.$$

It is established by Kuo-Lin in [21] and independently by Bartolucci-Tarantello in [4] that

(3.4) $$\lim_{k\to\infty}Q_l^k=\lim_{k\to\infty}p_l^k/\delta_k=e^{\frac{2l\pi i}{N+1}},\quad l=0,....,N.$$

Then it is proved in our previous work that (see (3.13) in [26])

$$Q_l^k-e^{\frac{2\pi li}{N+1}}=O(\mu_k e^{-\mu_k})+O(|\nabla\log\mathfrak{h}_k(0)|\delta_k).$$

Using the rate of $\nabla\mathfrak{h}_k(0)$ in [26] we have

(3.5) $$Q_l^k-e^{\frac{2\pi li}{N+1}}=O(\mu_k e^{-\mu_k})+O(\delta_k^2).$$

Choosing $3\varepsilon>0$ small and independent of k, we can make disks centered at Q_l^k with radius 3ε (denoted as $B(Q_l^k,3\varepsilon)$) mutually disjoint. Let

(3.6) $$\mu_k=\max_{B(Q_0^k,\varepsilon)}v_k.$$

Since Q_l^k are evenly distributed around ∂B_1, it is easy to use standard estimates for single Liouville equations ([30, 17, 12]) to obtain

$$\max_{B(Q_l^k,\varepsilon)}v_k=\mu_k+o(1),\quad l=1,...,N.$$

Let

(3.7) $$V_k(x)=\log\frac{e^{\mu_k}}{(1+\frac{e^{\mu_k}\mathfrak{h}_k(\delta_k e_1)}{8(1+N)^2}|y^{N+1}-e_1|^2)^2}.$$

Clearly V_k is a solution of

(3.8) $$\Delta V_k+\mathfrak{h}_k(\delta_k e_1)|y|^{2N}e^{V_k}=0,\quad \text{in}\quad \mathbb{R}^2,\quad V_k(e_1)=\mu_k.$$

This expression is based on the classification theorem of Prajapat-Tarantello [23].

The estimate of $v_k(x)-V_k(x)$ is important for the main theorem of this article. For convenience we use

$$\beta_l=\frac{2\pi l}{N+1},\quad \text{so } e_1=e^{i\beta_0}=Q_0^k,\quad e^{i\beta_l}=Q_l^k+E,\ \text{ for }\ l=1,...,N.$$

4. VANISHING OF THE FIRST DERIVATIVES

Our first goal is to prove the following vanishing rate for $\nabla\mathfrak{h}_k(0)$:

Theorem 4.1.

$$\nabla(\log\mathfrak{h}_k)(0) = O(\delta_k\mu_k) \tag{4.1}$$

Proof of Theorem 4.1:

Note that we have proved in [26] that

$$\nabla(\log\mathfrak{h}_k)(0) = O(\delta_k^{-1}\mu_k e^{-\mu_k}) + O(\delta_k).$$

If $\delta_k \geq C\varepsilon_k$, there is nothing to prove. So we assume that

$$\delta_k = o(\varepsilon_k). \tag{4.2}$$

By way of contradiction we assume that

$$|\nabla\mathfrak{h}_k(0)|/(\delta_k\mu_k) \to \infty. \tag{4.3}$$

Another observation is that based on (3.5) we have

$$\varepsilon_k^{-1}|Q_l^k - e^{i\beta_l}| \leq C\varepsilon_k^{\varepsilon}, \quad l = 0,...,N$$

for some small $\varepsilon > 0$. Thus ξ_k tends to U after scaling. We need this fact in our argument.

Under the assumption (4.2) we cite Proposition 3.1 of [27]:

Proposition 3.1 of [27]*: Let $l = 0,...,N$ and δ be small so that $B(e^{i\beta_l},\delta)\cap B(e^{i\beta_s},\delta) = \emptyset$ for $l \neq s$. In each $B(e^{i\beta_l},\delta)$*

$$|v_k(x) - V_k(x)| \leq \begin{cases} C\mu_k e^{-\mu_k/2}, & |x - e^{i\beta_l}| \leq Ce^{-\mu_k/2}, \\ C\frac{\mu_k e^{-\mu_k}}{|x-e^{i\beta_l}|} + O(\mu_k^2 e^{-\mu_k}), & Ce^{-\mu_k/2} \leq |x - e^{i\beta_l}| \leq \delta. \end{cases} \tag{4.4}$$

Remark 4.1. *We only need a re-scaled version of the Proposition above:*

$$|v_k(e^{i\beta_l} + \varepsilon_k y) - V_k(e^{i\beta_l} + \varepsilon_k y)| \leq C\varepsilon_k^{\varepsilon}(1+|y|)^{-1}, \quad 0 < |y| < \tau\varepsilon_k^{-1}. \tag{4.5}$$

for some small constants $\varepsilon > 0$ and $\tau > 0$ both independent of k,

One major step in the proof of Theorem 4.1 is the following estimate:

Proposition 4.1. *Let $w_k = v_k - V_k$, then*

$$|w_k(y)| \leq C\tilde{\delta}_k, \quad y \in \Omega_k := B(0, \tau\delta_k^{-1}),$$

where $\tilde{\delta}_k = |\nabla\mathfrak{h}_k(0)|\delta_k + \delta_k^2\mu_k$.

Proof of Proposition 4.1:

Obviously we can assume that $|\nabla\mathfrak{h}_k(0)|\delta_k > 2\delta_k^2\mu_k$ because otherwise there is nothing to prove. Now we recall the equation for v_k is (3.2), v_k is a constant on $\partial B(0, \tau\delta_k^{-1})$. Moreover $v_k(e_1) = \mu_k$. Recall that V_k defined in (3.7) satisfies

$$\Delta V_k + \mathfrak{h}_k(\delta_k e_1)|y|^{2N} e^{V_k} = 0, \quad \text{in} \quad \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |y|^{2N} e^{V_k} < \infty,$$

V_k has its local maximums at $e^{i\beta_l}$ for $l=0,...,N$ and $V_k(e_1)=\mu_k$. For $|y|\sim\delta_k^{-1}$,

$$V_k(y)=-\mu_k-4(N+1)\log\delta_k^{-1}+C+O(\delta_k^{N+1}).$$

Let $\Omega_k=B(0,\tau\delta_k^{-1})$, we shall derive a precise, point-wise estimate of w_k in $B_3\setminus\cup_{l=1}^N B(Q_l^k,\tau)$ where $\tau>0$ is a small number independent of k. Here we note that among $N+1$ local maximum points, we already have e_1 as a common local maximum point for both v_k and V_k and we shall prove that w_k is very small in B_3 if we exclude all bubbling disks except the one around e_1. Before we carry out more specific computation we emphasize the importance of

$$w_k(e_1)=|\nabla w_k(e_1)|=0. \tag{4.6}$$

Now we write the equation of w_k as

$$\Delta w_k+\mathfrak{h}_k(\delta_k y)|y|^{2N}e^{\xi_k}w_k=(\mathfrak{h}_k(\delta_k e_1)-\mathfrak{h}_k(\delta_k y))|y|^{2N}e^{V_k} \tag{4.7}$$

in Ω_k, where ξ_k is obtained from the mean value theorem:

$$e^{\xi_k(x)}=\begin{cases}\frac{e^{v_k(x)}-e^{V_k(x)}}{v_k(x)-V_k(x)}, & \text{if} \quad v_k(x)\neq V_k(x),\\ \\ e^{V_k(x)}, & \text{if} \quad v_k(x)=V_k(x).\end{cases}$$

An equivalent form is

$$e^{\xi_k(x)}=\int_0^1\frac{d}{dt}e^{tv_k(x)+(1-t)V_k(x)}dt=e^{V_k(x)}\big(1+\frac{1}{2}w_k(x)+O(w_k(x)^2)\big). \tag{4.8}$$

For convenience we write the equation for w_k as

$$\Delta w_k+\mathfrak{h}_k(\delta_k y)|y|^{2N}e^{\xi_k}w_k=\delta_k\nabla\mathfrak{h}_k(\delta_k e_1)\cdot(e_1-y)|y|^{2N}e^{V_k}+E_1 \tag{4.9}$$

where

$$E_1=O(\delta_k^2)|y-e_1|^2|y|^{2N}e^{V_k},\quad y\in\Omega_k.$$

Note that the oscillation of w_k on $\partial\Omega_k$ is $O(\delta_k^{N+1})$, which all comes from the oscillation of V_k.

Let $M_k=\max_{x\in\bar{\Omega}_k}|w_k(x)|$. We shall get a contradiction by assuming $M_k/\tilde{\delta}_k\to\infty$. This assumption implies

$$M_k/(\delta_k^2\mu_k)\to\infty. \tag{4.10}$$

Set

$$\tilde{w}_k(y)=w_k(y)/M_k,\quad x\in\Omega_k.$$

Clearly $\max_{x\in\Omega_k}|\tilde{w}_k(x)|=1$. The equation for $\tilde{w}_k$ is

$$\Delta\tilde{w}_k(y)+|y|^{2N}\mathfrak{h}_k(\delta_k e_1)e^{\xi_k}\tilde{w}_k(y)=a_k\cdot(e_1-y)|y|^{2N}e^{V_k}+\tilde{E}_1, \tag{4.11}$$

in Ω_k, where $a_k=\delta_k\nabla\mathfrak{h}_k(0)/M_k\to 0$,

$$\tilde{E}_1=o(1)|y-e_1|^2|y|^{2N}e^{V_k},\quad y\in\Omega_k. \tag{4.12}$$

Also on the boundary, since $M_k/\tilde{\delta}_k\to\infty$, we have

$$\tilde{w}_k=C+o(1/\mu_k),\quad \text{on} \quad \partial\Omega_k. \tag{4.13}$$

By Proposition 3.1 of [27]

$$\xi_k(e_1+\varepsilon_k z)=V_k(e_1+\varepsilon_k z)+O(\varepsilon_k^{\varepsilon})(1+|z|)^{-1} \tag{4.14}$$

Since V_k is not exactly symmetric around e_1, we shall replace the re-scaled version of V_k around e_1 by a radial function. Let U_k be solutions of

$$\Delta U_k+\mathfrak{h}_k(\delta_k e_1)e^{U_k}=0, \quad \text{in} \quad \mathbb{R}^2, \quad U_k(0)=\max_{\mathbb{R}^2} U_k=0. \tag{4.15}$$

By the classification theorem of Caffarelli-Gidas-Spruck [7] we have

$$U_k(z)=\log\frac{1}{(1+\frac{\mathfrak{h}_k(\delta_k e_1)}{8}|z|^2)^2}$$

and standard refined estimates yield (see [12, 30, 17])

$$V_k(e_1+\varepsilon_k z)+2\log\varepsilon_k=U_k(z)+O(\varepsilon_k)|z|+O(\mu_k^2\varepsilon_k^2). \tag{4.16}$$

Also we observe that

$$\log|e_1+\varepsilon_k z|=O(\varepsilon_k)|z|. \tag{4.17}$$

Thus, the combination of (4.14), (4.16) and (4.17) gives

$$\begin{aligned} &2N\log|e_1+\varepsilon_k z|+\xi_k(e_1+\varepsilon_k z)+2\log\varepsilon_k-U_k(z) \\ =&O(\varepsilon_k^{\varepsilon})(1+|z|) \quad 0\le |z|<\delta_0\varepsilon_k^{-1}. \end{aligned} \tag{4.18}$$

for a small $\varepsilon>0$ independent of k. Since we shall use the re-scaled version, based on (4.18) we have

$$\varepsilon_k^2|e_1+\varepsilon_k z|^{2N}e^{\xi_k(e_1+\varepsilon_k z)}=e^{U_k(z)}+O(\varepsilon_k^{\varepsilon})(1+|z|)^{-3} \tag{4.19}$$

Here we note that the estimate in (4.18) is not optimal. In the following we shall put the proof of Proposition 4.1 into a few estimates. In the first estimate we prove

Lemma 4.1. *For $\delta>0$ small and independent of k,*

$$\tilde{w}_k(y)=o(1), \quad \nabla\tilde{w}_k=o(1) \quad in \quad B(e_1,\delta)\setminus B(e_1,\delta/8) \tag{4.20}$$

where $B(e_1,3\delta)$ does not include other blowup points.

Proof of Lemma 4.1:

If (4.20) is not true, we have, without loss of generality that $\tilde{w}_k\to c>0$. This is based on the fact that $\tilde{w}_k$ tends to a global harmonic function with removable singularity. So $\tilde{w}_k$ tends to constant. Here we assume $c>0$ but the argument for $c<0$ is the same. Let

$$W_k(z)=\tilde{w}_k(e_1+\varepsilon_k z), \quad \varepsilon_k=e^{-\frac{1}{2}\mu_k}, \tag{4.21}$$

then if we use W to denote the limit of W_k, we have

$$\Delta W+e^U W=0, \quad \mathbb{R}^2, \quad |W|\le 1,$$

and U is a solution of $\Delta U+e^U=0$ in $\mathbb{R}^2$ with $\int_{\mathbb{R}^2}e^U<\infty$. Since 0 is the local maximum of U,

$$U(z)=\log\frac{1}{(1+\frac{1}{8}|z|^2)^2}.$$

Here we further claim that $W \equiv 0$ in $\mathbb{R}^2$ because $W(0) = |\nabla W(0)| = 0$, a fact well known based on the classification of the kernel of the linearized operator. Going back to W_k, we have

$$W_k(z) = o(1), \quad |z| \le R_k \text{ for some } \quad R_k \to \infty.$$

Based on the expression of $\tilde{w}_k$, (4.16) and (4.19) we write the equation of W_k as

$$\Delta W_k(z) + \mathfrak{h}_k(\delta_k e_1) e^{U_k(z)} W_k(z) = E_2^k, \tag{4.22}$$

for $|z| < \delta_0 \varepsilon_k^{-1}$ where a crude estimate of the error term E_2^k is

$$E_2^k(z) = o(1)\varepsilon_k^{\varepsilon}(1+|z|)^{-3}.$$

Let

$$g_0^k(r) = \frac{1}{2\pi}\int_0^{2\pi} W_k(r,\theta)d\theta. \tag{4.23}$$

Then clearly $g_0^k(r) \to c > 0$ for $r \sim \varepsilon_k^{-1}$. The equation for g_0^k is

$$\begin{aligned} &\frac{d^2}{dr^2} g_0^k(r) + \frac{1}{r}\frac{d}{dr} g_0^k(r) + \mathfrak{h}_k(\delta_k e_1) e^{U_k(r)} g_0^k(r) = \tilde{E}_0^k(r) \\ &g_0^k(0) = \frac{d}{dr} g_0^k(0) = 0. \end{aligned}$$

where $\tilde{E}_0^k(r)$ has the same upper bound as that of $E_2^k(r)$:

$$|\tilde{E}_0^k(r)| \le o(1)\varepsilon_k^{\varepsilon}(1+r)^{-3}.$$

For the homogeneous equation, the two fundamental solutions are known: g_{01}, g_{02}, where

$$g_{01} = \frac{1 - c_1 r^2}{1 + c_1 r^2}, \quad c_1 = \frac{\mathfrak{h}_k(\delta_k e_1)}{8}.$$

By the standard reduction of order process, $g_{02}(r) = O(\log r)$ for $r > 1$. Then it is easy to obtain, assuming $|W_k(z)| \le 1$, that

$$\begin{aligned} |g_0(r)| &\le C|g_{01}(r)| \int_0^r s|\tilde{E}_0^k(s) g_{02}(s)|ds + C|g_{02}(r)| \int_0^r s|g_{01}(s)\tilde{E}_0^k(s)|ds \\ &\le C\varepsilon_k^{\varepsilon}\log(2+r). \quad 0 < r < \delta_0 \varepsilon_k^{-1}. \end{aligned}$$

Clearly this is a contradiction to (4.23). We have proved $c = 0$, which means $\tilde{w}_k = o(1)$ in $B(e_1,\delta_0) \setminus B(e_1,\delta_0/8)$. Then it is easy to use the equation for $\tilde{w}_k$ and standard Harnack inequality to prove $\nabla \tilde{w}_k = o(1)$ in the same region. Lemma 4.1 is established. $\square$

The second estimate is a more precise description of $\tilde{w}_k$ around e_1:

Lemma 4.2. *For any given $\sigma \in (0,1)$ there exists $C > 0$ such that*

$$|\tilde{w}_k(e_1 + \varepsilon_k z)| \le C\varepsilon_k^{\sigma}(1+|z|)^{\sigma}, \quad 0 < |z| < \tau \varepsilon_k^{-1}. \tag{4.24}$$

for some $\tau > 0$.

Proof of Lemma 4.2: Let W_k be defined as in (4.21). In order to obtain a better estimate we need to write the equation of W_k more precisely than (4.22):

$$\Delta W_k + \mathfrak{h}_k(\delta_k e_1)e^{\Theta_k}W_k = E_3^k(z), \quad z \in \Omega_{Wk} \tag{4.25}$$

where Θ_k is defined by

$$e^{\Theta_k(z)} = |e_1 + \varepsilon_k z|^{2N} e^{\xi_k(e_1+\varepsilon_k z)+2\log\varepsilon_k},$$

$\Omega_{Wk} = B(0, \tau\varepsilon_k^{-1})$ and $E_3^k(z)$ satisfies

$$E_3^k(z) = O(\varepsilon_k)(1+|z|)^{-3}, \quad z \in \Omega_{Wk}.$$

Here we observe that by Lemma 4.1 $W_k = o(1)$ on $\partial\Omega_{Wk}$. Let

$$\Lambda_k = \max_{z\in\Omega_{Wk}} \frac{|W_k(z)|}{\varepsilon_k^\sigma(1+|z|)^\sigma}.$$

If (4.24) does not hold, $\Lambda_k \to \infty$ and we use z_k to denote where Λ_k is attained. Note that because of the smallness of W_k on $\partial\Omega_{Wk}$, z_k is an interior point. Let

$$g_k(z) = \frac{W_k(z)}{\Lambda_k(1+|z_k|)^\sigma\varepsilon_k^\sigma}, \quad z \in \Omega_{Wk},$$

we see immediately that

$$|g_k(z)| = \frac{|W_k(z)|}{\varepsilon_k^\sigma\Lambda_k(1+|z|)^\sigma} \cdot \frac{(1+|z|)^\sigma}{(1+|z_k|)^\sigma} \le \frac{(1+|z|)^\sigma}{(1+|z_k|)^\sigma}. \tag{4.26}$$

Note that σ can be as close to 1 as needed. The equation of g_k is

$$\Delta g_k(z) + \mathfrak{h}_k(\delta_k e_1)e^{\Theta_k}g_k = o(\varepsilon_k^{1-\sigma})\frac{(1+|z|)^{-3}}{(1+|z_k|)^\sigma}, \quad \text{in} \quad \Omega_{Wk}.$$

Then we can obtain a contradiction to $|g_k(z_k)| = 1$ as follows: If $\lim_{k\to\infty} z_k = P \in \mathbb{R}^2$, this is not possible because that fact that $g_k(0) = |\nabla g_k(0)| = 0$ and the sub-linear growth of g_k in (4.26) implies that $g_k \to 0$ over any compact subset of $\mathbb{R}^2$ (see [12, 30]). So we have $|z_k| \to \infty$. But this would lead to a contradiction again by using the Green's representation of g_k:

(4.27)

$$\begin{aligned} &\pm 1 = g_k(z_k) = g_k(z_k) - g_k(0) \\ &= \int_{\Omega_{k,1}} (G_k(z_k,\eta) - G_k(0,\eta))(\mathfrak{h}_k(\delta_k e_1)e^{\Theta_k}g_k(\eta) + o(\varepsilon_k^{1-\sigma})\frac{(1+|\eta|)^{-3}}{(1+|z_k|)^\sigma})d\eta + o(1). \end{aligned}$$

where $G_k(y,\eta)$ is the Green's function on Ω_{Wk} and $o(1)$ in the equation above comes from the smallness of W_k on $\partial\Omega_{Wk}$. Let $L_k = \tau\varepsilon_k^{-1}$, the expression of G_k is

$$G_k(y,\eta) = -\frac{1}{2\pi}\log|y-\eta| + \frac{1}{2\pi}\log(\frac{|\eta|}{L_k}|\frac{L_k^2\eta}{|\eta|^2} - y|).$$

$$G_k(z_k,\eta) - G_k(0,\eta) = -\frac{1}{2\pi}\log|z_k-\eta| + \frac{1}{2\pi}\log|\frac{z_k}{|z_k|} - \frac{\eta z_k}{L_k^2}| + \frac{1}{2\pi}\log|\eta|.$$

Using this expression in (4.27) we obtain from elementary computation that the right hand side of (4.27) is $o(1)$, a contradiction to $|g_k(z_k)|=1$. Lemma 4.2 is established. □

The smallness of $\tilde{w}_k$ around e_1 can be used to obtain the following third key estimate:

Lemma 4.3.

$$\tilde{w}_k = o(1) \quad in \quad B(e^{i\beta_l},\tau) \quad l=1,..,N. \tag{4.28}$$

Proof of Lemma 4.3: We abuse the notation W_k by defining it as

$$W_k(z)=\tilde{w}_k(e^{i\beta_l}+\varepsilon_k z), \quad z\in\Omega_{k,l}:=B(0,\tau\varepsilon_k^{-1}).$$

Here we point out that based on (3.5) and (4.2) we have $\varepsilon_k^{-1}|Q_l^k-e^{i\beta_l}|\to 0$. So the scaling around $e^{i\beta_l}$ or Q_l^k does not affect the limit function.

$$\varepsilon_k^2|e^{i\beta_l}+\varepsilon_k z|^{2N}\mathfrak{h}_k(\delta_k e_1)e^{\xi_k(e^{i\beta_l}+\varepsilon_k z)}\to e^{U(z)}$$

where $U(z)$ is a solution of

$$\Delta U+e^U=0, \quad \text{in} \quad \mathbb{R}^2, \quad \int_{\mathbb{R}^2}e^U<\infty.$$

Here we recall that $\lim_{k\to\infty}\mathfrak{h}_k(\delta_k e_1)=1$. Since W_k converges to a solution of the linearized equation:

$$\Delta W+e^U W=0, \quad \text{in} \quad \mathbb{R}^2.$$

W can be written as a linear combination of three functions:

$$W(x)=c_0\phi_0+c_1\phi_1+c_2\phi_2,$$

where

$$\phi_0=\frac{1-\frac{1}{8}|x|^2}{1+\frac{1}{8}|x|^2}$$

$$\phi_1=\frac{x_1}{1+\frac{1}{8}|x|^2}, \quad \phi_2=\frac{x_2}{1+\frac{1}{8}|x|^2}.$$

The remaining part of the proof consisting of proving $c_0=0$ and $c_1=c_2=0$. First we prove $c_0=0$.

Step one: $c_0=0$. First we write the equation for W_k in a convenient form. Since

$$|e^{i\beta_l}+\varepsilon_k z|^{2N}\mathfrak{h}_k(\delta_k e_1)=\mathfrak{h}_k(\delta_k e_1)+O(\varepsilon_k z),$$

and

$$\varepsilon_k^2 e^{\xi_k(e^{i\beta_l}+\varepsilon_k z)}=e^{U_k(z)}+O(\varepsilon_k^{\varepsilon})(1+|z|)^{-3}.$$

Based on (4.11) we write the equation for W_k as

$$\Delta W_k(z)+\mathfrak{h}_k(\delta_k e_1)e^{U_k}W_k=E_l^k(z) \tag{4.29}$$

where

$$E_l^k(z)=O(\varepsilon_k^{\varepsilon})(1+|z|)^{-3} \quad \text{in} \quad \Omega_{k,l}.$$

In order to prove $c_0=0$, the key is to control the derivative of $W_0^k(r)$ where

$$W_0^k(r)=\frac{1}{2\pi r}\int_{\partial B_r}W_k(re^{i\theta})dS,\quad 0<r<\tau\varepsilon_k^{-1}.$$

To obtain a control of $\frac{d}{dr}W_0^k(r)$ we use $\phi_0^k(r)$ as the radial solution of

$$\Delta\phi_0^k+\mathfrak{h}_k(\delta_k e_1)e^{U_k}\phi_0^k=0,\quad \text{in}\quad \mathbb{R}^2.$$

When $k\to\infty$, $\phi_0^k\to c_0\phi_0$. Thus using the equation for ϕ_0^k and W_k, we have

$$\int_{\partial B_r}(\partial_\nu W_k\phi_0^k-\partial_\nu\phi_0^k W_k)=o(\varepsilon_k^\varepsilon). \tag{4.30}$$

Thus from (4.30) we have

$$\frac{d}{dr}W_0^k(r)=\frac{1}{2\pi r}\int_{\partial B_r}\partial_\nu W_k=o(\varepsilon_k^\varepsilon)/r+O(1/r^3),\quad 1<r<\tau\varepsilon_k^{-1}. \tag{4.31}$$

Since we have known that

$$W_0^k(\tau\varepsilon_k^{-1})=o(1).$$

By the fundamental theorem of calculus we have

$$W_0^k(r)=W_0^k(\tau\varepsilon_k^{-1})+\int_{\tau\varepsilon_k^{-1}}^r(\frac{o(\varepsilon_k^\varepsilon)}{s}+O(s^{-3}))ds=O(1/r^2)+O(\varepsilon_k^\varepsilon\log\frac{1}{\varepsilon_k})$$

for $r\ge 1$. Thus $c_0=0$ because $W_0^k(r)\to c_0\phi_0$, which means when r is large, it is $-c_0+O(1/r^2)$.

Step two $c_1=c_2=0$. We first observe that Lemma 4.3 follows from this. Indeed, once we have proved $c_1=c_2=c_0=0$ around each $e^{i\beta_l}$, it is easy to use maximum principle to prove $\tilde{w}_k=o(1)$ in B_3 using $\tilde{w}_k=o(1)$ on ∂B_3 and the Green's representation of $\tilde{w}_k$. The smallness of $\tilde{w}_k$ immediately implies $\tilde{w}_k=o(1)$ in B_R for any fixed $R>>1$. Outside B_R, a crude estimate of v_k is

$$v_k(y)\le -\mu_k-4(N+1)\log|y|+C,\quad 3<|y|<\tau\delta_k^{-1}.$$

Using this and the Green's representation of w_k we can first observe that the oscillation on each ∂B_r is $o(1)$ ($R<r<\tau\delta_k^{-1}/2$) and then by the Green's representation of $\tilde{w}_k$ and fast decay rate of e^{V_k} we obtain $\tilde{w}_k=o(1)$ in $\overline{B(0,\tau\delta_k^{-1})}$. A contradiction to $\max|\tilde{w}_k|=1$.

There are $N+1$ local maximums with one of them being e_1. Correspondingly there are $N+1$ global solutions $V_{l,k}$ that approximate v_k accurately near Q_l^k for $l=0,...,N$. Note that $Q_0^k=e_1$. For $V_{l,k}$ the expression is

$$V_{l,k}=\log\frac{e^{\mu_l^k}}{(1+\frac{e^{\mu_l^k}}{D_l^k}|y^{N+1}-(e_1+p_l^k)|^2)^2},\quad l=0,...,N,$$

where $p_l^k=E$ and

$$D_l^k=8(N+1)^2/\mathfrak{h}_k(\delta_k Q_l^k). \tag{4.32}$$

The equation that $V_{l,k}$ satisfies is

$$\Delta V_{l,k}+|y|^{2N}\mathfrak{h}_k(\delta_k Q_l^k)e^{V_{l,k}}=0,\quad \text{in}\quad \mathbb{R}^2.$$

Since v_k and $V_{l,k}$ have the same common local maximum at Q_l^k, it is easy to see that

$$Q_l^k=e^{i\beta_l}+\frac{p_l^k e^{i\beta_l}}{N+1}+O(|p_l^k|^2),\quad \beta_l=\frac{2l\pi}{N+2}. \tag{4.33}$$

Let $M_{l,k}$ be the maximum of $|v_k-V_{l,k}|$ and we claim that all these $M_{l,k}$ are comparable:

$$M_{l,k}\sim M_{s,k},\quad \forall s\neq l. \tag{4.34}$$

The proof of (4.34) is as follows: We use $L_{s,l}$ to denote the limit of $(v_k-V_{l,k})/M_{l,k}$ around Q_s^k:

$$\frac{(v_k-V_{l,k})(Q_s^k+\varepsilon_k z)}{M_{l,k}}=L_{s,l}+o(1),\quad |z|\le \tau\varepsilon_k^{-1}$$

where

$$L_{s,l}=c_{1,s,l}\frac{z_1}{1+\frac{1}{8}|z|^2}+c_{2,s,l}\frac{z_2}{1+\frac{1}{8}|z|^2},\quad \text{and}\quad L_{l,l}=0,\quad s=0,...,N.$$

If all $c_{1,s,l}$ and $c_{2,s,l}$ are zero for a fixed l, we can obtain a contradiction just like the beginning of step two. So at least one of them is not zero. For each $s\neq l$, by Lemma 4.2 we have

$$v_k(Q_s^k+\varepsilon_k z)-V_{s,k}(Q_s^k+\varepsilon_k z)=O(\varepsilon_k^\sigma)(1+|z|)^\sigma M_{s,k},\quad |z|<\tau\varepsilon_k^{-1}. \tag{4.35}$$

Let $M_k=\max_i M_{i,k}$ ($i=0,...,N$) and we suppose $M_k=M_{l,k}$. Then to determine $L_{s,l}$ we see that

$$\begin{aligned}&\frac{v_k(Q_s^k+\varepsilon_k z)-V_{l,k}(Q_s^k+\varepsilon_k z)}{M_k}\\ =&o(\varepsilon_k^\sigma)(1+|z|)^\sigma+\frac{V_{s,k}(Q_s^k+\varepsilon_k z)-V_{l,k}(Q_s^k+\varepsilon_k z)}{M_k}.\end{aligned}$$

This expression says that $L_{s,l}$ is mainly determined by the difference of two global solutions $V_{s,k}$ and $V_{l,k}$. In order to obtain a contradiction to our assumption we will put the difference in several terms. The main idea in this part of the reasoning is that "first order terms" tell us what the kernel functions should be, then the "second order terms" tell us where the pathology is.

We write $V_{s,k}(y)-V_{l,k}(y)$ as

$$V_{s,k}(y)-V_{l,k}(y)=\mu_s^k-\mu_l^k+2A-A^2+O(|A|^3)$$

where

$$A(y)=\frac{\frac{e^{\mu_l^k}}{D_l^k}|y^{N+1}-e_1-p_l^k|^2-\frac{e^{\mu_s^k}}{D_s^k}|y^{N+1}-e_1-p_s^k|^2}{1+\frac{e^{\mu_s^k}}{D_s^k}|y^{N+1}-e_1-p_s^k|^2}.$$

Here for convenience we abuse the notation ε_k by assuming $\varepsilon_k = e^{-\mu_s^k/2}$. Note that $\varepsilon_k = e^{\mu_t^k/2}$ for some t, but it does not matter which t it is. From A we claim that

$$\begin{aligned} &V_{s,k}(Q_s^k+\varepsilon_k z) - V_{l,k}(Q_s^k+\varepsilon_k z) \\ =&\phi_1+\phi_2+\phi_3+\phi_4+\mathfrak{R}, \end{aligned} \tag{4.36}$$

where

$$\begin{aligned} \phi_1 &= (\mu_s^k-\mu_l^k)(1-\frac{(N+1)^2}{D_s^k}|z+O(\varepsilon_k)|z|^2|^2)/B, \\ \phi_2 &= \frac{2(N+1)^2}{D_s^k}\delta_k\nabla\mathfrak{h}_k(\delta_k Q_s^k)(Q_l^k-Q_s^k)|z|^2/B \\ \phi_3 &= \frac{4(N+1)}{D_s^k B}Re((z+O(\varepsilon_k|z|^2))(\frac{\bar p_s^k-\bar p_l^k}{\varepsilon_k}e^{-i\beta_s})) \\ \phi_4 &= \frac{|p_s^k-p_l^k|^2}{\varepsilon_k^2}\left(\frac{2}{D_s^k B}-\frac{2(N+1)^2|z|^2}{D_s^2B^2}-\frac{2(N+1)^2}{D_s^2B^2}|z|^2\cos(2\theta-2\theta_{st}-2\beta_s)\right), \\ B &= 1+\frac{(N+1)^2}{D_s^k}|z+O(\varepsilon_k|z|^2)|^2, \end{aligned}$$

and $\mathfrak{R}_k$ is the collections of other insignificant terms. Here we briefly explain the roles of each term. ϕ_1 corresponds to the radial solution in the kernel of the linearized operator of the global equation. In other words, ϕ_1^k/M_k should tend to zero. ϕ_2^k/M_k is the combination of the two other functions in the kernel. ϕ_4 is the second order term which will play a leading role later. ϕ_3^k comes from the difference of $\mathfrak{h}_k$ at Q_l^k and Q_s^k. The derivation of (4.36) is as follows: First by the expression of Q_s^k in (4.33) we have

$$y^{N+1} = 1+p_s^k+(N+1)\varepsilon_k z e^{-i\beta_s}+O(\varepsilon_k^2)|z|^2,$$

where $y = Q_s^k+\varepsilon_k z$. Then

$$|y^{N+1}-e_1-p_s^k|^2 = (N+1)^2\varepsilon_k^2|z+O(\varepsilon_k)z^2|^2+O(\varepsilon_k^3)|z|^3$$

$$|y^{N+1}-e_1-p_l^k|^2 = (N+1)^2\varepsilon_k^2|z+\frac{(p_s^k-p_l^k)e^{i\beta_s}}{(N+1)\varepsilon_k}+O(\varepsilon_k)|z|^2|^2+O(\varepsilon_k^3|z|^3).$$

Next by the definition of D_s^k in (4.32)

$$\frac{D_s^k-D_l^k}{D_l^k} = \delta_k\nabla(\log\mathfrak{h}_k)(0)\cdot(Q_l^k-Q_s^k)+O(\delta_k^2).$$

$$\begin{aligned} \frac{e^{\mu_l^k-\mu_s^k}}{D_l^k} &= \frac{1}{D_s^k}(1+\frac{D_s^k-D_l^k}{D_l^k}+\mu_l^k-\mu_s^k+O(\mu_l^k-\mu_s^k)^2+O(\delta_k^2)). \\ &= \frac{1}{D_s^k}(1+\delta_k\nabla\log\mathfrak{h}_k(0)\cdot(Q_l^k-Q_s^k)+\mu_l^k-\mu_s^k+O(\mu_l^k-\mu_s^k)^2+O(\delta_k^2)). \end{aligned} \tag{4.37}$$

Then the expression of A is (for simplicity we omit k in some notations)

$$A=\Big(\frac{e^{\mu_l-\mu_s}}{D_l}(N+1)^2\big(|z|^2+2Re(z\frac{\overline{p_s-p_l}}{\varepsilon_k(N+1)}e^{-i\beta_s})+\frac{|p_s-p_l|^2}{(N+1)^2\varepsilon_k^2}+O(\varepsilon_k|z|^3)\big)$$
$$-\frac{(N+1)^2}{D_s}(|z|^2+O(\varepsilon_k)|z|^3)\Big)/B$$

After using (4.37) we have

$$A=\Big(\frac{1}{D_s}(\delta_k\nabla(\log\mathfrak{h}_k(0)(Q_l-Q_s)+\mu_l-\mu_s+O(\mu_l-\mu_s)^2)(N+1)^2|z|^2$$
$$+2Re(z\frac{\bar p_s-\bar p_l}{\varepsilon_k}e^{-i\beta_s})(N+1)\frac{1}{D_s}+\frac{|p_s-p_l|^2}{\varepsilon_k^2 D_s}+O(\varepsilon_k)|z|^3+O(\delta_k^2)|z|^2\Big)/B. \tag{4.38}$$

$$A^2=(\frac{(N+1)^2}{D_s^2}4(Re(z\frac{\bar p_s-\bar p_l}{\varepsilon_k}e^{-i\beta_s})^2)/B^2+\text{other terms}. \tag{4.39}$$

The numerator of A^2 has the following leading term:

$$\frac{(N+1)^2}{D_s^2}\Big(2|z|^2(\frac{|p_s-p_l|}{\varepsilon_k})^2\big(1+2\cos(2\theta-2\theta_{sl})\big)\Big)$$

where $z=|z|e^{i\theta}$, $p_s-p_l=|p_s-p_l|e^{i\theta_{sl}}$. Using these expressions we can obtain (4.36) by direct computation. Here ϕ_1, ϕ_3 correspond to solutions to the linearized operator. Here we note that if we set $\varepsilon_{l,k}=e^{-\mu_l^k/2}$, there is no essential difference between $\varepsilon_{l,k}$ and $\varepsilon_k=e^{-\frac{1}{2}\mu_{1,k}}$ because $\varepsilon_{l,k}=\varepsilon_k+O(\varepsilon_k E)$. If $|\mu_{s,k}-\mu_{l,k}|/M_k\geq C$ there is no way to obtain a limit in the form of $L_{s,l}$ mentioned before. Thus we must have $|\mu_{s,k}-\mu_{l,k}|/M_k\to 0$. After simplification (see ϕ_3 of (4.36)) we have

$$c_{1,s,l}=\lim_{k\to\infty}\frac{|p_s^k-p_l^k|}{2(N+1)M_k\varepsilon_k}\cos(\beta_s+\theta_{sl}), \tag{4.40}$$
$$c_{2,s,l}=\lim_{k\to\infty}\frac{|p_s^k-p_l^k|}{2(N+1)\varepsilon_k M_k}sin(\beta_s+\theta_{sl})$$

We omit k for convenience. It is also important to observe that even if $M_k=o(\varepsilon_k)$ we still have $M_k\sim\max_s|p_s^k-p_l^k|/\varepsilon_k$. Since each $|p_l^k|=E$, an upper bound for M_k is

$$M_k\leq C\mu_k\varepsilon_k+C\delta_k^2\varepsilon_k^{-1}. \tag{4.41}$$

Equation (4.40) gives us a key observation: $|c_{1,s,l}|+|c_{2,s,l}|\sim|p_s^k-p_l^k|/(\varepsilon_k M_k)$. So whenever $|c_{1,s,l}|+|c_{2,s,l}|\neq 0$ we have $\frac{|p_s^k-p_l^k|}{\varepsilon_k}\sim M_k$. In other words for each l, $M_{l,k}\sim\max_{t\neq l}\frac{|p_t^k-p_l^k|}{\varepsilon_k}$. Hence for any t, if $\frac{|p_t^k-p_l^k|}{\varepsilon_k}\sim M_k$, let $M_{t,k}$ be the maximum of $|v_k-V_{t,k}|$, we have $M_{t,k}\sim M_k$. If all $\frac{|p_t^k-p_l^k|}{\varepsilon_k}\sim M_k$ (4.34) is proved. So we prove that even if some p_t^k is very close to p_l^k, M_t^k is still comparable to M_k. The reason is there exists q such that $\frac{|p_l^k-p_q^k|}{\varepsilon_k}\sim M_k$, if $\frac{|p_t^k-p_l^k|}{\varepsilon_k}=o(1)M_k$,

$$|p_t^k-p_q^k|\ge |p_l^k-p_q^k|-|p_t^k-p_l^k|\ge \frac{1}{2}|p_l^k-p_q^k|.$$

Thus $\frac{|p_t^k-p_q^k|}{\varepsilon_k}\sim M_k$ and $M_t^k\sim M_k$. (4.34) is established. From now on for convenience we shall just use M_k.

Set $w_{l,k}=(v_k-V_{l,k})$, then we have $w_{l,k}(Q_l^k)=|\nabla w_{l,k}(Q_l^k)|=0$. Correspondingly we set

$$\tilde{w}_{l,k}=w_{l,k}/M_k.$$

The equation of $w_{l,k}$ can be written as

$$\begin{aligned}\Delta w_{l,k}+|y|^{2N}\mathfrak{h}_k(\delta_kQ_l)e^{\xi_l}w_{l,k}\\ =&\delta_k\nabla\mathfrak{h}_k(\delta_kQ_l)(Q_l-y)|y|^{2N}e^{V_{l,k}}+O(\delta_k^2)|y|^{2N}|y-Q_l|^2e^{V_k}.\end{aligned}\tag{4.42}$$

where we omitted k in Q_l and ξ_l. ξ_l comes from the Mean Value Theorem and satisfies

$$e^{\xi_l}=e^{V_{l,k}}(1+\frac{1}{2}w_{l,k}+O(w_{l,k}^2)).\tag{4.43}$$

The function $\tilde{w}_{l,k}$ satisfies

$$\lim_{k\to\infty}\tilde{w}_{l,k}(Q_s^k+\varepsilon_kz)=\frac{c_{1,s,l}z_1+c_{2,s,l}z_2}{1+\frac{1}{8}|z|^2}\tag{4.44}$$

and around each Q_s^k (4.35) holds with $M_{s,k}$ replaced by M_k.

Now we use the estimates above to evaluate $\nabla\tilde{w}_{l,k}(Q_l^k)=0$. First we have

$$\begin{aligned}w_{l,k}(y)=&\int_{\Omega_k}(G_k(y,\eta)-G_k(Q_l,\eta))(\mathfrak{h}_k(\delta_kQ_l)|\eta|^{2N}e^{\xi_l}w_{l,k}(\eta)\\ &+\delta_k\nabla\mathfrak{h}_k(\delta_kQ_l)(\eta-Q_l)|\eta|^{2N}e^{V_{l,k}}\\ &+O(\delta_k^2)|\eta-Q_l|^2|\eta|^{2N}e^{V_k})d\eta+O(\delta_k^{N+1}).\end{aligned}$$

After differentiate the equation and divide the equation by M_k we have

$$\begin{aligned}&\nabla\tilde{w}_{l,k}(Q_l^k)\\ =&-\frac{1}{2\pi}\int_{\Omega_k}\frac{Q_l^k-\eta}{|Q_l^k-\eta|^2}\Big(\tilde{w}_{l,k}(\eta)\mathfrak{h}_k(\delta_kQ_l^k)|\eta|^{2N}e^{\xi_l}\\ &+\sigma_k\nabla\mathfrak{h}_k(\delta_kQ_l^k)(\eta-Q_l^k)|\eta|^{2N}e^{V_{l,k}}\Big)d\eta+o(\varepsilon_k).\end{aligned}\tag{4.45}$$

Note that we have used $\nabla H_k(Q_s^k,Q_l^k)=O(\delta_k^2)$ and the fact that the derivative of the harmonic function at Q_l^k is $O(\delta_k^{N+2}/M_k)$. Based on (4.2) and (4.10) we see this term is $o(\varepsilon_k)$.

Since the left hand side is 0, the right hand side gives

$$\sum_{s,s\neq l}\Big(\int_{B(Q_s^k,\tau)}\frac{Q_l^k-\eta}{|Q_l^k-\eta|^2}\tilde{w}_{l,k}(\eta)\mathfrak{h}_k(\delta_kQ_l^k)|\eta|^{2N}e^{V_{l,k}}(1+\frac{M_k}{2}\tilde{w}_{l,k})d\eta$$
$$+8\pi\sigma_k\frac{Q_l^k-Q_s^k}{|Q_l^k-Q_s^k|^2}\nabla\log\mathfrak{h}_k(\delta_kQ_l^k)(Q_s^k-Q_l^k)\Big)=o(\varepsilon_k), \tag{4.46}$$

where $\sigma_k=\delta_k/M_k$ and we used $|\tilde{w}_k|\leq 1$. Note that σ_k satisfies $\sigma_k|\nabla\mathfrak{h}_k(0)|=o(1)$. Around each Q_s^k we use

$$e^{V_{l,k}}=e^{V_{s,k}}(1-M_k\tilde{w}_{l,k}+O(M_k^2\tilde{w}_{l,k}^2)).$$

Then (4.46) can be written as

$$\sum_{s,s\neq l}\Big(\int_{B(Q_s^k,\tau)}\frac{Q_l^k-\eta}{|Q_l^k-\eta|^2}\tilde{w}_{l,k}(\eta)\mathfrak{h}_k(\delta_kQ_l^k)|\eta|^{2N}e^{V_{s,k}}d\eta$$
$$+8\pi\sigma_k\frac{Q_l^k-Q_s^k}{|Q_l^k-Q_s^k|^2}\nabla\log\mathfrak{h}_k(\delta_kQ_l^k)(Q_s^k-Q_l^k)\Big)=o(\varepsilon_k), \tag{4.47}$$

where the upper bound of M_k in (4.41) is used. To evaluate the first term in (4.46), around each Q_s^k we write

$$\frac{Q_l^k-\eta}{|Q_l^k-\eta|^2}=\frac{Q_l^k-Q_s^k}{|Q_l^k-Q_s^k|^2}+F_{ls}(\eta)$$

where

$$F_{ls}(\eta)=(F_{ls}^1(\eta),F_{ls}^2(\eta))=\frac{Q_l^k-\eta}{|Q_l^k-\eta|^2}-\frac{Q_l^k-Q_s^k}{|Q_l^k-Q_s^k|^2}.$$

We use (4.36) to obtain

$$\int_{B(Q_s^k,\tau)}(\tilde{w}_{l,k}(\eta)\mathfrak{h}_k(\delta_kQ_l^k)|\eta|^{2N}e^{V_k}d\eta$$
$$+8\pi\sigma_k\nabla\log\mathfrak{h}_k(\delta_kQ_s^k)(Q_s^k-Q_l^k)=O(\varepsilon_k^{\sigma}) \tag{4.48}$$

Note that in the computation above, the terms of ϕ_1 and ϕ_3 lead to $o(\varepsilon_k)$, the integration involving ϕ_2 cancels with the second term of (4.48). The computation of ϕ_2 is based on this equation:

$$\int_{\mathbb{R}^2}\frac{\frac{\mathfrak{h}_k(\delta_kQ_l^k)}{4}\sigma_k\nabla\mathfrak{h}_k(\delta_kQ_l^k)(Q_l^k-Q_s^k)|z|^2}{(1+\frac{\mathfrak{h}_k(\delta_kQ_l^k)}{8}|z|^2)^3}dz=8\pi\sigma_k\nabla(\log\mathfrak{h}_k)(\delta_kQ_l^k)(Q_l^k-Q_s^k),$$

and by (4.2)

$$\nabla\log\mathfrak{h}_k(\delta_kQ_l^k)-\nabla\log\mathfrak{h}_k(\delta_kQ_s^k)=O(\delta_k)=o(\varepsilon_k). \tag{4.49}$$

The integration involving ϕ_4 provides the leading term. More detailed information is the following: First for a global solution

$$V_{\mu,p}=\log\frac{e^{\mu}}{(1+\frac{e^{\mu}}{\lambda}|z^{N+1}-p|^2)^2}$$

of

$$\Delta V_{\mu,p}+\frac{8(N+1)^2}{\lambda}|z|^{2N}e^{V_{\mu,p}}=0,\quad \text{in}\quad \mathbb{R}^2,$$

by differentiation with respect to μ we have

$$\Delta(\partial_\mu V_{\mu,p})+\frac{8(N+1)^2}{\lambda}|z|^{2N}e^{V_{\mu,p}}\partial_\mu V_{\mu,p}=0,\quad \text{in}\quad \mathbb{R}^2.$$

By the expression of $V_{\mu,p}$ we see that

$$\partial_r\Big(\partial_\mu V_{\mu,p}\Big)(x)=O(|x|^{-2N-3}).$$

Thus we have

$$\int_{\mathbb{R}^2}\partial_\mu V_{\mu,p}|z|^{2N}e^{V_{\mu,p}}=\int_{\mathbb{R}^2}\frac{(1-\frac{e^\mu}{\lambda}|z^{N+1}-P|^2)|z|^{2N}}{(1+\frac{e^\mu}{\lambda}|z^{N+1}-P|^2)^3}dz=0. \tag{4.50}$$

From $V_{\mu,p}$ we also have

$$\int_{\mathbb{R}^2}\partial_P V_{\mu,p}|y|^{2N}e^{V_{\mu,p}}=\int_{\mathbb{R}^2}\partial_{\bar P} V_{\mu,p}|y|^{2N}e^{V_{\mu,p}}=0,$$

which gives

$$\int_{\mathbb{R}^2}\frac{\frac{e^\mu}{\lambda}(\bar z^{N+1}-\bar P)|z|^{2N}}{(1+\frac{e^\mu}{\lambda}|z^{N+1}-P|^2)^3}=\int_{\mathbb{R}^2}\frac{\frac{e^\mu}{\lambda}(z^{N+1}-P)|z|^{2N}}{(1+\frac{e^\mu}{\lambda}|z^{N+1}-P|^2)^3}=0. \tag{4.51}$$

Now we need more precise expressions of ϕ_1, ϕ_3 and B:

$$\begin{aligned}
&\phi_1=(\mu_s^k-\mu_l^k)(1-\frac{(N+1)^2}{D_s^k}|z+\frac{N}{2}\varepsilon_k z^2e^{-i\beta_s}|^2/B,\\
&\phi_3=\frac{4(N+1)}{D_s^kB}Re((z+\frac{N}{2}\varepsilon_k e^{-i\beta_s}z^2))(\frac{\bar p_s^k-\bar p_l^k}{\varepsilon_k}e^{-i\beta_s}))\\
&B=1+\frac{(N+1)^2}{D_s^k}|z+\frac{N}{2}z^2e^{-i\beta_s}\varepsilon_k|^2,
\end{aligned}$$

From here we use scaling and cancellation to have

$$\int_{B(0,\tau\varepsilon_k^{-1})}\frac{\phi_1}{M_k}B^{-3}=o(\varepsilon_k),\quad \int_{B(0,\tau\varepsilon_k^{-1})}\frac{\phi_3}{M_k}B^{-3}=o(\varepsilon_k).$$

Thus (4.48) holds.

Equation (4.48) also leads to a more accurate estimate of $\tilde w_{l,k}$ in regions between bubbling disks. By the Green's representation formula of $\tilde w_{l,k}$ it is easy to have

$$\begin{aligned}
&\tilde w_{l,k}(y)\\
=&-\frac{1}{2\pi}\int_{\Omega_k}\log\frac{|y-\eta|}{|Q_l^k-\eta|}\Big(\tilde w_{l,k}(\eta)\mathfrak{h}_k(\delta_kQ_l^k)|\eta|^{2N}e^{\xi_l}\\
&+\sigma_k\nabla\mathfrak{h}_k(\delta_kQ_l^k)(\eta-Q_l^k)|\eta|^{2N}e^{V_{l,k}}\Big)d\eta+o(\varepsilon_k)
\end{aligned}$$

Writing

$$\log\frac{|y-\eta|}{|Q_l^k-\eta|}=\log\frac{|y-Q_s^k|}{|Q_l^k-Q_s^k|}+(\log\frac{|y-\eta|}{|Q_l^k-\eta|}-\log\frac{|y-Q_s^k|}{|Q_l^k-Q_s^k|}),$$

the integration related to the second term is $O(\varepsilon_k)$. The integration involving the first term is $O(\varepsilon_k^\sigma)$ by (4.48). Therefore

$$|\tilde{w}_{l,k}(y)|=o(\varepsilon_k^\sigma),\quad y\in B_3\setminus\cup_{s=0}^N B(Q_s^k,\tau)$$

for $\sigma\in(0,1)$. Thus this extra control of $\tilde{w}_{l,k}$ away from bubbling disks gives a better estimate than (4.35) around Q_l^k: Using the same argument for Lemma 4.2 we have

$$|\tilde{w}_{l,k}(Q_l^k+\varepsilon_k z)|\le o(\varepsilon_k)(1+|z|)^\sigma,\quad |z|<\tau\varepsilon_k^{-1}. \tag{4.52}$$

From the decomposition in (4.36) we can now compute (4.45) in more detail:

$$\begin{aligned}\int_{B(Q_s^k,\tau)}\tilde{w}_{l,k}(\eta)\mathfrak{h}_k(\delta_k Q_l^k)|\eta|^{2N}e^{V_{l,k}}d\eta+8\pi\sigma_k\nabla\log\mathfrak{h}_k(\delta_k Q_s^k)(Q_s^k-Q_l^k)\\ =\frac{\pi}{(N+1)^2}\frac{|p_s^k-p_l^k|^2}{\varepsilon_k^2M_k^2}M_k+o(\varepsilon_k).\end{aligned}\tag{4.53}$$

$$\begin{aligned}&\int_{B(Q_s^k,\tau)}F_{ls}^1(\eta)\tilde{w}_{l,k}(\eta)\mathfrak{h}_k(\delta_k Q_l^k)|\eta|^{2N}e^{V_{l,k}}d\eta\\ &=\varepsilon_k\int_{B(0,\tau\varepsilon_k^{-1})}(\partial_1F_{ls}^1(Q_s^k)z_1+\partial_2F_{ls}^1(Q_s^k)z_2)\frac{\tilde{w}_{l,k}(Q_s^k+\varepsilon_k z)}{(1+\frac18|z|^2)^2}dz+o(\varepsilon_k).\\ &=4\pi\varepsilon_k\frac{-c_{1,s,l}\cos(\beta_s+\beta_l)-c_{2,s,l}\sin(\beta_l+\beta_s)}{\sin^2(\frac{\beta_l-\beta_s}{2})}+o(\varepsilon_k).\end{aligned}\tag{4.54}$$

Here is the detail of this computation:

$$\partial_1F_{ls}^1(Q_s)=\frac{(Q_l^1-Q_s^1)^2-(Q_l^2-Q_s^2)^2}{|Q_l-Q_s|^4}.$$

After simplification we have

$$\partial_1F_{ls}^1(Q_s)=-\frac14\frac{\cos(\beta_l+\beta_s)}{\sin^2(\frac{\beta_l-\beta_s}{2})}.$$

Similarly

$$\partial_2F_{ls}^1=\frac{2(Q_l^1-Q_s^1)(Q_l^2-Q_s^2)}{|Q_l-Q_s|^4}=-\frac14\frac{\sin(\beta_s+\beta_l)}{\sin^2(\frac{\beta_l-\beta_s}{2})}.$$

Besides these two identities we also used

$$\int_{\mathbb{R}^2}\frac{z_1^2}{(1+\frac18|z|^2)^3}dz=\int_{\mathbb{R}^2}\frac{z_2^2}{(1+\frac18|z|^2)^3}dz=16\pi.$$

$$
\begin{aligned}
(4.55)\qquad &\int_{B(Q_s^k,\tau)} F_{ls}^2(\eta)\tilde{w}_{l,k}(\eta)\mathfrak{h}_k(\delta_k Q_l^k)|\eta|^{2N}e^{V_{l,k}}d\eta \\
&= \varepsilon_k \int_{B(0,\tau\varepsilon_k^{-1})} (\partial_1 F_{ls}^2(Q_s^k)z_1 + \partial_2 F_{ls}^2(Q_s^k)z_2)\frac{\tilde{w}_{l,k}(Q_s^k+\varepsilon_k z)}{(1+\frac{1}{8}|z|^2)^2}dz + o(\varepsilon_k). \\
&= 4\pi\varepsilon_k \frac{-c_{1,s,l}\sin(\beta_s+\beta_l)+c_{2,s,l}\cos(\beta_l+\beta_s)}{\sin^2(\frac{\beta_l-\beta_s}{2})} + o(\varepsilon_k).
\end{aligned}
$$

The derivation of (4.55) is based on

$$\partial_1 F_{ls}^2(Q_s) = \frac{2(Q_l^1-Q_s^1)(Q_l^2-Q_s^2)}{|Q_l-Q_s|^4} = -\frac{1}{4}\frac{\sin(\beta_l+\beta_s)}{\sin^2(\frac{\beta_l-\beta_s}{2})}$$

and

$$\partial_2 F_{ls}^2(Q_s) = \frac{(Q_l^2-Q_s^2)^2-(Q_l^1-Q_s^1)^2}{|Q_l-Q_s|^4} = \frac{1}{4}\frac{\cos(\beta_l+\beta_s)}{\sin^2(\frac{\beta_l-\beta_s}{2})}.$$

Using (4.53),(4.54),(4.55) in (4.46), and by the expressions of $c_{1,s,l}, c_{2,s,l}$ in (4.40) we have

$$\sum_{s,s\neq l}\left(\frac{Q_l^k-Q_s^k}{|Q_l^k-Q_s^k|^2}\frac{\pi}{(N+1)^2}\frac{|p_s^k-p_l^k|^2}{\varepsilon_k^2 M_k^2}M_k - \frac{2\pi\varepsilon_k}{\sin^2(\frac{\beta_l-\beta_s}{2})}e^{i(\beta_l-\theta_{sl})}\frac{|p_s^k-p_l^k|}{\varepsilon_k M_k}\right) = o(\varepsilon_k).$$

Since $Q_s^k = e^{i\beta_s}+E$, by multiplying $e^{-i\beta_l}$ on both sides, we can simplify the equation above to

$$
\begin{aligned}
(4.56)\qquad &\sum_{s\neq l}\Bigg((\frac{1}{2} - \frac{i\sin(\beta_l-\beta_s)}{4\sin^2(\frac{\beta_s-\beta_l}{2})} + E)\frac{\pi}{(N+1)^2}(\frac{|p_l^k-p_s^k|}{\varepsilon_k M_k})^2 M_k \\
&\qquad - \frac{2\pi\varepsilon_k e^{-i\theta_{sl}}}{\sin^2(\frac{\beta_s-\beta_l}{2})}\frac{|p_s^k-p_l^k|}{\varepsilon_k M_k}\Bigg) = o(\varepsilon_k), \quad \text{for all } l.
\end{aligned}
$$

Note that $e^{-i\theta_{sl}}|p_s^k-p_l^k| = \bar{p}_s^k - \bar{p}_l^k$. Taking the sum of all l in (4.56) we have

$$CM_k\sum_{s\neq l}(|p_s^k-p_l^k|^2/(\varepsilon_k M_k)^2) = o(\varepsilon_k), \quad C>0.$$

If $M_k \geq C\varepsilon_k$ for some $C>0$, we have $|c_{1,s,l}|+|c_{2,s,l}| = 0$ for all s,l. Finally if $M_k = o(\varepsilon_k)$, the first term in (4.56) is ignored and (4.56) becomes

$$\sum_{s,s\neq l}\frac{\bar{p}_s^k-\bar{p}_l^k}{\sin^2(\frac{\beta_s-\beta_l}{2})\varepsilon_k M_k} = o(1), \quad l=0,1,...,N.$$

If we use $\eta_i = \bar{p}_i^k/(\varepsilon_k M_k)$ for $i=0,...,N$ and $D_i = 1/\sin^2(\frac{\pi i}{N+1})$ for $i=1,...,N$. Then we see that $\eta_0 = 0$ and our goal is to prove $\eta_i = o(1)$ for all $i=1,...,n$. By looking at the equation for η_i ($i=1,...,n$) we see that

$$(\sum_{m=1}^n D_m)\eta_l - \sum_{s=1,s\neq l}^N D_{|s-l|}\eta_s = 0, \quad i=1,...,N.$$

The corresponding coefficient matrix is invertible since it is strictly diagonally dominant. Thus all $|p_l^k|/(\varepsilon_k M_k) = o(1)$, which implies $c_{1,s,l} = c_{2,s,l} = 0$ for all s,l. Lemma 4.3 is established. □

Proposition 4.1 is an immediate consequence of Lemma 4.3. □.

Now we finish the proof of Theorem 4.1.

Let $\hat{w}_k = w_k/\tilde{\delta}_k$. (Recall that $\tilde{\delta}_k = \delta_k|\nabla\mathfrak{h}_k(0)| + \delta_k^2\mu_k$). If $|\nabla\mathfrak{h}_k(0)|/(\delta_k\mu_k) \to \infty$, we see that in this case $\tilde{\delta}_k \sim \delta_k\mu_k|\nabla\mathfrak{h}_k(0)|$. The equation of $\hat{w}_k$ is

$$\Delta\hat{w}_k + |y|^{2N}e^{\xi_k}\hat{w}_k = a_k\cdot(e_1-y)|y|^{2N}e^{V_k} + b_k e^{V_k}|y-e_1|^2|y|^{2N}, \tag{4.57}$$

in Ω_k, where $a_k = \delta_k\nabla\mathfrak{h}_k(0)/\tilde{\delta}_k$, $b_k = o(1/\mu_k)$. By Proposition 4.1, $|\hat{w}_k(y)| \le C$. Before we carry out the remaining part of the proof we observe that $\hat{w}_k$ converges to a harmonic function in $\mathbb{R}^2$ minus finite singular points. Since $\hat{w}_k$ is bounded, all these singularities are removable. Thus $\hat{w}_k$ converges to a constant. Based on the information around e_1, we shall prove that this constant is 0. However, looking at the right hand side the equation,

$$(e_1-y)|y|^{2N}e^{V_k} \rightharpoonup \sum_{l=1}^{N} 8\pi(e_1-e^{i\beta_l})\delta_{e^{i\beta_l}}.$$

we will get a contradiction by comparing the Pohozaev identities of v_k and V_k, respectively.

Now we use the notation W_k again and use Proposition 4.1 to rewrite the equation for W_k. Let

$$W_k(z) = \hat{w}_k(e_1+\varepsilon_k z), \quad |z| < \delta_0\varepsilon_k^{-1}$$

for $\delta_0 > 0$ small. Then from Proposition 4.1 we have

$$\mathfrak{h}_k(\delta_k y) = \mathfrak{h}_k(\delta_k e_1) + \delta_k\nabla\mathfrak{h}_k(\delta_k e_1)(y-e_1) + O(\delta_k^2)|y-e_1|^2, \tag{4.58}$$

$$|y|^{2N} = |e_1+\varepsilon_k z|^{2N} = 1 + O(\varepsilon_k)|z|, \tag{4.59}$$

$$V_k(e_1+\varepsilon_k z) + 2\log\varepsilon_k = U_k(z) + O(\varepsilon_k)|z| + O(\varepsilon_k^2)(\log(1+|z|))^2 \tag{4.60}$$

and

$$\xi_k(e_1+\varepsilon_k z) + 2\log\varepsilon_k = U_k(z) + O(\varepsilon_k)(1+|z|). \tag{4.61}$$

Using (4.58),(4.59),(4.60) and (4.61) in (4.57) we write the equation of W_k as

$$\Delta W_k + \mathfrak{h}_k(\delta_k e_1)e^{U_k(z)}W_k = -\varepsilon_k a_k\cdot z e^{U_k(z)} + E_w, \quad 0 < |z| < \delta_0\varepsilon_k^{-1} \tag{4.62}$$

where

$$E_w(z) = O(\varepsilon_k)(1+|z|)^{-3}, \quad |z| < \delta_0\varepsilon_k^{-1}. \tag{4.63}$$

Since $\hat{w}_k$ obviously converges to a global harmonic function with removable singularity, we have $\hat{w}_k \to \bar{c}$ for some $\bar{c} \in \mathbb{R}$. Then we claim that

Lemma 4.4. $\bar{c} = 0$.

Proof of Lemma 4.4:

If $\bar{c}\neq 0$, we use $W_k(z)=\bar{c}+o(1)$ on $B(0,\delta_0\varepsilon_k^{-1})\setminus B(0,\frac{1}{2}\delta_0\varepsilon_k^{-1})$ and consider the projection of W_k on 1:

$$g_0(r)=\frac{1}{2\pi}\int_0^{2\pi}W_k(re^{i\theta})d\theta.$$

If we use F_0 to denote the projection to 1 of the right hand side we have, using the rough estimate of E_w in (4.63)

$$g_0''(r)+\frac{1}{r}g_0'(r)+\mathfrak{h}_k(\delta_k e_1)e^{U_k(r)}g_0(r)=F_0,\quad 0<r<\delta_0\varepsilon_k^{-1}$$

where

$$F_0(r)=O(\varepsilon_k)(1+|z|)^{-3}.$$

In addition we also have

$$\lim_{k\to\infty}g_0(\delta_0\varepsilon_k^{-1})=\bar{c}+o(1).$$

For simplicity we omit k in some notations. By the same argument as in Lemma 4.1, we have

$$g_0(r)=O(\varepsilon_k)\log(2+r),\quad 0<r<\delta_0\varepsilon_k^{-1}.$$

Thus $\bar{c}=0$. Lemma 4.4 is established. □

Based on Lemma 4.4 and standard Harnack inequality for elliptic equations we have

(4.64) $\tilde{w}_k(x)=o(1),\ \nabla\tilde{w}_k(x)=o(1),\ x\in B_3\setminus(\cup_{l=1}^N(B(e^{i\beta_l},\delta_0)\setminus B(e^{i\beta_l},\delta_0/8))).$

Equation (4.64) is equivalent to $w_k=o(\tilde{\delta}_k)$ and $\nabla w_k=o(\tilde{\delta}_k)$ in the same region.

In the next step we consider the difference between two Pohozaev identities. For $s=1,...,N$ we consider the Pohozaev identity around Q_s^k. Let $\Omega_{s,k}=B(Q_s^k,r)$ for small $r>0$. For v_k we have

$$\int_{\Omega_{s,k}}\partial_\xi(|y|^{2N}\mathfrak{h}_k(\delta_k y))e^{v_k}-\int_{\partial\Omega_{s,k}}e^{v_k}|y|^{2N}\mathfrak{h}_k(\delta_k y)(\xi\cdot\nu)$$
$$=\int_{\partial\Omega_{s,k}}(\partial_\nu v_k\partial_\xi v_k-\frac{1}{2}|\nabla v_k|^2(\xi\cdot\nu))dS. \tag{4.65}$$

where ξ is an arbitrary unit vector. Correspondingly the Pohozaev identity for V_k is

$$\int_{\Omega_{s,k}}\partial_\xi(|y|^{2N}\mathfrak{h}_k(\delta_k e_1))e^{V_k}-\int_{\partial\Omega_{s,k}}e^{V_k}|y|^{2N}\mathfrak{h}_k(\delta_k e_1)(\xi\cdot\nu)$$
$$=\int_{\partial\Omega_{s,k}}(\partial_\nu V_k\partial_\xi V_k-\frac{1}{2}|\nabla V_k|^2(\xi\cdot\nu))dS. \tag{4.66}$$

Using $w_k = v_k - V_k$ and $|w_k(y)| \le C\tilde{\delta}_k$ we have

$$\begin{aligned}&\int_{\partial\Omega_{s,k}} (\partial_\nu v_k \partial_\xi v_k - \frac{1}{2}|\nabla v_k|^2(\xi\cdot\nu))dS\\ =&\int_{\partial\Omega_{s,k}} (\partial_\nu V_k \partial_\xi V_k - \frac{1}{2}|\nabla V_k|^2(\xi\cdot\nu))dS\\ &+\int_{\partial\Omega_{s,k}} (\partial_\nu V_k \partial_\xi w_k + \partial_\nu w_k \partial_\xi V_k - (\nabla V_k\cdot\nabla w_k)(\xi\cdot\nu))dS + o(\tilde{\delta}_k).\end{aligned}$$

If we just use crude estimate: $\nabla w_k = o(\tilde{\delta}_k)$, we have

$$\begin{aligned}&\int_{\partial\Omega_{s,k}} (\partial_\nu v_k \partial_\xi v_k - \frac{1}{2}|\nabla v_k|^2(\xi\cdot\nu))dS\\ &-\int_{\partial\Omega_{s,k}} (\partial_\nu V_k \partial_\xi V_k - \frac{1}{2}|\nabla V_k|^2(\xi\cdot\nu))dS = o(\tilde{\delta}_k).\end{aligned}$$

The difference on the second terms is minor: If we use the expansion of $v_k = V_k + w_k$ and that of $\mathfrak{h}_k(\delta_k y)$ around e_1, it is easy to obtain

$$\int_{\partial\Omega_{s,k}} e^{v_k}|y|^{2N}\mathfrak{h}_k(\delta_k y)(\xi\cdot\nu) - \int_{\partial\Omega_{s,k}} e^{V_k}|y|^{2N}\mathfrak{h}_k(\delta_k e_1)(\xi\cdot\nu) = o(\tilde{\delta}_k).$$

To evaluate the first term, we use

(4.67)

$$\begin{aligned}&\partial_\xi(|y|^{2N}\mathfrak{h}_k(\delta_k y))e^{v_k}\\ =&\partial_\xi(|y|^{2N}\mathfrak{h}_k(\delta_k e_1) + |y|^{2N}\delta_k\nabla\mathfrak{h}_k(\delta_k e_1)(y-e_1) + O(\delta_k^2))e^{V_k}(1 + w_k + O(\delta_k^2\mu_k))\\ =&\partial_\xi(|y|^{2N})\mathfrak{h}_k(\delta_k e_1)e^{V_k} + \delta_k\partial_\xi(|y|^{2N}\nabla\mathfrak{h}_k(\delta_k e_1)(y-e_1))e^{V_k}\\ &+\partial_\xi(|y|^{2N}\mathfrak{h}_k(\delta_k e_1))e^{V_k}w_k + O(\delta_k^2\mu_k)e^{V_k}.\end{aligned}$$

For the third term on the right hand side of (4.67) we use the equation for w_k:

$$\Delta w_k + \mathfrak{h}_k(\delta_k e_1)e^{V_k}|y|^{2N}w_k = -\delta_k\nabla\mathfrak{h}_k(\delta_k e_1)\cdot(y-e_1)|y|^{2N}e^{V_k} + O(\delta_k^2)e^{V_k}|y|^{2N}.$$

From integration by parts we have

$$\begin{aligned}&\int_{\Omega_{s,k}} \partial_\xi(|y|^{2N})\mathfrak{h}_k(\delta_k e_1)e^{V_k}w_k\\ =&2N\int_{\Omega_{s,k}} |y|^{2N-2}y_\xi\mathfrak{h}_k(\delta_k e_1)e^{V_k}w_k\\ =&2N\int_{\Omega_{s,k}} \frac{y_\xi}{|y|^2}(-\Delta w_k - \delta_k\nabla\mathfrak{h}_k(\delta_k e_1)(y-e_1)|y|^{2N}e^{V_k} + O(\delta_k^2)e^{V_k}|y|^{2N})\\ =&-2N\delta_k\int_{\Omega_{s,k}} \frac{y_\xi}{|y|^2}\nabla\mathfrak{h}_k(\delta_k e_1)(y-e_1)|y|^{2N}e^{V_k}\\ &+2N\int_{\partial\Omega_{s,k}} (\partial_\nu(\frac{y_\xi}{|y|^2})w_k - \partial_\nu w_k\frac{y_\xi}{|y|^2}) + o(\tilde{\delta}_k)\\ =&\nabla\mathfrak{h}_k(\delta_k e_1)\Big(-16N\delta_k\pi(e^{i\beta_s}\cdot\xi)(e^{i\beta_s}-e_1) + O(\mu_k\varepsilon_k^2)\Big) + o(\tilde{\delta}_k), \qquad (4.68)\end{aligned}$$

where we have used $\nabla w_k, w_k = o(\tilde{\delta}_k)$ on $\partial\Omega_{s,k}$. For the second term on the right hand side of (4.67), we have

$$\begin{aligned}(4.69)\quad &\int_{\Omega_{s,k}} \delta_k \partial_\xi (|y|^{2N}\nabla \mathfrak{h}_k(\delta_k e_1)(y-e_1))e^{V_k}\\ =&2N\delta_k \int_{\Omega_{s,k}} y_\xi |y|^{2N-2}\nabla \mathfrak{h}_k(\delta_k e_1)(y-e_1)e^{V_k} + \delta_k \int_{\Omega_{s,k}} |y|^{2N}\partial_\xi \mathfrak{h}_k(\delta_k e_1)e^{V_k}\\ =&\nabla \mathfrak{h}_k(\delta_k e_1)\big(16N\pi\delta_k(e^{i\beta_s}\cdot\xi)(e^{i\beta_s}-e_1)+O(\mu_k\varepsilon_k^2)\big)\\ &+\delta_k\partial_\xi \mathfrak{h}_k(\delta_k e_1)(8\pi+O(\mu_k\varepsilon_k^2))+o(\tilde{\delta}_k).\end{aligned}$$

Using (4.68) and (4.69) in the difference between (4.65) and (4.66), we have

$$\delta_k\partial_\xi \mathfrak{h}_k(\delta_k e_1)(1+O(\mu_k\varepsilon_k^2)) = o(\tilde{\delta}_k).$$

Thus $\nabla \mathfrak{h}_k(\delta_k e_1) = O(\delta_k\mu_k)$. Theorem 4.1 is established. □

5. Proof of Theorem 1.1

First we prove the case $N \geq 2$. In [26] we have already proved that

$$\Delta(\log \mathfrak{h}_k)(0) = O(\delta_k^{-2}\mu_k e^{-\mu_k}) + O(\delta_k).$$

So if $\delta_k/(\mu_k^{\frac{1}{2}}\varepsilon_k) \to \infty$ there is nothing to prove. So we only consider the case that $\delta_k \leq C\mu_k^{\frac{1}{2}}\varepsilon_k$. By in this case $\varepsilon_k^{-1}\delta_k^2 \leq C\varepsilon_k^{\varepsilon}$. The whole argument of Proposition 4.1 can be employed to prove

$$(5.1)\qquad |w_k(y)| \leq C\delta_k^2\mu_k$$

for some $\varepsilon \in (0,1)$. In order to employ the same strategy of proof, one needs to have three things: first $\varepsilon_k^{-1}\delta_k^2 = O(\varepsilon_k^{\varepsilon})$. This is clear from the definition of δ_k. Second, in the proof of Lemma 4.3 we need

$$O(\delta_k^{N+2}/M_k) = o(\varepsilon_k),$$

where $M_k = \delta_k^2\mu_k$. Since $\delta_k \leq C\mu_k^{\frac{1}{2}}\varepsilon_k$ and $N \geq 1$, the required inequality holds. Thirdly, we need to have $O(\delta_k)/M_k = o(\varepsilon_k)$. This is used in (4.49). This clearly also holds. The proof of Proposition 4.1 follows, except that we don't need to consider $M_k >> \varepsilon_k$, because in this case $M_k = o(\varepsilon_k)$. Thus for $N \geq 2$ we also have (5.1).

The precise upper bound of w_k in (5.1) leads to the vanishing rate of the Laplacian estimate for $N \geq 2$ and some cases of $N = 1$: If we use

$$W_k(z) = w_k(e_l+\varepsilon_k z)/(\delta_k^2\mu_k), \quad |z| < \tau\varepsilon_k^{-1}$$

where $e_l \neq e_1$. We shall show that the projection of W_k over 1 is not bounded when $|z| \sim \varepsilon_k^{-1}$, which gives the desired contradiction.

We write the equation of w_k as

$$\Delta w_k + |y|^{2N}e^{\xi_k}w_k = (\mathfrak{h}_k(\delta_k e_1) - \mathfrak{h}_k(\delta_k y))|y|^{2N}e^{V_k}.$$

Then for $l \neq 1$,

$$
\begin{aligned}
&\Delta W_k(z) + e^{U_k} W_k(z) \\
=& a_0 e^{U_k} + a_1 z e^{U_k} + \frac{1}{2\mu_k} \Delta \mathfrak{h}_k(0) |z|^2 e^{U_k} + \frac{1}{\mu_k} R_2(\theta) |z|^2 e^{U_k} + O(\varepsilon_k^{\varepsilon}(1+|z|)^{-3}).
\end{aligned}
$$

where $a_0 = (\mathfrak{h}_k(\delta_k e_1) - \mathfrak{h}_k(\delta_k e_l))/(\delta_k^2 \mu_k)$, $a_1 = -\nabla \mathfrak{h}_k(\delta_k e_l)/(\delta_k \mu_k)$, R_2 is the collection of spherical harmonic functions of degree 2.

Let $g_k(r)$ be the projection of W_k on 1, by the same ODE analysis as before, we see that g_k satisfies

$$
g_k'' + \frac{1}{r} g_k'(r) + e^{U_k} g_k = E_k
$$

where

$$
E_k(r) = O(\varepsilon_k^{\varepsilon})(1+r)^{-3} + \frac{1}{2\mu_k} \Delta(\log \mathfrak{h}_k)(0) r^2 e^{U_k}.
$$

Using the same argument as in Lemma 4.1, we have

$$
g_k(r) \sim \Delta(\log \mathfrak{h}_k)(0)(\log r)^2 \mu_k^{-1}, \quad r > 10.
$$

Clearly if $\Delta(\log \mathfrak{h}_k(0) \neq 0$ we obtain a violation of the bound of w_k for $r \sim \varepsilon_k^{-1}$. Theorem 1.1 for $N \geq 2$ is proved under the assumption

$$
\varepsilon_k^{-1} |Q_s^k - e^{i\beta_s}| \leq \varepsilon_k^{\varepsilon}, \quad s = 1, \ldots, N. \tag{5.2}
$$

We need this assumption because the ξ_k function that comes from the equation of w_k needs to tend to U after scaling. From (3.13) in [26], $|Q_s^k - e^{i\beta_s}| = O(\delta_k^2) + O(\mu_k e^{-\mu_k})$. If $\delta_k^2 \varepsilon_k^{-1} \geq C$, the argument in Theorem 4.1 cannot be used because either ξ_k does not tend U or $c_0 = 0$ cannot be proved. For $N \geq 2$, this is not a problem because we only consider $\delta_k \leq C \mu_k^{\frac{1}{2}} \varepsilon_k$.

Next we prove Theorem 1.1 for $N = 1$ and $\delta_k \leq \mu_k \varepsilon_k$. The reader could see immediately that the same proof for the case $N \geq 2$ still works. So the only remaining case is

Proof of Theorem 1.1 for $N = 1$ and $\delta_k \geq \mu_k \varepsilon_k$.

In this case we write the equation of w_k as

$$
\Delta w_k + |y|^2 \mathfrak{h}_k(\delta_k y) e^{v_k} - |y|^2 \mathfrak{h}_k(\delta_k e_1) e^{V_k} = 0.
$$

From $0 = \nabla w_k(e_1)$ we have

$$
0 = \int_{\Omega_k} \nabla_1 G_k(e_1, \eta) |\eta|^2 (\mathfrak{h}_k(\delta_k \eta) e^{v_k} - \mathfrak{h}_k(\delta_k e_1) e^{V_k}) d\eta + O(\delta_k^3) \tag{5.3}
$$

Note that v_k is close to another global solution $\bar{V}_k$ which matches with a local maximum of v_k at Q_2^k. Evaluating the right hand side of (5.3) we have

$$
\nabla_1 G_k(e_1, Q_2^k) - \nabla_1 G_k(e_1, e^{i\pi}) = O(\varepsilon_k^2 \mu_k) + O(\delta_k^3).
$$

This expression gives

$$
Q_2^k - e^{i\pi} = O(\delta_k^3) + O(\mu_k \varepsilon_k^2).
$$

This estimate will lead to a better estimate of w_k outside the two bubbling disks. From the Green's representation for w_k we now obtain

$$w_k(y)=\int_{\Omega_k}(G_k(y,\eta)-G_k(e_1,\eta))|\eta|^2(\mathfrak{h}_k(\delta_k\eta)e^{v_k(y)}-\mathfrak{h}_k(\delta_k e_1)e^{V_k})d\eta+O(\delta_k^2)$$

where the last term $O(\delta_k^2)$ comes from the oscillation of w_k on $\partial\Omega_k$. Then we have

$$\begin{aligned}w_k(y)&=-\frac{1}{2\pi}\int_{\Omega_k}\log\frac{|y-\eta|}{|e_1-\eta|}|\eta|^2(\mathfrak{h}_k(\delta_k\eta)-\mathfrak{h}_k(\delta_k e_1)e^{V_k})+O(\delta_k^2)\\&=-4\log\frac{|y-Q_2^k|}{|e_1-Q_2^k|}+4\log\frac{|y-e^{i\pi}|}{2}+O(\delta_k^2\mu_k).\end{aligned}$$

By $|Q_2^k-e^{i\pi}|=O(\delta_k^2)$ we see that $w_k(y)=O(\delta_k^2)$ on $|y-e^{i\pi}|=\tau$.

The standard point-wise estimate for singular equation (see [30, 17]) gives

$$\begin{aligned}&v_k(Q_2^k+\varepsilon_k z)+2\log\varepsilon_k\\=&\log\frac{e^{\mu_k}}{(1+\frac{e^{\mu_k}}{8\mathfrak{h}_k(\delta_k Q_k)}|z|^2)^2}+\phi_1^k+C\delta_k^2\Delta(\log\mathfrak{h}_k)(0)(\log(1+|z|))^2,\quad |z|\sim\varepsilon_k^{-1}.\end{aligned}$$

$$\begin{aligned}&V_k(e^{i\pi}+\varepsilon_k z)+2\log\varepsilon_k\\=&\log\frac{e^{\mu_k}}{(1+\frac{e^{\mu_k}}{8\mathfrak{h}_k(\delta_k e_1)}|z|^2)^2}+\phi_2^k+O(\varepsilon_k^2(\log\varepsilon_k)^2),\quad |z|\sim\varepsilon_k^{-1}.\end{aligned}$$

Thus

$$w_k(Q_2^k+\varepsilon_k z)=O(\varepsilon_k^2(\log\varepsilon_k)^2)+\phi_1^k-\phi_2^k+C\Delta(\log\mathfrak{h}_k)(0)\delta_k^2(\log(1+|z|)^2),$$

for $|z|\sim\varepsilon_k^{-1}$. Taking the average around the origin, the spherical averages of the two harmonic functions are zero and $O(\delta_k^2)$ respectively, since they take zero at the origin and a point at most $O(\delta_k^2)$ from the origin. So the spherical average of w_k is comparable to

$$\Delta(\log\mathfrak{h}_k)(0)\delta_k^2(\log\varepsilon_k)^2$$

for $|z|\sim\varepsilon_k^{-1}$. Thus we know $\Delta(\log\mathfrak{h}_k)(0)=o(1)$ because $w_k=O(\delta_k^2\mu_k)$ in this region, Theorem 1.1 is established for all the cases. $\Box$

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