MATH 516-101, 2021-2022 Homework Six

Due Date: By 6pm on December 8, 2021

10pts each unless otherwise stated.

1. (20pts) Let u be a weak sub-solution of

$$-\sum_{i,j} \partial_{x_j}(a^{ij}\partial_{x_i} u) + c(x)u = f$$

where $\theta \le (a^{ij}) \le C_2 < +\infty$. Suppose that $c(x) \in L^{\frac{n}{2}}(B_1), f \in L^q(B_1)$ where $q > \frac{n}{2}$. Show that there exists a generic constant $\epsilon_0 > 0$ such that if $\int_{B_1} |c|^{\frac{n}{2}} dx \le \epsilon_0$, then

$$\sup_{B_{1/2}} u^+ \le C(\|u^+\|_{L^2(B_1)} + \|f\|_{L^q(B_1)})$$

Hint: following the De Giorgi-Nash-Moser's iteration procedure.

2. Show that $u = \log|x|$ is in $H^1(B_1)$, where $B_1 = B_1(0) \subset R^3$ and that it is a weak solution of

$$-\Delta u + c(x)u = 0$$

for some $c(x) \in L^{\frac{3}{2}}(B_1)$ but u is not bounded.

3. Assume that $u \in H^1(\Omega)$ is a bounded weak solution of

$$-\sum_{i,j=1}^{n} \partial_j(a^{ij}\partial_i u) = 0 \text{ in } \Omega$$

Show that $\mathbf{u^2}$ is a weak sub-solution.

4. Let $u \in H_0^1(\Omega)$ be a weak solution of

$$-\Delta u = |u|^{q-1}u \ \text{ in } \Omega; u = 0 \text{ on } \partial\Omega$$

where $1 < q < \frac{n+2}{n-2}$. Without using Moser's iteration Lemma, use the $W^{2,p}-$ theory only to show that $u \in L^\infty$ and then use Schauder theory to show that $u \in C^{2,\alpha}(\Omega)$.

5. Assume that u is a smooth solution of

$$Lu = -a^{ij}u_{ij} = f \ \text{ in } U$$

$$\mathbf{u = 0} \ \text{ on } \partial\mathbf{U}$$

A barrier at x^0 is a C^2 function w such that

$$Lw \ge 1 \text{ in } U \, w(x^0) = 0, w \ge 0 \text{ on } \partial U$$

Show that if w is a barrier at x^0, there exists a constant C such that

$$|Du(x^0)| \le C|\frac{\partial w}{\partial \nu}(x^0)|$$

Hint: Since $u = 0$ on $\partial\Omega$, $|Du(x^0)| = |\frac{\partial u}{\partial \nu}(x^0)|$.

6. Let u be a smooth function satisfying

$$-\Delta u + u = f(x), |u| \le e^{\frac{1}{2}|x|}, \ \text{ in } R^n$$

where

$$|f(x)| \le Ce^{-|x|}$$

Deduce from maximum principle that u actually decays

$$|u(x)| \leq Ce^{-\frac{1}{2}|x|}$$

7. Let Ω be a smooth bounded domain in R^n. Assume that u is a smooth solution of

$$\partial_t u - \Delta u = 0 \text{ in } \Omega \times (0,T); u\|_{\partial\Omega} = 0, u(x,0) = g$$

Prove the exponential decay estimate

$$\|u(\cdot,t)\|_{L^2(\Omega)} \leq e^{-\lambda_1 t}\|g\|_{L^2(\Omega)}, t \geq 0$$

where $\lambda_1 > 0$ is the principle eigenvalue of $-\Delta$ with Dirichlet boundary condition.

8. (20pts) Let Ω be a smooth bounded domain in R^n, and $\{w_k\}_{k=1}^{\infty}$ be an orthonormal basis of $H_0^1(\Omega)$. Show that the set of functions

$$\mathcal{F} = \{v(t) = \sum_{k=1}^{m} d^k(t)w_k \mid d^k(t) \in C^1([0,T]), m \in \mathbb{N}\}$$

is dense in $L^2(0,T;H_0^1(\Omega))$.