MATH 516-101 Homework Four 2021-2022
Due Date: by 6pm, November 3, 2021
10pts each

1. Let $\Omega = (-1,1)$ and $u(x) = x_+$. Show that $u$ can not be approximated by $C^\infty(\Omega)$ in $W^{1,\infty}(\Omega)$ norm. (SO density theorems are not true for $p + +\infty$.)

2. Let $u \in C^\infty(\bar{R}^n_+)$. Extend $u$ to $Eu$ on $R^n$ such that

$$Eu = u, x \in R^n_+; Eu \in C^{3,1}(R^n) \cap W^{4,p}(R^n); \|Eu\|_{W^{4,p}} \leq \|u\|_{W^{4,p}}$$

Here $R^n_+ = \{(x', x_n); x_n > 0\}$ and $C^{3,1} = \{u \in C^3, D^\alpha u \text{ is Lipschitz}, |\alpha| = 3\}$.

3. Let $\Omega = \{(x,y) \mid x^2 + y^2 < 1\}$ and $\Omega_0 = \Omega \backslash \{(0,0)\}$. Consider the function $f(x) = 1 - |x|$. Prove that $f \in W^{1,p}_0(\Omega)$, if $1 \leq p < \infty$; $f \in W^{1,p}_0(\Omega_0)$, if $1 \leq p \leq 2$; $f \notin W^{1,p}_0(\Omega_0)$, if $2 < p \leq \infty$.

4. (a) If $n = 1$ and $u \in W^{1,1}(\Omega)$ then $u \in L^\infty$ and $u$ is continuous. (b) If $n > 1$, find an example of $u \in W^{1,n}(B_1)$ and $u \notin L^\infty$.

5. Let $u \in H^1_0((0,1))$. Show that there is $w \in C([0,1])$ with $w(0) = w(1) = 0$ such that $u = w$ almost everywhere in $[0,1]$.

6. Show that for any $u \in C^\infty_0(R^n)$ we have Hardy's inequality

$$\int_{R^n} \frac{u^2}{|x|^2} dx \leq \frac{(n-2)^2}{4} \int_{R^n} |\nabla u|^2$$

Hint: integrate $|\nabla| \lambda \frac{x}{|x|} u|^2$.

7. Fix $\alpha > 0, 1 < p < +\infty$ and let $U = B_1(0)$. Show that there exists a constant C, depending on $n$ and $\alpha$ such that

$$\int_U u^p dx \leq C \int_U |\nabla u|^p$$

provided

$$u \in W^{1,p}(U),\ |\{x \in U | u(x) = 0\}| \geq \alpha$$

8. (a) Show that $W^{1,2}(R^N) \subset L^2(R^N)$ is not compact. (b). Let $n > 4$. Show that the embedding $W^{2,2}(U) \to L^{\frac{2n}{n-4}}(U)$ is not compact; (b) Describe the embedding of $W^{3,p}(U)$ in different dimensions. State if the embedding is continuous or compact.

9. (a) Let $u \in W^{1,2}_r = H^1_r = \{u \in W^{1,2}(R^n) \mid u = u(r)\}$. Show that $|u(r)| \leq C\|u\|_{W^{1,2}} r^{-\frac{n-1}{2}}$. ( b) Show that for $n \geq 2$, the embedding $W^{1,2}_r \subset L^p$ is compact when $2 < p < \frac{2n}{n-2}$. (c) Let $u = \mathcal{D}^{1,2}_r = \{\int |\nabla u|^2 < +\infty; u = u(r)\}$. Show that $D^{1,2}_r \subset L^{\frac{2n}{n-2}}$ and $|u(r)| \leq C\|\nabla u\|_{L^2} r^{-\frac{n-2}{2}}$. However $D^{1,2}_r \subset L^{\frac{2n}{n-2}}$ is not compact.

10. Let $U = (-1,1)$. Show that the dual space of $H^1(U)$ is isomorphic to $H^{-1}(U) + E^*$ where $E^*$ is the two dimensional subspace of $H^1(U)$ spanned by the orthogonal vectors $\{e^x, e^{-x}\}$.