MATH 516-101 2018-2019 Homework THREE
Due Date: by 5pm, October 26, 2018

1. Consider the following function

$$u(x) = \frac{1}{|x|^\gamma}$$

in $\Omega = B_1(0)$. Show that if $\gamma + 1 < n$, the weak derivatives are given by

$$\partial_j u = -\gamma \frac{x_j}{|x|^{\gamma+2}}$$

i.e., you need to show rigorously that

$$\int u \partial_j \phi = \int \phi \gamma \frac{x_j}{|x|^{\gamma+2}}$$

For the condition on γ such that $u \in W^{1,p}$ or $u \in W^{2,p}$.

2. Find the weak derive for the following function $u : R \to R$, if exists:

$$(a) \quad u(x) = \begin{cases} 1 - |x|, & \text{for } |x| \le 1 \\ 0, & \text{for } |x| > 1 \end{cases} \qquad (b) \quad u(x) = \begin{cases} 1, & \text{for } x = 0 \\ 0, & \text{for } x \neq 0, \end{cases} \qquad (c) \quad u(x) = \begin{cases} 1, & \text{for } |x| \le 1 \\ 0, & \text{for } |x| > 1 \end{cases}$$

3. Let $\eta(t) = 1$ for $t \le 0$ and $\eta(t) = 0$ for $t > 1$. Let $f \in W^{k,p}(R^n)$ and $f_k = f\eta(|x| - k)$. Show that $\|f_k - f\|_{W^{k,p}} \to 0$ as $k \to +\infty$. As a consequence show that $W^{k,p}(R^n) = W_0^{k,p}(R^n)$.

4. Let $u \in C^\infty(\bar{R}^n_+)$. Extend u to Eu on R^n such that

$$Eu = u, x \in R^n_+; Eu \in C^{3,1}(R^n) \cap W^{4,p}(R^n); \|Eu\|_{W^{4,p}} \le \|u\|_{W^{4,p}}$$

Here $R^n_+ = \{(x', x_n); x_n > 0\}$ and $C^{3,1} = \{u \in C^3, D^\alpha u \text{ is Lipschitz}, |\alpha| = 3\}$.

5. (a) If $n = 1$ and $u \in W^{1,1}(\Omega)$ then $u \in L^\infty$ and u is continuous. (b) If $n > 1$, find an example of $u \in W^{1,n}(B_1)$ and $u \notin L^\infty$.

6. Prove the following Poincare type inequality: Suppose that $\Omega \subset \{a < x_1 < b\}$. Then for $u \in W_0^{1,2}(\Omega)$ it holds that

$$\|u\|_{L^2(\Omega)} \le 2(b - a)\|\partial_{x_1} u\|_{L^2(\Omega)}$$

7. Suppose $u : (a, b) \to R$ and the weak derivative exists and satisfies

$$Du = 0 \ a.e. \text{ in } (a, b)$$

Prove that u is constant a.e. in (a, b).