MATH400-101-2022 Homework Assignment 4 (Due Date: October 16, 2022, by 11pm)

Homework is admitted until 11pm on October 16, 2022. Graded homework is placed in a cardboard box outside my office for one week for you to pick up. Afterwards unclaimed homework is moved to a drawer of a file cabinet near my office. Your assignments are organized in the alphabetic order of last names. For other people's convenience, please do not change this order when you pick up your assignment.

1. (10pts) Show that the following differential equation

$$u_{xx} = f(x), 0 < x < 1; \quad u(0) = a, u^{'}(1) = b$$

is well-posed.

2. (10pts) (a) State the well-posedness criteria for the following backward heat equation

$$\begin{cases} u_t = -u_{xx}, t > 0, -\infty < x < +\infty, \\ u(x,0) = \phi(x). \end{cases}$$

(b) Is the above problem well-posed? For stability, try functions of the type $\frac{1}{n}e^{n^2 t}\sin(nx)$.

3. (20pts) For the following PDEs, determine its type and use the change of variable to transform it into one of the three standard form

(a) $u_{xx} + 9u_{xy} = 0$, (b) $9u_{xx} - 6u_{xy} + u_{yy} - 2u_x + u_y = 0$, (c) $4u_{xx} - u_{xy} + u_{yy} = 0$.

4.(10pts) Solve the following wave equation:

$$u_{tt} - 4u_{xx} = 0$$

$$(a) u(x,0) = e^x, u_t(x,0) = \sin(x), \quad (b)\ u(x,0) = \sin(x),\ u_t(x,0) = \cos(2x)$$

5. (20pts) For the following wave equation:

$$u_{tt} - u_{xx} = 0$$

$$u(x,0) = 0, u_t(x,0) = \begin{cases} 1, & 0 < x < 1; \\ 0, & \text{otherwise} \end{cases}$$

Find $u(x, \frac{1}{2}), u(x, \frac{5}{2})$.

6. (10pts) Solve the following wave equation

$$u_{tt} = 4u_{xx}, x > 0, t > 0$$

$$u(x,0) = x, u_t(x,0) = e^x, x > 0$$

$$u(0,t) = 1, t > 0$$

7. (15pts) Find the solution for

$$\begin{cases} u_{tt} + u_{tx} - 2u_{xx} = 0, t > 0, -\infty < x < +\infty, \\ u(x,0) = \sin(x), u_t(x,0) = e^x, -\infty < x < +\infty \end{cases}$$

8. (15pts) Find the solution for

$$\begin{cases} u_{tt} + 3u_{tx} - 4u_{xx} = 0, t > 0, 0 < x < +\infty, \\ u(x,0) = 0, u_t(x,0) = \cos x, \mathbf{0} < x < +\infty \\ u(0,t) = 0, 0 < t < +\infty \end{cases}$$

The following two formulas can be used: 1) the general solutions to

$$au_{tt} + bu_{tx} + cu_{xx} = 0$$

where

$$a\partial_t^2 + b\partial_t\partial_x + c\partial_x^2 = (a_1\partial_x + b_1\partial_t)(a_2\partial_x + b_2\partial_t)$$

are

$$u(x,t) = F(b_1x - a_1t) + G(b_2x - a_2t)$$

2) d'ALembert's formula: the solution to

$$u_{tt} = c^2u_{xx}$$

$$u(x,0) = \phi(x), u_t(x,0) = \psi(x)$$

is

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds$$