**MATH400-101 Homework Assignment 7, 2019-2020 (Due Date: by 6pm, December 3, 2019)**

Last homework. Good luck on finals

Please either hand in to my office or send it by email by 6pm of December 3rd, 2019. The solutions will be posted on my website on December 3rd. You can pick it up after December 9th.

1. (10pts) Use the method of separation of variables to find solutions to

$$u_t = k(u_{xx} + u_{yy}), 0 < x < \pi, 0 < y < \pi$$

$$u(x, y, 0) = \sin(x), 0 < x < \pi, 0 < y < \pi$$

$$u(0, y, t) = u(\pi, y, t) = u_y(x, 0, t) = u_y(x, \pi, t) = 0$$

2. (10pts) Consider the following Laplace equation in a disk

$$u_{xx} + u_{yy} = 0 \ \text{ in } \{x^2 + y^2 < 4\}$$

$$u(x, y) = 2y^2 - 3xy \text{ for } \ x^2 + y^2 = 4,$$

(i) Without solving the equation, find out the maximum and minimum values of $u$, and the value of $u(0, 0)$.
(ii) Use the method of separation of variables to find $u$.

3. (10pts) Consider the following Laplace equation in an annulus

$$u_{xx} + u_{yy} = 0 \ \text{ in } \{1 < x^2 + y^2 < 4\}$$

$$u(x, y) = xy \text{ for } \ x^2 + y^2 = 1,$$

$$u(x, y) = x, \text{ for } \ x^2 + y^2 = 4.$$

Use the method of separation of variables to find $u$.

4. (20pts) Solve the following exterior domain problem

$$\begin{cases} u_{xx} + u_{yy} = 0, \quad \text{for } x^2 + y^2 > 1 \\ u(x, y) = 2y^2 - 3x \quad \text{for } x^2 + y^2 = 1 \\ u(x, y) \text{ is bounded} \end{cases}$$

What can you say about uniqueness?

5. (20pts) Use the method of separation of variables to solve the Laplace equation in the quarter plane

$$u_{xx} + u_{yy} = 0 \ \text{ in } \{x^2 + y^2 > 1, x > 0, y > 0\}$$

$$u_x(x, 0) = 0, \text{ for } \ x^2 + y^2 > 1, x > 0,$$

$$u(0, y) = 0, \text{ for } \ x^2 + y^2 > 1, y > 0,$$

$$u = 1 \ \text{ on } x^2 + y^2 = 1, x > 0, y > 0$$

$$u \text{ is bounded}$$

Hint: use polar coordinates

6. (10pts) Solve the wave equation using the Bessel functions of order $n$

$$\begin{cases} u_{tt} = c^2(u_{rr} + \frac{1}{r}u_r + \frac{u_{\theta\theta}}{r^2}), \ 0 \le r < 1, 0 \le \theta < 2\pi, t > 0 \\ u(1, \theta, t) = 0, \ t > 0, 0 \le \theta < 2\pi \\ u(r, \theta, 0) = r\cos(\theta), \quad u_t(r, \theta, 0) = 1 \end{cases}$$

Here the Bessel function of order $n$ of first kind, denoted by $J_n(z)$, is given as the solution to

$$J'' + \frac{1}{z}J' + J - \frac{n^2}{z^2}J = 0, z > 0, \quad J(z) \sim z^n, \text{ as } z \to 0$$

7. (10pts) Solve the diffusion equation using the Bessel functions of order $n$

$$\begin{cases} u_t = k\Delta u, \ x^2 + y^2 < 1 t > 0 \\ u(x, y, t) = 0, t \geq 0, \ \text{on } x^2 + y^2 = 1 \\ u(x, y, 0) = 1 + xy \end{cases}$$

Hint: use polar coordinate $\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{u_{\theta\theta}}{r^2}$.

8. (10pts) Find an infinite series representation (in terms of the Bessel function) for the diffusion equation problem

$$\begin{cases} u_t = k(u_{rr} + \frac{1}{r} u_r + u_{zz}), \ 0 \leq r < a, 0 < z < b, \ t > 0 \\ u(a, z, t) = 0, u(r, 0, t) = 0, \ u(r, b, t) = 0 \\ u(r, z, 0) = \phi(r, z) \end{cases}$$