MATH400-101, 2019-2020 Homework Assignment 6 (Due Date: by 6pm of November 25, 2019)

1. (30pts) Put the following three problems in standard Sturm-Liouville form, identify the weight function $w(x)$, and calculate the eigenvalues and eigenfunctions. (You can use Bessel function of order zero $J_0(r)$: $J_0^{''} + \frac{1}{r} J_0^{'} + J_0 = 0, J_0(0) = 1$.) Write down the general formula for the expansion of a general function $f(x)$ in terms of the eigenfunctions.

$$(a) \quad x^2 X^{''} + x X^{'} + \lambda X = 0, 1 < x < 2, \quad X(1) = 0, \quad X(2) = 0$$

$$(b) \quad X^{''} - 2X^{'} + \lambda X = 0, 0 < x < 1; \quad X(0) = 0, \quad X(1) = 0; \qquad (c) \quad X^{''} + \frac{1}{x} X^{'} + \lambda X = 0, 0 < x < 1, X^{'}(1) + X(1) = 0$$

2. (10pts) Solve the following diffusion equation

$$\begin{cases} u_t = x^2 u_{xx} + x u_x, \ 1 < x < 2 \\ u(x,0) = 1, \\ u(1,t) = u(2,t) = 0 \end{cases}$$

You are allowed to use results in Problem 1.

3. (10pts) Solve the following wave equation

$$\begin{cases} u_{tt} = u_{xx} - 2u_x, \ 0 < x < 1 \\ u(x,0) = x, u_t(x,0) = 0 \\ u(0,t) = u(1,t) = 0 \end{cases}$$

You are allowed to use results in Problem 1.

4. (10pts) Solve the following wave equation

$$\begin{cases} u_t = u_{rr} + \frac{1}{r} u_r \ 0 < r < 1 \\ u(r,0) = r^2 \\ u_r(0,t) = 0, u_r(1,t) + u(1,t) = 0 \end{cases}$$

You are allowed to use results in Problem 1. Write your solution in terms of Bessel function of order zero J_0.

5.(10pts) Use the method of separation of variables to solve

$$\begin{cases} u_t = u_{xx} + e^t \sin(3x), \ 0 < x < \pi \\ u(x,0) = \sin(2x) \\ u(0,t) = t, \ u(\pi,t) = 0 \end{cases}$$

6.(10pts) Use the method of separation of variables to solve

$$\begin{cases} u_{tt} = u_{xx} + e^{-t} \sin(x), \ 0 < x < \pi \\ u(x,0) = \sin(3x), u_t(x,0) = 0 \\ u(0,t) = 1, \ u(\pi,t) = 0 \end{cases}$$

7.(20pts) (a) Use the method of separation of variables to solve

$$\begin{cases} u_{xx} + u_{yy} = 0, 0 < x < \pi, \ 0 < y < \pi, \\ u(0,y) = u_x(\pi,y) = u(x,0) = 0, \\ u(x,\pi) = \sin \frac{x}{2} - 2 \sin \frac{3x}{2} \end{cases}$$

(b) Use divergence theorem to show the solutions to (a) are unique.