Lecture 19: Heat conduction with distributed sources/sinks

(*Compiled 2 November 2017*)

In this lecture we consider heat conduction problems in which there is a distributed source or sink function $s(x,t)$ that applies throughout the domain. We first consider the case in which the source is time-independent, i.e., $s(x,t) = s(x)$. In this case the effect of the source can be dealt with entirely by determining an appropriate steady-state solution. Using this particular solution, we can reduce the problem to one of the standard homogeneous boundary value problems we encountered when we introduced separation of variables. Secondly, we consider a fully time-dependent source. In this case we have to resort to the method of eigenfunction expansions.

Key Concepts: Distributed sources or sinks, Particular Solutions, Steady state Solutions; Separation of variables, Eigenvalues and Eigenfunctions, Method of Eigenfunction Expansions.

19 Heat Conduction Problems with distributed sources

19.1 Heat Conduction Problems with distributed time-independent sources

Example 19.1 *A bar with an external heat source $s(x) = x$.*

$$u_t = \alpha^2 u_{xx} + x \qquad 0 < x < L \tag{19.1}$$

$$\text{BC: } u(0,t) = 0 \quad u(L,t) = B \tag{19.2}$$

$$\text{IC: } u(x,0) = g(x). \tag{19.3}$$

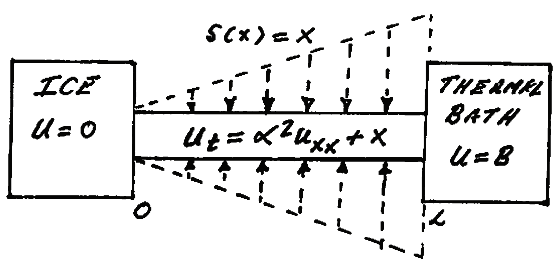

FIGURE 1. Bar subject to a time-independent heat source distributed along its length with inhomogeneous Dirichlet BC

Steady state problem $u_t = 0$:

$$\begin{aligned} &0 = \alpha^2 u''_\infty + x \\ &u_\infty(0) = 0 \quad u_\infty(L) = B \end{aligned} \tag{19.4}$$

$$\begin{aligned} &u''_\infty = -\frac{x}{\alpha^2} \quad u'_\infty = -\frac{x^2}{2\alpha^2} + a \quad u_\infty = -\frac{x^3}{6\alpha^2} + ax + b \\ &u_\infty(0) = b = 0 \quad u_\infty(L) = -\frac{L^3}{6\alpha^2} + aL = B \Rightarrow a = \frac{B}{L} + \frac{L^2}{6\alpha^2} \end{aligned} \tag{19.5}$$

Therefore

$$u_\infty(x) = -\frac{x^3}{6\alpha^2} + \left(\frac{B}{L} + \frac{L^2}{6\alpha^2}\right)x = x\left\{\frac{B}{L} + \frac{1}{6\alpha^2}(L^2 - x^2)\right\}. \tag{19.6}$$

Let $u(x,t) = u_\infty(x) + v(x,t)$.

$$\begin{array}{lllll} u_t = \alpha^2 u_{xx} + x & \Rightarrow & (\cancel{u_\infty} + v)_t = \alpha^2(\cancel{u_\infty} + v)_{xx} + \cancel{x} & \Rightarrow & v_t = \alpha^2 v_{xx} \\ u(0,t) = 0 & \Rightarrow & \cancel{u_\infty}(0) + v(0,t) = 0 & \Rightarrow & v(0,t) = 0 \\ u(L,t) = B & \Rightarrow & u_\infty(L) + v(L,t) = B & \Rightarrow & v(L,t) = 0 \\ u(x,0) = g(x) & \Rightarrow & u_\infty(x) + v(x,0) = g(x) & \Rightarrow & v(x,0) = g(x) - u_\infty(x). \end{array} \tag{19.7}$$

Separation of variables yields:

$$v(x,t) = \sum_{n=1}^{\infty} b_n \mathrm{e}^{-\left(\frac{n\pi}{L}\right)^2 \alpha^2 t} \sin\left(\frac{n\pi x}{L}\right) \tag{19.8}$$

where

$$b_n = \frac{2}{L}\int_0^L \{g(x) - u_\infty(x)\} \sin\left(\frac{n\pi x}{L}\right) dx. \tag{19.9}$$

Therefore

$$\begin{array}{c} u(x,t) = x\left\{\frac{B}{L} + \frac{1}{6\alpha^2}(L^2 - x^2)\right\} + \sum_{n=1}^{\infty} b_n \mathrm{e}^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right) \\ \uparrow \qquad\qquad\qquad\qquad \uparrow \\ \text{steady} \qquad\qquad\qquad \text{transient} \end{array} \tag{19.10}$$

Note:

$$\lim_{x\to\infty} u(x,t) = x\left\{\frac{B}{L} + \frac{1}{6\alpha^2}(L^2 - x^2)\right\}. \tag{19.11}$$

19.2 Distributed time-dependent heat sources/sinks - eigenfunction expansions

Example 19.2 *A Bar with a Time-Varying External Heat Source:*

$$u_t = \alpha^2 u_{xx} + \mathrm{e}^{-t}\sin\left(\frac{2\pi x}{L}\right) \quad 0 < x < L, \quad t > 0 \tag{19.12}$$

$$\textit{BC: } u(0,t) = 0; \quad u(L,t) = L \tag{19.13}$$

$$\textit{IC: } u(x,0) = x. \tag{19.14}$$

Consider the function $w(x) = x$ which satisfies the BC as well as the homogeneous version of the PDE.

Now let $u(x,t) = w(x) + v(x,t)$.

$$u_t = (\cancel{w} + v)_t = \alpha^2(\cancel{w} + v)_{xx} + \mathrm{e}^{-t}\sin\left(\frac{2\pi x}{L}\right) \tag{19.15}$$

$$\Rightarrow v_t = \alpha^2 v_{xx} + \mathrm{e}^{-t}\sin\left(\frac{2\pi x}{L}\right) \tag{19.16}$$

$$u(0,t) = \cancel{w(0)} + v(0,t) = 0 \Rightarrow v(0,t) = 0 \tag{19.17}$$

$$u(L,t) = \cancel{w(L)} + v(L,t) = \cancel{L} \Rightarrow v(L,t) = 0 \tag{19.18}$$

$$x = u(x,0) = w(x) + v(x,0) = x + v(x,0) \Rightarrow v(x,0) = 0. \tag{19.19}$$

Now assume that $v(x,t) = \displaystyle\sum_{n=1}^{\infty} \hat{v}_n(t)\sin\left(\frac{n\pi x}{L}\right)$.

$$\frac{\partial v}{\partial t} = \sum_{n=1}^{\infty} \frac{d\hat{v}_n}{dt}(t)\sin\left(\frac{n\pi x}{L}\right), \quad \frac{\partial^2 v}{\partial x^2} = \sum_{n=1}^{\infty} \hat{v}_n(t)\left\{-\left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi x}{L}\right)\right\}. \tag{19.20}$$

Therefore

$$v_t - \alpha^2 v_{xx} - \mathrm{e}^{-t}\sin\left(\frac{2\pi x}{L}\right) = \sum_{n=1}^{\infty}\left\{\frac{d\hat{v}_n}{dt} + \alpha^2\left(\frac{n\pi}{L}\right)^2\hat{v}_n - \mathrm{e}^{-t}\delta_{2n}\right\}\sin\left(\frac{n\pi x}{L}\right) = 0. \tag{19.21}$$

Therefore

$$\frac{d\hat{v}_n}{dt} + \alpha^2\left(\frac{n\pi}{L}\right)^2\hat{v}_n = \mathrm{e}^{-t}\delta_{2n} \tag{19.22}$$

$$\frac{d}{dt}\left[\mathrm{e}^{\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\hat{v}_n\right] = \mathrm{e}^{\left[\alpha^2\left(\frac{n\pi}{L}\right)^2 - 1\right]t}\delta_{2n}. \tag{19.23}$$

Therefore

$$\mathrm{e}^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\hat{v}_n = \frac{\mathrm{e}^{\left[\alpha^2\left(\frac{n\pi}{L}\right)^2 - 1\right]t}}{\alpha^2\left(\frac{n\pi}{L}\right)^2 - 1}\delta_{2n} + c_n \qquad c_n \text{ arbitrary} \tag{19.24}$$

$$\hat{v}_n(t) = \frac{\mathrm{e}^{-t}\delta_{2n}}{\alpha^2\left(\frac{n\pi}{L}\right)^2 - 1} + \mathrm{e}^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}c_n \tag{19.25}$$

$$v(x,0) = 0 \Rightarrow \hat{v}_n(0) = 0 = \frac{\delta_{2n}}{\alpha^2\left(\frac{n\pi}{L}\right)^2 - 1} + c_n \Rightarrow \tag{19.26}$$

$$c_n = \begin{cases} 0 & n \neq 2 \\ -\frac{1}{\alpha^2\left(\frac{2\pi}{L}\right)^2 - 1} & n = 2 \end{cases}$$

$$v(x,t) = \frac{1}{\alpha^2\left(\frac{2\pi}{L}\right)^2 - 1}\left\{\mathrm{e}^{-t} - \mathrm{e}^{-\alpha^2\left(\frac{2\pi}{L}\right)^2 t}\right\}\sin\left(\frac{2\pi x}{L}\right) \tag{19.27}$$

$$u(x,t) = x + v(x,t) = x + \left(\frac{\mathrm{e}^{-t} - \mathrm{e}^{-\alpha^2\left(\frac{2\pi}{L}\right)^2 t}}{\alpha^2\left(\frac{2\pi}{L}\right)^2 - 1}\right)\sin\left(\frac{2\pi x}{L}\right).$$

Example 19.3 *A bar with a general external heat source $s(x,t)$*

$$u_t = \alpha^2 u_{xx} + s(x,t) \tag{19.28}$$

$$\text{BC: } u(0,t) = A \quad u(L,t) = B \tag{19.29}$$

$$\text{IC: } u(x,0) = f(x). \tag{19.30}$$

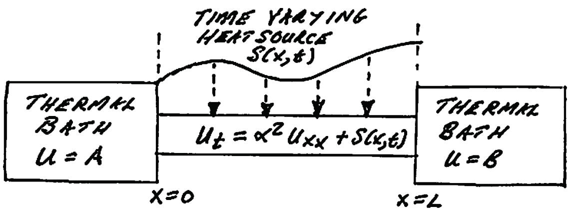

FIGURE 2. Bar subject to a time dependent heat source distributed along its length with inhomogeneous Dirichlet BC

We look for a particular solution: $w(x,t)$ by expanding $s(x,t)$ as a Sine Series. Note that the sine functions are the eigenfunctions that correspond to the homogeneous form of the BC in (19.29). Thus if we add $w(x,t)$ to a solution of (19.28)-(19.29) without the source (i.e. with $s(x,t) = 0$) we will not affect the BC.

(1) Eigenfunction Expansion:

Let

$$s(x,t) = \sum_{n=1}^{\infty} \hat{s}_n(t) \sin\left(\frac{n\pi x}{L}\right) \tag{19.31}$$

where

$$\hat{s}_n(t) = \frac{2}{L} \int_0^L s(x,t) \sin\left(\frac{n\pi x}{L}\right) dx. \tag{19.32}$$

If we assume

$$w(x,t) = \sum_{n=1}^{\infty} \hat{w}_n(t) \sin\left(\frac{n\pi x}{L}\right) \tag{19.33}$$

then

$$w_t = \sum_{n=1}^{\infty} \hat{w}_n'(t) \sin\left(\frac{n\pi x}{L}\right) \tag{19.34}$$

$$w_{xx} = -\sum_{n=1}^{\infty} \hat{w}_n \left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi x}{L}\right). \tag{19.35}$$

Therefore substituting these expansions into $w_t = \alpha^2 w_{xx} + s(x,t)$ we obtain:

$$\sum_{n=1}^{\infty} \left\{ \hat{w}_n' + \alpha^2 \left(\frac{n\pi}{L}\right)^2 \hat{w}_n - \hat{s}_n(t) \right\} \sin\left(\frac{n\pi x}{L}\right) = 0. \tag{19.36}$$

Therefore

$$\hat{w}_n'(t) = -\alpha^2 \left(\frac{n\pi}{L}\right)^2 \hat{w}_n(t) + \hat{s}_n(t). \tag{19.37}$$

This is a linear 1st order ODE with integrating factor $e^{\alpha^2 \left(\frac{n\pi}{L}\right)^2 t}$.

Therefore

$$w_n(t) = \int_0^t e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 (t-\tau)} \hat{s}_n(\tau)\, d\tau + c_n e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} \tag{19.38}$$

where the c_n are arbitrary constants. Since we are only looking for a particular solution we choose $c_n \equiv 0$.

Therefore

$$w(x,t) = \sum_{n=1}^{\infty} \left(\int_0^t e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 (t-\tau)} \hat{s}_n(\tau)\, d\tau \right) \sin\left(\frac{n\pi x}{L}\right). \tag{19.39}$$

(2) Now that we have a particular solution we exploit the fact that the Problem (19.28)-(19.29) is linear and use superposition. Let

$$u(x,t) = w(x,t) + v(x,t) \tag{19.40}$$

$$u_t = \cancel{w_t} + v_t = \alpha^2 (\cancel{w_{xx}} + v_{xx}) + \cancel{s}(x,t) \tag{19.41}$$

$$\Rightarrow v_t = \alpha^2 v_{xx}. \tag{19.42}$$

$$\begin{array}{lllllllll} A & = & u(0,t) & = & w(0,t) + v(0,t) & = & v(0,t) & \text{since} & w(0,t) = 0 \\ B & = & u(0,t) & = & w(L,t) + v(L,t) & = & v(L,t) & \text{since} & w(L,t) = 0. \end{array}$$

$$f(x) = u(x,0) = w(x,0) + v(x,0) \Rightarrow v(x,0) = f(x) - w(x,0) \tag{19.43}$$

thus $v(x,t)$ satisfies:

$$\left.\begin{array}{ll} & v_t = \alpha^2 v_{xx} \\ \text{BC:} & v(0,t) = A \quad v(L,t) = B \\ \text{IC:} & v(x,0) = f(x) - w(x,0) \end{array}\right\} \tag{19.44}$$

Now the boundary value problem(19.44) was solved in lecture 19 of the notes.

Therefore

$$u(x,t) = \left(\frac{B-A}{L}\right) x + A + \sum_{n=1}^{\infty} e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} \left\{ b_n + \int_0^t e^{\alpha^2 \left(\frac{n\pi}{L}\right)^2 \tau} \hat{s}_n(\tau)\, dx \right\} \sin\left(\frac{n\pi x}{L}\right) \tag{19.45}$$

where

$$b_n = \frac{2}{L} \int_0^L \left\{ f(x) - w(x,0) - \left[(B-A)\frac{x}{L} + A \right] \right\} \sin\left(\frac{n\pi x}{L}\right)\, dx. \tag{19.46}$$

Example 19.4 *A bar with an external heat source dependent on x and t: $S(x,t) = xt$*

$$u_t = \alpha^2 u_{xx} + xt$$
$$u(0,t) = A \quad u(L,t) = B$$
$$u(x,t) = f(x)$$

Let $q(x) = \left(\frac{B-A}{L}\right)x + A$ and $u(x,t) = q(x) + v(x,t)$ then

$$v_t = \alpha^2 v_{xx} + xt$$
$$v(0,t) = 0 = v(L,t)$$
$$v(x,0) = f(x) - q(x).$$

Expanding the source $S(x,t)$ in terms of the eigenfunctions of the problem with homogeneous boundary conditions

$$S(x,t) = xt = \sum_{n=1}^{\infty} \hat{S}_n(t)\sin(\lambda_n x)$$
$$\hat{S}_n(t) = \frac{2}{L}\int_0^L xt\sin\left(\frac{n\pi x}{L}\right)\,dx$$
$$= \frac{2t}{L}\left\{ -x\frac{\cos\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)}\Bigg|_{0\nearrow}^{L} + \left(\frac{L}{n\pi}\right)\int_0^L \cos\left(\frac{n\pi x}{L}\right)\,dx \right\}$$
$$\hat{S}_n(t) = \frac{2t}{L}\left\{(-1)^{n+1}\left(\frac{L^2}{n\pi}\right)\right\} = \left(\frac{2L}{n\pi}\right)(-1)^{n+1}t$$

$$0 = v_t - \alpha^2 v_{xx} - xt = \sum_{n=1}^{\infty}\left\{\dot{\hat{v}}_n + \alpha^2\lambda_n^2\hat{v}_n - \hat{S}_n(t)\right\}\sin x\lambda_n x?$$
$$\dot{\hat{v}}_n + \alpha^2\lambda_n^2\hat{v}_n = \left(\frac{2L}{n\pi}\right)(-1)^{n+1}t$$
$$\left(e^{\alpha^2\lambda_n^2 t}\hat{v}_n\right) = \left(\frac{2L}{n\pi}\right)(-1)^{n+1}\left[\frac{te^{\alpha^2\lambda_n^2 t}}{\alpha^2\lambda_n^2} - \frac{e^{\alpha^2\lambda_n^2 t}}{(\alpha^2\lambda_n^2)^2}\right]_0^t + c_n$$
$$= \left(\frac{2L}{n\pi}\right)(-1)^{n+1}\left[\frac{te^{\alpha^2\lambda_n^2 t}}{\alpha^2\lambda_n^2} - \frac{(e^{\alpha^2\lambda_n^2 t} - 1)}{\alpha^4\lambda_n^4}\right] + c_n$$
$$\hat{v}_n(t) = \left(\frac{2L}{n\pi}\right)(-1)^n\left[\frac{t(\alpha^2\lambda_n^2) + e^{-\alpha^2\lambda_n^2 t} - 1}{(\alpha^2\lambda_n^2)^2}\right] + c_n e^{-\alpha^2\lambda_n^2 t}$$
$$v(x,t) = \sum_{n=1}^{\infty}\left\{\left(\frac{2L}{n\pi}\right)(-1)^n\left[\frac{t(\alpha^2\lambda_n^2) - 1 + e^{-\alpha^2\lambda_n^2 t}}{(\alpha^4\lambda_n^4)}\right] + c_n e^{-\alpha^2\lambda_n^2 t}\right\}\sin(\lambda_n x)$$
$$f(x) - q(x) = v(x,0) = \sum_{n=1}^{\infty}\left(\frac{2L}{n\pi}\right)(-1)^n 0 + c_n\sin(\lambda_n x)$$
$$c_n = \frac{2}{L}\int_0^L\left[f(x) - \left\{\left(\frac{B-A}{L}\right)x + A\right\}\right]\sin(\lambda_n x)\,dx$$