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SOLUTION TO MATH 256 ASSIGNMENT 2

- 10 points each.

(1) (a)(3pts) Characteristic equation $r^2 + 3r - 4 = 0 \implies r_1 = 1, r_2 = -4$. Then the general solution is $y = C_1e^t + C_2e^{-4t}$. By $y(0) = 1$ and $y'(0) = 0$, $C_1 = \frac{1}{5}$ and $C_2 = \frac{4}{5}$. So the solution is $\boxed{y = \frac{4}{5}e^t + \frac{1}{5}e^{-4t}}$.

(b)(3pts) Characteristic equation $r^2 + 2r + 2 = 0 \implies r_1 = -1 + i, r_2 = -1 - i$. Then the general solution is $y = e^{-t}(C_1cost + C_2sint)$. By $y(0) = 1$ and $y'(0) = 0$, $C_1 = C_2 = 1$. So the solution is $\boxed{y = e^{-t}(cost + sint)}$.

(c)(4pts) Characteristic equation $r^2 - 4r + 4 = 0 \implies r_1 = r_2 = 2$. Then the general solution is $y = C_1e^{2t} + C_2te^{2t}$. By $y(0) = 1$ and $y'(0) = 0$, $C_1 = 1$ and $C_2 = -2$. So the solution is $\boxed{y = e^{2t} - 2te^{2t}}$.

(2) We are using Abel $W(x) = W(x_0)e^{-\int_{x_0}^x p(s)ds}$ for this question.

(a)(2pts) $W(x) = W(x_0)e^{-\int_{x_0}^x p(s)ds} = W(0)e^{-\int_0^x sds} \implies \boxed{W(x) = e^{-\frac{x^2}{2}}}$. So $W(1) = e^{-\frac{1}{2}}$.

(b)(2pts) First we divide both sides by x^2. Then $W(x) = W(x_0)e^{-\int_{x_0}^x p(s)ds} = W(4)e^{-\int_4^x \frac{1}{s}ds} \implies \boxed{W(x) = \frac{8}{x}}$. So $W(1) = 8$.

(c)(3pts) First we divide both sides by x. Then $W(x) = W(x_0)e^{-\int_{x_0}^x p(s)ds} = W(2)e^{-\int_2^x \frac{2}{s}ds} \implies \boxed{W(x) = \frac{12}{x^2}}$. So $W(1) = 12$.

(d)(3pts) First we divide both sides by $sinx$. Then $W(x) = W(x_0)e^{-\int_{x_0}^x p(s)ds} = W(\frac{\pi}{2})e^{\int_{\frac{\pi}{2}}^x cotsds} \implies \boxed{W(x) = sinx}$. So $W(1) = sin1$.

(3) (a)(4pts) Yes. It is easy to check that $y_1(x) = x$ and $y_2(x) = x^{-4}$ are solutions to our ODE. Since

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^{-4} \\ 1 & -4x^{-5} \end{vmatrix} = -5x^{-4} \neq 0, \text{ for } x > 0,$$

we know that y_1 and y_2 are linearly independent. So they form a fundamental set of solutions.
(b)(3pts) No. Directly compute

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & 2x \\ 1 & 2 \end{vmatrix} = 0.$$

So they are linearly dependent.
(c)(3pts) No. Because $y_2(x) = x^{-1}$ is not a solution to the original ODE.
An alternative way is to compute the indicial roots for the Euler-type equation.

(4) (a)(3pts) By Abel, we know that

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = e^{-\int \frac{2}{s}ds}.$$

$y_1(x) = x \implies y_2$ satisfies $y_2' - \frac{y_2}{x} = \frac{1}{x^3}$. By integrating factor, we know $\boxed{y_2(x) = -\frac{1}{3x^2}}$.

(b)(3pts) By Abel, we know that

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = e^{\int \frac{s}{s-1}ds}.$$

$y_1(x) = e^x \implies y_2$ satisfies $y_2' - y_2 = x - 1$. By integrating factor, we know $\boxed{y_2(x) = -x}$.

(c)(4pts) By Abel, we know that

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = e^{\int \frac{s+2}{s}ds}.$$

$y_1(x) = e^x \implies y_2$ satisfies $y_2' - y_2 = x^2$. By integrating factor, we know $\boxed{y_2(x) = -x^2 - 2x - 2}$.

(5) The homogeneous part for these three ODEs is the same as Q1 (a), so we only need to find a particular solution for each one. One can use the method of variation of parameters to get the general solution for the

(c)(4pts) Try $y_p = axe^x$. Then $5ae^x = 5e^x \implies a = 1$. So the general solution is

$$\boxed{y = C_1e^x + C_2e^{-4x} + xe^x}.$$

(6) Similar to Q5, we only need to find a particular solution for each inhomogeneous ODE.
(a)(3pts) Try $y_p = ae^x$. Then $5ae^x = e^x \implies a = \frac{1}{5}$. So the general solution is

$$\boxed{y = e^{-x}(C_1cosx + C_2sinx) + \frac{1}{5}e^x}.$$

(b)(3pts) Try $y_p = ae^x + bx$. Then $5ae^x + 2bx + 2b = e^x + x + 1 \implies a = \frac{1}{5}, b = \frac{1}{2}$. So the general solution is

$$\boxed{y = e^{-x}(C_1cosx + C_2sinx) + \frac{1}{5}e^x + \frac{1}{2}x}.$$

(c)(4pts) Try $y_p = e^{-x}(axsinx + bxcosx)$. Direct computation gives

$$-2be^{-x}sinx = e^{-x}sinx \text{ and } 2ae^{-x}cosx = 0.$$

So $a = 0$ and $b = -\frac{1}{2}$. The general solution is

$$\boxed{y = e^{-x}(C_1cosx + C_2sinx) - \frac{1}{2}xe^{-x}cosx}.$$

(7) (a)(3pts) Try $y_p = (ax + b)e^x$. Then direct computation gives $(ax - 2a + b)e^x = xe^x$. So $a = 1, b = 2$. The general solution is

$$\boxed{y = C_1e^{2x} + C_2xe^{2x} + (x + 2)e^x}.$$

(b)(3pts) Try $y_p = acos2x + bsin2x + ccosx + dsinx$. By direct computations, we obtain

$$-8bcos2x + 4asin2x + (3c - 4d)cosx + (3d + 4c)sinx = 4cos2x + 5sinx.$$

So,

$$\begin{cases} -8b = 4 \\ 4a = 0 \\ 3c - 4d = 0 \\ 3d + 4c = 5 \end{cases} \implies \begin{cases} a = 0 \\ b = -\frac{1}{2} \\ c = \frac{4}{5} \\ d = \frac{3}{5} \end{cases}.$$

The general solution is

$$\boxed{y = C_1e^{2x} + C_2xe^{2x} - \frac{1}{2}sin2x + \frac{4}{5}cosx + \frac{3}{5}sinx}.$$

(c)(4pts) Try $y = ax^2e^{2x}$. Plugging in, we get $2ae^{2x} = e^{2x}$, which implies $a = \frac{1}{2}$. So the general solution is

$$\boxed{y = C_1e^{2x} + C_2xe^{2x} + \frac{1}{2}x^2e^{2x}}.$$