SOLUTION TO MATH 256 ASSIGNMENT 1

- Full mark: 90. 10 points each.
- *Warning 1*: The constant of integration appears when the integral is evaluated. It has to be carried along, not added anew in the next line (so $\frac{1}{y} = -\cos x + C$ is the same as $y = \frac{1}{-\cos x + C}$ and **not** $y = \frac{1}{-\cos x} + C$).
- *Warning 2*: $e^{(1-a)t}$ is not an exponential function when $a = 1$. It is a constant 1 and integrates to $t + C$.
- If you see :(, you seem to be at risk of (i) not knowing **basic algebra**, (ii) not having enough **integration skills**, (iii) **writing nonsense**, or (iv) all of the above. This could be a sign of eventually failing the course. Your TA strongly suggests you to catch up by seeking help (office hours, MLC, private tutor, etc.)

(1) (a) $\frac{dv}{dt} = 9.8 - v \implies \frac{dv}{dt} + v = 9.8 \implies e^t\frac{dv}{dt} + e^t v = 9.8e^t \implies (e^t v)' = 9.8e^t \implies e^t v = 9.8e^t + C \implies$ $v = 9.8 + Ce^{-t}$. From $v(0) = 0$, $C = -9.8$ so $\boxed{v = 9.8(1 - e^{-t})}$.

Alternatively, you can separable variables. However, you will need to know how to deal with absolute values and positivity of the constants. See Q3(a).

(b) $\frac{dx}{dt} = 9.8(1 - e^t)$, $x(0) = 0 \implies x = 9.8\int_0^t (1 - e^{-s})ds \implies \boxed{x = 9.8(t + e^{-t} - 1)}$.

Or you can compute indefinite integrals.

(c) $x(T) = 10 \implies 9.8(T + e^{-T} - 1) = 10$.

$\boxed{\text{$T$ is determined by $T + e^{-T} = \frac{99}{49}$. Since the function $T \mapsto T + e^{-T}$ is strictly increasing for $T > 0$, there exists a unique solution $T \in (1, 2)$.}}$

(2) (a) $r^2 - 2r - 3 = 0 \implies r = 3, -1 \implies \boxed{y = ae^{3t} + be^{-t}}$.

(b) $1 = y(0) = a + b$, $-1 = y'(0) = 3a - b \implies a = 0, b = 1 \implies \boxed{y = e^{-t}}$.

(3) (a) $\frac{dy}{dt} = 2y - 5 \implies \int \frac{dy}{y - \frac{5}{2}} = \int 2dt \implies \log|y - \frac{5}{2}| = 2t + C_1$ for some constant $C_1 \in \mathbb{R}$. Then $|y - \frac{5}{2}| = e^{C_1}e^{2t} = C_2e^{2t}$, for $C_2 = e^{C_1} > 0$. We can include $C_2 = 0$ (corresponding to $y \equiv \frac{5}{2}$, the stationary solution). Then $y - \frac{5}{2} = Ce^{2t}$, for $C = \pm C_2 \in \mathbb{R}$. From $y(0) = y_0$, $\boxed{y = \frac{5}{2} + (y_0 - \frac{5}{2})e^{2t}}$. As $t \to +\infty$, we either have (i) $\boxed{y \to +\infty \text{ if } y_0 > \frac{5}{2}}$, (ii) $\boxed{y \to -\infty \text{ if } y_0 < \frac{5}{2}}$, or (iii) $\boxed{y \equiv \frac{5}{2} \text{ if } y_0 = \frac{5}{2}}$.

(b) Separating variables as above, $\frac{dy}{dt} + 8y = 10 \implies |y - \frac{5}{4}| = Ce^{-8t} \implies \boxed{y = \frac{5}{4} + (y_0 - \frac{5}{4})e^{-8t}}$. As $t \to +\infty$, $\boxed{y \to \frac{5}{4} \text{ for any } y_0}$.

(4) (a) $y' + y = e^{-t} \implies (e^t y)' = 1 \implies y = (t + C)e^{-t}$. From $y(0) = 1$, $\boxed{y = (t + 1)e^{-t}}$.

(b) $ty' + y = 3t\cos 2t \implies (ty)' = 3t\cos 2t \implies ty = 3\int t\cos 2t dt = 3[(t)(\frac{\sin 2t}{2}) - (1)(-\frac{\cos 2t}{4})] + C \implies$ $\boxed{y = \frac{3}{2}\sin 2t + \frac{3}{4t}\cos 2t + \frac{C}{t}}$. We used integration by parts in the form: $\int uv'' = uv' - u'v + C$ for $u'' = 0$.

(c) $2y' + y = \frac{3}{2}t^2 \implies (e^{t/2}y)' = 3t^2e^{t/2} \implies e^{t/2}y = \frac{3}{2}[(t^2)(2e^{t/2}) - (2t)(4e^{t/2}) + (2)(8e^{t/2})] + C$ so $\boxed{y = 3(t^2 - 4t + 8) + Ce^{-t/2}}$.

(d) $ty' + (t + 1)y = t \implies y' + (1 + \frac{1}{t})y = 1$. The integrating factor (which is not obvious) is $e^{\int(1+\frac{1}{t})dt} = e^{t + \log t} = te^t$. So $(te^t y)' = te^t$, $te^t y = (t - 1)e^t + C$, $\boxed{y = \frac{t-1}{t} + \frac{C}{t}e^{-t}}$.

(5) We'll need $\int e^{ax}\cos bx dx = \dfrac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2} + C$ and $\int e^{ax}\sin bx dx = \dfrac{e^{ax}(-b\cos bx + a\sin bx)}{a^2+b^2} + C$.
Digression: To get it without integrating by parts twice, one can use complex variable ($Re \int e^{(a+ib)x}dx = Re\frac{e^{(a+ib)x}}{a+ib} + C$ etc.) or by integrating the two equations

$$\begin{pmatrix} (e^{ax}\cos bx)' \\ (e^{ax}\sin bx)' \end{pmatrix} = \begin{pmatrix} e^{ax}(a\cos bx - b\sin bx) \\ e^{ax}(a\sin bx + b\cos bx) \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} e^{ax}\cos bx \\ e^{ax}\sin bx \end{pmatrix}.$$

(a) $y' - y = 2te^t \implies (e^{-t}y)' = 2t \implies (t^2+C)e^t$. From $y(0) = 1$, $\boxed{y = (t^2+1)e^t}$. The interval of existence is $\boxed{(-\infty, +\infty)}$.

(b) $y' + \frac{2}{t}y = \frac{\cos t}{t^2} \implies (t^2y)' = \cos t \implies y = \dfrac{\sin t + C}{t^2}$. From $y(\pi) = 0$, $\boxed{y = \dfrac{\sin t}{t^2}}$. Interval of existence: $\boxed{(0, +\infty)}$. (Note that the initial point $\pi > 0$ and we need $t \neq 0$ on the interval of existence.)

(c) $y' - y = t - \sin t + e^{2t} \implies (e^{-t}y)' = te^{-t} - e^{-t}\sin t + e^t \implies e^{-t}y = -(t+1)e^{-t} + \frac{e^{-t}(\sin t + \cos t)}{2} + e^t + C$.
From $y(0) = 0$, $C = -\frac{1}{2}$. So $\boxed{y = -t - 1 + \dfrac{\sin t + \cos t}{2} + e^{2t} - \dfrac{1}{2}e^t}$. It exists on $\boxed{(-\infty, +\infty)}$.

(d) $\sin t\, y' - 2\cos t\, y = \sin^3 t \implies (\sin^{-2} t\, y)' = 1 \implies y = (t + C)\sin^2 t$. $y(\frac{\pi}{2}) = 1$ yields $C = 1 - \frac{\pi}{2}$, so $\boxed{y = (t + 1 - \dfrac{\pi}{2})\sin^2 t}$. Interval of existence: $\boxed{(0, \pi)}$ (because we need to keep $\sin t \neq 0$, near $\frac{\pi}{2}$).

(6) $y' + 2y = 3 + 2\cos 2t \implies (e^{2t}y)' = 3e^{2t} + 3e^{2t}\cos 2t \implies e^{2t}y = \frac{3}{2}e^{2t} + 3e^{2t}\dfrac{2\cos 2t + 2\sin 2t}{4} + C \implies y = \frac{3}{2} + \frac{1}{2}(\cos 2t + \sin 2t) + Ce^{-2t}$. By $y(0) = 0$, $C = -2$ so $\boxed{y = \dfrac{3}{2} + \dfrac{1}{2}(\cos 2t + \sin 2t) - 2e^{-2t}}$.
$\boxed{\text{As } t \to +\infty,\ y \text{ oscillates between } \frac{3-\sqrt{2}}{2} \text{ and } \frac{3+\sqrt{2}}{2}}$. This can be seen by rewriting

$$\cos 2t + \sin 2t = \sqrt{2}\sin(2t + \frac{\pi}{4}).$$

(7) $\boxed{\begin{array}{l} y' + y = e^{-at} \implies (e^ty)' = e^{(1-a)t}. \text{ When } a \neq 1,\ y = \frac{1}{1-a}e^{-at} + Ce^{-t}. \text{ When } a = 1, \text{ we have} \\ y = (t + C)e^{-t} \text{ as in Q4(a). In both cases we have } y(t) \to 0 \text{ as } t \to +\infty. \end{array}}$
Note that the "answer" is the complete argument. The case $a = 1$ must be treated separately.

(8) (a) $y' + y^2\sin x = 0 \implies -\frac{y'}{y^2} = \sin x \implies (\frac{1}{y})' = \sin x \implies \frac{1}{y} = -\cos x + C \implies \boxed{y = \dfrac{1}{-\cos x + C}}$.

(b) $y' = \frac{x - e^{-x}}{y + e^y} \implies (y + e^y)y' = x - e^{-x} \implies \boxed{\dfrac{y^2}{2} + e^y = \dfrac{x^2}{2} + e^{-x} + C}$.

(c) $y' = \frac{1+y^2}{2+x^2} \implies \frac{1}{1+y^2}y' = \frac{1}{2+x^2} \implies \boxed{\tan^{-1} y = \dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{x}{\sqrt{2}} + C}$.
Note that $(\tan^{-1} x)' = \frac{1}{1+x^2}$ while $(\frac{1}{2}\log(1+x^2))' = \frac{x}{1+x^2}$.

(d) $y' = \frac{x}{y} \implies 2yy' = 2x \implies \boxed{y^2 = x^2 + C}$.

(9) (a) $y' = (1 - 2x)y^2 \implies (\frac{1}{y})' = 2x - 1 \implies \frac{1}{y} = x^2 - x + C$. From $y(0) = 2$, $C = \frac{1}{2}$. Since $x^2 - x + \frac{1}{2} = (x - \frac{1}{2})^2 + \frac{1}{4} \geq \frac{1}{4} > 0$, we have $\boxed{y = \frac{1}{x^2 - x + \frac{1}{2}}, \text{ existing on } (-\infty, +\infty).}$

(b) $y' = xy^3(1 + x^2)^{-\frac{1}{2}} \implies -\frac{1}{2y^2} = \sqrt{1 + x^2} + C$. From $y(0) = 1$, $C = -\frac{3}{2}$. So $y^2 = \frac{1}{3 - 2\sqrt{1+x^2}}$, or $\boxed{y = \sqrt{\dfrac{1}{3 - 2\sqrt{1 + x^2}}}}$. The positive square root is taken because the $y = 1 > 0$ at $x = 0$. Solving $3 - 2\sqrt{1 + x^2} > 0$, $\boxed{\text{the interval of existence is } (-\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}).}$

(c) $y' = \frac{3x^2-e^x}{2y-5} \implies (2y-5)y' = 3x^2 - e^x \implies y^2 - 5y = x^3 - e^x + C$. The initial condition $y(0) = 1$ gives $C = -3$, i.e. $y^2 - 5y = x^3 - e^x - 3$. y can be solved explicitly, by completing the square (say), as

$$\boxed{y = \frac{5}{2} - \frac{1}{2}\sqrt{x^3 - e^x + \frac{13}{4}}}.$$

The negative square root is chosen because of the initial condition $y(0) = 1$. Since the initial value is given, you should always choose the branch of the square root.
The solution exists as long as $2y - 5 \neq 0$, i.e. $y \neq \frac{5}{2}$. The corresponding x-interval is determined by solving $x^3 - e^x - \frac{13}{4} > 0$. By intermediate value theorem, it has solutions in $(-2, 0)$ and in $(0, 5)$. Let x_- and x_+ be respectively the negative and positive solutions closest to 0, the initial value. The interval of existence is then $\boxed{(x_-, x_+)}$.

This is how to deal with the question without wolfram alpha, or whatever solver. (But, of course, they help.)
(d) $\sin(2x)dx + \cos(3y)dy = 0 \implies -\frac{\cos 2x}{2} + \frac{\sin 3y}{3} = C$. By $y(\frac{\pi}{2}) = \frac{\pi}{3}$, $C = \frac{1}{2}$. So

$$\boxed{\frac{\sin 3y}{3} = \frac{1 + \cos 2x}{2} = \cos^2 x.}$$

Extra discussions concerning the interval of existence:
We need to ensure that $\cos(3y) \neq 0$ on the interval of existence. Near $y = \frac{\pi}{3}$, the condition for y is $\frac{\pi}{2} < 3y < \frac{3\pi}{2}$. This corresponds to requiring $|\sin 3y| < 1$ (strictly!). Thus the interval of existence is determined by the equation $\cos^2 x < \frac{1}{3}$ near $x = \frac{\pi}{2}$, giving $-\frac{1}{\sqrt{3}} < \cos x < \frac{1}{\sqrt{3}}$. Taking the principal branch of $\cos^{-1}(\cdot)$ so that $\cos^{-1}(\frac{1}{\sqrt{3}}) \in (0, \frac{\pi}{2})$, the interval of existence is given by

$$\boxed{\left(\cos^{-1}(\frac{1}{\sqrt{3}}), \pi - \cos^{-1}(\frac{1}{\sqrt{3}})\right)}.$$

1 point has been awarded, even if you don't get the last part. This is done in our database (you won't see it but you get the point).