Chapter 7

$$A = \begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix}, \quad g = \begin{pmatrix} 3t-2 \\ e^{-t} \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} t - \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$$

eigenvalues $\det(A - \lambda I) = \lambda^2 - 5\lambda - 6 = 0$

$$(\lambda-6)(\lambda+1) = 0 \quad \lambda_1 = 6, \ \lambda_2 = -1$$

$$\lambda_p = 6 \Rightarrow \begin{pmatrix} -5 & 2 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \xi^{(1)} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$\lambda_2 = -1 \quad \begin{pmatrix} 2 & 2 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \xi^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

1a $\quad T = \begin{pmatrix} 2 & 1 \\ 5 & -1 \end{pmatrix}$

$$x = Ty$$

$$y' = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} y + T^{-1} \begin{pmatrix} 3t-2 \\ e^{-t} \end{pmatrix}$$

$$\det T = -7 \quad T^{-1} = \frac{1}{-7} \begin{pmatrix} -1 & -1 \\ -5 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{5}{7} & -\frac{2}{7} \end{pmatrix}$$

$$T^{-1} \begin{pmatrix} 3t-2 \\ e^{-t} \end{pmatrix} = \begin{pmatrix} \frac{1}{7}(3t-2) + \frac{1}{7} e^{-t} \\ \frac{5}{7}(3t-2) - \frac{2}{7} e^{-t} \end{pmatrix}$$

$$y_1' = 6y_1 + \frac{1}{7}(3t-2) + \frac{1}{7} e^{-t}$$

$$y_1 = e^{6t} \int e^{-6t} \left(\frac{1}{7}(3t-2) + \frac{1}{7} e^{-t} \right)$$

$$\boxed{\begin{aligned} & y' + ay = g \\ & y = e^{-at}\left(c + \int e^{at} g\right) \end{aligned}}$$

$$= -\frac{t}{14} - \frac{1}{84} + \frac{1}{21} - \frac{1}{49} e^{-t}$$

$$= -\frac{t}{14} + \frac{1}{28} - \frac{1}{49} e^{-t}$$

$$Y_2' = -Y_2 + \frac{5}{7}(3t-2) - \frac{2}{7}e^{-t}$$

$$Y_2 = e^{-t}\int e^{t}\left(\frac{5}{7}(3t-2) - \frac{2}{7}e^{-t}\right)$$

$$= e^{-t}\left[\frac{15}{7}(t-1)e^{t} - \frac{10}{7}e^{t} - \frac{2}{7}t\right]$$

$$= \frac{15}{7}(t-1) - \frac{10}{7} - \frac{2}{7}te^{-t}$$

$$x = Ty = \begin{pmatrix} 2 & 1 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2Y_1 + Y_2 \\ 5Y_1 - Y_2 \end{pmatrix} = \begin{pmatrix} -\frac{t}{7} + \frac{1}{14} - \frac{2}{49}e^{-t} + \frac{15}{7}(t-1) - \frac{10}{7} - \frac{10}{7}te^{-t} \\ -\frac{5}{14}t + \frac{25}{84} - \frac{5}{49}e^{-t} - \frac{15}{7}(t-1) + \frac{10}{7} + \frac{2}{7}te^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} 2t + \frac{1}{14} - \frac{25}{7} - \frac{2}{49}e^{-t} - \frac{10}{7}te^{-t} \\ -\frac{5}{14}t + \frac{5}{84} + \frac{25}{7} - \frac{5}{49}e^{-t} + \frac{2}{7}te^{-t} \end{pmatrix}$$

2\. For the first two, we use

$$x_{p_1} = \vec{a}t + \vec{b}, \quad x_{p_1}' = \vec{a}$$

$$\vec{a} = A\vec{a}t + A\vec{b} + \binom{3}{0}t - \binom{2}{0}$$

$$A\vec{a} + \binom{3}{0} = 0$$

$$A\vec{b} - \binom{2}{0} = \vec{a}$$

So $\vec{a} = \begin{pmatrix} 2 \\ -\frac{5}{2} \end{pmatrix}$ $b = \begin{pmatrix} -\frac{5}{6} \\ \frac{5}{12} \end{pmatrix}$

For the last one

$$x_{p_2} = t\vec{c}\,e^{-t} + \vec{d}\,e^{-t}$$

$$x_{p_2}' = -\vec{c}\,t e^{-t} + (\vec{c} - \vec{d})e^{-t}$$

$$A x_{p_2} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} = x_{p_2}'$$

$$A\vec{c} = -\vec{c} \quad \Rightarrow \vec{c} = k\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$Ad + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \vec{c} - \vec{d}$$

$$(A+I)\vec{d} = \vec{c} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 \\ 5 & 5 \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = k\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} k \\ -k+1 \end{pmatrix}$$

$$2d_1 + 2d_2 = k$$

$$5d_1 + 5d_2 = -k+1$$

$$5k = 2(-k+1) \qquad k = \frac{2}{7}$$

$$d_1 = \frac{1}{7}, \quad d_2 = 0$$

$$x_{p_2} = \frac{2}{7}\begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{-t} + \begin{pmatrix} \frac{1}{7} \\ 0 \end{pmatrix} e^{-t}$$

So $x_p = x_{p_1} + x_{p_2}$

$$= \begin{pmatrix} 2 \\ -\frac{5}{2} \end{pmatrix} t + \begin{pmatrix} -\frac{5}{6} \\ \frac{5}{12} \end{pmatrix} + \begin{pmatrix} \frac{2}{7} \\ -\frac{2}{7} \end{pmatrix} t e^{-t} + \begin{pmatrix} \frac{1}{7} \\ 0 \end{pmatrix} e^{-t}$$

$$x = c_1 \begin{pmatrix} 2 \\ 5 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + x_p$$

3. $x^{(1)} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} e^{6t} \qquad x^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$

$$\Psi(t) = \begin{pmatrix} 2e^{6t} & e^{-t} \\ 5e^{6t} & -e^{-t} \end{pmatrix}$$

$$x_p = \Psi(t)\, u(t)$$

$$\Psi(t)\, u'(t) = g(t)$$

$$\begin{pmatrix} 2e^{6t} & e^{-t} \\ 5e^{6t} & -e^{-t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 3t-2 \\ e^{-t} \end{pmatrix}$$

$$2e^{6t} u_1' + e^{-t} u_2' = 3t - 2$$

$$5e^{6t} u_1' - e^{-t} u_2' = e^{-t}$$

$$7e^{6t} u_1' = 3t - 2 + e^{-t}$$

$$u_1' = \frac{1}{7}(3t-2)\, e^{-6t} + \frac{1}{7} e^{-7t}$$

$$u_1 = \frac{3}{7}\int te^{-6t} - \frac{2}{7}\int e^{-6t} + \frac{1}{7}\int e^{-7t}$$

$$= \frac{3}{7}\left(-\frac{1}{6}te^{-6t} - \frac{1}{36}e^{-6t}\right) - \frac{2}{7}\left(-\frac{1}{6}\right)e^{-6t} - \frac{1}{49}e^{-7t}$$

$$e^{6t}u_1 = -\frac{1}{14}t - \frac{1}{84} + \frac{1}{21} - \frac{1}{49}e^{-t}$$

Similarly

$$7e^{-t}u_2' = 5(3t-2) - 2e^{-t}$$

$$u_2' = \frac{15}{7}te^{t} - \frac{10}{7}e^{t} - \frac{2}{7}$$

$$u_2 = \frac{15}{7}(t-1)e^{t} - \frac{10}{7}e^{t} - \frac{2}{7}t$$

$$e^{-t}u_2 = \frac{15}{7}(t-1) - \frac{10}{7} - \frac{2}{7}te^{-t}$$

$$x_p = \begin{pmatrix} 2\left(-\frac{1}{14}t + \frac{1}{28} - \frac{1}{49}e^{-t}\right) + \frac{15}{7}(t-1) - \frac{10}{7} - \frac{2}{7}te^{-t} \\ 5\left(-\frac{1}{14}t + \frac{1}{28} - \frac{1}{49}e^{-t}\right) - \frac{15}{7}(t-1) + \frac{10}{7} + \frac{2}{7}te^{-t} \end{pmatrix}$$

4. Now $\Psi(t) = \begin{pmatrix} 2e^{6t} & e^{-t} \\ 5e^{6t} & -e^{-t} \end{pmatrix}$, $\Psi(0) = \begin{pmatrix} 2 & 1 \\ 5 & -1 \end{pmatrix}$

$$(\Psi(0))^{-1} = \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{5}{7} & -\frac{2}{7} \end{pmatrix}$$

$$\Phi(t) = \Psi(t)\left(\Psi(0)\right)^{-1} = \begin{pmatrix} 2e^{6t} & e^{-t} \\ 5e^{6t} & -e^{-t} \end{pmatrix}\begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{5}{7} & -\frac{2}{7} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2e^{6t}}{7} + \frac{5e^{-t}}{7} & \frac{2}{7}e^{6t} - \frac{2}{7}e^{-t} \\ \frac{5}{7}e^{6t} - \frac{5}{7}e^{-t} & \frac{5}{7}e^{6t} + \frac{2}{7}e^{-t} \end{pmatrix}$$

5. $A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$

$$\det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{pmatrix} = \lambda^2 - 4 + 5 = \lambda^2 + 1 = 0$$

$\lambda = \pm i$

$$\lambda_1 = i \quad \Rightarrow \quad \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\xi_1 - (2+i)\xi_2 = 0 \qquad \xi_2 = 1, \ \xi_1 = (2+i)$$

$$\xi^{(1)} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

so $$\xi^{(1)} e^{\lambda_1 t} = \begin{pmatrix} (2+i)(\cos t + i \sin t) \\ \cos t + i \sin t \end{pmatrix}$$

$$x^{(1)} = \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix}, \quad x^{(2)} = \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$$

Use variation of parameter:

$$\Psi(t) = \begin{pmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ \cos t & \sin t \end{pmatrix}$$

$$x_p = \Psi(t) u(t)$$

$$\begin{pmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ \cos t & \sin t \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

$$\begin{cases} (2\cos t - \sin t)u_1' + (\cos t + 2\sin t)u_2' = \cos t & \text{①} \\ \cos t\, u_1' + \sin t\, u_2' = \sin t & \text{②} \end{cases}$$

① $\times \cos t$ − ② $\times (2\cos t - \sin t)$

$$\left(\cos^2 t + 2\cos t \sin t - \sin t(2\cos t - \sin t)\right) u_2' = \cos^2 t$$
$$- \sin t (2c$$

$$u_2' = 1 - \sin 2t$$

$$u_2 = t + \frac{1}{2}\cos 2t$$

① $\times \sin t$ − ② $\times (\cos t + 2\sin t)$

$$\left((2\cos t - \sin t)\sin t - \cos t(\cos t + 2\sin t)\right) u_1' = \cos t \sin t$$
$$- \sin t (\cos t +$$

$$-u_1' = -2\sin^2 t$$

$$u_1' = 2\sin^2 t = 1 - \cos 2t$$

$$u_1 = t - \frac{1}{2}\sin 2t$$

$$x_p = \begin{pmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ \cos t & \sin t \end{pmatrix} \begin{pmatrix} t + \frac{1}{2}\cos 2t \\ t - \frac{1}{2}\sin 2t \end{pmatrix}$$

6. $A = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}$

$\det(A - \lambda I) = \lambda^2 - 16 + 16 = \lambda^2 = 0.$

$\lambda_1 = \lambda_2 = 0. \rightarrow$ repeated

$\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \xi = 0 \Rightarrow \xi^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$x^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $x^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \eta$

$A\eta = \xi \Rightarrow \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \eta = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}$

$\psi(t) = \begin{pmatrix} x^{(1)} & x^{(2)} \end{pmatrix}$

$= \begin{pmatrix} 1 & t \\ 2 & 2t - \frac{1}{2} \end{pmatrix}$

$X_p = \Psi(t) u(t)$

$\Psi(t) u'(t) = \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}$

$\begin{pmatrix} 1 & t \\ 2 & 2t - \frac{1}{2} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}$

$u_1' + t u_2' = t^{-3}$

$2u_1' + 2tu_2' - \frac{1}{2} u_2' = -t^{-2}$

$-\frac{1}{2} u_2' = -t^{-2} - 2t^{-3}$

$u_2' = 2t^{-2} + 4t^{-3}$

$u_2 = -2t^{-1} - 2t^{-2}$

$$u_1' = -t u_2' + t^{-3}$$
$$= -t\left(2t^{-2} + 4t^{-3}\right) + t^{-3}$$
$$= -2t^{-1} - 4t^{-2} + t^{-3}$$
$$u_1 = -2\ln t + 4t^{-1} - \frac{1}{2}t^{-2}$$

$$X_p = \begin{pmatrix} 1 & t \\ 2 & 2t - \frac{1}{2} \end{pmatrix} \begin{pmatrix} -2t^{-1} - 2t^{-2} \\ -2\ln t + 4t^{-1} - \frac{1}{2}t^{-2} \end{pmatrix}$$

$$X = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \left(t\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \right) + X_p$$

7. Following 5

$$\Psi(t) = \begin{pmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ \cos t & \sin t \end{pmatrix}$$

$$\Psi(t)u' = \begin{pmatrix} \csc t \\ \sec t \end{pmatrix}$$

$$(2\cos t - \sin t)u_1' + (\cos t + 2\sin t)u_2' = \csc t \qquad (1)$$

$$(\cos t)u_1' + (\sin t)u_2' = \sec t \qquad (2)$$

$(1) \times \cos t - (2) \times (2\cos t - \sin t)$

$$u_2' = \csc t \cos t - \sec t\,(2\cos t - \sin t)$$
$$= -\frac{\cos t}{\sin t} - 2 + \frac{\sin t}{\cos t}$$

$$u_2 = -\ln \sin t - 2t - \ln \cos t$$

$$① \times \sin t - ② \times (\cos t + 2\sin t) \Rightarrow$$

$$-u_1' = 1 - \sec t\,(\cos t + 2\sin t)$$

$$= -\frac{2\sin t}{\cos t}$$

$$u_1 = -\ln \cos t$$

So $$x_p = \begin{pmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ \cos t & \sin t \end{pmatrix} \begin{pmatrix} -\ln \cos t \\ -\ln \sin t - \ln \cos t - 2t \end{pmatrix}$$

$$x = c_1 \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix} + x_p$$

8. $$W' = \mathrm{trace}(P(t))W = (t + \ln t)W$$

$$W = e^{\int (t + \ln t)\,dt} = c\,e^{\frac{t^2}{2} + t\ln t - t}$$