Final Review of MATH256-201-2018

1. First order equations
1.1. Homogeneous linear first order

$$y^{'} + p(t)y = 0, \qquad \text{is} \qquad y = Ce^{-\int p(t)dt}$$

1.2. Inhomogeneous linear first order

$$y^{'} + p(t)y = g(t)$$

$$\text{compute} \;\; \mu(t) = e^{\int p(t)dt}, \quad \int \mu(t)g(t)dt =$$

$$y(t) = \frac{1}{\mu(t)}(C + \int \mu(t)g(t)dt)$$

1.3. Separable equation

$$\frac{dy}{dt} = h(t)k(y), \qquad \int \frac{dy}{k(y)} = \int h(t)dt$$

1.4. Bernoulli equation

$$y^{'} + p(t)y = q(t)y^n, \quad \text{let } v = y^{1-n}$$

$$v^{'} + (1-n)p(t)v = (1-n)q(t)$$

1.5. Homogeneous equation

$$\frac{dy}{dx} = \frac{ax + by}{cx + dy}$$

$$\text{let } v = \frac{y}{x}, \; x\frac{dv}{dx} = \frac{a + bv}{c + dv} - v$$

1.6. Interval of Existence: three factors a) The solution, b) The Equation, c) the Initial Condition
1.7. Difference between linear and nonlinear: for linear equation, existence is global and uniqueness is guaranteed; for nonlinear equation, existence and uniqueness are nonlocal; for nonlinear nonsmooth $f(t, y)$ nonuniqueness

1.8. Applications in banking, falling objects, escaping velocity problem
1.9. Autonomous ODEs:

$$\frac{dy}{dt} = f(y)$$

Classification of critical points. Population models.

2. Linear Second Order Equations

$$y'' + p(t)y' + q(t)y = g(t)$$

2.1. Homogeneous case

$$y'' + p(t)y' + q(t)y = 0$$

2.1.1. Wronskian $W[y_1, y_2](t) = y_1 y_2' - y_1' y_2$.

$$\text{Abel's equation } W' + pW = 0$$

$$W(t) = W(t_0)e^{-\int_{t_0}^{t} p(s)ds}$$

2.1.2. Set of Fundamental Solutions y_1, y_2. All solutions are given by $y = c_1 y_1 + c_2 y_2$
2.1.3. Constant Coefficients:

$$ay'' + by' + cy = 0$$
$$\text{Characteristic equation } \; ar^2 + br + c = 0$$

- $b^2 - 4ac > 0$, two unequal real roots $r_1 \neq r_2$.

$$y_1 = e^{r_1 t}, y_2 = e^{r_2 t}$$

- $b^2 - 4ac < 0$, two complex roots: $r_1 = \lambda + i\mu, r_2 = \lambda - i\mu$

$$y_1 = e^{\lambda t}\cos(\mu t), \quad y_2 = e^{\lambda t}\sin(\mu t)$$

- $b^2 - 4ac = 0$, two equal roots: $r_1 = r_2 = r$

$$y_1 = e^{rt}, \; y_2 = te^{rt}$$

2.1.4. Euler's type equation

$$at^2y^{''} + bty^{'} + cy = 0$$

Characteristic equation $ar(r-1) + br + c = 0, \quad ar^2 + (b-a)r + c = 0$

- $(b-a)^2 - 4ac > 0$, two unequal real roots $r_1 \neq r_2$.

$$y_1 = t^{r_1}, y_2 = t^{r_2}$$

- $(b-a)^2 - 4ac = 0$, two equal roots: $r_1 = r_2 = r$

$$y_1 = t^r, \; y_2 = t^r \log t$$

- $(b-a)^2 - 4ac < 0$, two complex roots: $r_1 = \lambda + i\mu, r_2 = \lambda - i\mu$

$$y_1 = t^\lambda \cos(\mu \log t), \quad y_2 = t^\lambda \sin(\mu \log t)$$

2.1.5. Reduction of Order

$$y^{''} + p(t)y^{'} + q(t)y = 0$$

If y_1 is known, we can get y_2 by letting $y_2 = v(t)y_1$. Then v satisfies

$$v^{''} + (\frac{2y_1^{'}}{y_1} + p)v^{'} = 0$$

and

$$v^{'} = \frac{W}{y_1^2}$$

where $W = e^{-\int p(t)dt}$ is the Wronskian.

2.2 Inhomogeneous equations

$$y^{''} + py^{'} + qy = h(t)$$

$$y = y_p(t) + c_1y_1 + c_2y_2$$

where y_p is a particular solution and y_1, y_2–set of fundamental solutions of homogeneous problem.

2.2.1 Method One: Method of Undetermined Coefficients. Works only for

$$ay^{''} + by^{'} + cy = h(t)$$

- $h(t) = a_0 + a_1 t + ... + a_n t^n$

$$y_p = t^s (A_0 + A_1 t + ... + A_n t^n)$$

- $h(t) = e^{\alpha t}(a_0 + a_1 t + ... + a_n t^n)$

$$y_p = t^s e^{\alpha t}(A_0 + A_1 t + ... + A_n t^n)$$

- $h(t) = e^{\alpha t}(a_0 + a_1 t + ... + a_n t^n)\cos(\beta t)$ or $g(t) = e^{\alpha t}(a_0 + a_1 t + ... + a_n t^n)\sin(\beta t)$

$$y_p = t^s e^{\alpha t}[(A_0 + A_1 t + ... + A_n t^n)\cos(\beta t) + (B_0 + B_1 t + ... + B_n t^n)\sin(\beta t)]$$

- s equals either 0, or 1, or 2, is the least integer such that there are no solutions of the homogeneous problem in y_p

- $h(t) = h_1 + ... + h_m$

$$y_p = y_{p,1} + ... y_{p,m}$$

2.2.2. Method of Variation of Parameters

$$y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$$

where

$$\begin{cases} u_1^{'} y_1 + u_2^{'} y_2 = 0, \\ u_1^{'} y_1^{'} + u_2^{'} y_2 = h(t) \end{cases}$$

Formula:

$$u_1 = -\int \frac{y_2 g(t)}{W} dt, \quad u_2 = \int \frac{y_1 g(t)}{W} dt$$

$$y_p = -y_1(t) \int \frac{y_2 g(t)}{W} dt + y_2(t) \int \frac{y_1 g(t)}{W} dt$$

2.3. Applications: Spring-Mass System

$$mu^{''} + \gamma u^{'} + ku = F(t)$$

2.3.1. $\gamma = 0$, $F_0 = 0$

$$u = A\cos(\omega_0 t) + B\sin(\omega_0 t) = R\cos(\omega_0 t - \delta), R\cos\delta = A, R\sin\delta = B$$

2.3.2. $\gamma = 0, F_0 = F_0 \cos(\omega t)$. If $\omega = \omega_0$, resonance, solution becomes unbounded
2.3.3. $\gamma \neq 0$. All solutions approach zero as $t \to +\infty$.

3. Systems of Equations
   3.1. General Theory

$$\mathbf{x}^{'}(t) = A(t)\mathbf{x} + \mathbf{g}(t)$$

$$x = x_h + x_p$$

   3.2. Homogeneous Systems

$$\mathbf{x}^{'}(t) = A(t)\mathbf{x}$$

$$x_h = \sum_{j=1}^{n} c_j \mathbf{x}^{(\mathbf{j})}(t)$$

The Wronskian and its Abel's Formula

$$W(t) = det(x^{(1)}(t)...x^{(n)}(t)),\ W(t) = Ce^{\int \text{trace}(A(t))dt}$$

   3.3. Homogeneous Systems with Constant Coefficients

$$\mathbf{x}^{'} = A\mathbf{x}$$

(2 × 2 case only)
3.3.1. Two linearly independent eigenvectors

$$x = c_1 \xi^1 e^{r_1 t} + c_2 \xi^2 e^{r_2 t}$$

3.3.2. Complex eigenvalue

$$x = c_1 Re(\xi e^{(\lambda + i\mu)t}) + c_2 Im(\xi e^{(\lambda + i\mu)t})$$

3.3.3. Repeated eigenvalues, only on eigenvector

$$x^1 = \xi e^{rt}, x^2 = \xi t e^{rt} + \eta, (A - rI)\eta = \xi$$

3.3.4. Types and stabilities and trajectories.
Types: unstable saddle, stable sink, unstable source, stable spiral, unstable spiral, center
3.3.5. Euler type systems

$$t\mathbf{x}^{'} = A\mathbf{x}$$

$$\mathbf{x} = \xi t^r, A\xi = r\xi$$

3.4. Inhomogeneous Systems with Constant Coefficients

$$\mathbf{x}^{'} = A\mathbf{x} + \mathbf{g}(t)$$

Method I: Diagonalization

$$x = Ty, y^{'} = T^{-1}AT + T^{-1}\mathbf{g}$$

where $T^{-1}AT$ is a diagonal matrix.

Method II: Method of undetermined coefficients

If g is $ae^{\lambda t}\cos(\mu t)$ or polynomials or sum of these types, then

$$x_h = t^s(\text{the same type}) + \text{lower order terms}$$

Special case: r is an eigenvalue, $\mathbf{g} = e^{rt}\mathbf{a}_0$. $x_p = ate^{rt} + be^{rt}$

$$Aa = ra, Ab + \mathbf{a}_0 = rb + a$$

Method III: Method of variation of constants

$$x_p = \mathbf{\Psi}\mathbf{c}(t), \mathbf{\Psi}\mathbf{c}^{'} = \mathbf{g}$$

4. Laplace Transforms

4.1 The Laplace transform is defined by

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)dt$$

4.2 List of Formulas

$f(t)$	$\mathcal{L}[f](s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
e^{at}	$\frac{1}{s-a}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sin at$	$\frac{a}{s^2+a^2}$

$e^{\lambda t}\cos(\mu t)$	$\frac{s-\lambda}{(s-\lambda)^2+\mu^2}$
$e^{\lambda t}\sin(\mu t)$	$\frac{\mu}{(s-\lambda)^2+\mu^2}$
$e^{at}f(t)$	$\mathcal{L}[f](s-a)$
$tf(t)$	$-\frac{d}{ds}(\mathcal{L}[f](s))f(t-c)u_c(t) \quad e^{-cs}\mathcal{L}[f](s)$
$\delta(t-c)$	e^{-cs}
$\int_0^t f(t-\tau)g(\tau)d\tau$	$\mathcal{L}[f](s)\mathcal{L}[g](s)$
$f^{'}(t)$	$s\mathcal{L}[f](s)-f(0)$
$f^{''}(t)$	$s^2\mathcal{L}[f](s)-sf(0)-f^{'}(0)$

$$u_c(t) = H(t-c) = \begin{cases} 0, t < c \\ 1, t \geq c \end{cases}$$

4.3. Use of Laplace transforms to solve:

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y^{'} + a_0 y = g(t), y^{(j)}(0) = y_j, j = 0, ..., n-1$$

where

$$g(t) = e^{at}\cos(bt) \text{ or } e^{at}\sin(bt) \text{ or polynomials}$$

or

$$g(t) = \sum_{j=1}^{m} d_j(t)u_{c_j}(t), 0 \leq c_1 < c_2 < ... < c_m$$

or

$$g(t) = \delta(t-c)$$

5. Fourier Series and Method of Separation of Variables

5.1. Fourier Series

Let f be a function of $2L$ periodic, i.e., $f(x+2L) = f(x)$. Its Fourier Series is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(\frac{n\pi}{L}x) + b_n\sin(\frac{n\pi}{L}x))$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi}{L}x)dx, n = 0, 1, 2, ...$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi}{L}x)dx, n = 1, 2, ...$$

Convergence Formula:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(\frac{n\pi}{L}x) + b_n \sin(\frac{n\pi}{L}x)) = \frac{1}{2}(f(x-) + f(x+))$$

5.2. Fourier Sine Series

Let f be an odd function of $2L$ periodic, i.e. $f(x) = -f(-x), f(x+2L) = f(x)$. Its Fourier Sine Series is

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{L}x)$$

where

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin(\frac{n\pi}{L}x)dx, n = 1, 2, ...$$

5.3. Fourier Cosine Series

Let f be an even function of $2L$ periodic, i.e. $f(x) = f(-x), f(x+2L) = f(x)$. Its Fourier Sine Series is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi}{L}x)$$

where

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\sin(\frac{n\pi}{L}x)dx, n = 0, 1, 2, ...$$

5.4. Even or Odd Extension

Let $f(x)$ be defined in $[0, L)$. We can extend it to an even $2L$-periodic function, or an odd $2L$-periodic function as follows

$$f_{even}(x) = \begin{cases} f(x), 0 \leq x < L, \\ f(-x), -L < x < 0 \end{cases} \quad f_{even}(x+2L) = f_{even}(x);$$

$$f_{odd}(x) = \begin{cases} f(x), 0 \leq x < L, \\ -f(-x), -L < x < 0 \end{cases} \quad f_{odd}(x+2L) = f_{odd}(x)$$

5.5. Method of Separation of Variables applied to Heat Equation

5.5.1. The solution to

$$\begin{cases} u_t = \alpha^2 u_{xx}, 0 < x < L, t > 0, \\ u(0,t) = 0, u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

is

$$u(x,t)=\sum_{n=1}^{\infty}a_ne^{-\alpha^2(\frac{n\pi}{L})^2t}\sin(\frac{n\pi}{L}x),\ a_n=\frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)dx, n=1,2,...$$

5.5.2. The solution to

$$\begin{cases} u_t=\alpha^2u_{xx}, 0<x<L, t>0,\\ u_x(0,t)=0, u_x(L,t)=0\\ u(x,0)=f(x)\end{cases}$$

is

$$u(x,t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_ne^{-\alpha^2(\frac{n\pi}{L})^2t}\cos(\frac{n\pi}{L}x),\ a_n=\frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi}{L}x)dx,\ n=0,1,...$$

5.5.3. The solution to

$$\begin{cases} u_t=\alpha^2u_{xx}, 0<x<L, t>0,\\ u(0,t)=u(L,t), u_x(0,t)=u_x(L,t)\\ u(x,0)=f(x)\end{cases}$$

is

$$u(x,t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}e^{-\alpha^2(\frac{n\pi}{L})^2t}(a_n\cos(\frac{n\pi}{L}x)+b_n\sin(\frac{n\pi}{L}x),$$

$$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos(\frac{n\pi}{L}x)dx,\ n=0,1,...,b_n=\frac{1}{L}\int_{-L}^{L}\sin(\frac{n\pi}{L}x)dx, n=1,...$$

5.5.4. The solution to

$$\begin{cases} u_t=\alpha^2u_{xx}, 0<x<L, t>0,\\ u(0,t)=T_1, u(L,t)=T_2\\ u(x,0)=f(x)\end{cases}$$

is

$$u(x,t)=v(x)+\sum_{n=1}^{\infty}a_ne^{-\alpha^2(\frac{n\pi}{L})^2t}\sin(\frac{n\pi}{L}x)$$

$$v(x)=T_1+\frac{T_2-T_1}{L}x,\ a_n=\frac{2}{L}\int_0^L(f(x)-v(x))\sin(\frac{n\pi}{L}x)dx, n=1,2,...$$

5.5.5. The solution to

$$\begin{cases} u_t=\alpha^2u_{xx}, 0<x<L, t>0,\\ u_x(0,t)=T, u_x(L,t)=T\\ u(x,0)=f(x)\end{cases}$$

is

$$u(x,t)=v(x)+\frac{a_0}{2}+\sum_{n=1}^{\infty}a_ne^{-\alpha^2(\frac{n\pi}{L})^2t}\cos(\frac{n\pi}{L}x)$$

$$v(x) = Tx,\ a_n = \frac{2}{L}\int_0^L (f(x) - v(x))\sin(\frac{n\pi}{L}x)dx, n = 0, 1, 2, ...$$

5.6. Method of Separation of Variables applied to Wave Equation

5.6.1. The solution to

$$\begin{cases} u_{tt} = a^2 u_{xx}, 0 < x < L, t > 0, \\ u(0,t) = 0, u(L,t) = 0 \\ u(x,0) = f(x), u_t(x,0) = g(x) \end{cases}$$

is

$$u(x,t) = \sum_{n=1}^{\infty}(a_n \cos(\frac{n\pi}{L}at) + b_n \sin(\frac{n\pi}{L}at))\sin(\frac{n\pi}{L}x)$$

$$a_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)dx, b_n\frac{n\pi}{L}a = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)dx, n = 1, 2, ...,$$

5.6.2. The solution to

$$\begin{cases} u_{tt} = a^2 u_{xx}, 0 < x < L, t > 0, \\ u_x(0,t) = 0, u_x(L,t) = 0 \\ u(x,0) = f(x), u_t(x,0) = g(x) \end{cases}$$

is

$$u(x,t) = \frac{a_0 + b_0 t}{2} + \sum_{n=1}^{\infty}(a_n \cos(\frac{n\pi}{L}at) + b_n \sin(\frac{n\pi}{L}at))\cos(\frac{n\pi}{L}x)$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi}{L}x)dx, n = 0, 1, ..., b_0 = \frac{2}{L}\int_0^L g(x)dx, b_n\frac{n\pi}{L}a = \frac{2}{L}\int_0^L g(x)\cos(\frac{n\pi}{L}x)dx, n = 1, 2, ...,$$

5.7. Method of Separation of Variables applied to Laplace Equation

5.7.1. The solution to

$$\begin{cases} u_{xx} + u_{yy} = 0, 0 < x < a, 0 < y < b \\ u(x,0) = 0, u(x,b) = 0 \\ u(0,y) = 0, u(a,y) = g(y) \end{cases}$$

is

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sinh(\frac{n\pi}{b}x)\sin(\frac{n\pi}{b}y)$$

$$a_n \sinh(\frac{n\pi}{b}a) = \frac{2}{b}\int_0^b g(y)\sin(\frac{n\pi}{b}y)dy, n = 1, ...$$