Sol'ns to Written HW#4, MATH 256-103, 2018-2019.

1 (a). $y = y_p + y_h = \sin x + A\cos\sqrt{2}x + B\sin\sqrt{2}x$

(20 pts)

$y(0)=0 \Rightarrow A=0$

$y(\pi)=0 \Rightarrow 0 + B\sin\sqrt{2}\pi = 0 \Rightarrow B=0$

$y = \sin x$. Sol'n is unique

(b) $y = y_p + y_h = -\frac{1}{2}x\cos x + A\cos x + B\sin x$

$y(0)=0 \Rightarrow A=0$

$y'(\pi)=0 \Rightarrow \frac{\pi}{2} + B\sin\pi = 0 \Rightarrow \frac{\pi}{2}=0$. A contradiction

No sol'n exists

(c) $y = -\frac{1}{2}x\cos x - \frac{1}{2} + A\cos x + B\sin x$

$y(0)=0 \Rightarrow A=\frac{1}{2}$

$y(\pi)=0 \quad \frac{\pi}{2} - \frac{1}{2} + \frac{1}{2}\cos\pi + B\sin\pi = \frac{\pi}{2} - 1 + 0 = \frac{\pi}{2} - 1 = 0$

A contradiction

No sol'n exists

(d) $y = -\frac{1}{3}\sin 2x + A\cos x + B\sin x$

$y(0)=0 \Rightarrow A=0$

$y(\pi)=0 \Rightarrow 0 + B\sin\pi = 0 \Rightarrow$ true for all B.

$y = -\frac{1}{3}\sin 2x + B\sin x$

infinitely many solutions

2A. (a) Case 1. $\lambda = -a^2 < 0$, $y = c_1\cosh ax + c_2\sinh ax$, $y'(\pi)=0 \Rightarrow ac_2\cosh(a\pi)=0 \Rightarrow c_2=0$

(20 pts)

$y(0)=0 \Rightarrow c_1=0$, $y(0)=0 \Rightarrow c_1=0$

Case 2. $\lambda=0$, $y = c_1 + c_2 x$, $y(0)=0$, $c_1=c_2=0$

$y'(\pi)=0 \Rightarrow c_2=0$,

Case 3. $\lambda = \beta^2 > 0$, $y = c_1\cos\beta x + c_2\sin\beta x$

$y(0)=0 \Rightarrow c_1=0$, $y'(\pi)=0 \Rightarrow \cos\beta\pi=0$

$\beta\pi = \frac{(2n-1)\pi}{2} \Rightarrow \beta = \frac{2n-1}{2}$

$\lambda = \left(\frac{2n-1}{2}\right)^2$, $n=1,2,\dots$, $y = \sin\frac{2n-1}{2}x$.

(b). $y = x^r$, $r(r-1) + r + \lambda = r^2 + \lambda = 0$

Case 1. $\lambda=-a^2<0$, $r=\pm a$

$y=c_1x^a+c_2x^{-a}$, $y(1)=0 \Rightarrow c_1+c_2=0$
$y(2)=0 \Rightarrow c_1 2^a+c_2 2^{-a}=0$ $\Big\} \Rightarrow c_1=c_2=0$

Case 2. $\lambda=0$, $y=c_1+c_2\ln x$

$y(1)=0 \Rightarrow c_1=0$, $y(2)=0 \Rightarrow c_1+c_2\ln 2=0 \Rightarrow c_2=0$

Case 3. $\lambda=\beta^2>0$, $y=c_1\cos(\beta\ln x)+c_2\sin(\beta\ln x)$

$y(1)=0 \Rightarrow c_1=0$

$y(2)=0 \Rightarrow \sin(\beta\ln 2)=0$, $\beta\ln 2=n\pi$, $\beta=\frac{n\pi}{\ln 2}$

$\lambda=\left(\frac{n\pi}{\ln 2}\right)^2$, $n=1,2,\ldots$ $y=\sin\left(\frac{n\pi}{\ln 2}\ln x\right)$

28. (a). $a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi}{L}x\,dx$

(30 pts)

$=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{0}x\cos nx\,dx=\frac{1}{\pi}\left(\frac{1}{n}x\sin nx+\frac{1}{n^2}\cos nx\right)\Big|_{-\pi}^{0}$

$=\frac{1}{n^2\pi}(1-(-1)^n)$, $n\geq 2$ $\quad a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{0}x\,dx=-\frac{\pi}{2}$

$b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\left(\frac{n\pi}{L}x\right)dx=\frac{1}{\pi}\int_{-\pi}^{0}x\sin nx\,dx=\frac{1}{\pi}\left(-\frac{1}{n}x\cos nx+\frac{1}{n^2}\sin nx\right)\Big|_{-\pi}^{0}$

$=-\frac{1}{n}(-1)^n$, 0

(b) $a_n=\int_{-1}^{1}f(x)\cos n\pi x\,dx=\int_{-1}^{0}x^2\cos n\pi x\,dx+\int_0^1\cos n\pi x\,dx$

$=\frac{1}{n\pi}x^2\sin n\pi x+\frac{2}{(n\pi)^2}x\cos n\pi x-\frac{2}{(n\pi)^3}\sin n\pi x\Big|_{-1}^{0}+\frac{1}{n\pi}\sin n\pi x\Big|_0^1$

$=\frac{2}{(n\pi)^2}(-1)^n$, $n\geq 1$

$b_n=\int_{-1}^{0}x^2\sin n\pi x\,dx+\int_0^1\sin n\pi x\,dx$

$=\frac{1}{n\pi}x^2(-\cos n\pi x)+\frac{2}{(n\pi)^2}x\sin n\pi x+\frac{2}{(n\pi)^3}\cos n\pi x\Big|_{-1}^{0}+\frac{1}{n\pi}(-\cos n\pi x)\Big|_0^1$

$=\frac{1}{n\pi}(-1)^n-\frac{2}{(n\pi)^3}(-1)^n+\frac{1}{n\pi}(1-(-1)^n)=\frac{1}{n\pi}-\frac{2}{(n\pi)^3}(-1)^n$

$a_0=\int_{-1}^{0}x^2dx+\int_0^1dx=\frac{1}{3}+1=\frac{4}{3}$

(c) $a_n=\frac{1}{2}\int_{-2}^{2}f(x)\cos\frac{n\pi}{2}x\,dx=\frac{1}{2}\left[\int_{-2}^{2}x\cos\frac{n\pi}{2}x\,dx+\int_0^2\cos\frac{n\pi}{2}x\,dx\right]$

$=\frac{1}{2}\left\{\frac{2}{n\pi}x\sin\frac{n\pi}{2}+\left(\frac{2}{n\pi}\right)^2\cos\left(\frac{n\pi}{2}x\right)\Big|_{-2}^{2}+\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}x\right)\Big|_0^2\right]$

$=0$, $n\geq 1$

$$a_0 = \frac{1}{2}\int_0^2 dx = 1$$

$$b_n = \frac{1}{2}\left[\int_{-2}^{2} x\sin\left(\frac{n\pi}{2}x\right)dx + \int_0^2 \sin\left(\frac{n\pi}{2}x\right)dx\right]$$

$$= \frac{1}{2}\left[\frac{2}{n\pi}\left(-x\cos\left(\frac{n\pi}{2}x\right)\right) + \left(\frac{2}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}x\right)\Big|_{-2}^{2} + \left(\frac{2}{n\pi}\right)\left(-\cos\frac{n\pi}{2}x\right)\Big|_0^2\right]$$

$$= \frac{1}{n\pi}\left(-4(-1)^n\right) + \frac{2}{n\pi}\left(1-(-1)^n\right)$$

3. (1). Even extension

(30pts) $f_{ext} = \begin{cases} \sin\pi x, & 0<x<1 \\ \sin(\pi(-x)), & -1<x<0 \end{cases}$; $f(x+2) = f(x)$

$$a_0 = 2\int_0^1 \sin\pi x\,dx = \frac{2}{\pi}\left(-\cos\pi x\right)\Big|_0^1 = \frac{4}{\pi}$$

$$a_n = 2\int_0^1 \sin(\pi x)\cos(n\pi x)dx = \int_0^1 \left[\sin((n+1)\pi x) - \sin((n-1)\pi x)\right]dx$$

$$= \frac{1}{(n+1)\pi}\left(1-(-1)^{n+1}\right) - \frac{1}{(n-1)\pi}\left(1-(-1)^{n-1}\right)$$

Odd Extension

$f_{odd} = \sin\pi x$

$$b_n = 2\int_0^{\pi} \sin\pi x\sin(n\pi x)dx = \begin{cases} 1, & n=1 \\ 0, & n\geq 2 \end{cases}$$

(2) Even extension

$f_{even} = \cos\pi x$.

$a_0 = 0$, $a_1 = 1$, $a_n = 0$, $n \geq 2$

Odd extension

$f_{odd} = \begin{cases} \cos\pi x, & 0<x<1 \\ -\cos\pi x, & -1<x<0 \end{cases}$

$$b_n = 2\int_0^1 \cos\pi x\sin(n\pi x)dx = \int_0^1 \left[\sin((n+1)\pi x) + \sin((n-1)\pi x)\right]dx$$

$$= \frac{1}{(n+1)\pi}\left(1-(-1)^{n+1}\right) + \frac{1}{(n-1)\pi}\left(1-(-1)^{n-1}\right)$$

(3) Even Extension

$$f_{even} = \begin{cases} x+1, & 0<x<1 \\ -x+1, & -1<x<0 \end{cases} \qquad f(x+2) = f(x)$$

$$a_0 = 2\int_0^1 (x+1)\,dx = 2\cdot\left(\tfrac{1}{2}+1\right) = 3$$

$$a_n = 2\int_0^1 (x+1)\cos(n\pi x)\,dx = 2\left[\frac{1}{n\pi}x\sin n\pi x + \frac{1}{(n\pi)^2}\cos(n\pi x)\Big|_0^1 + \frac{1}{n\pi}\sin n\pi x\Big|_0^1\right]$$

$$= \frac{2}{(n\pi)^2}\left((-1)^n - 1\right)$$

Odd Extension

$$f_{odd} = \begin{cases} x+1, & 0<x<-1 \\ -(-x+1), & -1<x<0 \end{cases} = \begin{cases} x+1, & 0<x<-1 \\ x-1, & -1<x<0 \end{cases} ; \quad f(x+2) = f(x)$$

$$b_n = 2\int_0^1 (x+1)\sin(n\pi x)\,dx = 2\left[-\frac{1}{n\pi}x\cos(n\pi x) + \frac{1}{(n\pi)^2}\sin(n\pi x)\Big|_0^1 - \frac{1}{n\pi}\cos(n\pi x)\Big|_0^1\right]$$

$$= 2\left[-\frac{1}{n\pi}(-1)^n - \frac{1}{n\pi}\left((-1)^n - 1\right)\right]$$

4. (a) (10 pts)

$$a_0 = \int_{-1}^1 f(x)\,dx = \int_{-1}^0 x\,dx + \int_0^1 1\,dx = \frac{1}{2} + 1 = \frac{3}{2}$$

$$a_1 = \int_{-1}^1 f(x)\cos \pi x\,dx = \int_{-1}^0 x\cos\pi x + \int_0^1 \cos\pi x\,dx$$

$$= \frac{1}{\pi}x\sin\pi x + \frac{1}{\pi^2}\cos\pi x\Big|_{-1}^0 + \frac{1}{\pi}\sin\pi x\Big|_0^1$$

$$= \frac{2}{\pi^2}$$

$$b_1 = \int_{-1}^0 x\sin\pi x + \int_0^1 \sin\pi x\,dx$$

$$= -\frac{1}{\pi}x\cos\pi x + \frac{1}{\pi^2}\sin\pi x\Big|_{-1}^0 \quad - \frac{1}{\pi}\cos\pi x\Big|_0^1$$

$$= \frac{1}{\pi} - \frac{1}{\pi}(-1-1) = \frac{3}{\pi}$$

(b) $$\frac{a_0}{2} + \sum_{n=1}^{+\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right) = \frac{1}{2}\left(f(x-)+f(x+)\right) = \begin{cases} -\frac{1}{2} & x = -\frac{1}{2} \\ \frac{1}{2} & x = 0 \\ 1 & x = \frac{1}{2} \end{cases}$$

5. $L=\frac{\pi}{2}$, $k=2$, Dirichlet

(10pts)

$$u(x,t)=\sum a_n e^{-k\left(\frac{n\pi}{L}\right)^2 t}\sin\left(\frac{n\pi}{L}x\right)$$

$$=\sum_{n=1}^{+\infty} a_n e^{-2(2n)^2 t}\sin(2nx)$$

$$u(x,0)=\sin 2x+4\sin 8x=\sum a_n \sin(2nx)$$

$$\Rightarrow a_1=1,\ a_4=4$$

$$u(x,t)=e^{-2\cdot 2^2 t}\sin 2x+4e^{-2\cdot(2\cdot4)^2 t}\sin 8x$$

6. Solve steady state $2u^0_{xx}+4=0$

(20pts)

$$u^0(0)=1,\ u^0(\pi)=-1$$

$$u^0=-x^2+A+Bx \qquad u^0(0)=1 \Rightarrow A=1$$

$$u^0(\pi)=-1,\qquad -1=-\pi^2+1+B\cdot\pi \Rightarrow B=\frac{\pi^2-2}{\pi}$$

$$u^0=-x^2+1+\frac{\pi^2-2}{\pi}x$$

$$u=u^0+v$$

$$\begin{cases} v_t=2v_{xx} \\ v(x,0)=u(x,0)-u^0(x)=x^2-1-\frac{\pi^2-2}{\pi}x=\phi(x) \\ v(0,t)=v(\pi,t)=0 \end{cases}$$

$$v(x,t)=\sum_{n=1}^{+\infty} a_n e^{-2n^2 t}\sin(nx)$$

$$a_n=\frac{2}{\pi}\int_0^\pi \phi(x)\sin nx=\frac{2}{\pi}\int_0^\pi\left(x^2-1-\frac{\pi^2-2}{\pi}x\right)\sin nx\,dx$$

$$=\frac{2}{\pi}\left(\frac{-x^2}{n}\cos nx+\frac{2}{n^2}x\sin nx-\frac{2}{n^3}\cos nx+\frac{1}{n}\cos nx+\frac{\pi^2-2}{\pi}\cdot\frac{1}{n}x\cos nx-\frac{\pi^2-2}{\pi}\frac{1}{n^2}\sin nx\right)\Big|_0^\pi$$

$$=\frac{2}{\pi}\left(-\frac{\pi^2(-1)^n}{n}-\frac{2}{n^3}((-1)^n-1)+\frac{1}{n}((-1)^n-1)+\frac{\pi^2-2}{\pi}\frac{1}{n}(\pi(-1)^n)\right)$$

As $t\to+\infty$, $u(x,t)\longrightarrow u^0(x)=-x^2+1+\frac{\pi^2-2}{\pi}x$

7. Steady-state

(20pts)

$$4u^0_{xx}=0$$
$$u^0(0)=0,\ u^0(\pi)=\pi$$
$$u^0=A+Bx \Rightarrow u^0=x$$

$$u=u^0+v \Rightarrow \quad v_{tt}=4v_{xx}$$
$$v(0,t)=v(\pi,t)=0$$
$$v(x,0)=u(x,0)-u^0=\begin{cases}-x, & 0<x<\frac{\pi}{2}\\ 0, & \frac{\pi}{2}\le x<\pi\end{cases}$$
$$v_t(x,0)=\sin 3x$$

$$v=\sum_{n=1}^{+\infty}\sin(nx)\left(a_n\cos(2nt)+b_n\sin(2nt)\right)$$

$$v(x,0)=\sum_{n=1}^{+\infty}a_n\sin(nx) \Rightarrow a_n=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}(-x)\sin(nx)\,dx$$
$$=\frac{2}{\pi}\left(\frac{1}{n}x\cos nx-\frac{1}{n^2}\sin nx\right)\Big|_0^{\frac{\pi}{2}}$$
$$=\frac{2}{\pi}\left(\frac{1}{n}\cdot\frac{\pi}{2}\cos\frac{n\pi}{2}-\frac{1}{n^2}\sin\frac{n}{2}\pi\right)$$

$$v_t(x,0)=\sum(2n)b_n\sin nx=\sin 3x$$
$$6b_3=1 \Rightarrow b_3=\frac{1}{6},\quad b_n=0,\ \text{for } n\neq 3$$

8. ~~Steady-state~~ Neumann BC. wave

(10pts)

$$u(x,t)=\frac{a_0+b_0t}{2}+\sum\cos(nx)\left(a_n\cos nt+b_n\sin nt\right)$$
$$u(x,0)=\cos 2x \Rightarrow a_0=0,\ a_2=1,\ a_n=0 \text{ for } n\neq 2$$
$$u_t(x,0)=1 \quad 1=\frac{b_0}{2}+\sum n\,b_n\cos nx \Rightarrow b_0=2,\ b_n=0 \text{ for } n\neq 0$$

$$u(x,t)=t+\cos 2x\cos 2t$$

9. $u=X(x)Y(y)$
$$X''+\lambda_1X=0,\quad X(0)=X(\pi)=0$$
$$Y''+\lambda_2Y=0,\quad Y(0)=0$$

(20pts)

$$\lambda_1+\lambda_2=0$$
$$\lambda_1=\left(\frac{n\pi}{\pi}\right)^2=n^2,\quad X=\sin nx$$
$$\lambda_2=-n^2,\quad Y=\sinh ny$$

$$u(x,y) = \sum_{n=1}^{+\infty} a_n \sin nx \sinh ny$$

$$u(x,\pi) = x = \sum_{n=1}^{+\infty} a_n \sin nx \sinh(n\pi)$$

$$a_n \sinh(n\pi) = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi}\left(-\frac{x\cos nx}{n} + \frac{1}{n^2}\sin nx\right)\Big|_0^{\pi}$$

$$= \frac{2}{n}(-1)^{n+1}$$

$$a_n = \frac{2}{n}(-1)^{n+1}\frac{1}{\sinh(n\pi)}$$

$$u(x,y) = \sum_{n=1}^{+\infty} \frac{2}{n}(-1)^{n+1}\frac{1}{\sinh(n\pi)} \sin nx \sinh ny.$$