MATH256-103 Homework Assignment 4 (Due Date: by 6pm on December 4th 2018)

Hand in your homework either in class, or by 6pm at my office, on December 3th. Graded homework is placed in a cardboard box outside my office for one week for you to pick up. Afterwards unclaimed homework is moved to a drawer of a file cabinet near my office. Your assignments are organized in the alphabetic order of last names. For other people's convenience, please do not change this order when you pick up your assignment.

1. Find solutions to the following two-point boundary value problem:

$$(a)\ \ y^{''} + 2y = \sin(x), y(0) = 0, y(\pi) = 0$$

$$(b)\ \ y^{''} + y = \sin(x), y(0) = 0, y(\pi) = 0,$$

$$(c)\ \ y^{''} + y = \sin(x) - \frac{1}{2}, y(0) = 0, y(\pi) = 0,$$

$$(d)\ \ y^{''} + y = \sin(2x), y(0) = 0, y(\pi) = 0.$$

2. Find the eigenvalues to the following eigenvalue problem

$$(a)\ y^{''} + \lambda y = 0, 0 < x < \pi, y(0) = 0, y^{'}(\pi) = 0$$

$$(b)\ x^2 y^{''} + xy^{'} + \lambda y = 0, 1 < x < 2, y(1) = 0, y(2) = 0$$

2. Find the Fourier series for the following function

$$(a)\ f(x) = \begin{cases} x, -\pi \le x < 0; \\ 0, 0 \le x < \pi; \end{cases} \qquad f(x + 2\pi) = f(x)$$

$$(b)\ f(x) = \begin{cases} x^2, -1 \le x < 0; \\ 1, 0 \le x < 1; \end{cases} \qquad f(x + 2) = f(x);$$

$$(c)\ f(x) = \begin{cases} x, -2 \le x < 0; \\ x + 1, 0 \le x < 2; \end{cases} \qquad f(x + 4) = f(x).$$

3. The following functions are given on an interval of length 1. (a) Extend evenly to be 2 periodic function and find the Fourier series; (b) extend oddly to be 2 periodic function and find the Fourier series

$$(1)\ f(x) = \sin(\pi x), 0 < x < 1;\ \ (2)\ f(x) = \cos(\pi x), 0 < x < 1;\ \ (3)\ f(x) = x + 1, 0 < x < 1$$

4.Consider the following function

$$f(x) = \begin{cases} x, & -1 \le t < 0 \\ 1, & 0 \le x < 1 \end{cases} \qquad f(x + 2) = f(x)$$

(a) Compute the first three coefficients of full Fourier series expansion a_0, a_1, b_1.

(b) Find out the values of the full Fourier series expansion at $x = -\frac{1}{2},\ 0,\ \frac{1}{2}$. Hint: you can use the following theorem:

$$\frac{a_0}{2} + \sum_{j=1}^{\infty}(a_n \cos(\frac{\pi x}{L}) + b_n \sin(\frac{\pi x}{L})) = \frac{1}{2}(f(x-) + f(x+))$$

where f is a piecewise smooth function.

5. Use the method of separation of variables to solve the following heat equation

$$u_t = 2u_{xx}, \quad 0 < x < \frac{\pi}{2},\ t > 0;$$
$$u(x,0) = \sin(2x) + 4\sin(8x), \quad 0 < x < \frac{\pi}{2};$$
$$u(0,t) = 0, \quad u(\frac{\pi}{2}, t) = 0,\ t > 0.$$

6. Use the method of separation of variables to solve the following heat equation

$$u_t = 2u_{xx} + 4, \quad 0 < x < \pi,\ t > 0;$$
$$u(x,0) = 0, \quad 0 < x < \pi;$$
$$u(0,t) = 1, \quad u(\pi,t) = -1,\ t > 0.$$

What is the asymptotic behavior of $u(x,t)$ as $t \to +\infty$?

7. Use the method of separation of variables to solve the following wave equation

$$u_{tt} = 4u_{xx}, \quad 0 < x < \pi, \quad t > 0;$$
$$u(0,t) = 0, \quad u(\pi,t) = \pi;$$
$$u(x,0) = \begin{cases} 0, & 0 < x < \frac{\pi}{2} \\ x, & \frac{\pi}{2} \le x < \pi \end{cases}, \quad u_t(x,0) = \sin(3x).$$

8. Use the method of separation of variables to solve the following wave equation

$$u_{tt} = u_{xx}, 0 < x < \pi$$
$$u(x,0) = \cos(2x), u_t(x,0) = 1$$
$$u_x(0,t) = 0, u_x(\pi,t) = 0$$

9. Use the method of separation of variables to solve the following laplace equation

$$u_{xx} + u_{yy} = 0, 0 < x < \pi, 0 < y < \pi$$
$$u(x,0) = 0, u(x,\pi) = x, 0 < x < \pi$$
$$u(0,y) = 0, u(\pi,y) = 0, 0 < y < \pi$$

10. Use the method of separation of variables to solve the following laplace equation

$$u_{xx} + u_{yy} = 0, x^2 + y^2 < 1$$
$$u(x,y) = x^2 \text{ on } x^2 + y^2 = 1$$