LOCAL BEHAVIOR OF SOLUTIONS TO A FRACTIONAL EQUATION WITH ISOLATED SINGULARITY AND CRITICAL SERRIN EXPONENT

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Abstract. In this paper, we study the local behavior of positive singular solutions to the equation

$$(-\Delta)^\sigma u = u^{\frac{n}{n-2\sigma}} \quad \text{in } B_1\backslash\{0\}$$

where $(-\Delta)^\sigma$ is the fractional Laplacian operator, $0 < \sigma < 1$ and $\frac{n}{n-2\sigma}$ is the critical Serrin exponent. We show that either u can be extended as a continuous function near the origin or there exist two positive constants c_1 and c_2 such that

$$c_1|x|^{2\sigma-n}(-\ln|x|)^{-\frac{n-2\sigma}{2\sigma}} \le u(x) \le c_2|x|^{2\sigma-n}(-\ln|x|)^{-\frac{n-2\sigma}{2\sigma}} \quad \text{in } B_1\backslash\{0\}.$$

1. Introduction

In this paper, we study the local behavior of positive solutions to the equation

$$(-\Delta)^\sigma u = u^{\frac{n}{n-2\sigma}} \quad \text{in } B_1\backslash\{0\} \tag{1.1}$$

where the punctured ball $B_1\backslash\{0\} \subset \mathbb{R}^n$ with $n \ge 2, \sigma \in (0,1)$. The fractional Laplacian $(-\Delta)^\sigma$ is defined by

$$(-\Delta)^\sigma u(x) = c_{n,\sigma}\text{C.V.}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2\sigma}}dy = c_{n,\sigma}\lim_{\epsilon\to 0}\int_{\mathbb{R}^n\backslash B_\epsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2\sigma}}dy,$$

where C.V. stands for the Cauchy principal value and

$$c_{n,\sigma} = \frac{2^{2\sigma}\sigma\Gamma(\frac{n}{2}+\sigma)}{\pi^{\frac{n}{2}}\Gamma(1-\sigma)}$$

ia a normalization constant. Let

$$L_\sigma(\mathbb{R}^n) = \{u \in L^1_{\text{loc}}(\mathbb{R}^n) | \int_{\mathbb{R}^n} \frac{|u|}{1+|x|^{n+2\sigma}}dx < \infty\}.$$

It is well known that if $u \in C^2(\mathbb{R}^n) \cap L_\sigma(\mathbb{R}^n)$, then the function $(-\Delta)^\sigma u$ is well defined.

Before presenting our result, we first list some results concerning positive solutions of the equation

$$(-\Delta)^\sigma u = u^p \quad \text{in } B_1\backslash\{0\}. \tag{1.2}$$

When $\sigma = 1$, (1.2) was studied by Aviles [1, 2] when $p = n/(n-2)$–the critical Serrin exponent, by Gidas and Spruck [15] for $n/(n-2) < p < (n+2)/(n-2)$ and by Caffarelli, Gidas and Spruck [7] in the case of $p = (n+2)/(n-2)$–the critical Sobolev exponent. If $p > (n+2)/(n-2)$, then (1.2) was studied in [6].

If $\sigma \neq 1$, there are also a lot of results. In [12], the fractional equation

$$\begin{cases} (-\Delta)^\sigma u = u^p & \text{in } B_1 \backslash \{0\}, \\ u = 0 & \text{in } \mathbb{R}^n \backslash B_1 \end{cases} \tag{1.3}$$

when $p > 1$ and $\sigma \in (0,1)$ was considered. It was proved in [12] that every classical solution of (1.3) is a very weak solution of the equation

$$\begin{cases} (-\Delta)^\sigma u = u^p + k\delta_0 & \text{in } B_1, \\ u = 0 & \text{in } \mathbb{R}^n \backslash B_1 \end{cases}$$

for some $k \geq 0$, where δ_0 is the Dirac mass at the origin.

When $n \geq 2, \sigma \in (0,1)$ and $p = (n+2\sigma)/(n-2\sigma)$, the local behaviors of nonnegative solutions of (1.2) was considered in [8]. Among other things, it was proved in [8] that if u is a nonnegative solution of (1.2), then either u can be extended as a continuous function near 0, or there exist two positive constants c_1 and c_2 such that

$$c_1|x|^{-\frac{n-2\sigma}{2}} \leq u(x) \leq c_2|x|^{-\frac{n-2\sigma}{2}}.$$

When $\sigma \in (0,1)$ and $n/(n-2\sigma) < p < (n+2\sigma)/(n-2\sigma)$, (1.2) was studied in [21] and [22]. The main results in [21] and [22] can give a precise description of the exact behavior of the singular solutions.

Besides the classification of local behaviors of positive solutions, the existence of singular solutions is also a very important problem. When $\sigma = 1$, singular solutions to (1.2) were constructed in [1], [10], [11], [19], [18]. Recently, the existence of singular solutions to (1.2) with prescribed singularities was also considered for $\sigma \neq 1$. For some results concerning this problem, we refer to [4], [3], [13].

The main objective in this paper is to consider(1.1) when $n \geq 2, \sigma \in (0,1)$ and $p = n/(n-2\sigma)$. In [5], the authors point out that the positive solutions of (1.1) should have the asymptotic form $|x|^{2\sigma-n}(-\ln|x|)^{-\frac{n-2\sigma}{2\sigma}}$ (see Remark 1.3 in [5]). We will show that this is true. More precisely, we have the following result.

Theorem 1.1. *Let $n \geq 2, \sigma \in (0,1)$ and let $u \in C^2(B_1 \backslash \{0\}) \cap L_\sigma(\mathbb{R}^n)$ be a positive solution of (1.1), then either u can be extended as a continuous function near the origin or there exist two positive constants c_1 and c_2 such that*

$$c_1|x|^{2\sigma-n}(-\ln|x|)^{-\frac{n-2\sigma}{2\sigma}} \leq u(x) \leq c_2|x|^{2\sigma-n}(-\ln|x|)^{-\frac{n-2\sigma}{2\sigma}} \quad \text{in } B_1 \backslash \{0\}.$$

We analyze (1.1) via the extension formulas established in [9]. Let $X = (x,t)$ be points in $\mathbb{R}^{n+1}$. We denote $\mathcal{B}_R^+$ as the upper half ball $\mathcal{B}_R \cap \mathbb{R}_+^{n+1}$, where $\mathcal{B}_R$ is the ball in $\mathbb{R}^{n+1}$ with radius R and its center at the origin. We also denote $\partial^{'}\mathcal{B}_R = \partial\mathcal{B}_R^+ \cap \partial\mathbb{R}_+^{n+1}$ and $\partial^{''}\mathcal{B}_R = \partial\mathcal{B}_R \cap \mathbb{R}_+^{n+1}$. For $u \in C^2(B_1 \backslash \{0\}) \cap L_\sigma(\mathbb{R}^n)$, we define

$$U(x,t) = \int_{\mathbb{R}^n} P_\sigma(x-y,t)u(y)dy, \tag{1.4}$$

where

$$P_\sigma(x,t) = p_{n,\sigma}\frac{t^{2\sigma}}{(|x|^2+t^2)^{\frac{n+2\sigma}{2}}}$$

with a constant $p_{n,\sigma}$ such that $\int_{\mathbb{R}^n} P_\sigma(x,1)dx = 1$. By the main results in [9], we know that $U(x,t)$ satisfies the equation

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla U) = 0 & \text{in } \mathbb{R}^{n+1}, \\ U(x,0) = u. \end{cases}$$

Moreover, up to a constant, $U(x,t)$ satisfies the Neumann boundary condition

$$\frac{\partial U}{\partial \nu^\sigma}(x,0) = (-\Delta)^\sigma u,$$

where

$$\frac{\partial U}{\partial \nu^\sigma}(x,0) = -\lim_{t\to 0} t^{1-2\sigma}\partial_t U(x,t).$$

Therefore, instead of (1.1), we will study the extension problem

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla U) = 0 & \text{in } \mathcal{B}_1^+, \\ \frac{\partial U}{\partial \nu^\sigma}(x,0) = U^{\frac{n}{n-2\sigma}}(x,0) & \text{on } \partial'\mathcal{B}_1\backslash\{0\}. \end{cases} \tag{1.5}$$

In terms of (1.5), we will prove Theorem 1.1 by proving the following result.

Theorem 1.2. *Let $n \geq 2, \sigma \in (0,1)$ and let u be a positive solution of (1.1). If U is the function given by (1.4), then either U can be extended as a continuous function near the origin or there exist two positive constants c_1 and c_2 such that*

$$c_1|X|^{2\sigma-n}(-\ln|X|)^{-\frac{n-2\sigma}{2\sigma}} \leq U(X) \leq c_2|X|^{2\sigma-n}(-\ln|X|)^{-\frac{n-2\sigma}{2\sigma}} \quad in\ \mathcal{B}_1^+\backslash\{0\}. \tag{1.6}$$

This paper will be organized as follows. In section 2, we give some preliminary results. In section3, we derive an upper bound for solutions of (1.1) near the isolated singularity. In section 4, we give the proof of Theorem 1.1 and Theorem 1.2.

Notation. In the rest of the paper, c will denote a strictly positive constant which may vary from line to line.

2. Preliminaries

In this section, we recall some results which will be used later.

Theorem 2.1 ([17]). *Let $n \geq 2, \sigma \in (0,1), 1 < p < (n+2\sigma)/(n-2\sigma)$ and let $u \in C^2(B_2\backslash\{0\}) \cap L_\sigma(\mathbb{R}^n)$ be a positive solution of the equation*

$$(-\Delta)^\sigma u = u^p \quad in\ B_1\backslash\{0\},$$

then there exists a positive constant $c = c(n,\sigma,p)$ such that

$$u(x) \leq c|x|^{-\frac{2\sigma}{p-1}} \quad near\ x = 0. \tag{2.1}$$

One consequence of the blow up rate (2.1) is the following Harnack inequality, which will be used very frequently in this rest of the paper.

Proposition 2.2. *Let $n \geq 2, \sigma \in (0,1)$ and let $U \in C^2(\overline{\mathcal{B}_1^+}\backslash\{0\})$ be a nonnegative solution of the equation*

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla U) = 0 & in\ \mathcal{B}_1^+, \\ \frac{\partial U}{\partial \nu^\sigma}(x,0) = U^p(x,0) & on\ \partial'\mathcal{B}_1\backslash\{0\}, \end{cases}$$

for $1 < p < (n+2\sigma)/(n-2\sigma)$, then for all $0 < r < \frac{1}{4}$, we have

$$\sup_{\mathcal{B}_{2r}^+\backslash\overline{\mathcal{B}_{\frac{r}{2}}^+}} U \leq c \inf_{\mathcal{B}_{2r}^+\backslash\overline{\mathcal{B}_{\frac{r}{2}}^+}} U,$$

where c is a positive constant independent of r.

Proof. The proof is essentially the same as the proof of Lemma 3.2 in [8]. □

As a direct application of Proposition 2.2, we can obtain the following result.

Corollary 2.3. *Let $n \geq 2, \sigma \in (0,1)$ and let $U \in C^2(\overline{\mathcal{B}_1^+}\backslash\{0\})$ be a nonnegative solution of the equation (1.5), then either $U \equiv 0$ or U is strictly positive.*

3. Upper bound near a singularity

In this section, we first prove an upper bound for positive solutions of (1.1) with a possible isolated singularity. The upper bound obtained in this section will also be used in deriving the lower bound.

Lemma 3.1. *Let $n \geq 2, \sigma \in (0,1)$ and let u be a positive solution of (1.1). If U is the function given by (1.4), then*

$$\lim_{|X|\to 0} |X|^{n-2\sigma} U(X) = 0. \tag{3.1}$$

Proof. Let $X = (x_1, x_2, \cdots, x_n, t) \in \mathbb{R}^{n+1}$ and let $(r, \xi, \theta_{n-1}, \cdots, \theta_2, \phi)$ be the corresponding spherical coordinates given by

$$\begin{cases} x_1 = r \sin\xi \sin\theta_{n-1} \cdots \sin\theta_2 \sin\phi, \\ x_2 = r \sin\xi \sin\theta_{n-1} \cdots \sin\theta_2 \cos\phi, \\ x_3 = r \sin\xi \sin\theta_{n-1} \cdots \cos\theta_2, \\ \quad \cdots, \\ t = r\cos\xi, \end{cases}$$

where $\xi \in [0,\pi), \theta_k \in [0,\pi)$ for $k = 2, 3, \cdots n-1$ and $\phi \in [0, 2\pi)$. We denote

$$\theta = (\xi, \theta_{n-1}, \cdots, \theta_2, \phi), \quad \theta' = (0, \theta_{n-1}, \cdots, \theta_2, \phi).$$

We also use θ_1 to denote $\cos\xi$. Let us consider the classical change of variable

$$U(r,\theta) = r^{2\sigma - n} V(s,\theta), \quad s = -\ln r. \tag{3.2}$$

By (1.5), Proposition 2.2 and Theorem 2.1, we know that V is a bounded solution of the equation

$$\left\{ \begin{array}{ll} \partial_{ss} V + (n-2\sigma)\partial_s V + \theta_1^{2\sigma-1} \mathrm{div}(\theta_1^{1-2\sigma} \nabla_{S^n_+} V) = 0 & \text{in } \mathbb{R}_+ \times S^n_+, \\ -\lim\limits_{\theta_1 \to 0} \theta_1^{1-2\sigma} \partial_{\theta_1} V = V^{\frac{n}{n-2\sigma}}(s, 0, \theta') & \text{on } \mathbb{R}_+ \times \partial S^n_+, \end{array} \right. \tag{3.3}$$

where

$$S^n_+ = \{X \in \mathbb{R}^{n+1} : r = 1, \theta_1 > 0\}.$$

Multiplying the both sides of (3.3) by $\partial_s V$ and using integration by part, we can get that

$$\begin{aligned} &\frac{1}{2}\frac{d}{ds}\int_{S^n_+} \theta_1^{1-2\sigma} (\partial_s V)^2 d\theta - \frac{1}{2}\frac{d}{ds}\int_{S^n_+} \theta_1^{1-2\sigma} |\nabla_{S^n_+} V|^2 d\theta \\ &= -(n-2\sigma)\int_{S^n_+} \theta_1^{1-2\sigma} (\partial_s V)^2 d\theta + \frac{2\sigma - n}{2n-2\sigma}\frac{d}{ds}\int_{\partial S^n_+} V(s,0,\theta')^{\frac{2n-2\sigma}{n-2\sigma}} d\theta'. \end{aligned} \tag{3.4}$$

where $d\theta'$ is the volume form of $\partial S^n_+ = S^{n-1}$. Let T_1, T_2 be two positive numbers such that $T_2 > T_1 > 1$. Integrating the both sides of (3.4) from T_1 to T_2 we can get that

$$\begin{aligned}
&\frac{1}{2}\int_{S^n_+}\theta_1^{1-2\sigma}(\partial_s V)^2(T_2,\theta)d\theta-\frac{1}{2}\int_{S^n_+}\theta_1^{1-2\sigma}(\partial_s V)^2(T_1,\theta)d\theta\\
&+(n-2\sigma)\int_{T_1}^{T_2}\int_{S^n_+}\theta_1^{1-2\sigma}(\partial_s V)^2 d\theta ds\\
=&\frac{1}{2}\int_{S^n_+}\theta_1^{1-2\sigma}|\nabla_{S^n_+}V|^2(T_2,\theta)d\theta-\frac{1}{2}\int_{S^n_+}\theta_1^{1-2\sigma}|\nabla_{S^n_+}V|^2(T_1,\theta)d\theta\\
&+\frac{2\sigma-n}{2n-2\sigma}[\int_{\partial S^n_+}V^{\frac{2n-2\sigma}{n-2\sigma}}(T_2,0,\theta')d\theta'-\int_{\partial S^n_+}V^{\frac{2n-2\sigma}{n-2\sigma}}(T_1,0,\theta')d\theta'].
\end{aligned}\tag{3.5}$$

The elliptic estimates in [16] imply that $\partial_s V$ and $\partial_{ss}V$ are uniformly bounded. Then

$$\int_{T_1}^{T_2}\int_{S^n_+}\theta_1^{1-2\sigma}(\partial_s V)^2 d\theta ds<\infty.$$

Let T_2 tend to $+\infty$ in (3.5), then

$$\int_{T}^{\infty}\int_{S^n_+}\theta_1^{1-2\sigma}(\partial_s V)^2 d\theta ds<+\infty.$$

Similar to the proof of Theorem 1.4 in [15], we can obtain that

$$\lim_{s\to+\infty}\int_{S^n_+}\theta_1^{1-2\sigma}(\partial_s V)^2 d\theta=0. \tag{3.6}$$

For any sequence $\{s_k\}$ such that $s_k\to\infty$ as $k\to\infty$, we consider the translation of V defined by $V_k(s,\theta)=V(s+s_k,\theta)$. Then there exist a subsequence $\{V_{l_k}(s,\theta)\}$ and a function $V_\infty(s,\theta)$ such that $V_{l_k}(s,\theta)\to V_\infty(s,\theta)$ in $C^2([-1,1]\times S^n_+)$. By (3.6) and the dominated convergence theorem, we know that $\partial_s V_\infty(s,\theta)=0$ in $[-1,1]\times S^n_+$. Therefore, there exists a function $\phi(\theta)$ such that $V_\infty(s,\theta)=\phi(\theta)$. Moreover, $\phi(\theta)$ satisfies the equation

$$\begin{cases}\operatorname{div}(\theta_1^{1-2\sigma}\nabla_{S^n_+}\phi) &=0 \quad\quad \text{in } S^n_+,\\ -\lim\limits_{\theta_1\to 0}\theta_1^{1-2\sigma}\partial_{\theta_1}\phi &=\phi^{\frac{n}{n-2\sigma}}(0,\theta')\quad \text{on } \partial S^n_+.\end{cases}\tag{3.7}$$

Integrating the both sides of (3.7) over S^n_+ and using integration by part, we get that

$$\int_{\partial S^n_+}\phi^{\frac{n}{n-2\sigma}}(0,\theta')d\theta'=0.$$

It follows that

$$\phi=0 \quad \text{on} \quad \theta_1=0. \tag{3.8}$$

Multiplying the both sides of (3.7) by ϕ and integrating over S^n_+, we get that

$$\int_{S^n_+}\theta_1^{1-2\sigma}|\nabla_{S^n_+}\phi|^2 d\theta=0. \tag{3.9}$$

By (3.8) and (3.9), we know that $\phi(\theta)\equiv 0$. Since $\{s_k\}$ can be any sequence, we get that

$$\lim_{s\to\infty}V(s,\theta)=0. \tag{3.10}$$

Then (3.1) follows from (3.10) and the definition of V. □

Proposition 3.2. *Let $n \geq 2, \sigma \in (0,1)$ and let u be a positive solution of (1.1). If U is the function given by (1.4), then there exists a positive constant c such that*

$$U(X) \leq c|X|^{2\sigma-n}(-\ln(|X|))^{-\frac{n-2\sigma}{2\sigma}} \quad in\ \mathcal{B}_1\backslash\{0\}. \tag{3.11}$$

Proof. We define

$$U(r,\theta) = r^{2\sigma-n}W(s,\theta), \quad s = \frac{r^{n-2\sigma}}{n-2\sigma},$$

then $W(s,\theta)$ satisfies the equation

$$\begin{cases} \theta_1^{1-2\sigma}\partial_{ss}W + \frac{1}{(n-2\sigma)^2s^2}\operatorname{div}((\theta_1^{1-2\sigma}\nabla_{S_+^n}W) = 0, \\ -\lim\limits_{\theta_1\to 0}\theta_1^{1-2\sigma}\partial_{\theta_1}W = W^{\frac{n}{n-2\sigma}}(s,0,\theta'). \end{cases} \tag{3.12}$$

Let

$$\overline{W}(s) = \frac{1}{\gamma_n}\int_{S_+^n}\theta_1^{1-2\sigma}W(s,\theta)d\theta,$$

where

$$\gamma_n = \int_{S_+^n}\theta_1^{1-2\sigma}d\theta. \tag{3.13}$$

Then $\overline{W}(s)$ satisfies the equation

$$\partial_{ss}\overline{W} + \frac{1}{\gamma_n(n-2\sigma)^2s^2}\int_{\partial S_+^n}W^{\frac{n}{n-2\sigma}}(s,0,\theta')d\theta' = 0. \tag{3.14}$$

By the Harnack inequality in Proposition 2.2, we can get that

$$\int_{\partial S_+^n}W^{\frac{n}{n-2\sigma}}(s,0,\theta')d\theta' \geq c(\max_{\theta\in S_+^n}W(s,\theta))^{\frac{n}{n-2\sigma}}. \tag{3.15}$$

Since $\int_{S_+^n}\theta_1^{1-2\sigma}d\theta < \infty$, then there exists a constant $c > 0$ such that

$$\max_{\theta\in S_+^n}W(s,\theta) \geq \frac{c}{\gamma_n}\int_{S_+^n}\theta_1^{1-2\sigma}W(s,\theta)d\theta = c\overline{W}(s). \tag{3.16}$$

We deduce from (3.14), (3.22) and (3.16) that there exists a constant $c > 0$ such that

$$\partial_{ss}\overline{W} + \frac{c}{s^2}\overline{W}^{\frac{n}{n-2\sigma}} \leq 0. \tag{3.17}$$

Since (3.1) holds, it is easy to see that

$$\lim_{s\to 0}W(s,\theta) = \lim_{s\to 0}\overline{W}(s) = 0. \tag{3.18}$$

By combining (3.17) and (3.18), we conclude that

$$\partial_s\overline{W} > 0 \quad \text{in a neighborhood of } 0.$$

Let ρ_0 be a positive constant such that

$$\partial_s\overline{W} > 0 \quad \text{in } (0,\rho_0).$$

If $\rho < \rho_0$, then

$$\begin{aligned}\partial_s \overline{W}(\rho_0) &= \partial_s \overline{W}(\rho) + \int_\rho^{\rho_0} \partial_{ss}\overline{W}(s)ds \\ &\le \partial_s \overline{W}(\rho) - c\int_\rho^{\rho_0} \frac{\overline{W}^{\frac{n}{n-2\sigma}}}{s^2}ds \\ &\le \partial_s \overline{W}(\rho) - c\frac{\overline{W}^{\frac{n}{n-2\sigma}}(\rho)}{\rho} + c\frac{\overline{W}^{\frac{n}{n-2\sigma}}(\rho)}{\rho_0}.\end{aligned} \tag{3.19}$$

By (3.19), we deduce that

$$\partial_s \overline{W} - c\frac{\overline{W}^{\frac{n}{n-2\sigma}}}{s} > 0 \quad \text{in a neighborhood of } 0. \tag{3.20}$$

Integrating the both sides of (3.20), we can get that

$$\overline{W}(s) \le c(-\ln s)^{-\frac{n-2\sigma}{2\sigma}} \quad \text{in a neighborhood of } 0. \tag{3.21}$$

By (3.21) and Proposition 2.2, we know that

$$W(s,\theta) \le c(-\ln s)^{-\frac{n-2\sigma}{2\sigma}} \quad \text{in a neighborhood of } 0. \tag{3.22}$$

Then (3.11) follows from the definition of W and (3.22). □

4. Lower bound near a singularity

In this section, we complete the proof of Theorem 1.2. Similar to [2], we will transform (1.5) into a time dependent equation. But contrary to [2], the occurrence of the boundary term in our situation will led to a lot of new difficulties.

Lemma 4.1. *Let $n \ge 2, \sigma \in (0,1)$ and let u be a positive solution of (1.1). Suppose U is the function given by (1.4) and V is the function given by (3.2), then there exists a positive constant c such that*

$$|\partial_s V(s,\theta)| + |\partial_{ss} V(s,\theta)| + |\partial_{sss} V(s,\theta)| \le cs^{-\frac{n-2\sigma}{2\sigma}}. \tag{4.1}$$

Proof. Let $|X_0|$ be a point such that $0 < |X_0| < 1/4$. We define

$$U^\lambda(X) = \lambda^{2\sigma-n}U(\lambda X)$$

with $\lambda = |X_0|/2$, then U^λ satisfies

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla U^\lambda) = 0 & \text{in } \mathcal{B}^+_{\frac{3}{2}}\backslash\mathcal{B}^+_{\frac{1}{2}}, \\ \frac{\partial U^\lambda}{\partial \nu^\sigma}(x,0) = (U^\lambda)^{\frac{n}{n-2\sigma}}(x,0) & \text{on } \partial'\mathcal{B}^+_{\frac{3}{2}}\backslash\partial'\mathcal{B}^+_{\frac{1}{2}}. \end{cases}$$

By Proposition 2.13 in [16], Lemma 2.18 in [16] and the standard elliptic estimates for uniformly elliptic equations, we have

$$\frac{X_0}{\lambda}\cdot\nabla U^\lambda(\frac{X_0}{\lambda}) \le c\|U^\lambda\|_{L^\infty(\mathcal{B}^+_{\frac{3}{2}}\backslash\mathcal{B}^+_{\frac{1}{2}})} \le c(-\ln(\lambda))^{-\frac{n-2\sigma}{2\sigma}}. \tag{4.2}$$

It follows that

$$|\partial_r U(|X_0|, \frac{X_0}{|X_0|})| \le c|X_0|^{2\sigma-n-1}(-\ln(|X_0|))^{-\frac{n-2\sigma}{2\sigma}}. \tag{4.3}$$

By the definition of V and (4.3), we can get that

$$|\partial_s V(s,\theta)| \le cs^{-\frac{n-2\sigma}{2\sigma}}. \tag{4.4}$$

In order to estimate $\partial_{ss}V$, we consider

$$\tilde{U}^\lambda(X) = (n-2\sigma)\tilde{U}^\lambda + X\cdot\nabla\tilde{U}^\lambda(X).$$

It is easy to check that $\tilde{U}^\lambda$ satisfies

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla\tilde{U}^\lambda) = 0 & \text{in } \mathcal{B}^+_{\frac{3}{2}}\backslash\mathcal{B}^+_{\frac{1}{2}}, \\ \frac{\partial\tilde{U}^\lambda}{\partial\nu^\sigma}(x,0) = \frac{n}{n-2\sigma}(U^\lambda)^{\frac{2\sigma}{n-2\sigma}}\tilde{U}^\lambda(x,0) & \text{on } \partial'\mathcal{B}^+_{\frac{3}{2}}\backslash\partial'\mathcal{B}^+_{\frac{1}{2}}. \end{cases}$$

By Proposition 2.13 in [16], Lemma 2.18 in [16] and the standard elliptic estimates for uniformly elliptic equations, we have

$$|\partial_{rr}U(|X_0|,\frac{X_0}{|X_0|})| \le c|X_0|^{2\sigma-n-2}(-\ln(|X_0|))^{-\frac{n-2\sigma}{2\sigma}}. \tag{4.5}$$

By (4.4), (4.5) and the definition of V, we can get that

$$|\partial_{ss}V(s,\theta)| \le cs^{-\frac{n-2\sigma}{2\sigma}}. \tag{4.6}$$

The term $|\partial_{sss}V(s,\theta)|$ can be estimated similarly, hence (4.1) is proved. □

Lemma 4.2. *Let $n\ge 2, \sigma\in(0,1)$ and let V be a solution of (3.3). Let $\overline{V}$ be the function defined by*

$$\overline{V}(s) = \frac{1}{\gamma_n}\int_{S^n_+}\theta_1^{1-2\sigma}V(s,\theta)d\theta, \tag{4.7}$$

where γ_n is the constant given by (3.13), then there exists a constant c such that

$$\int_{\partial S^n_+}(V-\overline{V})^2 d\theta' \le c\int_{S^n_+}\theta_1^{1-2\sigma}|\nabla_{S^n_+}V|^2 d\theta. \tag{4.8}$$

Proof. By Lemma 2.2 in [14], we know that there exists a constant c such that

$$\int_{\partial S^n_+}(V-\overline{V})^2 d\theta' \le c\int_{S^n_+}\theta_1^{1-2\sigma}((V-\overline{V})^2 + |\nabla_{S^n_+}V|^2)d\theta. \tag{4.9}$$

On the other hand, since

$$\int_{S^n_+}\theta_1^{1-2\sigma}(V-\overline{V})d\theta = 0,$$

we get from Corollary 4.15 that

$$\int_{S^n_+}\theta_1^{1-2\sigma}(V-\overline{V})^2 d\theta \le \tilde{\lambda}_1\int_{S^n_+}\theta_1^{1-2\sigma}|\nabla_{S^n_+}V|^2 d\theta \tag{4.10}$$

with $\tilde{\lambda}_1 = n+1-2\sigma$. By (4.9) and (4.10), we can get (4.8). □

Lemma 4.3. *Let $n\ge 2, \sigma\in(0,1)$ and let u be a positive solution of (1.1). Suppose U is the function given by (1.4) and $\overline{V}$ is the function defined by (4.7), then there exist two constants c and s_0 such that*

$$|\partial_s\overline{V}(s)| \le cs^{-\frac{n}{2\sigma}} \quad \text{in } (s_0,+\infty). \tag{4.11}$$

Proof. Integrating the both sides of (3.3) over S^n_+ and using integration by part, we can get that $\overline{V}$ satisfies the equation

$$\partial_{ss}\overline{V} + (n-2\sigma)\partial_s\overline{V} + \int_{\partial S^n_+}V^{\frac{n}{n-2\sigma}}(s,0,\theta')d\theta' = 0. \tag{4.12}$$

By (3.11), we know that there exist two constants c and s_0 such that

$$f(s)=\int_{\partial S^n_+} V^{\frac{n}{n-2\sigma}}(s,0,\theta')d\theta' \leq cs^{-\frac{n}{2\sigma}} \quad \text{in } (s_0,+\infty). \tag{4.13}$$

A direct computation shows that, for some $\alpha_0,\beta_0\in\mathbb{R}$,

$$\begin{aligned}\overline{V}(s)=&\alpha_0+\frac{1}{2\sigma-n}\int_{s_0}^{s}f(\tau)d\tau+\beta_0 e^{(2\sigma-n)s}\\ &-\frac{1}{2\sigma-n}\int_{s_0}^{s}e^{(n-2\sigma)(\tau-s)}f(\tau)d\tau.\end{aligned} \tag{4.14}$$

Since

$$\lim_{s\to\infty}V(s,\theta)=\lim_{s\to\infty}\overline{V}(s)=0,$$

then

$$\alpha_0=\frac{1}{n-2\sigma}\int_{s_0}^{+\infty}f(\tau)d\tau. \tag{4.15}$$

We take (4.15) into (4.14), then $\overline{V}$ can be rewritten as

$$\begin{aligned}\overline{V}(s)=&\frac{1}{n-2\sigma}\int_{s}^{+\infty}f(\tau)d\tau+\beta_0 e^{(2\sigma-n)s}\\ &-\frac{1}{2\sigma-n}\int_{s_0}^{s}e^{(n-2\sigma)(\tau-s)}f(\tau)d\tau.\end{aligned} \tag{4.16}$$

Taking the derivative with respect to s in (4.16), we can get that

$$\begin{aligned}\partial_s\overline{V}(s)=&-\frac{2}{n-2\sigma}f(s)+(2\sigma-n)\beta_0 e^{(2\sigma-n)s}\\ &-\int_{s_0}^{s}e^{(n-2\sigma)(\tau-s)}f(\tau)d\tau.\end{aligned} \tag{4.17}$$

If $s>4s_0$, then the term $\int_{s_0}^{s}e^{(n-2\sigma)(\tau-s)}f(\tau)d\tau$ can be estimated as follows .

$$\begin{aligned}&\int_{s_0}^{s}e^{(n-2\sigma)(\tau-s)}f(\tau)d\tau\\ =&\int_{s_0}^{\frac{s}{2}}e^{(n-2\sigma)(\tau-s)}f(\tau)d\tau+\int_{\frac{s}{2}}^{s}e^{(n-2\sigma)(\tau-s)}f(\tau)d\tau\\ \leq&\|f\|_{L^\infty((s_0,\frac{s}{2}))}e^{-\frac{(n-2\sigma)s}{2}}(\frac{s}{2}-s_0)+cs^{-\frac{n}{2\sigma}}\int_{\frac{s}{2}}^{s}e^{(n-2\sigma)(\tau-s)}d\tau\\ \leq& cs^{-\frac{n}{2\sigma}}.\end{aligned}$$

It follows easily that (4.11) holds. □

Lemma 4.4. *Let $n\geq 2,\sigma\in(0,1)$ and let u be a positive solution of (1.1). Suppose U is the function given by (1.4) and V is the function defined by (3.2), then there exist two positive constants c and $\tilde{s_0}$ such that*

$$\int_{S^n_+}\theta_1^{1-2\sigma}|\nabla_{S^n_+}V|^2d\theta\leq cs^{-\frac{2n-3\sigma}{2\sigma}} \quad \text{in } (\tilde{s_0},\infty). \tag{4.18}$$

Proof. Let us consider

$$Y(s)=\int_{S^n_+}\theta_1^{1-2\sigma}(V-\overline{V})^2(s,\theta)d\theta,$$

where $\overline{V}$ is the function defined by (4.7). By (3.3) and some computations, we know that there exist two constants c and s_0 such that the function Y satisfies

$$Y''+(n-2\sigma)Y'-(n+1-2\sigma)Y\geq -cs^{-\frac{n-\sigma}{\sigma}}\quad \text{in } (s_0,+\infty). \tag{4.19}$$

The homogeneous equation associated to (4.19) admits two linearly independent solutions

$$\begin{cases} Y_1(s)=e^{(2\sigma-1-n)s},\\ Y_2(s)=e^{s}.\end{cases}$$

A particular solution of

$$Y''+(n-2\sigma)Y'-(n+1-2\sigma)Y=-cs^{-\frac{n-\sigma}{\sigma}}$$

is given by

$$\begin{aligned} Y_p(s)=&\frac{c}{n-2-2\sigma}\int_s^{+\infty}e^{s-\tau}\tau^{-\frac{n-\sigma}{\sigma}}d\tau\\ &+Me^{(2\sigma-1-n)s}-\frac{c}{2\sigma-2-n}\int_{s_0}^{s}e^{(n+1-2\sigma)(\tau-s)}\tau^{-\frac{n-\sigma}{\sigma}}d\tau,\end{aligned}$$

where M is a fixed constant. Similar to the arguments used in Lemma 4.3, we know that there exist two positive constants c and $\tilde{s}_0$ such that

$$Y_p(s)\leq cs^{-\frac{n-\sigma}{\sigma}}\quad \text{in } (\tilde{s}_0,+\infty).$$

Since $\lim_{s\to\infty}Y(s)=0$, basic comparison principles imply

$$Y(s)\leq Y_p(s)\leq cs^{-\frac{n-\sigma}{\sigma}}\quad \text{in } (\tilde{s}_0,+\infty) \tag{4.20}$$

for some constant c which is sufficiently large. Multiplying the both sides of (3.3) by $V-\overline{V}$ and using integration by part, we can get that

$$\begin{aligned}\int_{S^n_+}\theta_1^{1-2\sigma}|\nabla_{S^n_+}V|^2d\theta=&\int_{S^n_+}\theta_1^{1-2\sigma}[\partial_{ss}V+(n-2\sigma)\partial_sV](V-\overline{V})d\theta\\ &+\int_{\partial S^n_+}V^{\frac{n}{n-2\sigma}}(V-\overline{V})d\theta'.\end{aligned}$$

Since

$$\begin{aligned}\int_{S^n_+}\theta_1^{1-2\sigma}\partial_{ss}V(V-\overline{V})d\theta&\leq cs^{-\frac{n-2\sigma}{2\sigma}}Y(s)^{\frac12}\leq cs^{-\frac{2n-3\sigma}{2\sigma}},\\ \int_{S^n_+}\theta_1^{1-2\sigma}\partial_{s}V(V-\overline{V})d\theta&\leq cs^{-\frac{n-2\sigma}{2\sigma}}Y(s)^{\frac12}\leq cs^{-\frac{2n-3\sigma}{2\sigma}},\\ \int_{\partial S^n_+}V^{\frac{n}{n-2\sigma}}(V-\overline{V})d\theta'&\leq cs^{-\frac{n}{2\sigma}}(\int_{S^n_+}\theta_1^{1-2\sigma}|\nabla_{S^n_+}V|^2d\theta)^{\frac12},\end{aligned}$$

we conclude that

$$\int_{S^n_+}\theta_1^{1-2\sigma}|\nabla_{S^n_+}V|^2d\theta\leq cs^{-\frac{2n-3\sigma}{2\sigma}}+cs^{-\frac{n}{2\sigma}}(\int_{S^n_+}\theta_1^{1-2\sigma}|\nabla_{S^n_+}V|^2d\theta)^{\frac12} \tag{4.21}$$

for some constant c. It follows from (4.21) that

$$\int_{S^n_+} \theta_1^{1-2\sigma} |\nabla_{S^n_+} V|^2 d\theta \leq c s^{-\frac{2n-3\sigma}{2\sigma}} \quad \text{in } (\tilde{s}_0, +\infty)$$

for some constant c. □

Remark 4.5. *In the process of deriving (4.19), we have applied Corollary 4.15.*

Lemma 4.6. *Let $n \geq 2, \sigma \in (0,1)$ and let u be a positive solution of (1.1). Suppose U is the function given by (1.4) and $\overline{V}$ is the function defined by (4.7), then there exist two constants c and s_0 such that*

$$\partial_{ss} \overline{V}(s) \leq c s^{-\frac{2n+\sigma}{4\sigma}} \quad \textit{in } (s_0, +\infty). \tag{4.22}$$

Proof. Taking the derivative with respect to s in (3.3), we can get that

$$\begin{cases} \partial_{sss} V + (n-2\sigma)\partial_{ss} V + \theta_1^{2\sigma-1} \mathrm{div}(\theta_1^{1-2\sigma} \nabla_{S^n_+} \partial_s V) = 0, \\ \qquad - \lim_{\theta_1 \to 0} \theta_1^{1-2\sigma} \partial_{\theta_1} \partial_s V = \frac{n}{n-2\sigma} V^{\frac{2\sigma}{n-2\sigma}} \partial_s V, \end{cases} \tag{4.23}$$

Similar to the arguments used in Lemma 4.4, we can get that there exist two constant c and $\tilde{s}_0$ such that

$$\int_{S^n_+} \theta_1^{1-2\sigma} |\nabla_{S^n_+} \partial_s V|^2 d\theta \leq c s^{-\frac{2n-3\sigma}{2\sigma}} \quad \text{in } (\tilde{s}_0, +\infty). \tag{4.24}$$

By Lemma 2.2 in [14] and Lemma 4.3, we know that there exists a constant c such that

$$\begin{aligned} \int_{\partial S^n_+} (\partial_s V)^2 d\theta' &\leq c \int_{S^n_+} \theta_1^{1-2\sigma} ((\partial_s V)^2 + |\nabla_{S^n_+} \partial_s V|^2) d\theta \\ &\leq c \int_{S^n_+} \theta_1^{1-2\sigma} ((\partial_s \overline{V})^2 + |\nabla_{S^n_+} \partial_s V|^2) d\theta \\ &\leq c s^{-\frac{2n-3\sigma}{2\sigma}}. \end{aligned} \tag{4.25}$$

In the process of obtaining (4.25), we have applied (4.24), Lemma 4.3, Corollary 4.15 and the fact that

$$\int_{S^n_+} \theta_1^{1-2\sigma} (\partial_s V)^2 d\theta \leq 2 \int_{S^n_+} \theta_1^{1-2\sigma} ((\partial_s \overline{V})^2 + (\partial_s V - \partial_s \overline{V})^2) d\theta.$$

Integrating the both sides of (4.23) and using integration by part, we can get that $\overline{V}$ satisfies the equation

$$\partial_{sss} \overline{V} + (n-2\sigma)\partial_{ss} \overline{V} + \frac{n}{n-2\sigma} \int_{\partial S^n_+} V^{\frac{2\sigma}{n-2\sigma}} \partial_s V(s,0,\theta') d\theta' = 0, \tag{4.26}$$

We denote

$$\tilde{f}(s) = \frac{n}{n-2\sigma} \int_{\partial S^n_+} V^{\frac{2\sigma}{n-2\sigma}} \partial_s V(s,0,\theta') d\theta'.$$

Since

$$\int_{\partial S^n_+} V^{\frac{2\sigma}{n-2\sigma}} \partial_s V(s,0,\theta') d\theta' \leq c s^{-1} \left(\int_{\partial S^n_+} (\partial_s V)^2 (s,0,\theta') d\theta'\right)^{\frac{1}{2}},$$

we get from (4.25) that

$$\tilde{f}(s) \leq c s^{-\frac{2n+\sigma}{4\sigma}} \quad \text{in a neighborhood of } +\infty. \tag{4.27}$$

Then 4.22 can be obtained by repeating the arguments used in the last part of the proof of Lemma 4.3. □

Lemma 4.7. *Let $n \geq 2, \sigma \in (0,1)$ and let u be a positive solution of* (1.1)*. Suppose the function given by* (1.4) *satisfies*

$$\liminf_{|X|\to 0} |X|^{n-2\sigma}(-\ln|X|)^{\frac{n-2\sigma}{2\sigma}}U(X) < \limsup_{|X|\to 0} |X|^{n-2\sigma}(-\ln|X|)^{\frac{n-2\sigma}{2\sigma}}U(X), \quad (4.28)$$

then

$$\limsup_{|X|\to 0} |X|^{n-2\sigma}(-\ln|X|)^{\frac{n-2\sigma}{2\sigma}}U(X) \leq (\frac{(2\sigma-n)^2\gamma_n}{2\sigma\omega_{n-1}})^{\frac{n-2\sigma}{2\sigma}}, \quad (4.29)$$

where γ_n is given by (3.13) *and ω_{n-1} is the volume of $S^{n-1} = \partial S^n_+$.*

Proof. Let

$$U(r,\theta) = r^{2\sigma-n}(-\ln r)^{-\frac{n-2\sigma}{2\sigma}}\tilde{V}(s,\theta), \quad s = -\ln r, \quad (4.30)$$

then $\tilde{V}$ satisfies the equation

$$\begin{cases} \partial_{ss}\tilde{V} - (2\sigma-n)(1-\frac{1}{\sigma s})\partial_s\tilde{V} - \chi(s)\tilde{V} + \theta_1^{2\sigma-1}\mathrm{div}(\theta_1^{1-2\sigma}\nabla_{\mathrm{S}^n_+}\tilde{\mathrm{V}}) = 0, \\ -\lim_{\theta_1\to 0}\theta_1^{1-2\sigma}\partial_{\theta_1}\tilde{V} = \frac{\tilde{V}^{\frac{n}{n-2\sigma}}}{s}(s,0,\theta'), \end{cases} \quad (4.31)$$

where $\chi(s)$ is given by

$$\chi(s) = \frac{(2\sigma-n)^2}{2\sigma s} - \frac{n(n-2\sigma)}{4\sigma^2 s^2}.$$

Multiplying the both sides of (4.31) by $\partial_s\tilde{V}$ and integrating over S^n_+, we can get that

$$\begin{aligned} &\frac{1}{2}\frac{d}{ds}\int_{S^n_+}\theta_1^{1-2\sigma}(\partial_s\tilde{V})^2 d\theta - \frac{1}{2}\frac{d}{ds}\int_{S^n_+}\theta_1^{1-2\sigma}\chi(s)\tilde{V}^2 d\theta \\ &-\int_{S^n_+}\theta_1^{1-2\sigma}[(2\sigma-n)(1-\frac{1}{\sigma s})(\partial_s\tilde{V})^2 - \frac{1}{2}\tilde{V}^2\frac{d\chi}{ds}(s)]d\theta \\ =&\frac{n-2\sigma}{2n-2\sigma}\frac{d}{ds}\int_{\partial S^n_+}\frac{1}{s}\tilde{V}^{\frac{2n-2\sigma}{n-2\sigma}}(s,0,\theta')d\theta' + \frac{1}{2}\frac{d}{ds}\int_{S^n_+}\theta_1^{1-2\sigma}|\nabla_{S^n_+}\tilde{V}|^2 d\theta \\ &+\frac{n-2\sigma}{2n-2\sigma}\int_{\partial S^n_+}\frac{1}{s^2}\tilde{V}^{\frac{2n-2\sigma}{n-2\sigma}}(s,0,\theta')d\theta'. \end{aligned} \quad (4.32)$$

Let T_1, T_2 be two positive constants such that $1 \ll T_1 < T_2$. Integrating the both sides of (4.32) from T_1 to T_2 and using the fact that $\tilde{V}, \partial_s\tilde{V}$ and $\partial_{ss}\tilde{V}$ are uniformly bounded, we get that

$$\int_{T_1}^{T_2}\int_{S^n_+} -(2\sigma-n)(1-\frac{1}{\sigma s})\theta_1^{1-2\sigma}(\partial_s\tilde{V})^2 d\theta ds < \infty.$$

Let T_2 tend to ∞, then

$$\int_{T_1}^{\infty}\int_{S^n_+} -(2\sigma-n)(1-\frac{1}{\sigma s})\theta_1^{1-2\sigma}(\partial_s\tilde{V})^2 d\theta ds < \infty.$$

Similar to the proof of Lemma 4 in [2], we can get that

$$\lim_{s\to\infty}\int_{S^n_+}\theta_1^{1-2\sigma}(\partial_s\tilde{V})^2 d\theta = 0. \quad (4.33)$$

For any sequence $\{s_k\}$ such that $s_k \to \infty$ as $k \to \infty$, we consider the translation of $\tilde{V}$ defined by $\tilde{V}_k(s,\theta) = \tilde{V}(s+s_k,\theta)$, then there exists a function $\tilde{\phi}(\theta)$ such that $\tilde{V}_k(s,\theta) \to \tilde{\phi}(\theta)$ in $C^2([-1,1]\times S^n_+)$. Moreover, $\tilde{\phi}(\theta)$ satisfies the equation

$$\begin{cases} \mathrm{div}(\theta_1^{1-2\sigma}\nabla_{S^n_+}\tilde{\phi}) & = 0 \quad \text{in } S^n_+, \\ -\lim\limits_{\theta_1\to 0}\theta_1^{1-2\sigma}\partial_{\theta_1}\tilde{\phi}(0,\theta') & = 0. \end{cases} \tag{4.34}$$

Integrating the both sides of (4.34) over S^n_+, we can get that $\tilde{\phi}(\theta)$ equals a constant. In order to continue the proof, we define

$$\overline{\tilde{V}}(s) = \frac{1}{\gamma_n}\int_{S^n_+}\theta_1^{1-2\sigma}\tilde{V}(s,\theta)d\theta \tag{4.35}$$

with γ_n be the constant given by (3.13), then

$$\overline{\tilde{V}}(s) = s^{\frac{n-2\sigma}{2\sigma}}\overline{V}(s),$$

where $\overline{V}$ is the function given by (4.7). Since

$$\partial_{ss}\overline{\tilde{V}} = \frac{n-2\sigma}{2\sigma}(\frac{n-2\sigma}{2\sigma}-1)s^{\frac{n-2\sigma}{2\sigma}-2}\overline{V} + \frac{n-2\sigma}{\sigma}s^{\frac{n-2\sigma}{2\sigma}-1}\partial_s\overline{V} + s^{\frac{n-2\sigma}{2\sigma}}\partial_{ss}\overline{V},$$

we know from Lemma 4.3 and Lemma 4.6 that

$$|\partial_{ss}\overline{\tilde{V}}(s)| \le cs^{-\frac{5}{4}} \quad \text{in a neighborhood of } +\infty. \tag{4.36}$$

Integrating the both sides of (4.34) over S^n_+ and using integration by part, we can get that $\overline{\tilde{V}}(s)$ satisfies

$$\partial_{ss}\overline{\tilde{V}} - (2\sigma-n)(1-\frac{1}{\sigma s})\partial_s\overline{\tilde{V}} - \chi(s)\overline{\tilde{V}} + \frac{1}{\gamma_n s}\int_{\partial S^n_+}\tilde{V}^{\frac{n}{n-2\sigma}}(s,0,\theta')d\theta' = 0. \tag{4.37}$$

By (4.28) and the above analysis, we know that there exist two sequence $\{s_{n_k}\}, \{s_{l_k}\}$ such that

$$\lim_{k\to\infty}\overline{\tilde{V}}(s_{n_k}) = \limsup_{|X|\to 0}|X|^{n-2\sigma}(-\ln|X|)^{\frac{n-2\sigma}{2\sigma}}U(X) = \alpha_1$$

and

$$\lim_{k\to\infty}\overline{\tilde{V}}(s_{l_k}) = \liminf_{|X|\to 0}|X|^{n-2\sigma}(-\ln|X|)^{\frac{n-2\sigma}{2\sigma}}U(X) = \alpha_2.$$

By taking subsequences if necessary, we can assume that

$$s_{n_k} < s_{l_k} < s_{n_{k+1}} < s_{l_{k+1}}.$$

In view of our assumptions, it is easy to see that there exists a sequence $\{s_{p_k}\}$ such that

$$s_{p_k} < s_{l_k} < s_{p_{k+1}} < s_{l_{k+1}}$$

and

$$\overline{\tilde{V}}(s_{p_{k+1}}) = \max_{s\in(s_{l_k},s_{l_{k+1}})}\overline{\tilde{V}}(s), \quad \lim_{k\to\infty}\overline{\tilde{V}}(s_{n_k}) = \alpha_1. \tag{4.38}$$

By (4.36), (4.38) and (4.37), we deduce that

$$\frac{1}{\gamma_n s_{p_{k+1}}}\int_{\partial S^n_+}\tilde{V}^{\frac{n}{n-2\sigma}}(s_{p_{k+1}},0,\theta')d\theta' - \frac{(2\sigma-n)^2}{2\sigma s_{p_{k+1}}}\overline{\tilde{V}}(s_{p_{k+1}}) - \frac{c}{(s_{p_{k+1}})^{\frac{5}{4}}} \le 0 \tag{4.39}$$

for some constant c. Let $k\to\infty$ in (4.39), we can get that

$$\frac{w_{n-1}}{\gamma_n}\alpha_1^{\frac{2\sigma}{n-2\sigma}} - \frac{(2\sigma-n)^2}{2\sigma} \le 0.$$

In terms of the definition of α_1, we know that (4.29) holds. $\square$

Lemma 4.8. *Let $n \geq 2, \sigma \in (0,1)$ and let u be a positive solution of (1.1). If U is the function given by (1.4), then*

$$\lim_{|X|\to 0} |X|^{n-2\sigma}(-\ln|X|)^{\frac{n-2\sigma}{2\sigma}} U(X) \quad \textit{exists.} \tag{4.40}$$

Proof. The equation (4.37) can be rewritten as

$$\begin{aligned} &\partial_{ss}\overline{\tilde{V}} - (2\sigma-n)(1-\frac{1}{\sigma s})\partial_s\overline{\tilde{V}} - \frac{(2\sigma-n)^2}{2\sigma s}\overline{\tilde{V}} + \frac{\omega_{n-1}}{\gamma_n s}\overline{\tilde{V}}^{\frac{n}{n-2\sigma}} \\ &+ \frac{n(n-2\sigma)}{4\sigma^2 s^2}\overline{\tilde{V}} + \frac{1}{\gamma_n s}\int_{\partial S^n_+}(\tilde{V}^{\frac{n}{n-2\sigma}} - \overline{\tilde{V}}^{\frac{n}{n-2\sigma}})d\theta' = 0. \end{aligned} \tag{4.41}$$

If (4.40) does not hold, then (4.28) holds. It follows that there exist two sequences $(s_{n_k}, \theta_{n_k}), (s_{l_k}, \theta_{l_k})$ such that

$$\lim_{k\to\infty} \tilde{V}(s_{n_k}, \theta_{n_k}) = \limsup_{|X|\to 0} |X|^{n-2\sigma}(-\ln|X|)^{\frac{n-2\sigma}{2\sigma}} U(X) = \alpha_1$$

and

$$\lim_{k\to\infty} \tilde{V}(s_{l_k}, \theta_{l_k}) = \liminf_{|X|\to 0} |X|^{n-2\sigma}(-\ln|X|)^{\frac{n-2\sigma}{2\sigma}} U(X) = \alpha_2.$$

By the analysis used in the proof of Lemma 4.7, we know that

$$\lim_{k\to\infty} \overline{\tilde{V}}(s_{n_k}) = \alpha_1, \quad \lim_{k\to\infty} \overline{\tilde{V}}(s_{l_k}) = \alpha_2.$$

Without loss of generality, we can assume

$$s_{n_k} < s_{l_k} < s_{n_{k+1}} < s_{l_{k+1}}.$$

Integrating the both sides of (4.41) from s_{n_k} to s_{l_k}, we have

$$\begin{aligned} &(2\sigma-n)(1-\frac{1}{\sigma s_{l_k}})\overline{\tilde{V}}(s_{l_k}) - (2\sigma-n)(1-\frac{1}{\sigma s_{n_k}})\overline{\tilde{V}}(s_{n_k}) \\ &= \partial_s\overline{\tilde{V}}(s_{l_k}) - \partial_s\overline{\tilde{V}}(s_{n_k}) + \frac{2\sigma-n}{\sigma}\int_{s_{n_k}}^{s_{l_k}} \frac{\overline{\tilde{V}}}{s^2}ds \\ &+ \int_{s_{n_k}}^{s_{l_k}} \frac{\overline{\tilde{V}}}{s}[\frac{\omega_{n-1}}{\gamma_n}\overline{\tilde{V}}^{\frac{2\sigma}{n-2\sigma}} - \frac{(2\sigma-n)^2}{2\sigma}]ds + \frac{n(n-2\sigma)}{4\sigma^2}\int_{s_{n_k}}^{s_{l_k}} \frac{\overline{\tilde{V}}}{s^2}ds \\ &+ \frac{1}{\gamma_n}\int_{s_{n_k}}^{s_{l_k}}\int_{\partial S^n_+} \frac{1}{s}(\tilde{V}^{\frac{n}{n-2\sigma}} - \overline{\tilde{V}}^{\frac{n}{n-2\sigma}})d\theta' ds = 0. \end{aligned} \tag{4.42}$$

Since (4.28) holds, we know from Lemma 4.7 that

$$-\frac{(2\sigma-n)^2}{2\sigma s} + \frac{\omega_{n-1}}{\gamma_n s}\overline{\tilde{V}}^{\frac{2\sigma}{n-2\sigma}} \leq 0. \tag{4.43}$$

By Lemma 4.2, Lemma 4.4 and the mean value theorem, we can get that

$$\begin{aligned}
&\frac{1}{s}\int_{\partial S^n_+}(\tilde{V}^{\frac{n}{n-2\sigma}}-\overline{\tilde{V}}^{\frac{n}{n-2\sigma}})d\theta' \\
&\le \frac{c}{s}(\int_{\partial S^n_+}(\tilde{V}-\overline{\tilde{V}})^2)^{\frac{1}{2}} \\
&\le \frac{c}{s}(\int_{\partial S^n_+}\theta_1^{1-2\sigma}|\nabla_{S^n_+}\tilde{V}|^2 d\theta)^{\frac{1}{2}} \\
&\le cs^{-\frac{5}{4}}.
\end{aligned} \tag{4.44}$$

We take (4.43) and (4.44) into (4.42), then

$$\begin{aligned}
&(2\sigma-n)(1-\frac{1}{\sigma s_{l_k}})\overline{\tilde{V}}(s_{l_k})-(2\sigma-n)(1-\frac{1}{\sigma s_{n_k}})\overline{\tilde{V}}(s_{n_k}) \\
&\le \partial_s\overline{\tilde{V}}(s_{l_k})-\partial_s\overline{\tilde{V}}(s_{n_k})+c\int_{s_{n_k}}^{s_{l_k}}\frac{1}{s^{\frac{5}{4}}}ds.
\end{aligned} \tag{4.45}$$

By taking $k\to+\infty$ in (4.45), we can get that

$$(2\sigma-n)(\alpha_2-\alpha_1)\le 0.$$

Because of our assumptions, we get a contradiction. □

Corollary 4.9. *Let $n\ge 2,\sigma\in(0,1)$ and let u be a positive solution of (1.1). If the function given by (1.4) satisfies*

$$\liminf_{|X|\to 0}|X|^{n-2\sigma}(-\ln|X|)^{\frac{n-2\sigma}{2\sigma}}U(X)=0,$$

then

$$\lim_{|X|\to 0}|X|^{n-2\sigma}(-\ln|X|)^{\frac{n-2\sigma}{2\sigma}}U(X)=0.$$

Proposition 4.10. *Let $n\ge 2,\sigma\in(0,1)$ and let U be a positive solution of (1.5) such that*

$$\lim_{|X|\to 0}|X|^{n-2\sigma}(-\ln|X|)^{\frac{n-2\sigma}{2\sigma}}U(X)=0, \tag{4.46}$$

then the singularity of U at the origin is removable.

Proof. Let $\tilde{V}(s,\theta)$ be the function defined by (4.30) and let $\overline{\tilde{V}}(s,\theta)$ be the function defined by (4.34). Since (4.46) holds, then

$$\lim_{s\to\infty}\tilde{V}(s,\theta)=\lim_{s\to\infty}\overline{\tilde{V}}(s)=0. \tag{4.47}$$

By (4.47), (4.37) and Proposition 2.2, we know that there exists a positive number $s_1>0$ such that

$$\partial_{ss}\overline{\tilde{V}}-(2\sigma-n)(1-\frac{1}{\sigma s})\partial_s\overline{\tilde{V}}>0 \quad \text{in } (s_1,+\infty), \tag{4.48}$$

Let ϵ, s_2 be two positive constants such that

$$\epsilon^2+(2\sigma-n)(1-\frac{1}{\sigma s})\epsilon<0 \quad \text{in } (s_2,+\infty).$$

Let $s_3=\max\{s_1,s_2\}$ and let

$$\Psi(s)=\overline{\tilde{V}}(s)-Me^{-\epsilon s},$$

where M is a large constant such that $\overline{\tilde{V}}(s_3) < Me^{-\epsilon s_3}$. Then $\Psi(s)$ satisfies

$$\begin{cases} \partial_{ss}\Psi - (2\sigma - n)(1 - \frac{1}{\sigma s})\partial_s \Psi > 0 \quad \text{in } (s_3, +\infty), \\ \Psi(s_3) < 0, \\ \lim_{s\to\infty} \Psi(s) = 0. \end{cases}$$

By the maximum principle, we can get that

$$\Psi(s) \leq 0 \quad \text{in } (s_3, +\infty).$$

Therefore,

$$\overline{\tilde{V}}(s) \leq Me^{-\epsilon s} \quad \text{in } (s_3, +\infty).$$

The Harnack inequality in Proposition 2.2 implies that

$$\tilde{V}(s, \theta) < Me^{-\epsilon s} \quad \text{for some } M > 0.$$

It follows that

$$U(r, \theta) < Mr^{\epsilon + 2\sigma - n} \quad \text{for some } M > 0$$

and

$$U^{\frac{2\sigma}{n-2\sigma}}(r, 0, \theta') = U(x, 0)^{\frac{2\sigma}{n-2\sigma}} \in L^q(B_1) \quad \text{for some } q > \frac{n}{2\sigma}.$$

Proposition 2.6 in [16] implies that U is Hölder continuous at the origin. □

Proof of Theorem 1.2. The proof of Theorem 1.2 is now just a combination of Proposition 3.2, Corollary 4.9 and Proposition 4.10. □

Finally, we describe the exact local behavior of positive solutions of (1.1) with a nonremovable singularity at the origin.

Proposition 4.11. *Let $n \geq 2, \sigma \in (0, 1)$ and let u be a positive solution of (1.1). Suppose the singularity at the origin is not removable and suppose U is the function given by (1.4), then*

$$\lim_{|X|\to 0} |X|^{n-2\sigma}(-\ln|X|)^{\frac{n}{n-2\sigma}} U(X) = \left(\frac{(2\sigma - n)^2 \gamma_n}{2\sigma\omega_{n-1}}\right)^{\frac{n-2\sigma}{2\sigma}}. \tag{4.49}$$

Proof. By Lemma 4.8, we know that

$$\lim_{|X|\to 0} |X|^{n-2\sigma}(-\ln|X|)^{\frac{n}{n-2\sigma}} U(X) \quad \text{exists.}$$

Since the singularity at the origin is not removable, we know from Proposition 4.10 that

$$\lim_{|X|\to 0} |X|^{n-2\sigma}(-\ln|X|)^{\frac{n}{n-2\sigma}} U(X) = \beta > 0.$$

By integrating the both sides of (4.37) over (s_0, s_1), where s_0 is a fixed number and s_1 is a number which is large enough, we can get that there is a constant c independent of s_1 such that

$$-\frac{(2\sigma - n)^2}{2\sigma}\int_{s_0}^{s_1} \frac{\overline{\tilde{V}}(s)}{s} ds + \frac{1}{\gamma_n}\int_{s_0}^{s_1}\int_{\partial S^n_+} \frac{\tilde{V}^{\frac{n}{n-2\sigma}}(s, 0, \theta')}{s} ds d\theta' < c. \tag{4.50}$$

Since s_1 can be arbitrary, it follows that β should be given by $\left(\frac{(2\sigma-n)^2\gamma_n}{2\sigma\omega_{n-1}}\right)^{\frac{n-2\sigma}{2\sigma}}$. □

Appendix: An eigenvalue problem

Let us consider the eigenvalue problem

$$\begin{cases} \mathrm{div}_{S^n}(|\theta_1|^{1-2\sigma}\nabla_{S^n}\Phi) + \lambda|\theta_1|^{1-2\sigma}\Phi = 0 & \text{in } S^n, \\ \Phi \in H^1(S^n, |\theta_1|^{1-2\sigma}), \end{cases} \tag{4.51}$$

where $H^1(S^n, |\theta_1|^{1-2\sigma})$ is the completion of $C^\infty(S^n)$ with respect to the norm

$$\|\psi\|_{H^1(S^n,|\theta_1|^{1-2\sigma})} = (\int_{S^n} |\theta_1|^{1-2\sigma}(|\nabla_{S^n}\psi|^2 + |\psi|^2)d\theta)^{\frac{1}{2}}.$$

From classical spectral theory, problem (4.51) admits a diverging sequence of real eigenvalues with finite multiplicity

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots .$$

Remark 4.12. *We notice that*

$$\lambda_k \ge 0 \textit{ for } k = 0, 1, 2, \cdots . \tag{4.52}$$

Indeed, multiplying the both sides of (4.51) *by* Φ *and using integration by part, we can get that*

$$-\int_{S^n} |\theta_1|^{1-2\sigma}|\nabla_{S^n}\Phi|^2 d\theta + \lambda\int_{S^n} |\theta_1|^{1-2\sigma}\Phi^2 d\theta = 0.$$

It follows that (4.52) *holds.*

Proposition 4.13. *The eigenvalues of* (4.51) *are in fact*

$$\tilde{\lambda}_k = k(k + n - 2\sigma). \tag{4.53}$$

Moreover, the multiplicity of the eigenvalue $\tilde{\lambda}_k$ *is*

$$\tilde{m}_k = \frac{(n-1+2k)(n-2+k)!}{k!(n-1)!}. \tag{4.54}$$

Proof. It is known from [20] that the eigenvalues of $-\Delta_{S^{n-1}}$ are given by

$$\mu_k = k(k + n - 2) \tag{4.55}$$

with the multiplicity

$$m_k = \frac{(n-2+2k)(n-3+k)!}{k!(n-2)!}. \tag{4.56}$$

Let $\Psi_k^j(\theta'), j = 1, 2\cdots, m_k$ be the eigenfunctions of $-\Delta_{S^{n-1}}$ associated to the eigenvalue μ_k and let

$$\Phi(\theta) = \sum_{k=0}^{\infty}\sum_{j=1}^{m_k} a_k^j(\xi)\Psi_k^j(\theta'),$$

then each $a_k^j(\xi)$ satisfies the equation

$$\frac{|\theta_1|^{2\sigma-1}}{\sin^{n-1}\xi}\frac{\partial}{\partial\xi}(|\theta_1|^{1-2\sigma}\sin^{n-1}\xi\frac{\partial a_k^j}{\partial\xi}(\xi)) - \frac{\mu_k}{\sin^2\xi}a_k^j(\xi) + \lambda a_k^j(\xi) = 0. \tag{4.57}$$

Let $\tau = \cos\xi$ and let $\phi_k^j(\tau) = a_k^j(\xi)$, then $\phi_k^j(\tau)$ satisfies

$$(1-\tau^2)\partial_{\tau\tau}\phi_k^j - [(n+1-2\sigma)\tau + \frac{2\sigma-1}{\tau}]\partial_\tau\phi_k^j + (\lambda - \frac{\mu_k}{1-\tau^2})\phi_k^j = 0 \quad \text{in } (-1,1). \tag{4.58}$$

We find solutions of (4.58) with the form $\phi_k^j(\tau)=(1-\tau^2)^\mu F_k^j(\tau)$, where

$$\mu=\frac{2-n}{4}+\frac{\sqrt{(n-2)^2+4\mu_k}}{4}=\frac{k}{2}$$

or

$$\mu=\frac{2-n}{4}-\frac{\sqrt{(n-2)^2+4\mu_k}}{4}=-\frac{k}{2}.$$

then $F_k^j(\tau)$ satisfies

$$(1-\tau^2)\partial_{\tau\tau}F_k^j-[(n+1+4\mu-2\sigma)\tau+\frac{2\sigma-1}{\tau}]\partial_\tau F_k^j-(\mu_k+4\mu-4\mu\sigma-\lambda)F_k^j=0. \quad (4.59)$$

By the method of solution in series, we may assume, at the regular singular point $\tau=0$ the solution of (4.59), the solution to be

$$F_k^j(\tau)=\sum_{l=0}^{\infty}b_l\tau^l.$$

Substituting in (4.59), we obtain the recurrence relation between the coefficients:

$$b_{l+2}=\frac{(k+l)(k+l+n-2\sigma)-\lambda}{(l+2)(l+2-2\sigma)}b_l. \quad (4.60)$$

Since we want to find solutions of (4.58) which is regular near $\tau=1$, then

$$(k+l)(k+l+n-2\sigma)-\lambda=0 \quad \text{for some } l$$

and we need to take $\mu=\frac{2-n}{4}+\frac{\sqrt{(n-2)^2+4\mu_k}}{4}$ in $\phi_k(\tau)=(1-\tau^2)^\mu F_k(\tau)$.

By the above analysis, we know that the eigenvalues of (4.51) are in fact given by $(k+l)(k+l+n-2\sigma), k=0,1,\cdots,l=0,1,\cdots$. Let

$$\tilde\lambda_{j'}=(k+l)(k+l+n-2\sigma),$$

where $j'=k+j$, then we have obtained all the eigenvalues of (4.51). It is easy to see that the multiplicity of the eigenvalue $\tilde\lambda_{j'}$ is

$$\tilde m_{j'}=\sum_{k=0}^{j'}m_k=\frac{(n-1+2k)(n-2+k)!}{k!(n-1)!}.$$

Therefore, (4.53) and (4.54) hold. □

Let us define $H^1(S_+^n;\theta_1^{1-2\sigma})$ as the completion of $C^\infty(\overline{S_+^n})$ with respect to the norm

$$\|\psi\|_{H^1(S_+^n;\theta_1^{1-2\sigma})}=(\int_{S_+^n}\theta_1^{1-2\sigma}(|\nabla_{S_+^n}\psi|^2+|\psi|^2)d\theta)^{\frac12}.$$

We also denote

$$L^2(S_+^n;\theta_1^{1-2\sigma})=\{\psi:S_+^n\to\mathbb{R}\text{ measurable such that }\int_{S_+^n}\theta_1^{1-2\sigma}\psi^2d\theta<\infty\}.$$

Corollary 4.14. *Let us consider the eigenvalue problem*

$$\begin{cases}\operatorname{div}_{S_+^n}(\theta_1^{1-2\sigma}\nabla_{S_+^n}\Phi)+\lambda\theta_1^{1-2\sigma}\Phi=0 & \text{in } S_+^n,\\ -\lim_{\theta_1\to0}\partial_{\theta_1}\Phi=0 & \text{on } \partial S_+^n,\end{cases} \quad (4.61)$$

in $H^1(S_+^n;\theta_1^{1-2\sigma})$, then the eigenvalues of (4.61) are given by (4.53).

Proof. If Φ satisfies (4.61), then the even extension of Φ to S^n satisfies (4.51). Therefore, if λ is an eigenvalue of (4.61), then there exists some $k \in \mathbb{N}$ such that $\lambda = \tilde{\lambda}_k$. On the other hand, for each $k \in \mathbb{N}$, there exists an eigenfunction Φ_k^j of (4.51) which is symmetric with respect to the equator $\theta_1 = 0$. Therefore, $\tilde{\lambda}_k$ is also an eigenvalue of (4.61). By the above analysis, we know that Corollary 4.14 holds. □

Corollary 4.15. *Let $\Phi \in H^1(S^n_+; \theta_1^{1-2\sigma})$ be a function such that*

$$\int_{S^n_+} \Phi(\theta) d\theta = 0, \tag{4.62}$$

then

$$\int_{S^n_+} \theta_1^{1-2\sigma} \Phi^2 d\theta \leq \tilde{\lambda}_1 \int_{S^n_+} \theta_1^{1-2\sigma} |\nabla_{S^n_+} \Phi|^2 d\theta. \tag{4.63}$$

Proof. For all $k \geq 0$, let $\tilde{\Phi}_k^j(\theta), j = 1, 2 \cdots, \tilde{\tilde{m}}_k$ be the eigenfunctions of (4.61) associated to the eigenvalue $\tilde{\lambda}_k$, where $\tilde{\tilde{m}}_k$ is the multiplicity of $\tilde{\lambda}_k$. We normalize $\tilde{\Phi}_k^j$ so that

$$\int_{S^n_+} \theta_1^{1-2\sigma} \tilde{\Phi}_k^j(\theta) \tilde{\Phi}_k^j(\theta) d\theta = 1,$$

then $\{\tilde{\Phi}_k^j(\theta)\}$ forms a orthogonal base of $L^2(S^n_+; \theta_1^{1-2\sigma})$. Let us expand Φ as

$$\Phi(\theta) = \sum_{k=0}^{\infty} \sum_{j=1}^{\tilde{\tilde{m}}_k} \phi_k^j \tilde{\Phi}_k^j(\theta),$$

where

$$\phi_k^j = \int_{S^n_+} \Phi(\theta) \tilde{\Phi}_k^j(\theta) d\theta.$$

Since (4.62) holds, then $\phi_0^1 = 0$. Therefore,

$$\begin{aligned}
\int_{S^n_+} \theta_1^{1-2\sigma} |\nabla_{S^n_+} \Phi|^2 d\theta &= \sum_{k=1}^{\infty} \sum_{j=1}^{\tilde{\tilde{m}}_k} (\phi_k^j)^2 \int_{S^n_+} \theta_1^{1-2\sigma} |\nabla_{S^n_+} \tilde{\Phi}_k^j|^2 d\theta \\
&= \sum_{k=1}^{\infty} \sum_{j=1}^{\tilde{\tilde{m}}_k} \tilde{\lambda}_k (\phi_k^j)^2 \\
&\geq \sum_{k=1}^{\infty} \sum_{j=1}^{\tilde{\tilde{m}}_k} \tilde{\lambda}_1 (\phi_k^j)^2 \\
&= \tilde{\lambda}_1 \int_{S^n_+} \theta_1^{1-2\sigma} \Phi^2 d\theta.
\end{aligned}$$

Hence (4.63) holds. □

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