MULTI-VORTEX TRAVELING WAVES FOR THE GROSS-PITAEVSKII EQUATION AND THE ADLER-MOSER POLYNOMIALS*

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Abstract. For each positive integer $n \leq 34$, we construct traveling waves with small speed for the Gross-Pitaevskii equation, by gluing $n(n+1)/2$ pairs of degree ± 1 vortice of the Ginzburg-Landau equation. The location of these vortice is symmetric in the plane and determined by the roots of a special class of Adler-Moser polynomials, which are originated from the study of Calogero-Moser system and rational solutions of the KdV equation. The construction still works for $n > 34$, under the additional assumption that the corresponding Adler-Moser polynomials have no repeated roots. It is expected that this assumption holds for any $n \in \mathbb{N}$.

Key words. Gross-Pitaevskii equation, Ginzburg-Landau equation, Adler-Moser polynomial

AMS subject classifications. 35B08, 35Q40, 37K35

1. Introduction and statement of the main results.

The Gross-Pitaevskii (GP for short) equation arises as a model equation in Bose-Einstein condensate as well as various other related physical contexts. It has the form

$$i\partial_t \Phi = \Delta\Phi + \Phi\left(1 - |\Phi|^2\right), \text{ in } \mathbb{R}^2 \times (0, +\infty), \tag{1.1}$$

where Φ is complex valued and i represents the imaginary unit. For traveling wave solutions of the form $U(x, y - \varepsilon t)$, the GP equation becomes

$$-i\varepsilon\partial_y U = \Delta U + U\left(1 - |U|^2\right), \text{ in } \mathbb{R}^2. \tag{1.2}$$

In this paper, we would like to construct multi-vortex type solutions of (1.2) when the speed ε is close to zero. Note that when the parameter $\varepsilon = 0$, equation (1.2) reduces to the well-known Ginzburg-Landau equation:

$$\Delta U + U\left(1 - |U|^2\right) = 0, \text{ in } \mathbb{R}^2. \tag{1.3}$$

Let us use (r, θ) to denote the polar coordinate of $\mathbb{R}^2$. For each $d \in \mathbb{Z}\setminus\{0\}$, it is known that the Ginzburg-Landau equation (1.3) has a degree d vortex solution, of the form $S_d(r)e^{id\theta}$. The function S_d is real valued and vanishes exactly at $r = 0$. It satisfies

$$-S_d'' - \frac{1}{r}S_d' + \frac{d^2}{r^2}S_d = S_d\left(1 - S_d^2\right), \text{ in } (0, +\infty).$$

This equation has a unique solution S_d satisfying $S_d(0) = 0$ and $S_d(+\infty) = 1$ and $S'(r) > 0$. See [22, 27] for a proof. The "standard" degree ± 1 solutions $S_1(r)e^{\pm i\theta}$ are global minimizers of the Ginzburg-Landau energy functional(For uniqueness of the global minimizer, see [37, 45]). When $|d| > 1$, these standard vortice are unstable([36,

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31]). It is also worth mentioning that for $|d| > 1$, the uniqueness of degree d vortex $S_d(r)e^{id\theta}$ in the class of solutions with degree d is still an open problem. We refer to [7, 43, 44] and the references therein for more discussion on the Ginzburg-Landau equation.

Obviously the constant 1 is a solution to the equation (1.2). We are interested in those solutions U with

$$U(z) \to 1, \text{ as } |z| \to +\infty.$$

The existence or nonexistence of solutions to (1.2) with this asymptotic behavior has been extensively studied in the literature. Jones, Putterman, Roberts([28, 29]) studied it from the physical point of view, both in dimension two and three. It turns out that the existence of solutions is related to the traveling speed ε. When $\varepsilon \geq \sqrt{2}$ (the sound speed in this context), nonexistence of traveling wave with *finite energy* is proved by Gravejat in [24, 25]. On the other hand, for $\varepsilon \in (0, \sqrt{2})$, the existence of traveling waves as constrained minimizer is studied by Bethuel, Gravejat, Saut [10, 12], by variational arguments. For ε close to 0, these solutions have two vortice. The existence issue in higher dimension is studied in [11, 15, 16]. We also refer to [9] for a review on this subject. Recently, Chiron-Scheid [14] performed numerical simulation on this equation. We also mention that as ε tends to $\sqrt{2}$, a suitable rescaled traveling waves will converge to solutions of the KP-I equation([8]), which is a classical integrable system. In a forthcoming paper, we will construct transonic traveling waves based on the lump solution of the KP-I equation.

Another motivation for studying (1.2) arises in the study of super-fluid passing an obstacle. Equation (1.2) is the limiting equation in the search of vortex nucleation solution. We refer to the recent paper [33] for references and detailed discussion.

To simplify notations, we write the degree ± 1 vortex solutions of the Ginzburg-Landau equation (1.3) as

$$v_+ = e^{i\theta} S_1(r), v_- = e^{-i\theta} S_1(r).$$

In this paper, we construct new traveling waves for ε close to 0, using v_+, v_- as basic blocks. Our main result is

THEOREM 1.1. *For each $n \leq 34$, there exists $\varepsilon_0 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$, the equation (1.2) has a solution U_ε which has the form*

$$U_\varepsilon = \prod_{k=1}^{n(n+1)/2} \left(v_+ \left(z - \varepsilon^{-1} p_k \right) v_- \left(z + \varepsilon^{-1} p_k \right) \right) + o(1),$$

where p_k, $k = 1, ..., n(n+1)/2$ are the roots of the Adler-Moser polynomial A_n defined in the next section, and $o(1)$ is a term converging to zero as $\varepsilon \to 0$.

Remark 1.2. The case $n = 1$ corresponds to the two-vortex solutions constructed by variational method ([12]) as well as reduction method ([32]). For large n, U_ε are higher energy solutions which have been observed numerically in [14]. It is also possible to construct families of traveling wave solutions using higher degree vortice of the Ginzburg-Landau equation under suitable nondegeneracy assumption of these vortice.

Remark 1.3. For general n, the theorem remains true under the additional assumption that A_n has no repeated roots. The condition $n \leq 34$ is only technical. In this case, we can verify, using computer software, that the Adler-Moser polynomial

A_n has no repeated roots. We also know that if A_{n-1} and A_n have no common roots, then A_n has no repeated roots. On a usual personal laptop, it takes around 5 hours to compute the common factors of A_{33} and A_{34} using Maple. It is possible to develop faster algorithms to verify this for large n(for instance, using the recursive identity (2.5) to compute the Adler-Moser polynomials, instead of computing the Wronskian (2) directly), but we will not pursue this here. We conjecture that the special Adler-Moser polynomial A_n(as constructed in this paper) has only simple roots for all n.

Remark 1.4. If A_n has repeated roots(For instance, suppose p is a root of multiplicity $j > 1$, and other roots are simple), to do the construction, we then have to put a degree j vortex at the point $\varepsilon^{-1}p$. However, we still don't know the nondegeneracy of higher degree vortice(although they are believed to be nondegenerated). Hence in this paper we need the assumption that A_n has no repeated roots.

Our method is based on finite dimensional Lyapunov-Schmidt reduction. We show that the existence of multi-vortex solutions is essentially reduced to the study of the nondegeneracy of a symmetric vortex-configuration. To show this nondegeneracy, we use the theory of Adler-Moser polynomials and the Darboux transformation. An interesting feature of the solutions in Theorem 1.1 is that the vortex location has a ring-shaped structure for large n, see Figure 1. The emergence of this remarkable property still remains mysterious.

In Section 2, we introduce the Adler-Moser polynomials and prove the nondegeneracy of the symmetric configuration. In Section 3, we recall the linear theory of the degree one vortex of the Ginzburg-Landau equation. In Section 4, we use Lyapunov-Schmidt reduction to glue the standard degree one vortice together and get a traveling wave solution for sufficiently small $\varepsilon > 0$.

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2. Vortex location and the Adler-Moser polynomials. Adler-Moser[1] has studied a set of polynomials corresponding to rational solutions of the KdV equation. Around the same time, it is found that these polynomials are related to the Calogero-Moser system [2]. It turns out that the Adler-Moser polynomials also have deep connections to the vortex dynamics with logarithmic interaction energy. This connection is first observed in [6], and later studied in [3, 4, 5, 17, 30]. It is worth pointing out that Vortex configuration for more general systems have been studied in [21, 34, 38, 39, 40] using polynomial method and from integrable system point of view. On the other hand, periodic vortex patterns have been investigated in [26]. See also the references cited in the above mentioned papers. While the above mentioned results mainly focus on the generating polynomials of those point vortice, we haven't seen much work on the application of these results to a PDE problem, such as GP equation. One of our aims in this paper is to fill this gap. In this section, we will first recall some basic facts of these polynomials and then analyze some of their properties, which will be used in our construction of the traveling wave for the GP equation.

Let $p_1, ..., p_k$ designate the position of the positive vortice and $q_1, ..., q_m$ be that of the negative ones. In general, p_j and q_j are complex numbers. Let $\mu \in \mathbb{R}$ be a fixed parameter. As we will see later, the vortex location of the traveling waves will be determined by the following system of equations

$$\left\{\begin{array}{l}\sum_{j\neq\alpha}\frac{1}{p_\alpha-p_j}-\sum_j\frac{1}{p_\alpha-q_j}=\mu,\ \text{for }\alpha=1,...,k,\\ \sum_{j\neq\alpha}\frac{1}{q_\alpha-q_j}-\sum_j\frac{1}{q_\alpha-p_j}=-\mu,\ \text{for }\alpha=1,...,m.\end{array}\right. \tag{2.1}$$

Adding all these equation together, we find that if $\mu\neq 0$, then $m=k$(In the case of $\mu=0$, this is no longer true). That is, the number of positive vortice has to equal that of the negative vortice. Solutions of this system(see for instances [5]) are related to the Adler-Moser polynomials. To explain this, let us define the generating polynomials

$$P(z)=\prod_j(z-p_j),\ Q(z)=\prod_j(z-q_j).$$

If p_j,q_j satisfy (2.1), then we have(see equation (68) of [5], or equation (3.8) of [17])

$$P''Q-2P'Q'+PQ''=-2\mu(P'Q-PQ'). \tag{2.2}$$

This equation is usually called generalized Tkachenko equation. Setting $\psi(z)=\frac{P}{Q}e^{\mu z}$, we derive from (2.2) that

$$\psi''+2(\ln Q)''\psi=\mu^2\psi.$$

This is a one dimensional Schrodinger equation with the potential $2(\ln Q)''$. It is well known that this equation appears in the Lax pair of the KdV equation. Hence equation (2.2) is naturally related to the theory of integrable systems.

For any $z\in\mathbb{C}$, we use $\bar z$ to denote its complex conjugate. To simplify the notation, we also write $-\bar z$ as z^*. Note that this is just the reflection of z across the y axis. Let $K=(k_2,...)$, where k_i are complex parameters. Following [17], we define functions θ_n, depending on K, by

$$\sum_{n=0}^{+\infty}\theta_n(z;K)\lambda^n=\exp\left(z\lambda-\sum_{j=2}^{\infty}\frac{k_j\lambda^{2j-1}}{2j-1}\right).$$

Note that θ_n is a degree n polynomial in z and $\theta'_{n+1}=\theta_n$. Let $c_n=\prod_{j=1}^{n}(2j+1)^{n-j}$. For each $n\in\mathbb{N}$, the Adler-Moser polynomials are then defined by

$$\Theta_n(z,K):=c_nW(\theta_1,\theta_3,...,\theta_{2n-1}),$$

where $W(\theta_1,\theta_3,...,\theta_{2n-1})$ is the Wronskian of $\theta_1,...,\theta_{2n-1}$. In particular, the degree of Θ_n is $n(n+1)/2$. The constant c_n is chosen such that the leading coefficient of Θ_n is 1. Note that this definition is slightly different from that of Adler-Moser[1](The parameter τ_i in that paper is different from k_i here). We observe that for a given μ, Θ_n depends on $n-1$ complex parameters $k_2,...,k_n$. This together with the translation in z give us a total of n complex parameters.

Let μ be another parameter, the modified Adler-Moser polynomial $\tilde\Theta$ is defined by

$$\tilde\Theta_n(z,\mu,K):=c_ne^{-\mu z}W(\theta_1,\theta_3,...,\theta_{2n-1},e^{\mu z}).$$

It is still a polynomial in z with degree $n(n+1)/2$.

Let $\tilde K=(k_2+\mu^{-3},k_3+\mu^{-5},...,k_n+\mu^{-2n+1})$. The following result, pointed out without proof in [17], will play an important role in our later analysis.

LEMMA 2.1. *The Adler-Moser and modified Adler-Moser polynomials are related by*

$$\tilde{\Theta}_n(z,\mu,K)=\mu^n\Theta_n\left(z-\mu^{-1},\tilde{K}\right).$$

Proof. We sketch the proof for completeness. First of all, direction computation shows that

$$\sum_{n=0}^{+\infty}\theta_n(z;K)\lambda^n=\sqrt{\frac{1+\mu^{-1}\lambda}{1-\mu^{-1}\lambda}}\sum_{n=0}^{+\infty}\theta_n\left(z-\mu^{-1};\tilde{K}\right)\lambda^n.$$

From this we obtain

$$\mu^{-1}\sum_{n=0}^{+\infty}\theta_{n-1}(z;K)\lambda^n=\mu^{-1}\lambda\sqrt{\frac{1+\mu^{-1}\lambda}{1-\mu^{-1}\lambda}}\sum_{n=0}^{+\infty}\theta_n\left(z-\mu^{-1};\tilde{K}\right)\lambda^n.$$

Hence using the fact that $\theta_n'=\theta_{n-1}$, we get

$$\begin{aligned}&\sum_{n=0}^{+\infty}\left(\theta_n(z;K)-\mu^{-1}\theta_n'(z;K)-\theta_n\left(z-\mu^{-1};\tilde{K}\right)\right)\lambda^n\\&=\left(\sqrt{\frac{1+\mu^{-1}\lambda}{1-\mu^{-1}\lambda}}-1-\mu^{-1}\lambda\sqrt{\frac{1+\mu^{-1}\lambda}{1-\mu^{-1}\lambda}}\right)\sum_{n=0}^{+\infty}\theta_n\left(z-\mu^{-1};\tilde{K}\right)\lambda^n.\end{aligned}$$

We observe that

$$\sqrt{\frac{1+\mu^{-1}\lambda}{1-\mu^{-1}\lambda}}-1-\mu^{-1}\lambda\sqrt{\frac{1+\mu^{-1}\lambda}{1-\mu^{-1}\lambda}}=\sqrt{1-\mu^{-2}\lambda^2}-1.$$

The Taylor expansion of this function contains only even powers of λ. Hence for odd n, $\theta_n(z;K)-\mu^{-1}\theta_n'(z;K)-\theta_n\left(z-\mu^{-1};\tilde{K}\right)$ can be written as a linear combination of $\theta_k\left(z-\mu^{-1};\tilde{K}\right)$ with k being odd. The desired identity then follows. □

The next result, which essentially follows from Crum type theorem, reveals the relation of the Adler-Moser polynomial with the vortex dynamics([5], see also Theorem 3.3 in [17]).

LEMMA 2.2. *The functions* $Q=\Theta_n(z,K)$, $P=\tilde{\Theta}_n(z,\mu,K)$ *satisfy* (2.2).

By definition, θ_n is a polynomial in z. A general degree m term in this polynomial has the form $k_2^{l_2}\cdots k_j^{l_j}z^m$. We define the index of this term to be $(-1)^{l_2+\ldots+l_j+m}$. We now prove the following

LEMMA 2.3. *For each term of* θ_{2n+1}*, its index is* -1.

Proof. Let $k_2^{l_2}\cdots k_j^{l_j}z^m$ be a degree m term in θ_{2n+1}. By Taylor expansion of the generating function and using the fact that $2n+1$ is odd, this term comes from functions of the form,

$$\frac{1}{\alpha!}\left(z\lambda-\sum_{j=2}^{\infty}\frac{k_j\lambda^{2j-1}}{2j-1}\right)^{\alpha},$$

where α is an odd integer. Hence $l_2+...+l_j=\alpha-m$. Then the index is $(-1)^{\alpha}=-1$. □

LEMMA 2.4. *For each term of Θ_n, its index is equal to $(-1)^{\frac{n(n+1)}{2}}$.*

Proof. Let us consider a typical term of Θ_n, say $\theta_1\theta_3'...\theta_{2n-1}^{(n-1)}$, where the notation $(n-1)$ represents taking $n-1$-th derivatives. By Lemma 2.3, terms in $\theta_k^{(j)}$ have index $(-1)^{1+j}$. Hence the index of terms in $\theta_1\theta_3'...\theta_{2n-1}^{(n-1)}$ is $(-1)^{1+2+...+n} = (-1)^{\frac{n(n+1)}{2}}$. This finishes the proof. ∎

Let t be another parameter, we introduce the notation

$$\Theta_{n,t}(z, K) := \Theta_n(z - t, K).$$

For any polynomial ϕ (with argument z), we use $R(\phi)$ to denote the set of roots of ϕ. We have the following

LEMMA 2.5. *Suppose μ is a real number. Assume $t = -\frac{\mu}{2}$ and $k_j = -\frac{1}{2}\mu^{2j-1}$ for $j = 2, ...$ Then*

$$(\Theta_{n,t}(z, K))^* = (-1)^{\frac{n(n+1)}{2}+1} \tilde{\Theta}_{n,t}\left(z^*, \mu^{-1}, K\right).$$

As a consequence, in this case, the reflection of $R(\Theta_{n,t}(z, K))$ across the y axis is $R\left(\tilde{\Theta}_{n,t}\left(z, \mu^{-1}, K\right)\right)$, and $R(\Theta_{n,t}(z, K))$ is invariant respect to the reflection across the x axis.

Proof. By Lemma 2.4, for each term $f = k_1^{i_1} \cdots k_j^{i_j}(z - t)^m$ of the function $\Theta_{n,t}(z, K)$, there is a corresponding term $\tilde{k}_1^{i_1} \cdots \tilde{k}_j^{i_j}(z^* - t - \mu)^m$ in $\tilde{\Theta}_{n,t}\left(z^*, \mu^{-1}, K\right)$, denoted by g. Due to the choice of k_j, we have

$$\tilde{k}_j = -k_j.$$

By Lemma 2.4, the index of $k_1^{i_1} \cdots k_j^{i_j} z^m$ is $(-1)^{\frac{n(n+1)}{2}}$. Hence using the fact that μ is real, we get

$$\begin{aligned} f^* &= -k_1^{i_1} \cdots k_j^{i_j}(-z^* - t)^m \\ &= (-1)^{1+i_1+...+i_j+m} \tilde{k}_1^{i_1} \cdots \tilde{k}_j^{i_j}(z^* + t)^m \\ &= (-1)^{\frac{n(n+1)}{2}+1} g. \end{aligned}$$

This completes the proof. ∎

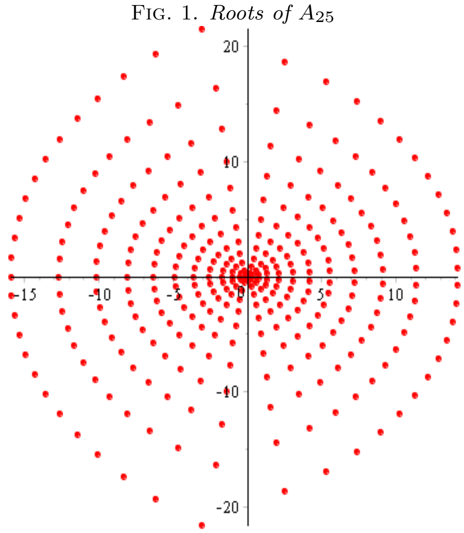
In the sequel, for simplicity, we shall choose $\mu = 1$ and $t = k_j = -\frac{1}{2}$. Let us denote the corresponding polynomial $\Theta_{n,t}(z, K)$ by $A_n(z)$. Then $A_n(z)$ is a polynomial with real coefficients. In particular, the roots of $A_n(z)$ is symmetric with respect to the x axis. Then from Lemma 2.5, we infer that the polynomial $\tilde{\Theta}_{n,t}\left(z, \mu^{-1}, K\right)$ and $A_n(-z)$ have the same roots. Hence in view of their leading coefficients, $\tilde{\Theta}_{n,t}\left(z, \mu^{-1}, K\right)$ is equal to $(-1)^{n(n+1)/2}A_n(-z)$, which we denote by $B_n(z)$. We observe that since A_n is a polynomial with real coefficients, automatically we have $-(A_n(z^*))^* = A_n(-z)$. See Figure 1 for the location of the roots of A_{25}.

Since our traveling wave solutions will roughly speaking have vortice at the roots of A_n, it is natural to ask that whether all the roots of A_n are simple. This question seems to be nontrivial. Following similar ideas as that of [13], we have

LEMMA 2.6. *Let $P(z), Q(z)$ be two polynomials satisfying*

$$P''Q - 2P'Q' + PQ'' = -2\mu(P'Q - PQ'), \tag{2.3}$$

Fig. 1. *Roots of* A_{25}



or

$$P''Q - 2P'Q' + PQ'' = 0. \tag{2.4}$$

Suppose $P(\xi) = 0$ *and* $Q(\xi) \neq 0$ *at a point* ξ. *Then* ξ *is a simple root of* P.

Proof. We prove the lemma assuming (2.3). The case of (2.4) is similar.

Suppose ξ is root of P with multiplicity $k \geq 2$. We have

$$P''Q = 2P'Q' - PQ'' - 2\mu(P'Q - PQ').$$

Then ξ is a root of the right hand side polynomial with multiplicity at least $k-1$. But its multiplicity in $P''Q$ is $k-2$. This is a contradiction. ∎

Lemma 2.7. *Suppose* $P(z), Q(z)$ *are two polynomials satisfying* (2.3) *or* (2.4). *Let* ξ *be a common root of* P *and* Q. *Assume* ξ *is a simple root of* Q. *Then* ξ *can not be a simple root of* P.

Proof. We prove this lemma assuming (2.4). The case of (2.3) is similar.

Assume to the contrary that ξ is a simple root of P. Then

$$2P'(\xi)Q'(\xi) \neq 0.$$

But this contradicts with the equation (2.4). This finishes the proof. ∎

Lemma 2.8. *Suppose* A_n *and* A_{n-1} *have no common roots. Then* A_n *has no repeated roots. Moreover,* $A_n(z)$ *and* $A_n(-z)$ *have no common roots.*

Proof. We know(See [17], Theorem 3.1) that the sequence of Adler-Moser polynomials satisfy the following recursion relation

$$A_n''A_{n-1} - 2A_n'A_{n-1}' + A_nA_{n-1}'' = 0. \tag{2.5}$$

By Lemma 2.6, any root of A_n is a simple root. Similarly, any root of $A_n(-z)$ is a simple root.

Now suppose to the contrary that ξ is a common root of $A_n(z)$ and $A_n(-z)$. Note that$(-1)^{n(n+1)/2} A_n(-z) = B_n(z)$. We have

$$A_n'' B_n - 2A_n' B_n' + A_n B_n'' = -2\mu \left(A_n' B_n - A_n B_n' \right).$$

Then by Lemma 2.7, either ξ is a repeated root of $A_n(z)$, or it is a repeated root of $A_n(-z)$. This is a contradiction. ∎

2.1. Linearization of the symmetric configuration. Our construction of traveling wave solutions requires that the vortex configuration we found is nondegenerated in the symmetric setting(in the sense of Lemma 2.5). For small number of vortice, the nondegeneracy can be proved directly. To explain this, we now consider the case of $n = 2$. Let p_1, p_2, p_3 be the three roots of the Adler-Moser polynomial A_2. Here p_1 is the real root and $p_3 = \bar{p}_2$. Note that p_1, p_2, p_3 lie on the vertices of a regular triangle. Let $q_i = p_i^*$. For $z_1 \in \mathbb{R}, z_2 \in \mathbb{C}$, we define the force map

$$F_1(z_1, z_2) := \frac{1}{z_1 - z_2} + \frac{1}{z_1 - \bar{z}_2} - \frac{1}{2z_1} - \frac{1}{z_1 + z_2} - \frac{1}{z_1 - z_2^*},$$
$$F_2(z_1, z_2) := \frac{1}{z_2 - z_1} + \frac{1}{z_2 - \bar{z}_2} - \frac{1}{z_2 + z_1} - \frac{1}{2z_2} - \frac{1}{z_2 - z_2^*}.$$

We have in mind that z_1 represents the vortex on the real axis and z_2 represents the one lying in the second quadrant. Note that by symmetry, $F_1(z_1, z_2) \in \mathbb{R}$. The name "force map" comes from the fact that if $z_1 = p_1, z_2 = p_2$, then

$$F_1(z_1, z_2) = 1, F_2(z_1, z_2) = 1,$$

which reduces to the equation (2.1).

Writing $z_1 = a_1, z_2 = a_2 + b_2 i$, where $a_i, b_i \in \mathbb{R}$, we can define

$$F(a_1, a_2, b_2) := (F_1, \operatorname{Re} F_2, \operatorname{Im} F_2).$$

The configuration $(p_1, p_2, p_3, q_1, q_2, q_3)$ is called nondegenerated, if

$$\det DF(p_1, \operatorname{Re} p_2, \operatorname{Im} p_2) \neq 0.$$

Numerical computation shows that $\det DF(p_1, \operatorname{Re} p_2, \operatorname{Im} p_2) \neq 0$. Hence it is nondegenerated. It turns out for n large, this procedure is very tedious and we have to find other ways to overcome this difficulty.

In the general case, let $\mathbf{p} = \left(p_1, ..., p_{n(n+1)/2}\right), \mathbf{q} = \left(q_1, ..., q_{n(n+1)/2}\right)$. Define the map F :

$$(\mathbf{p}, \mathbf{q}) \to \left(F_1, ..., F_{n(n+1)/2}, G_1, ..., G_{n(n+1)/2}\right),$$

where

$$F_k = \sum_{j \neq k} \frac{1}{p_k - p_j} - \sum_j \frac{1}{p_k - q_j},$$
$$G_k = \sum_{j \neq k} \frac{1}{q_k - q_j} - \sum_j \frac{1}{q_k - p_j}.$$

Let $a = \left(a_1, ..., a_{n(n+1)/2}\right)$, where a_j are the roots of A_n. Set $b = -\left(\bar{a}_1, ..., \bar{a}_{n(n+1)/2}\right)$. Moreover, we assume that there exists i_0 such that for $j = 1, ..., i_0$,

$$a_{2j-1} = \bar{a}_{2j},$$

while for $j = 2i_0 + 1, ..., n(n+1)/2$, $\operatorname{Im} a_j = 0$. We consider the linearization of F at $(\mathbf{p}, \mathbf{q}) = (a, b)$. Denote it by $DF|_{(a,b)}$. It is a map from $\mathbb{C}^{n(n+1)}$ to $\mathbb{C}^{n(n+1)}$.

The map $DF|_{(a,b)}$ always has kernel. Indeed, for any parameter $K = (k_2, ..., k_n)$, $\Theta_n(z, K)$ and $\tilde{\Theta}_n(z, K)$ satisfy

$$\Theta_n''\tilde{\Theta}_n - 2\Theta_n'\tilde{\Theta}_n' + \Theta_n\tilde{\Theta}_n'' = -2\mu\left(\Theta_n'\tilde{\Theta}_n - \Theta_n\tilde{\Theta}_n'\right).$$

Differentiating this equation with respect to the parameters t, k_j, $j = 2, ..., n-1$, we get correspondingly n linearly independent elements of the kernel. Denote them by

$$\varpi_1, ..., \varpi_n. \tag{2.6}$$

Let $\xi = \left(\xi_1, ..., \xi_{n(n+1)/2}\right) \in \mathbb{C}^{n(n+1)/2}$, $\eta = \left(\eta_1, ..., \eta_{n(n+1)/2}\right) \in \mathbb{C}^{n(n+1)/2}$. The pair (ξ, η), with $\eta = \xi^*$, is called symmetric if for $j = 1, ..., i_0$,

$$\xi_{2j-1} = \bar{\xi}_{2j},$$

while for $j = 2i_0 + 1, ..., n(n+1)/2$, $\operatorname{Im}\xi_j = 0$.

The main result of this section is the nondegeneracy of the vortex configuration given by A_n :

PROPOSITION 2.9. *Suppose $DF|_{(a,b)}(\xi, \eta) = 0$ and (ξ, η) is symmetric. Then $\xi = \eta = 0$.*

The rest of this section will be devoted to the proof of this result.

2.2. Darboux transformation and nondegeneracy of the symmetric configuration. Before going to the details of the proof of Proposition 2.9, let us explain the main idea of the proof. We would like to investigate the relation between the n-th and $(n-1)$-th Adler-Moser polynomials A_n, A_{n-1}. This will enable us to transform elements of the kernel of DF for A_n to that of A_{n-1}, and finally to that of A_0, which is much easier to be handled.

We first recall the following classical result on Darboux transformation([35], Theorem 2.1).

THEOREM 2.10. *Let λ, λ_1 be two constants. Suppose*

$$-\Psi'' + u\Psi = \lambda\Psi,$$
$$-\Psi_1'' + u\Psi_1 = \lambda_1\Psi_1.$$

Then the function $\Phi := W(\Psi_1, \Psi)/\Psi_1$ satisfies

$$-\Phi'' + \tilde{u}\Phi = \lambda\Phi,$$

where $\tilde{u} := u - 2(\ln\Psi_1)''$.

The function Φ is called the Darboux transformation of Ψ. Since later on we need a linearized version of this result, we sketch its proof below. For more detailed computation, we refer to Sec. 2.1 of [35].

Proof. We compute

$$-\Phi'' + \tilde{u}\Phi - \lambda\Phi = -\left(\Psi' - \frac{\Psi_1'}{\Psi_1}\Psi\right)'' + (\tilde{u} - \lambda)\left(\Psi' - \frac{\Psi_1'}{\Psi_1}\Psi\right)$$

$$= \left(-\Psi'' + (u-\lambda)\Psi\right)' + \left(\tilde{u} - u + 2\left(\frac{\Psi_1'}{\Psi_1}\right)'\right)\Psi'$$

$$+ \left(-u' + \left(\frac{\Psi_1'}{\Psi_1}\right)'' + \frac{\Psi_1'}{\Psi_1}(u - \tilde{u})\right)\Psi.$$

For later applications, we write this equation as

$$-\Phi'' + \tilde{u}\Phi - \lambda\Phi = \left(-\Psi'' + (u-\lambda)\Psi\right)'$$

$$+ \left(\tilde{u} - u + 2\left(\frac{\Psi_1'}{\Psi_1}\right)'\right)\left(\Psi' - \frac{\Psi_1'}{\Psi_1}\Psi\right)$$

$$+ \left(\frac{\Psi_1'' - u\Psi_1 + \lambda_1\Psi_1}{\Psi_1}\right)'\Psi. \tag{2.7}$$

The theorem follows directly from this identity. ∎

Let $\phi_n = \frac{A_{n+1}}{A_n}$ and $\psi_n(z) = \frac{B_n}{A_n}e^{\mu z}$, where $\mu = 1$. Note that ψ_n has the Wronskian representation:

$$\psi_n = \frac{W(\theta_1, ..., \theta_{2n-1}, e^{\mu z})}{W(\theta_1, ..., \theta_{2n-1})}.$$

An application of the repeated Dauboux transformation tells us that(See [17])

$$\psi_n'' + 2(\ln A_n)''\psi_n = \mu^2\psi_n. \tag{2.8}$$

Moreover, the Darboux transformation between ψ_n and ψ_{n+1} is given by

$$\psi_{n+1} = \frac{W(\phi_n, \psi_n)}{\phi_n}. \tag{2.9}$$

As we mentioned before, our main idea is to transform the kernel of DF at (A_n, B_n) to (A_0, B_0). To do this, we need the following identities. The first one is the equation (2.9), which connects ψ_j to ψ_{j+1}, hence connect B_j to B_{j+1}. The second one is the recursive identity (2.5) between A_j and A_{j+1}:

$$A_j''A_{j+1} - 2A_j'A_{j+1}' + A_jA_{j+1}'' = 0. \tag{2.10}$$

This equation can also be written in terms of ϕ_j as

$$\phi_j'' + 2(\ln A_j)''\phi_j = 0.$$

Note that this is an equation has the form appeared in Theorem 2.9. The third one is the relation between A_j and B_j:

$$A_j''B_j - 2A_j'B_j' + A_jB_j'' + 2\mu\left(A_j'B_j - A_jB_j'\right) = 0. \tag{2.11}$$

This equation implies (2.8). In certain sense, the linearization of equation (2.11) corresponds to the kernel of DF. As we will see later on, the linearized version of these three identities together with (2.7) will enable us to transform the kernel of DF at the j-th step to $j-1$-th step.

To proceed, we would like to analyze the linearized equations of (2.9), (2.10) and (2.11). First of all, linearizing the equation (2.11) at (A_j, B_j), we obtain the following equation(ξ_j, η_j are the infinitesimal variations of A_j, B_j):

$$\begin{aligned} &\xi_j'' B_j - 2\xi_j' B_j' + \xi_j B_j'' + 2\mu\left(\xi_j' B_j - \xi_j B_j'\right) \\ &+ A_j'' \eta_j - 2A_j' \eta_j' + A_j \eta_j'' + 2\mu\left(A_j' \eta_j - A_j \eta_j'\right) \\ &= 0. \end{aligned} \tag{2.12}$$

Next we need to connect (ξ_{j+1}, η_{j+1}) to (ξ_j, η_j). Linearizing the equation (2.10) at (A_j, A_{j+1}), we obtain

$$\xi_j'' A_{j+1} - 2\xi_j' A_{j+1}' + \xi_j A_{j+1}'' + A_j'' \xi_{j+1} - 2A_j' \xi_{j+1}' + A_j \xi_{j+1}'' = 0. \tag{2.13}$$

It will be more convenient to introduce a new function

$$f_j = \left(\frac{\xi_j}{A_j}\right)'. \tag{2.14}$$

The equation (2.13) then becomes

$$f_j' + \left(\ln \frac{A_j^2}{A_{j+1}^2}\right)' f_j + f_{j+1}' + \left(\ln \frac{A_{j+1}^2}{A_j^2}\right)' f_{j+1} = 0.$$

Given function f_{j+1}, "formally" we can solve this equation and get a solution

$$\begin{aligned} f_j(z) &= -\frac{A_{j+1}^2}{A_j^2} \int_a^z \frac{A_j^2}{A_{j+1}^2} \left(f_{j+1}' + \left(\ln \frac{A_{j+1}^2}{A_j^2}\right)' f_{j+1}\right) ds \\ &= f_{j+1} - 2\frac{A_{j+1}^2}{A_j^2} \int_c^z \frac{A_j^2}{A_{j+1}^2} f_{j+1}' ds. \end{aligned} \tag{2.15}$$

The last equality follows from integrating by parts for the second term. Here a, c are two numbers and we intentionally haven't specified the integration paths, because the integrands may have singularities, depending on the form of the function f_{j+1}.

Linearizing the equation (2.9) yields the equation(with σ_j being the infinitesimal variation of ψ_j):

$$\sigma_{j+1} = -\sigma_j (\ln \phi_j)' + \sigma_j' - \psi_j \left(\frac{\xi_{j+1}}{A_{j+1}} - \frac{\xi_j}{A_j}\right)'.$$

Inserting (2.14) into this equation, we get

$$\sigma_j' - \sigma_j (\ln \phi_j)' = (f_{j+1} - f_j)\, \psi_j + \sigma_{j+1}.$$

For given functions f_j, f_{j+1}, σ_j, we can solve this equation and get a solution

$$\sigma_j(z) = \phi_j \int_c^z \left(\psi_j (f_{j+1} - f_j) + \sigma_{j+1}\right) \phi_j^{-1} ds. \tag{2.16}$$

Note that the infinitesimal variation σ_j should be related to ξ_j and η_j. Indeed, linearizing the relation $\psi_j = \frac{B_j}{A_j} e^{\mu z}$, we get

$$\sigma_j e^{-\mu z} = -\frac{B_j \xi_j}{A_j^2} + \frac{\eta_j}{A_j}. \tag{2.17}$$

With all these preparations, we are now ready to prove the following

PROPOSITION 2.11. *For any n, the elements of the kernel of the map $DF|_{(a,b)}$ are given by linear combinations of $\varpi_j, j = 1, ..., n,$ defined in* (2.6)*.*

Proof. Suppose we have an element of the kernel of the map $DF|_{(a,b)}$, with the form

$$\left(\tau_1, ..., \tau_{n(n+1)/2}, \delta_1, ..., \delta_{n(n+1)/2}\right).$$

Consider the generating functions $\prod_j (z - a_j - \rho\tau_j)$ and $\prod_j (z - b_j - \rho\delta_j)$, where ρ is a small parameter. Differentiating these two functions with respect to ρ at $\rho = 0$, we get two polynomials ξ_n, η_n, with degree less than $n(n+1)/2$, satisfying

$$\begin{aligned} &\xi_n'' B_n - 2\xi_n' B_n' + \xi_n B_n'' + 2\mu\left(\xi_n' B_n - \xi_n B_n'\right) \\ &+ A_n''\eta_n - 2A_n'\eta_n' + A_n\eta_n'' + 2\mu\left(A_n'\eta_n - A_n\eta_n'\right) \\ &= 0. \end{aligned} \tag{2.18}$$

Consider the function $f_n = \left(\frac{\xi_n}{A_n}\right)'$. It is a rational function with possible poles at the roots of A_n. Using (2.15), for each $j \le n-1$, we can define functions

$$f_j = f_{j+1} - 2\frac{A_{j+1}^2}{A_j^2}\int_c^z \frac{A_j^2}{A_{j+1}^2} f_{j+1}' ds. \tag{2.19}$$

Here c is to be determined later on. With this definition, we see that f_j has possible poles at the roots of $A_j, A_{j+1}, ..., A_n$. In particular,

$$f_0 = f_1 - 2\left(z + \frac{1}{2}\right)^2 \int_c^z \frac{f_1'}{\left(s + \frac{1}{2}\right)^2} ds. \tag{2.20}$$

We remark that as a complex valued function with poles, at this stage, f_j may be multiple-valued.

On the other hand, we can define σ_n through

$$\sigma_n e^{-\mu z} = -\frac{B_n \xi_n}{A_n^2} + \frac{\eta_n}{A_n},$$

and then define $\sigma_j, j \le n-1$, in terms of relation (2.16). Finally, we define $\eta_j, j \le n-1$, using (2.17). We recall that $\phi_0 = \frac{A_1}{A_0} = z + \frac{1}{2}$ and $\psi_0 = e^{\mu z}$. Hence

$$\sigma_0 = \left(z + \frac{1}{2}\right)\int_c^z \frac{1}{s + \frac{1}{2}}\left(e^{\mu s}\left(f_1 - f_0\right) + \sigma_1\right) ds. \tag{2.21}$$

Since equation (2.12) holds for ξ_n, η_n(see equation (2.18)), then by linearizing the identity (2.7) (with Ψ_1 being ϕ_j, Ψ being ψ_j), we find that (2.12) also holds for $j \le n-1$. Therefore, using $A_0 = B_0 = 1$, we get

$$\xi_0'' + 2\mu\xi_0' + \eta_0'' - 2\mu\eta_0' = 0. \tag{2.22}$$

That is, $(\xi_0+\eta_0)'+2\mu(\xi_0-\eta_0)$ is locally a constant, say C. By (2.17), $\eta_0=\sigma_0 e^{-\mu z}+\xi_0$. It follows that

$$\left(\sigma_0 e^{-\mu z}+2\xi_0\right)'-2\mu\left(\sigma_0 e^{-\mu z}\right)=C. \tag{2.23}$$

Recall that $f_0=\xi_0'$. Thus by (2.21),

$$f_0+f_1+\sigma_1 e^{-\mu z}+\left(1-3\mu\left(z+\frac{1}{2}\right)\right)e^{-\mu z}\int_c^z\frac{1}{s+\frac{1}{2}}\left(e^{\mu s}\left(f_1-f_0\right)+\sigma_1\right)ds=C. \tag{2.24}$$

Our next aim is to show that f_1 has no singularity except the root of A_1, that is, $-\frac{1}{2}$.

Assume to the contrary that $d_0\neq-\frac{1}{2}$ is a singularity of f_1. Let c be a number close to d_0. Note that d_0 has to be a root of some A_k. Integrating by parts in (2.19) yields

$$f_j=-f_{j+1}+2\frac{A_{j+1}^2}{A_j^2}\int_c^z\left(\frac{A_j^2}{A_{j+1}^2}\right)'f_{j+1}ds+c_1\frac{A_{j+1}^2}{A_j^2}, \tag{2.25}$$

for some constant c_1.

We first consider the case that A_j has no repeated roots for any $j\leq n$. Actually numerical computation tells us that this holds if $n=34$.

Since ξ_n, f_n are polynomials with degree less than $n(n-1)/2$, by (2.19), we can assume that the main order(non-analytic part) of f_1 around the singularity d_0 has the form

$$\beta_1(z-d_0)^{-1}+\beta_2(z-d_0)^{-2}+\beta_3(z-d_0)^2\ln(z-d_0),$$

where at least one of the constants β_j is nonzero.

Let us first consider the case that β_2 is nonzero and d_0 is not a root of A_2.

By (2.20), around d_0, at the main order, f_0 has the form $-\beta_2(z-d_0)^{-2}$. From (2.16), we deduce that

$$\sigma_1=\frac{A_2}{A_1}\int_c^z\frac{A_1}{A_2}\left(\frac{B_1e^{\mu s}}{A_1}\left(f_2-f_1\right)+\sigma_2\right)ds. \tag{2.26}$$

Since σ_2 has no $(z-d_0)^{-2}$ term and $f_2\sim-\beta_2(z-d_0)^2$, we infer from (2.26) that the main order term of σ_1 is $\frac{2d_0-1}{2d_0+1}2\beta_2e^{d_0}(z-d_0)^{-1}$. Inserting this into (2.24) and applying (2.25), we find that the $(z-d_0)^{-1}$ order terms in (2.24) satisfy

$$\frac{4}{d_0+\frac{1}{2}}\beta_2(z-d_0)^{-1}+\frac{2d_0-1}{2d_0+1}2\beta_2(z-d_0)^{-1}-\frac{1-3\left(d_0+\frac{1}{2}\right)}{d_0+\frac{1}{2}}2\beta_2(z-d_0)^{-1}=0. \tag{2.27}$$

This equation has no solution and we thus get a contradiction. Hence $\beta_2=0$. Similarly, we have $\beta_1=\beta_3=0$. Thus we know that f_1 has no singularity other other $-\frac{1}{2}$.

Now we choose the base point c to be $-\infty$. We would like to show that $f_0=0$. Using the recursive relation and the fact that f_1 has no singularities other than $-\frac{1}{2}$, we deduce that f_1 is actually single valued and $f_1=a_1\frac{1}{z+\frac{1}{2}}+a_2\frac{1}{\left(z+\frac{1}{2}\right)^2}$. Recall that

$$\sigma_1=\phi_1\int_c^z\phi_1^{-1}\left(\psi_1\left(f_2-f_1\right)-\sigma_2\right)ds.$$

Putting this into (2.24), we find that $a_1 = 0$. This implies that $f_0 = 0$ and $\sigma_0 = 0$. Once this is proved, we can show that ξ_n, η_n actually come from the differentiation with respect to the parameters t and $k_j, j = 2, ..., n$.

Next we consider the general case that A_j has repeated roots for some $j \leq n$.(We conjecture that this case does no happen).

Let $d \neq -\frac{1}{2}$ be a repeated root of some A_j, $j \leq n$, with highest multiplicity r. We still would like to show that $d_0 \neq d$. Assume to the contrary that $d_0 = d$. Then around d_0, by (2.19), the main order terms of the function f_1 has the form

$$\beta_1 (z - d_0)^{-1} + \beta_2 (z - d_0)^{-2} + ...\beta_{2r} (z - d_0)^{-2r} + \beta_{2r+1} (z - d_0)^2 \ln (z - d_0).$$

Then same arguments above tell us that all the β_j are zero, which is a contradiction. Hence the only pole of f_1 is $-\frac{1}{2}$ and the claim of the proposition follows. ∎

Let $K = \left(-\frac{1}{2}, -\frac{1}{2}, ...\right)$. We also need the following uniqueness result about the symmetric configuration.

Lemma 2.12. *Suppose $\hat{K}$ is an $n-1$ dimensional vector and $\left|\hat{K} - K\right| + t + \frac{1}{2} < \delta$ for some small $\delta > 0$, with $\hat{K} \neq K$. Then*

$$\Theta_n \left(-z - t, \hat{K}\right) \neq (-1)^{n(n+1)/2} \tilde{\Theta}_n \left(z - t, \hat{K}\right).$$

Proof. We prove this statement using induction argument. This is true for $n = 1$. Assume it is true for $n = j$, we shall prove that it is also true for $n = j + 1$.

Suppose to the contrary that

$$\Theta_{j+1} \left(-z - t, \hat{K}\right) = (-1)^{(j+1)(j+2)/2} \tilde{\Theta}_{j+1} \left(z - t, \hat{K}\right).$$

We know that

$$\begin{aligned} &\Theta''_{j+1} \left(z - t, \hat{K}\right) \Theta_j \left(z - t, \hat{K}\right) - 2\Theta'_{j+1} \left(z - t, \hat{K}\right) \Theta'_j \left(z - t, \hat{K}\right) \\ &+ \Theta_{j+1} \left(z - t, \hat{K}\right) \Theta''_j \left(z - t, \hat{K}\right) = 0. \end{aligned}$$

Replacing z by $-z$, we get

$$\begin{aligned} &\tilde{\Theta}''_{j+1} \left(z - t, \hat{K}\right) \Theta_j \left(-z - t, \hat{K}\right) - 2\tilde{\Theta}'_{j+1} \left(z - t, \hat{K}\right) \Theta'_j \left(-z - t, \hat{K}\right) \\ &+ \tilde{\Theta}_{j+1} \left(z - t, \hat{K}\right) \Theta''_j \left(-z - t, \hat{K}\right) = 0. \end{aligned} \tag{2.28}$$

On the other hand,

$$\begin{aligned} &\tilde{\Theta}''_{j+1} \left(z - t, \hat{K}\right) \tilde{\Theta}_j \left(z - t, \hat{K}\right) - 2\tilde{\Theta}'_{j+1} \left(z - t, \hat{K}\right) \tilde{\Theta}'_j \left(z - t, \hat{K}\right) \\ &+ \tilde{\Theta}_{j+1} \left(z - t, \hat{K}\right) \tilde{\Theta}''_j \left(z - t, \hat{K}\right) = 0. \end{aligned}$$

This together with (2.28) imply that

$$\Theta_j \left(-z - t, \hat{K}\right) = (-1)^{j(j+1)/2} \tilde{\Theta}_j \left(z - t, \hat{K}\right).$$

Hence by assumption $t = -\frac{1}{2}$, and the first $j - 1$ components of $\hat{K}$ is $-\frac{1}{2}$. It then follows that the last component of $\hat{K}$ is also $-\frac{1}{2}$. This is a contradiction. ∎

Now we can prove Proposition 2.9. By Proposition 2.11, elements of the kernel of the map $DF|_{(a,b)}$ is given by linear combination of $\varpi_j, j = 1, ..., n$. But on the other hand, for $\mu = 1$, we know from Lemma 2.12 that $t = -\frac{1}{2}, k_j = -\frac{1}{2}, j = 1, ..., n-1$, is the only set of parameters for which Θ_n and $\tilde{\Theta}_n$ give arise to symmetric configuration. Hence the configuration determined by A_n and B_n is nondegenerated. We remark that by the same method, it is also possible to show that the balancing configuration given by other Adler-Moser polynomials are also nondegenerated.

3. Preliminaries on the Ginzburg-Landau equation. In this section, we recall some results on the Ginzburg-Landau equation. Most of the materials in this section can be found in the book [43](possibly with different notations though).

Stationary solutions of the GP equation (1.1) solve the following Ginzburg-Landau equation

$$-\Delta\Phi = \Phi\left(1 - |\Phi|^2\right) \text{ in } \mathbb{R}^2, \tag{3.1}$$

where Φ is a complex valued function. We have mentioned in the first section that equation (3.1) has degree $\pm d$ vortice of the form $S_d(r)e^{\pm id\theta}$. It is also known that as $r \to +\infty$,

$$S_d(r) = 1 - \frac{d^2}{2r^2} + O\left(r^{-4}\right). \tag{3.2}$$

On the other hand, as $r \to 0$, there is a constant $\kappa = \kappa_d > 0$ such that

$$S_d(r) = \kappa r\left(1 - \frac{r^2}{8} + O\left(r^4\right)\right). \tag{3.3}$$

See [22] for detailed proof of these facts.

In the case of $d = \pm 1$, the solution will be denoted by $v_\pm$, and S_1 will simply be written as S. The linearized operator of the Ginzburg-Landau equation around v_+ will be denoted by L :

$$\eta \to \Delta\eta + \left(1 - |v_+|^2\right)\eta - 2v_+ \operatorname{Re}\left(\eta\bar{v}_+\right). \tag{3.4}$$

It turns out to be more convenient to study the operator

$$\mathcal{L}\eta := e^{-i\theta}L\left(e^{i\theta}\eta\right).$$

If we write the complex function η as $w_1 + iw_2$ with w_1, w_2 being real valued functions, then explicitly

$$\begin{aligned}\mathcal{L}\eta &= e^{-i\theta}\Delta\left(e^{i\theta}\eta\right) + \left(1 - S^2\right)\eta - 2S^2w_1 \\ &= \Delta w_1 + \left(1 - 3S^2\right)w_1 - \frac{1}{r^2}w_1 - \frac{2}{r^2}\partial_\theta w_2 \\ &\quad + i\left(\Delta w_2 + \left(1 - S^2\right)w_2 - \frac{1}{r^2}w_2 + \frac{2}{r^2}\partial_\theta w_1\right).\end{aligned}$$

Invariance of the equation (3.1) under rotation and translation gives us three linearly independent elements of the kernel of the operator $\mathcal{L}$, called Jacobi fields. Rotational invariance yields the solution

$$\Phi^0 := ie^{-i\theta}v_+ = iS, \tag{3.5}$$

while the translational invariance along x and y directions leads to the solutions

$$\Phi^{+1} := S' \cos\theta - \frac{S}{r}\sin\theta,$$

$$\Phi^{-1} := S' \sin\theta + \frac{S}{r}\cos\theta.$$

Note that these elements of the kernel are bounded but decay slowly at infinity, hence not in $L^2(\mathbb{R}^2)$. As a consequence, the analysis of the mapping property of $\mathcal{L}$ is quite delicate. An important fact is that v_+ is nondegenerated in the sense that all the bounded solutions of $\mathcal{L}\eta = 0$ are given by linear combinations of Φ^0 and Φ^+, Φ^-([43], Theorem 3.2). Similar results hold for the degree -1 vortex v_-. It is also worth mentioning that the nondegeneracy of those higher degree vortice $e^{id\theta}S_d(r)$, $|d| > 1$, is still an open problem. Actually this is the main reason that we only deal with the degree ± 1 vortice in this paper. One can indeed construct solutions of GP equation by gluing higher degree vortices under the additional assumption that they are nondegenerated in suitable sense.

The analysis of the asymptotic behavior of the elements of the kernel of $\mathcal{L}$ near 0 and ∞ is crucial in understanding the mapping properties of the linearized operator $\mathcal{L}$. In doing this, the main strategy is to decompose the elements of the kernel into different Fourier modes. Let us now briefly describe the results in the sequel. Lemma 3.1, Lemma 3.2 and Lemma 3.3 below can be found in Section 3.3 of [43].

We start the discussion with the lowest Fourier mode, which is the simplest case and plays an important role in analyzing the mapping property of the linearized operator.

LEMMA 3.1. *Suppose a is a complex valued solution of the equation $\mathcal{L}a = 0$, depending only on r.*
(I) As $r \to 0$, either $|a|$ blows up at least like r^{-1}, or a can be written as a linear combination of two linearly independent solutions $w_{0,1}, w_{0,2}$, with

$$w_{0,1}(r) = r\left(1 + O\left(r^2\right)\right),$$

$$w_{0,2}(r) = ir\left(1 + O\left(r^2\right)\right).$$

(II) As $r \to +\infty$, if a is an imaginary valued function, then $a = c_1 + c_2 \ln r + O\left(r^{-2}\right)$; if a is real valued, then it either blows up or decays exponentially.

Proof. We sketch the proof for completeness.

If $\mathcal{L}a = 0$ and the complex function a depends only on r, then a will satisfy

$$a'' + \frac{1}{r}a' - \frac{1}{r^2}a = S^2\bar{a} - \left(1 - 2S^2\right)a. \tag{3.6}$$

Note that this equation is not complex linear and its solution space is a 4-dimensional real vector space. The Jacobi field Φ^0 defined by (3.5) is a purely imaginary solution of (3.6). Writing $a = a_1 + a_2 i$, where a_i are real valued functions, we get from (3.6) two *decoupled* equations:

$$a_1'' + \frac{1}{r}a_1' - \frac{1}{r^2}a_1 + \left(1 - 3S^2\right)a_1 = 0,$$

$$a_2'' + \frac{1}{r}a_2' - \frac{1}{r^2}a_2 + \left(1 - S^2\right)a_2 = 0. \tag{3.7}$$

Observe that due to (3.2), as $r \to +\infty$,

$$1 - 3S^2 - r^{-2} = -2 + O\left(r^{-2}\right),$$
$$1 - S^2 - r^{-2} = O\left(r^{-4}\right).$$

The results of this lemma then follow from a perturbation argument. ∎

For each integer $n \geq 1$, we consider element of the kernel of $\mathcal{L}$ the form $a(r)e^{in\theta} + b(r)e^{-in\theta}$. The complex valued functions a, b will satisfy the following coupled ODE system in $(0, +\infty)$:

$$\begin{cases} a'' + \frac{1}{r}a' - \frac{(n+1)^2}{r^2}a = S^2\bar{b} - \left(1 - 2S^2\right)a \\ b'' + \frac{1}{r}b' - \frac{(n-1)^2}{r^2}b = S^2\bar{a} - \left(1 - 2S^2\right)b. \end{cases} \tag{3.8}$$

By analyzing this coupled ODE system, one gets the precise asymptotic behavior of its solutions. The next lemma deals with the $n = 1$ case.

Lemma 3.2. *Suppose $w = a(r)e^{i\theta} + b(r)e^{-i\theta}$ solves $\mathcal{L}w = 0$.*
(I) As $r \to 0$, either $|w|$ blows up at least like $-\ln r$, or w can be written as a linear combination of four linearly independent solutions $w_{1,i}, i = 1, ..., 4$, satisfying: As $r \to 0$,

$$w_{1,1} = r^2\left(1 + O\left(r^2\right)\right)e^{i\theta} + O\left(r^6\right)e^{-i\theta},$$
$$w_{1,2} = ir^2\left(1 + O\left(r^2\right)\right)e^{i\theta} + O\left(r^6\right)e^{-i\theta},$$
$$w_{1,3} = \left(1 + O\left(r^2\right)\right)e^{-i\theta} + O\left(r^4\right)e^{i\theta},$$
$$w_{1,4} = i\left(1 + O\left(r^2\right)\right)e^{-i\theta} + O\left(r^4\right)e^{i\theta}.$$

(II) As $r \to +\infty$, either $|w|$ is unbounded(blows up exponentially or like r), or $|w|$ decays to zero(exponentially or like r^{-1}).

For the $n \geq 2$ case, we have the following

Lemma 3.3. *Suppose $w = a(r)e^{in\theta} + b(r)e^{-in\theta}$ solves $\mathcal{L}w = 0$.*
(I) As $r \to 0$, either $|w|$ blows up at least like r^{1-n}, or w can be written as a linear combination of four linearly independent solutions $w_{1,i}, i = 1, ..., 4$, satisfying: As $r \to 0$,

$$w_{n,1} = r^{n+1}\left(1 + O\left(r^2\right)\right)e^{in\theta} + O\left(r^{n+5}\right)e^{-in\theta},$$
$$w_{n,2} = ir^{n+1}\left(1 + O\left(r^2\right)\right)e^{in\theta} + O\left(r^{n+5}\right)e^{-in\theta},$$
$$w_{n,3} = r^{n-1}\left(1 + O\left(r^2\right)\right)e^{-in\theta} + O\left(r^{n+3}\right)e^{in\theta},$$
$$w_{n,4} = ir^{n-1}\left(1 + O\left(r^2\right)\right)e^{-in\theta} + O\left(r^{n+3}\right)e^{in\theta}.$$

(II) As $r \to +\infty$, either $|w|$ is unbounded(blows up exponentially or like r^n), or $|w|$ decays to zero(exponentially or like r^{-n}).

By Lemma 3.3, for $n \geq 3$, if $\mathcal{L}w = 0$ and w is bounded near 0, then decays at least like r^2 as $r \to 0$, hence decaying faster than the vortex solution itself. For $n \leq 2$, solutions of $\mathcal{L}w = 0$ bounded near 0 behaves like $O(r)$ or $O(1)$. Note that $\Phi_0, \Phi_{+1}, \Phi_{-1}$ have this property. Let $\Psi_0 = \kappa w_{0,2}$,

$$\Psi_{+1} = \kappa w_{1,3} + \frac{\kappa}{8}w_{1,1}, \Psi_{-1} = \kappa w_{1,4} - \frac{\kappa}{8}w_{1,2},$$
$$\Psi_{+2} = w_{2,3}, \Psi_{-2} = w_{2,4}.$$

Then they behave like $O(r)$ or $O(1)$ near 0, but blow up as $r \to +\infty$.

From the above lemmas, we know that for r large, the imaginary part of the linearized operator essentially behaves like Δ, while the real part looks like $\Delta - 2$.

4. Construction of multi-vortex solutions.

4.1. Approximate solutions and estimate of the error. We would like to construct traveling wave solutions by gluing together $n(n+1)/2$ pairs of degree ± 1 vortice. Let us simply choose $n = 2$, the proof of the general case is almost the same, but notations will be more involved.

For $k = 1, 2, 3$, Let $p_k, q_k \in \mathbb{C}$. We have in mind that p_k are close to roots of the Adler-Moser polynomial A_2. We define the translated vortice

$$u_k = v_+\left(z - \varepsilon^{-1} p_k\right), u_{3+k} = v_-\left(z - \varepsilon^{-1} q_k\right).$$

We then define the approximate solution

$$u := \prod_{j=1}^{6} u_j.$$

Note that as $r \to +\infty$, $u \to 1$. Hence the degree of u is 0. Let us denote the function $z \to \overline{u(z)}$ by $\bar{u}$. The next lemma states that the real part of u is even both in the x and y variables, while the imaginary part is even in x and odd in y.

LEMMA 4.1. *The approximate solution u has the following symmetry:*

$$u(\bar{z}) = \bar{u}(z), \ u(z^*) = u(z).$$

Proof. Observe that the standard vortex $v_+ = S(r) e^{i\theta}$ satisfies

$$v_+(\bar{z}) = \bar{v}_+(z), v_+(z^*) = (v_+(z))^*.$$

The opposite(degree -1) vortex v_- has similar properties. Hence using the fact that the set $\{p_1, p_2, p_3\}$ is invariant with respect to the reflection across the x axis, we get

$$\begin{aligned} u(\bar{z}) &= \prod_{k=1}^{3} \left(v_+\left(\bar{z} - \varepsilon^{-1} p_k\right) v_-\left(\bar{z} - \varepsilon^{-1} q_k\right)\right) \\ &= \prod_{k=1}^{3} \left(\bar{v}_+\left(z - \varepsilon^{-1} \bar{p}_k\right) \bar{v}_-\left(z - \varepsilon^{-1} \bar{q}_k\right)\right) = \bar{u}(z). \end{aligned}$$

Moreover, since $v_- = \bar{v}_+$, we have

$$\begin{aligned} u(z^*) &= \prod_{k=1}^{3} \left(v_+\left(z^* - \varepsilon^{-1} p_k\right) v_-\left(z^* - \varepsilon^{-1} q_k\right)\right) \\ &= \prod_{k=1}^{3} \left(\left(v_+\left(z - \varepsilon^{-1} q_k\right)\right)^* \left(v_-\left(z - \varepsilon^{-1} p_k\right)\right)^*\right) \\ &= \prod_{k=1}^{3} \left(\bar{v}_+\left(z - \varepsilon^{-1} q_k\right) \left(\bar{v}_-\left(z - \varepsilon^{-1} p_k\right)\right)\right) = u(z). \end{aligned}$$

This finishes the proof. ∎

We use $E(u)$ to denote the error of the approximate solution:

$$E(u) := \varepsilon i \partial_y u + \Delta u + u\left(1 - |u|^2\right).$$

We have

$$\begin{aligned}\Delta u &= \Delta(u_1 ... u_6)\\ &= \sum_k \left(\Delta u_k \prod_{j\neq k} u_j\right) + \sum_{k\neq j}\left((\nabla u_k \cdot \nabla u_j)\prod_{l\neq k,j} u_l\right),\end{aligned}$$

where $\nabla u_k \cdot \nabla u_j := \partial_x u_k \partial_x u_j + \partial_y u_k \partial_y u_j$. On the other hand, writing $|u_k|^2 - 1 = \rho_k$, we obtain

$$|u|^2 - 1 = \prod_k (1+\rho_k) - 1 = \sum_k \rho_k + \sum_{k=2}^{6} Q_k,$$

where $Q_k = \sum_{i_1<i_2<\cdots<i_k}(\rho_{i_1}\cdots\rho_{i_k})$. Using the fact that u_k solves the Ginzburg-Landau equation, we get

$$\begin{aligned}E(u) &= \varepsilon i \sum_k \left(\partial_y u_k \prod_{j\neq k} u_j\right)\\ &\quad + \sum_{k,j,k\neq j}\left((\nabla u_k \cdot \nabla u_j)\prod_{l\neq k,j} u_l\right) - u\sum_{k=2}^{6} Q_k.\end{aligned} \tag{4.1}$$

We have in mind that the main order terms are $\partial_y u_k \prod_{j\neq k} u_j$ and $(\nabla u_k \cdot \nabla u_j)\prod_{l\neq k,j} u_l$.

Throughout the paper (r_j, θ_j) will denote the polar coordinate with respect to the point $\varepsilon^{-1}p_j$. Note that

$$\partial_x\left(e^{i\theta}\right) = -\frac{yie^{i\theta}}{r^2}, \partial_y\left(e^{i\theta}\right) = \frac{xie^{i\theta}}{r^2}.$$

Moreover, $\partial_x r = x/r$, $\partial_y r = y/r$. Hence we have, for $k \leq 3$,

$$\begin{aligned}\partial_x u_k &= -\frac{iy_k e^{i\theta_k}}{r_k^2} S(r_k) + \frac{x_k}{r_k} S'(r_k) e^{i\theta_k},\\ \partial_y u_k &= \frac{ix_k e^{i\theta_k}}{r_k^2} S(r_k) + \frac{y_k}{r_k} S'(r_k) e^{i\theta_k}.\end{aligned}$$

Now we study the projection of the error of the approximate solution on the kernel of the linearized operator at the approximate solutions. Lyapunov-Schmidt reduction arguments require that these projections are "small", in suitable sense(See Proposition 4.5 below).

In the region where $\left|z - \varepsilon^{-1}p_k\right| \leq C_{k,j}\varepsilon^{-1}$, with $C_{k,j} = \frac{1}{2|p_k - p_j|}$, using $S'(r) = O\left(r^{-3}\right)$, we get

$$\begin{aligned}\nabla u_k \cdot \nabla u_j &= \partial_x u_k \partial_x u_j + \partial_y u_k \partial_y u_j\\ &= \partial_x u_k \left(-\frac{y_j i e^{i\theta_j}}{r_j^2}\right) + \partial_y u_k \left(\frac{x_j i e^{i\theta_j}}{r_j^2}\right) + O\left(\varepsilon^3\right).\end{aligned}$$

Note that $\operatorname{Im}\left(\partial_y u_k \overline{(\partial_x u_k)}\right) = \frac{SS'}{r_k}$. It follows that for $k, j \leq 3$,

$$\begin{aligned}
&\operatorname{Re}\int_{|z-\varepsilon^{-1}p_k|\leq C_{k,j}\varepsilon^{-1}} e^{-i\theta_j}\left(\nabla u_k\cdot\nabla u_j\right)\left(\overline{\partial_x u_k}\right)dxdy\\
&= -\operatorname{Re}\left(\frac{\varepsilon}{p_k-p_j}\right)\int_{|z-p_k|\leq C_{k,j}\varepsilon^{-1}}\operatorname{Im}\left(\partial_y u_k\overline{(\partial_x u_k)}\right)+O\left(\varepsilon^2\right)\\
&= -\operatorname{Re}\left(\frac{\varepsilon}{p_k-p_j}\right)\int_{|z-\varepsilon^{-1}p_k|\leq C_{k,j}\varepsilon^{-1}}\frac{SS'}{r_k}+O\left(\varepsilon^2\right)\\
&= -\pi\operatorname{Re}\left(\frac{\varepsilon}{p_k-p_j}\right)+O\left(\varepsilon^2\right).
\end{aligned}$$

In general, for $t>0$, we also have

$$\begin{aligned}
&\operatorname{Re}\int_{|z-\varepsilon^{-1}p_k|\leq A} e^{-i\theta_j}\left(\nabla u_k\cdot\nabla u_j\right)\left(\overline{\partial_x u_k}\right)dxdy\\
&= -\pi S^2(t)\operatorname{Re}\left(\frac{\varepsilon}{p_k-p_j}\right)+O\left(\varepsilon^2\right). \qquad (4.2)
\end{aligned}$$

Now we compute

$$\begin{aligned}
&\operatorname{Re}\int_{|z-\varepsilon^{-1}p_k|\leq C_{k,j}\varepsilon^{-1}} e^{-i\theta_j}\left(\nabla u_k\cdot\nabla u_j\right)\overline{(\partial_y u_k)}dxdy\\
&= \pi\operatorname{Im}\left(\frac{\varepsilon}{p_k-p_j}\right)+O\left(\varepsilon^2\right).
\end{aligned}$$

Next, if $l, j \neq k$, we estimate that for $\left|z-\varepsilon^{-1}p_k\right| \leq \min_{j\neq k} C_{k,j}\varepsilon^{-1}$,

$$\begin{aligned}
\left(\nabla u_l\cdot\nabla u_j\right)\overline{(\partial_x u_k)} &\sim e^{-i\theta_k}\left(\frac{y_l}{r_l^2}e^{i\theta_l}\frac{y_j}{r_j^2}e^{i\theta_j}+\frac{x_l}{r_l^2}e^{i\theta_l}\frac{x_j}{r_j^2}e^{i\theta_j}\right)\left(-\frac{y_kS}{r_k^2}+\frac{x_kS'}{r_k}\right)\\
&= O\left(\varepsilon^2\right).
\end{aligned}$$

Finally, we compute

$$\begin{aligned}
&\operatorname{Re}\int_{|z-\varepsilon^{-1}p_k|\leq C_{k,j}\varepsilon^{-1}} i\varepsilon\partial_y u_k\overline{(\partial_y u_k)} = O\left(\varepsilon^2\right),\\
&\operatorname{Re}\int_{|z-\varepsilon^{-1}p_k|\leq C_{k,j}\varepsilon^{-1}} i\varepsilon\partial_y u_k\overline{(\partial_x u_k)} = \pi\varepsilon+O\left(\varepsilon^2\right).
\end{aligned}$$

Note that if the integrating region is replaced by the ball radius t centered at $\varepsilon^{-1}p_k$, then we get a corresponding estimate like (4.2) with π replaced by $\pi S^2(t)$.

We can do similar estimates as above for $k \leq 3$ and $j \geq 4$, with a possible different sign before the main order term. Combining all these estimates, we find that the projected equation at the main order is (2.1) with $\mu = 1$.(See also system (4.26)).

4.2. Solving the nonlinear problem and proof of Theorem 1.1. In this subsection, we would like to construct solutions of the GP equation stated in Theorem 1.1, near the family of approximate solutions u analyzed in Section 4.1. To this aim, we shall use the finite dimensional Lyapunov-Schmidt reduction method to reduce the original problem to the nondegeneracy of the roots of the Adler-Moser polynomials. This nondegeneracy result has already been proved in Section 2, see Proposition 2.9.

Applying finite or infinite dimensional Lyapunov-Schmidt reduction to construct solutions of nonlinear elliptic PDEs is by now more or less standard. There exists vast literature on this subject. It is well known that one of the steps in the Lyapunov-Schmidt reduction is to establish the solvability of the projected linear problem, in suitable functional spaces. In our case, this will be accomplished in Proposition 4.5.

For each $\varepsilon > 0$ sufficiently small, we look for a traveling wave solution U of the GP equation:

$$-i\varepsilon\partial_y U = \Delta U + U\left(1 - |U|^2\right). \tag{4.3}$$

Let u be the approximate solution. Then around each vortex point(it is a root of the associated Adler-Moser polynomial), u is close to the standard degree one vortex solution of the Ginzburg-Landau equation, described in Section 3. Recall that by $E(u)$ we mean the error of u, which has the form

$$E(u) = \varepsilon i\partial_y u + \Delta u + u\left(1 - |u|^2\right).$$

If u is written as $w + iv$, where w, v are its real and imaginary parts, then we know from Lemma 4.1 that u has the following symmetry:

$$w(x,y) = w(-x,y) = w(x,-y); v(x,y) = v(-x,y) = -v(x,-y).$$

The following lemma states that $E(u)$ has the same symmetry as u.

LEMMA 4.2. *The real part of $E(u)$ is even in both x and y variables. The imaginary part of $E(u)$ is even in x and odd in y.*

Proof. This follows from the symmetry of the approximate solution u and the fact that $E(u)$ consists of terms which are suitable derivatives of u. Note that taking second order derivatives of u in x or y does not change this symmetry. On the other hand, the term $\varepsilon i\partial_y u$ is obtained by taking the y derivative and multiplying by i. This operation also preserves the symmetry stated in this lemma. ∎

Let $\tilde{\chi}$ be a smooth cutoff function such that $\tilde{\chi}(s) = 1$ for $s \leq 1$ and $\tilde{\chi}(s) = 0$ for $s \geq 2$. Let χ be the cutoff function localized near the vortice defined by:

$$\chi(z) := \sum_{j=1}^{3} \tilde{\chi}\left(\left|z - \varepsilon^{-1}p_j\right|\right) + \sum_{j=1}^{3} \tilde{\chi}\left(\left|z - \varepsilon^{-1}q_j\right|\right).$$

Following [18], we seek a true solution of the form

$$U := (u + u\eta)\chi + (1-\chi)ue^{\eta}, \tag{4.4}$$

where $\eta = \eta_1 + \eta_2 i$ is complex valued function close to 0 in suitable norm which will be introduced below. We also assume that η has the same symmetry as u. We see that near the vortice, U is obtained from u by an additive perturbation; while away from the vortice, U is of the form ue^{η}. The reason of choosing the perturbation η in the form (4.4) is explained in Section 3 of [18]. Roughly speaking, away from the vortex points, this specific form simplifies the higher order error terms when solving the nonlinear problem, compared to the usual additive perturbation. In view of (4.4), we can write $U = ue^{\eta} + \epsilon$, where

$$\epsilon := \chi u \left(1 + \eta - e^{\eta}\right).$$

Note that ϵ is localized near the vortex points and of the order $o(\eta)$, for η small.

Let us set $A := \left(\chi + (1-\chi) e^{\eta}\right) u$. Then U can also be written as $U = u\eta\chi + A$. We have

$$U\left(1 - |U|^2\right) = (u\eta\chi + A)\left(1 - |ue^{\eta} + \epsilon|^2\right).$$

By this formula, computing $\varepsilon i \partial_y U + \Delta U$ using (4.4), we find that the GP equation becomes

$$-A\mathbb{L}(\eta) = (1+\eta)\chi E(u) + (1-\chi) e^{\eta} E(u) + N_0(\eta), \tag{4.5}$$

where $E(u)$ represents the error of the approximate solution, and

$$\mathbb{L}\eta := i\varepsilon \partial_y \eta + \Delta \eta + 2u^{-1}\nabla u \cdot \nabla \eta - 2|u|^2 \eta_1, \tag{4.6}$$

while N_0 is $o(\eta)$, and explicitly given by

$$\begin{aligned} N_0(\eta) &:= (1-\chi) ue^{\eta} |\nabla \eta|^2 + i\varepsilon \left(u\left(1+\eta - e^{\eta}\right)\right) \partial_y \chi \\ &+ 2\nabla\left(u\left(1+\eta-e^{\eta}\right)\right) \cdot \nabla \chi + u\left(1+\eta-e^{\eta}\right)\Delta\chi \\ &- 2u|u|^2 \eta\eta_1\chi - (A + u\eta\chi)\left[|u|^2\left(e^{2\eta_1} - 1 - 2\eta_1\right) + |\epsilon|^2 + 2\operatorname{Re}\left(ue^{\eta}\bar{\epsilon}\right)\right]. \end{aligned}$$

Note that in the region away from the vortex points, the real part of the operator $\mathbb{L}$ is modeled on $\Delta\eta_1 - 2\eta_1 - \varepsilon\partial_y\eta_2$, while the imaginary part is like $\Delta\eta_2 + \varepsilon\partial_y\eta_1$.

Dividing equation (4.5) by A, we obtain

$$\begin{aligned} &-\mathbb{L}(\eta) \\ &= u^{-1}E(u) - |u|^2\left(e^{2\eta_1} - 1 - 2\eta_1\right) + |\nabla\eta|^2 \\ &+ i\varepsilon A^{-1}\left(u\left(1+\eta-e^{\eta}\right)\right)\partial_y\chi + 2A^{-1}\nabla\left(u\left(1+\eta-e^{\eta}\right)\right)\cdot\nabla\chi \\ &+ A^{-1}u\left(1+\eta-e^{\eta}\right)\Delta\chi - A^{-1}u\chi|\nabla\eta|^2 - |\epsilon|^2 - 2\operatorname{Re}\left(ue^{\eta}\bar{\epsilon}\right) \\ &+ A^{-1}u\eta\chi\left[u^{-1}E(u) - 2|u|^2\eta_1 - |u|^2\left(e^{2\eta_1}-1-2\eta_1\right) - |\epsilon|^2 - 2\operatorname{Re}\left(ue^{\eta}\bar{\epsilon}\right)\right]. \end{aligned}$$

Let us write this equation as

$$\mathbb{L}(\eta) = -u^{-1}E(u) + N(\eta).$$

This nonlinear equation, equivalent to the original GP equation, is the one we eventually want to solve. Observe that in $N(\eta)$, except $|u|^2\left(e^{2\eta_1} - 1 - 2\eta_1\right) - |\nabla\eta|^2$, other terms are all localized near the vortex points. As we will see later, the terms $|u|^2\left(e^{2\eta_1} - 1 - 2\eta_1\right)$ and $|\nabla\eta|^2$ are well suited to the functional setting below.

Now let us introduce the functional framework which we will work with. It is adapted to the mapping property of the linearized operator $\mathbb{L}$. Note that one of our purpose is to solve a linear equation of the form $\mathbb{L}(\eta) = h$, where h is a given function with suitable smooth and decaying properties away from the vortex points.

Recall that $r_j, j = 1, \cdots, 6$, represent the distance to the j-th vortex point. Let w be a weight function defined by

$$w(z) := \left(\sum_{j=1}^{6}\left(1+r_j\right)^{-1}\right)^{-1}.$$

This function measures the minimal distance from the point z to those vortex points. We use $B_a(z)$ to denote the ball of radius a centered at z. Let $\gamma, \sigma \in (0,1)$ be small positive numbers. For complex valued function $\eta = \eta_1 + \eta_2 i$, we define the following weighted $C^{2,\gamma}$ norm.

$$
\begin{aligned}
&\|\eta\|_* \\
&= \|u\eta\|_{C^{2,\gamma}(w<3)} + \left\|w^{1+\sigma}\eta_1\right\|_{L^\infty(w>2)} + \left\|w^{2+\sigma}\left(|\nabla\eta_1| + |\nabla^2\eta_1|\right)\right\|_{L^\infty(w>2)} \\
&+ \sup_{z\in\{w>2\}} \sup_{z_1,z_2\in B_{w/3}(z)} \left(\frac{|\nabla\eta_1(z_1) - \nabla\eta_1(z_2)| + \left|\nabla^2\eta_1(z_1) - \nabla^2\eta_1(z_2)\right|}{w(z)^{-2-\sigma-\gamma}|z_1 - z_2|^\gamma} \right) \\
&+ \|w^\sigma\eta_2\|_{L^\infty(w>2)} + \left\|w^{1+\sigma}\nabla\eta_2\right\|_{L^\infty(w>2)} + \left\|w^{2+\sigma}\nabla^2\eta_2\right\|_{L^\infty(w>2)} \\
&+ \sup_{z\in\{w>2\}} \sup_{z_1,z_2\in B_{w/3}(z)} \left(w(z)^{1+\sigma+\gamma} \frac{|\nabla\eta_2(z_1) - \nabla\eta_2(z_2)|}{|z_1 - z_2|^\gamma} \right) \\
&+ \sup_{z\in\{w>2\}} \sup_{z_1,z_2\in B_{w/3}(z)} \left(w(z)^{2+\sigma+\gamma} \frac{\left|\nabla^2\eta_2(z_1) - \nabla^2\eta_2(z_2)\right|}{|z_1 - z_2|^\gamma} \right).
\end{aligned}
$$

Although this definition of norm seems to be complicated, its meaning is rather clear: The real part of η decays like $w^{-1-\sigma}$ and its first and second derivatives decay like $w^{-2-\sigma}$. Moreover, the imaginary part of η only decays as $w^{-\sigma}$, but its first and second derivative decay as $w^{-1-\sigma}$ and $w^{-2-\sigma}$ respectively. As a consequence, real and imaginary parts of the function η behave in different ways away from the vortex points. It is worth mentioning that the Hölder norms are taken into account in the definition because eventually we shall use the Schauder estimates. We remark that it is also possible to work in suitable weighted L^p spaces and then use the L^p estimates, as is done in [20] for the Allen-Cahn equation.

On the other hand, for complex valued function $h = h_1 + ih_2$, we define the following weighted Hölder norm

$$
\begin{aligned}
\|h\|_{**} :=& \|uh\|_{C^{0,\gamma}(w<3)} + \left\|w^{1+\sigma}h_1\right\|_{L^\infty(w>2)} \\
&+ \left\|w^{2+\sigma}\nabla h_1\right\|_{L^\infty(w>2)} + \left\|w^{2+\sigma}h_2\right\|_{L^\infty(w>2)} \\
&+ \sup_{z\in\{w>2\}} \sup_{z_1,z_2\in B_{w/3}(z)} \left(w(z)^{2+\sigma+\gamma} \frac{|\nabla h_1(z_1) - \nabla h_1(z_2)|}{|z_1 - z_2|^\gamma} \right) \\
&+ \sup_{z\in\{w>2\}} \sup_{z_1,z_2\in B_{w/3}(z)} \left(w(z)^{2+\sigma+\gamma} \frac{|h_2(z_1) - h_2(z_2)|}{|z_1 - z_2|^\gamma} \right).
\end{aligned}
$$

This definition tells us that the real and imaginary parts of h have different decay rates. Moreover, intuitively we require h_1 to gain one more power of decay at infinity after taking one derivative. The choice of this norm is partly decided by the decay and smooth properties of $E(u)$.

As was already mentioned at the beginning of this subsection, to carry out the Lyapunov-Schmidt reduction procedure, we need the projected linear theory for the linearized operator $\mathbb{L}$. We now know that the imaginary part of $\mathbb{L}$ behaves like the Laplacian operator at infinity. To deal with it, we need the following result(Lemma 4.2 in [32]):

LEMMA 4.3. *Let $\sigma \in (0,1)$. Suppose η is a real valued function satisfying*

$$\Delta\eta = h(z),\ \eta(\bar{z}) = -\eta(z), |\eta| \le C,$$

where

$$|h(z)| \leq \frac{C}{(1+|z|)^{2+\sigma}}.$$

Then we have

$$|\eta(z)| \leq \frac{C}{(1+|z|)^{\sigma}}.$$

It is well known that without any assumption on h, the solution η may grow at a logarithmic rate at infinity. This result tells us that if h is odd in the y variable, then η will not have the log part, due to cancellation. For completeness, we give the detailed proof of this fact in the sequel.

Proof of Lemma 4.3. Let $Z = X + iY$. By Poisson's formula, we have

$$\eta(z) = \frac{1}{2\pi}\int_{Y>0} \ln\left(\frac{\bar{z}-Z}{z-Z}\right) h(Z)\, dX dY.$$

Using the decay assumption of h, we find that $\eta(z) \to 0$, as $z \to +\infty$.

Let us construct suitable supersolution in the upper half plane. Define

$$g(z) := r^{\beta} y^{\alpha},$$

where $r = |z|$ and β, α are chosen such that

$$\beta + \alpha = -\sigma, 0 < \sigma < \alpha < 1.$$

We compute

$$\begin{aligned} \Delta g &= r^{\beta} y^{\alpha}\left(\left(\beta^2 + 2\beta\alpha\right) r^{-2} + \alpha(\alpha-1) y^{-2}\right) \\ &\leq -C r^{\beta} y^{\alpha}\left(r^{-2} + y^{-2}\right) \\ &\leq -C r^{\beta-1} y^{\alpha-1} \leq -C r^{\beta+\alpha-2} = -C r^{\sigma-2}. \end{aligned}$$

Hence by maximum principle,

$$|\eta(z)| \leq C g(z) \leq \frac{C}{(1+|z|)^{\sigma}}.$$

The proof is then completed. □

We also need the following

LEMMA 4.4. *Let $\sigma \in (0,1)$. Suppose η is a real valued function satisfying*

$$\Delta\eta - 2\eta = h, |\eta| \leq C,$$

where

$$|h(z)| \leq \frac{C}{(1+|z|)^{2+\sigma}}.$$

Then we have

$$|\eta(z)| \leq \frac{C}{(1+|z|)^{2+\sigma}}.$$

The proof of this lemma is easier than that of Lemma 4.3. Indeed, one can directly construct a supersolution of the form $1/r^{2+\sigma}$ for the operator $-\Delta + 2$, in the region $\{z : |z| > a\}$, where a is a fixed large constant. We omit the details.

With all these preparations, now we are ready to prove the following a priori estimate for solutions of the equation $\mathbb{L}(\eta) = h$.

PROPOSITION 4.5. *Let $\varepsilon > 0$ be small. Suppose $\|\eta\|_* < \infty$, $\|h\|_{**} < \infty$ and*

$$\begin{cases} \mathbb{L}\eta = h, \\ \operatorname{Re}\left(\int_{|z-\varepsilon^{-1}p_k|\le 1} \bar{u}\bar{\eta}\partial_x u\right) = 0, \ \text{for } k = 1, ..., 3, \\ \operatorname{Re}\left(\int_{|z-\varepsilon^{-1}p_k|\le 1} \bar{u}\bar{\eta}\partial_y u\right) = 0, \ \text{for } k = 1, ..., 3, \\ u\eta \text{ and } u\eta \text{ have the same symmetry as } E(u) \text{ stated in Lemma 4.2.} \end{cases}$$

Then $\|\eta\|_* \le C\varepsilon^{-\sigma} |\ln \varepsilon| \|h\|_{**}$, where C is a constant independent of ε and h.

Proof. The mapping properties of $\mathbb{L}$ are closely related to that of the operator L, which is the linearized operator of the standard degree one vortex solution v_+ of the Ginzburg-Landau equation analyzed in Section 3(See (3.4)). We would like to point out that one of the difficulties in the proof of this proposition is that L has three bounded linearly independent elements of the kernel, corresponding respectively to translation in the x variable($\partial_x v_+$), translation in the y variable($\partial_y v_+$), and rotation($\partial_\theta v_+$). But here a priori we only assume in the statement of this proposition that $u\eta$ is orthogonal to two of them($\partial_x u$ and $\partial_y u$) in a certain sense. This is quite different from the situation(only one pair of vortice, located on the x axis) considered in [32], where by symmetry the functions are automatically orthogonal to the kernels corresponding to y translation and rotation.

It is also worth mentioning that comparing with the Ginzburg-Landau equation, we have the term $\varepsilon i \partial_y \eta$ in the linearized operator $\mathbb{L}$. However, in our context, due to the fact that ε is small, essentially we can deal with it as a "perturbation term". To take care of this additional term, we need to analyze the decay rate of the real and imaginary parts of the involved functions a little bit more precisely than the Ginzburg-Landau case. This issue is already reflected in the definition of the norms $\|\cdot\|_*$ and $\|\cdot\|_{**}$.

The proof given below is actually a straightforward modification of the proof of Lemma 4.1 in [18]. The ideas of the proof are almost the same. As we mentioned above, the norms defined here are slightly different with the one appeared in [18], in particular regarding the decay rate of the first derivatives of the imaginary part of η and real part of h. This is the reason why we have a negative power of ε in the bound, instead of $|\ln \varepsilon|$ in [18]. Interested readers can compare the proof of Lemma 4.1 in [18] and the one presented here to see these minor differences.

Recall that the vortex points of our approximate solution u are located at $\varepsilon^{-1}p_j$, $\varepsilon^{-1}q_j, j = 1, 2, 3$. Let us choose a large constant d_0 such that all the points $p_j, q_j, j = 1, 2, 3$, are contained inside the ball of radius $d_0/2$ centered at the origin of the complex plane. We will split the proof into several steps.

Step 1. *Estimates in the exterior domain Ξ, assuming a priori the required bound of η in the interior region.*

To emphasize the main idea of how to take care of the term $\varepsilon i \partial_y \eta$, let us assume for the moment that we have already established the desired weighted estimate of η and its derivatives in terms of $\varepsilon^{-\sigma} |\ln \varepsilon| \|h\|_{**}$, in the interior region $\left\{z : |z| \le d_0\varepsilon^{-1}\right\}$. This assumption will be justified later on.

Let us now estimate η and its derivatives in the exterior domain

$$\Xi := \left\{z : |z| > d_0\varepsilon^{-1}\right\}.$$

In view of the decay rates in the definition of the norms, the main task is to estimate the weighted norm of $\nabla\eta_1$. The estimate of η itself will be relatively easier.

In Ξ, by (4.6), the equation $\mathbb{L}\eta = h$ takes the form

$$i\varepsilon\partial_y\eta + \Delta\eta + 2u^{-1}\nabla u \cdot \nabla\eta - 2|u|^2\eta_1 = h.$$

Splitting into real and imaginary parts, we can write this equation as

$$\begin{cases} -\Delta\eta_1 + 2\eta_1 + \varepsilon\partial_y\eta_2 = -h_1 + 2\operatorname{Re}(u^{-1}\nabla u \cdot \nabla\eta) - 2(|u|^2 - 1)\eta_1, \\ -\Delta\eta_2 - \varepsilon\partial_y\eta_1 = -h_2 + 2\operatorname{Im}(u^{-1}\nabla u \cdot \nabla\eta),\ \eta_2(\bar{z}) = -\eta_2(z). \end{cases} \tag{4.7}$$

In Ξ, the terms in the right hand side containing η are small in suitable sense. Indeed, due to the asymptotic behavior $S - 1 = O(r^{-2})$, we have

$$\left|\left(|u|^2 - 1\right)\eta_1\right| \leq Cr^{-2}|\eta_1|.$$

Moreover, using the formula

$$\nabla f \cdot \nabla g = \partial_r f\partial_r g + \frac{\partial_\theta f\partial_\theta g}{r^2},$$

we obtain,

$$\left|\operatorname{Re}\left(u^{-1}\nabla u \cdot \nabla\eta\right)\right| \leq Cr^{-1}|\nabla\eta_2| + Cr^{-2}|\nabla\eta_1|,$$
$$\left|\operatorname{Im}\left(u^{-1}\nabla u \cdot \nabla\eta\right)\right| \leq Cr^{-1}|\nabla\eta_1| + Cr^{-2}|\nabla\eta_2|.$$

Consider any point $z_0 \in \Xi$. To estimate η_2 around z_0, we denote $|z_0|$ by R and define the rescaled function $g(z) := \eta_2(Rz)$. Then by the second equation of (4.7), g satisfies

$$\Delta g(z) = -\varepsilon R^2\partial_y\eta_1 + R^2h_2 - 2R^2\operatorname{Im}\left(u^{-1}\nabla u \cdot \nabla\eta\right),$$

where the right hand side is evaluated at the point Rz. Applying Lemma 4.3 and the Schauder estimates to the rescaled function g, using the assumed bound of η in the interior domain, we find that

$$\begin{aligned} \|g\|_{C^{2,\gamma}(1<|z|<2)} &\leq C\varepsilon R^2\|\nabla\eta_1(R\cdot)\|_{C^{0,\gamma}(2/3<|z|<3)} \\ &\quad + C\varepsilon R^2\left\||z|^{2+\sigma}\nabla\eta_1(R\cdot)\right\|_{L^\infty(2/3<|z|)} \\ &\quad + CR^{-\sigma}\varepsilon^{-\sigma}|\ln\varepsilon|\,\|h\|_{**}. \end{aligned}$$

Rescaling back, we find that in particular,

$$\begin{aligned} &\left\|w^{2+\sigma}\nabla^2\eta_2\right\|_{L^\infty(\Xi)} \\ &\leq C\varepsilon\left\|w^{2+\sigma}\nabla\eta_1\right\|_{L^\infty(\mathbb{R}^2)} \\ &\quad + C\varepsilon\sup_{z\in\Xi}\sup_{z_1,z_2\in B_{w/3}(z)}\left(w(z)^{2+\sigma+\gamma}\frac{|\nabla\eta_1(z_1) - \nabla\eta_1(z_2)|}{|z_1 - z_2|^\gamma}\right) \\ &\quad + C\varepsilon^{-\sigma}|\ln\varepsilon|\,\|h\|_{**}. \end{aligned} \tag{4.8}$$

We also have corresponding estimate for the weighted Hölder norm of $\nabla^2\eta_2$. Note that in the right hand side, we have the small constant ε before the norm of η_1. Similar estimates hold for $\nabla\eta_2$.

To get the desired weighted estimate of $\partial_y \eta_1$, instead of working directly with the first equation of (4.7), we shall differentiate it with respect to y. This yields

$$\begin{aligned} & -\Delta\left(\partial_y \eta_1\right)+2 \partial_y \eta_1 \\ & =-\varepsilon \partial_y^2 \eta_2-\partial_y h_1+2 \partial_y\left(\operatorname{Re}\left(u^{-1} \nabla u \cdot \nabla \eta\right)\right)-2 \partial_y\left(\left(|u|^2-1\right) \eta_1\right) . \end{aligned} \tag{4.9}$$

Note that by the definition of the norm $\|\cdot\|_{**}$, $\partial_y h_1$ decays one more power faster than h_1. Applying the standard estimate for the operator $-\Delta+2$(Lemma 4.4), we find that

$$\begin{aligned} \left\|w^{2+\sigma} \partial_y \eta_1\right\|_{L^{\infty}(\Xi)} \leq & C \varepsilon\left\|w^{2+\sigma} \nabla^2 \eta_2\right\|_{C^{0, \gamma}(\Xi)} \\ & +C \varepsilon\left\|w^{1+\sigma} \nabla \eta_2\right\|_{C^{0, \gamma}(\Xi)}+C \varepsilon^{-\sigma}|\ln \varepsilon|\|h\|_{**} . \end{aligned} \tag{4.10}$$

Given any pair of points z_1, z_2, we define the difference quotient of ϕ as

$$Q(\phi)(z):=\frac{\phi\left(z+z_1\right)-\phi\left(z+z_2\right)}{\left|z_1-z_2\right|^\gamma} .$$

Then from equation (4.9) , we find that $Q\left(\partial_y \eta_1\right)$ satisfies

$$\begin{aligned} & -\Delta\left(Q\left(\partial_y \eta_1\right)\right)+2 Q\left(\partial_y \eta_1\right) \\ & =-\varepsilon Q\left(\partial_y^2 \eta_2\right)-Q\left(\partial_y h_1\right)+2 Q\left(\partial_y\left(\operatorname{Re}\left(u^{-1} \nabla u \cdot \nabla \eta\right)\right)\right) \\ & -2 Q\left(\partial_y\left(\left(|u|^2-1\right) \eta_1\right)\right) . \end{aligned}$$

Same argument as (4.10) applied to the function G yields the weighted Hölder norm of $\partial_y \eta_1$. Similar estimate can be derived for $\partial_x \eta_1$, by taking the x-derivative in the equation (4.7) .

From (4.8) , (4.10) , and the corresponding weighted Hölder estimates, we deduce

$$\begin{aligned} & \left\|w^{2+\sigma} \partial_y \eta_1\right\|_{L^{\infty}(\Xi)}+\sup _{z \in \Xi} \sup _{z_1, z_2 \in B_{w / 3}(z)}\left(w(z)^{2+\sigma+\gamma} \frac{\left|\partial_y \eta_1\left(z_1\right)-\partial_y \eta_1\left(z_2\right)\right|}{\left|z_1-z_2\right|^\gamma}\right) \\ & \leq C \varepsilon^{-\sigma}|\ln \varepsilon|\|h\|_{**} . \end{aligned}$$

With this desired decay estimate of $\partial_y \eta_1$ at hand, we can use the second equation of (4.7) and the mapping property of the Laplacian operator to get the estimates of η_2 and its derivatives, and then use the first equation of (4.7) to get the estimates of η_1 and its derivatives.

Step 2. *Estimates in the interior region.*

Let us estimate η in the interior region

$$\Gamma_{\varepsilon}:=\left\{z:|z| \leq d_0 \varepsilon^{-1}\right\} .$$

We will choose $d_1>0$ such that the balls centered at points $p_j, q_j, j=1,2,3$, with radius d_1 are disjoint to each other. Denote the union of these balls by Ω. We then define Ω_{ε} to be the union of the balls of radius $d_1 \varepsilon^{-1}$ centered at vortex points $\varepsilon^{-1} p_j, \varepsilon^{-1} q_j, j=1,2,3$. Note that $\Omega_{\varepsilon} \subset \Gamma_{\varepsilon}$.

To prove the bound of η, we assume to the contrary that there were sequence $\varepsilon_k \rightarrow 0$, sequences $h^{(k)}, \eta^{(k)}$, with $\eta^{(k)}$ satisfying the orthogonality condition, $\mathbb{L} \eta^{(k)}=h^{(k)}$, and as k tends to infinity,

$$\left\|\eta^{(k)}\right\|_*=\varepsilon_k^{-\sigma}, \quad\left|\ln \varepsilon_k\right|\left\|h^{(k)}\right\|_{**} \rightarrow 0, \tag{4.11}$$

We will also write ε_k as ε for simplicity. According to the definition of our norms, this implies

$$\left\|\eta_2^{(k)}\right\|_{L^\infty(\Gamma_\varepsilon\setminus\Omega_\varepsilon)}+\varepsilon^{-1}\left\|\nabla\eta_2^{(k)}\right\|_{L^\infty(\Gamma_\varepsilon\setminus\Omega_\varepsilon)}\leq C.$$

Moreover, we have

$$\left\|\eta_1^{(k)}\right\|_{L^\infty(\Gamma_\varepsilon\setminus\Omega_\varepsilon)}+\varepsilon^{-1}\left\|\nabla\eta_1^{(k)}\right\|_{L^\infty(\Gamma_\varepsilon\setminus\Omega_\varepsilon)}\leq C\varepsilon.$$

Substep A. *The $L^\infty(\mathbb{R}^2)$ norm of $u\eta^{(k)}$ is uniformly bounded with respect to k.*

Before starting the proof, we point out that the main task is to estimate η_2. The reason is that the near the vortex points, the operator $\mathbb{L}(\cdot)$ resembles $\mathcal{L}(S\cdot)$, where $\mathcal{L}$ is the conjugate operator of L defined in Section 3. Due to rotational symmetry of the Ginzburg-Landau equation, the constant i is a bounded kernel of the operator $\mathcal{L}(S\cdot)$. One can also check directly that $\mathbb{L}(i)=0$. As we will see later on, the presence of this purely imaginary kernel implies that the L^∞ norm of η near the vortex points is essentially determined by the L^∞ norm of η at the boundary of Ω_ε.

Let ρ be a real valued smooth cutoff function satisfying

$$\rho(s)=\begin{cases}1, s<\frac{1}{2},\\ 0, s>1.\end{cases}$$

Consider the function

$$\tilde{\eta}^{(k)}(z):=\eta^{(k)}(z)\,\rho\left(\frac{\varepsilon}{d_1}\left(z-\varepsilon^{-1}p_1\right)\right).$$

This function is localized in the $\frac{d_1}{\varepsilon}$ neighborhood of the vortex point $\varepsilon^{-1}p_1$. We shall fix a large constant R_0 independent of ε_k. For notational simplicity, we will drop the superscript k if there is no confusion. In form of real and imaginary parts, we have $\tilde{\eta}=\tilde{\eta}_1+i\tilde{\eta}_2$.

Claim 1: *We have the following(the decay here is not optimal) estimate of η away from the vortex points:*

$$\begin{aligned}&\|\tilde{\eta}_2\|_{L^\infty(r_1>2R_0)}+\|r_1\nabla\tilde{\eta}_2\|_{L^\infty(r_1>2R_0)}\\&+\left\|r_1^{1+\sigma}\tilde{\eta}_1\right\|_{L^\infty(r_1>2R_0)}+\left\|r_1^{1+\sigma}\nabla\tilde{\eta}_1\right\|_{L^\infty(r_1>2R_0)}\\&\leq C\left(\|\tilde{\eta}\|_{L^\infty(r_1<2R_0)}+1\right).\end{aligned}\tag{4.12}$$

The proof of this claim is same as the proof of Lemma 4.1 in [18](although notations here are different). We repeat their arguments for completeness.

Let us estimate $\tilde{\eta}_1$. First of all, in the region $r_1>R_0$, using the fact that $\partial_y\tilde{\eta}_2\leq C\varepsilon^{-\sigma}r_1^{-1-\sigma}$, we obtain from the first equation of (4.7) that

$$-\Delta\tilde{\eta}_1+2S^2\tilde{\eta}_1=O\left(\frac{1}{r_1}\right)\nabla\tilde{\eta}_2+o(1)\frac{1}{r_1^{1+\sigma}}.\tag{4.13}$$

Here $O(1/r_1)$ is bounded by C/r_1, and $o(1)$ represents a term tending to 0 as k goes to infinity. The right hand side of (4.13) is then bounded by $Br_1^{-1-\sigma}$, where

$$B:=\|r_1^\sigma\nabla\tilde{\eta}_2\|_{L^\infty(r_1>R_0)}+o(1).$$

Since S converges to 1 at infinity, it is easy to check that the function $r_1^{-1-\sigma}$ is a supersolution of the operator $-\Delta+2S^2$ in this region. Using maximum principle and elliptic estimates, we infer from equation (4.13) that

$$|\nabla\tilde{\eta}_1|+|\tilde{\eta}_1|\leq C\left(B+\|\tilde{\eta}_1\|_{L^\infty(r_1=R_0)}\right)r_1^{-1-\sigma},\ r_1\geq 2R_0. \tag{4.14}$$

On the other hand, in the region $r_1>2R_0$, using the fact that $\partial_y\tilde{\eta}_1\leq C\varepsilon^{-\sigma}r_1^{-2-\sigma}$, we know that the imaginary part $\tilde{\eta}_2$ satisfies an equation of the form

$$-\Delta\tilde{\eta}_2=O\left(\frac{1}{r_1}\right)\nabla\tilde{\eta}_1+o(1)\frac{1}{r_1^{2+\sigma}}+C\varepsilon^2. \tag{4.15}$$

Using the estimate (4.14) of $\tilde{\eta}_1$, we find that the right hand side of the equation (4.15) is bounded by $CB'r_1^{-2-\sigma}+C\varepsilon^2$, where $B':=\|\tilde{\eta}_1\|_{L^\infty(r_1=R_0)}+o(1)$, and C is a universal constant. Consider the function

$$M(z):=C_0B'\left(1-r_1^{-\sigma}\right)+C_0\left(d_1^2-r_1^2\varepsilon^2\right)+\|\tilde{\eta}_2\|_{L^\infty(r_1=2R_0)}.$$

If C_0 is a fixed large constant, then

$$-\Delta\left(M-\tilde{\eta}_2\right)\geq 0.$$

Moreover,

$$\tilde{\eta}_2\leq M,\ \text{if}\ r_1=2R_0\ \text{or}\ r_1=d_1/\varepsilon.$$

Hence by the maximum principle, $\tilde{\eta}_2\leq M$. That is,

$$\begin{aligned}\|\tilde{\eta}_2\|_{L^\infty(r_1>2R_0)}&\leq CB'\left(1-r_1^{-\sigma}\right)+C\left(d_1^2-r_1^2\varepsilon^2\right)+\|\tilde{\eta}_2\|_{L^\infty(r_1=2R_0)}\\&\leq C+\|\tilde{\eta}\|_{L^\infty(r_1<2R_0)}.\end{aligned} \tag{4.16}$$

Given $R>0$, to obtain the decay estimate of $\nabla\tilde{\eta}_2$ near any point of the form $\varepsilon^{-1}p_1+Rz_0$, where $|z_0|=1$, we use the scaling argument again and define the rescaled function $\eta^*:=\tilde{\eta}_2\left(\varepsilon^{-1}p_1+R(z+z_0)\right)$. Elliptic estimates for the equation satisfied by η^* together with (4.16) yield

$$\|r_1\nabla\tilde{\eta}_2\|_{L^\infty(r_1>R_0)}\leq C\left(1+\|\tilde{\eta}\|_{L^\infty(r_1<2R_0)}\right). \tag{4.17}$$

Inserting this estimate back to (4.14), we finally deduce

$$|\nabla\tilde{\eta}_1|+|\tilde{\eta}_1|\leq Cr_1^{-1-\sigma}\left(1+\|\tilde{\eta}\|_{L^\infty(r_1<2R_0)}\right). \tag{4.18}$$

Claim 1 then follows.

To proceed, we need to pay special attention to the projection of $\tilde{\eta}$ onto the lowest Fourier mode(the constant mode, with respect to the angle). In the (r_1,θ_1) coordinate, we still use v_+ to denote the standard degree one vortex solution $S(r_1)e^{i\theta_1}$ of the Ginzburg-Landau equation, and L will be the linearized Ginzburg-Landau operator around v_+. The linear operator $\mathcal{L}$ is its conjugate operator, as is defined in Section 3. In the lowest Fourier mode, $\mathcal{L}$ has a bounded kernel of the form iS, which tends to the constant i as r_1 goes to infinity. This kernel arises from rotation. We define the projection onto the constant mode as:

$$\beta(r_1):=\frac{S(r_1)}{2\pi r_1}\int_{|z|=r_1}\tilde{\eta}\left(\varepsilon^{-1}p_1+z\right).$$

We also write β into its real and imaginary form: $\beta=\beta_1+\beta_2 i$. We recall that $\mathcal{L}$ is decoupled in this Fourier mode. Let us use $\mathbb{L}_1$ to denote the operator obtained from $\mathbb{L}-i\varepsilon\partial_y\eta$, replacing u by $S(r_1)e^{i\theta_1}$. Note that in Ω_ε, u is close to $S(r_1)e^{i\theta_1}$, up to an error of the order $O(\varepsilon^2)$. The operator $\mathbb{L}_1$ and $\mathcal{L}$ are equivalent under the transformation $g\to Sg$: If $\mathbb{L}_1 g=0$, then $\mathcal{L}(Sg)=0$. Hence using the assumption that $\|\eta\|_*=\varepsilon^{-\sigma}$ and $\|h\|_{**}\le o(1)|\ln\varepsilon|^{-1}$, where $o(1)$ means a term tending to 0 as $\varepsilon_k\to 0$, we infer from the explicit form of the operator $\mathcal{L}$(see (3.7)) that in the region $1<r_1<d_1\varepsilon^{-1}$,

$$\beta_1''+\frac{1}{r_1}\beta_1'-\frac{1}{r_1^2}\beta_1+\left(1-3S^2\right)\beta_1=o(1)|\ln\varepsilon|^{-1}r_1^{-1-\sigma}, \tag{4.19}$$

$$\beta_2''+\frac{1}{r_1}\beta_2'-\frac{1}{r_1^2}\beta_2+\left(1-S^2\right)\beta_2=o(1)|\ln\varepsilon|^{-1}r_1^{-2-\sigma}. \tag{4.20}$$

Note that due to the asymptotic behavior of S, the left hand side of the equation (4.20) essentially behaves like $\beta_2''+\frac{1}{r_1}\beta_2'$ for r_1 large. Since S is the unique bounded solution of (4.20), variation of parameter formula(See Lemma 3.1 for the asymptotic behavior of the homogeneous equation) together with the fact that β_2 is bounded by a constant at the point $r_1=\frac{d_1}{\varepsilon}$ tell us that indeed $|\beta_2|\le C$. Similarly, from (4.19), we deduce that $|\beta_1|\le C$.

We remark that the estimate of β can also be obtained directly(and actually will be easier, especially if we are going to deal with higher order vortex solutions) from the explicit form of the operator $\mathbb{L}$, without using $\mathcal{L}$. The reason that we choose the arguments above is to fit the linear theory cited in Section 3.

Claim 2: $\left\|u\tilde{\eta}^{(k)}\right\|_{L^\infty(r_1<2R_0)}$ *is uniformly bounded with respect to* k

Let us assume to the contrary that, up to a subsequence, $\left\|u\tilde{\eta}^{(k)}\right\|_{L^\infty(r_1<2R_0)}\to+\infty$. Then we define the renormalized function

$$\xi^{(k)}:=\left\|u\tilde{\eta}^{(k)}\right\|_{L^\infty(r_1<2R_0)}^{-1}u\tilde{\eta}^{(k)}.$$

Using (4.12) and elliptic estimates, we see that this sequence of functions will converge to a bounded solution ξ of the equation $L(\xi)=0$. By the nondegeneracy of degree one vortex v_+, we have $\xi=c_1 i v_+ + c_2\partial_x v_+ + c_3\partial_y v_+$. The fact that β is bounded implies $c_1=0$. The orthogonality of $\xi^{(k)}$ with $\partial_x u$ and $\partial_y u$ tells that $c_2=c_3=0$. Hence $\xi=0$. This contradicts with the fact that $\|\xi\|_{L^\infty(r_1<2R_0)}\ge 1$. Claim 2 is thereby proved.

We observe that similar estimates as above are also valid near other vortex points $\varepsilon^{-1}p_j,\varepsilon^{-1}q_j,\ j=1,2,3$. Hence we have proved that $\left\|u\eta^{(k)}\right\|_{L^\infty(\mathbb{R}^2)}$ is uniformly bounded with respect to k.

Substep B. $\left\|u\eta^{(k)}\right\|_{L^\infty(\mathbb{R}^2\setminus\Omega_\varepsilon)}$ *tends to zero as* k *goes to infinity.*

We assume to the contrary that up to a subsequence, $\left\|u\eta^{(k)}\right\|_{L^\infty(\mathbb{R}^2\setminus\Omega_\varepsilon)}\ge C_1>0$, for a universal constant C_1. With the estimates (4.18) of $\nabla\eta_1$ at hand, we find that the rescaled function $\eta_2^{(k)}\left(\varepsilon^{-1}z\right)$ will converge to a bounded solution of the problem

$$\Delta g=0,\ \text{in } \mathbb{R}^2\setminus\{p_1,p_2,p_3,q_1,q_2,q_3\},\ g \text{ is odd in } y.$$

By the removable singularity theorem of harmonic functions, g is smooth and has to be zero. This contradict with the fact that for k large,

$$\left\|u\eta_2^{(k)}\right\|_{L^\infty(\mathbb{R}^2\setminus\Omega_\varepsilon)}\ge C_1/2.$$

Therefore, we conclude that

$$\left\| u\eta^{(k)} \right\|_{L^\infty(\mathbb{R}^2\setminus\Omega_\varepsilon)} \to 0, \text{ as } k \to 0. \tag{4.21}$$

Substep C. $\left\|u\eta^{(k)}\right\|_{L^\infty(\Omega_\varepsilon)}$ *tends to zero as k goes to infinity.*

The proof of Claim 2 tells us that the L^∞ bound of η is determined by the value of η_2 at $\partial\Omega_\varepsilon$. In view of the estimate (4.21), we can repeat the arguments in Claim 2 to infer that actually

$$\left\| u\eta^{(k)} \right\|_{L^\infty(r_1<2R_0)} \to 0. \tag{4.22}$$

It then follows from Claim 1 that $\left\|u\eta^{(k)}\right\|_{L^\infty(\Omega_\varepsilon)}$ tends to zero as k goes to infinity.

Once we obtain (4.22) for the L^∞ norm, we can estimate $\nabla^2\eta$, $\nabla\eta$ and their weighted Hölder norms using inequalities like (4.8) and (4.10), and deduce that $\varepsilon^\sigma \left\|\eta^{(k)}\right\|_* \to 0$. But this will contradict with the assumption (4.11). This contradiction finally tells us that actually $\|\eta\|_* \leq C\varepsilon^{-\sigma} |\ln\varepsilon| \, \|h\|_{**}$, for some universal constant C. The proof is then completed. ∎

Now we would like to turn to estimate the error of the approximate solution in the exterior region Ξ, which is far away from the vortex points. Let r be the distance of z to the origin. We have the following

Lemma 4.6. *In Ξ, we have*

$$|E(u)| \leq Cr^{-2}. \tag{4.23}$$

Moreover,

$$\left| \operatorname{Im}\left(e^{-i\tilde{\theta}} E(u)\right) \right| \leq C\varepsilon r^{-3}. \tag{4.24}$$

Proof. Recall that $u = \prod_j u_j = \prod_j \left(S(r_j) e^{i\theta_j}\right)$. For $r \geq d_0\varepsilon^{-1}$, we have

$$|\partial_y(\theta_j - \theta_{j+3})| \leq C\varepsilon^{-1} r^{-2}, \ j = 1, 2, 3.$$

Hence $|\partial_y u| \leq C\varepsilon^{-1} r^{-2}$. Next,

$$\begin{aligned} |\nabla u_k \cdot \nabla u_j| &= |\partial_x u_k \partial_x u_j + \partial_y u_k \partial_y u_j| \\ &\leq |\partial_x u_k| \, |\partial_x u_j| + |\partial_y u_k| \, |\partial_y u_j| \\ &\leq Cr^{-2}. \end{aligned}$$

Finally, since $\rho_k \leq Cr^{-2}$, we have $Q_k \leq Cr^{-4}$. Combining these estimates, we get (4.23).

Now we prove (4.24). For each k, using the fact that $S'(r) = O\left(r^{-3}\right)$, we have

$$\operatorname{Im}\left(e^{-i\tilde{\theta}} i\varepsilon\partial_y u_k \prod_{j\neq k} u_j\right) = O\left(r^{-3}\right).$$

Moreover, for $k \neq j$, with $k, j \leq 3$, we know that u_k and u_j are vortex of degree one. Then we compute

$$\begin{aligned}
&\operatorname{Im}\left(e^{-i\tilde{\theta}} \left(\nabla u_k \cdot \nabla u_j \right) \prod_{l \neq k,j} u_l \right) \\
&= S\left(r_k\right) \partial_x \theta_k S'\left(r_j\right) \partial_x r_j + S'\left(r_k\right) \partial_x r_k S\left(r_j\right) \partial_x \theta_j \\
&+ S\left(r_k\right) \partial_y \theta_k S'\left(r_j\right) \partial_y r_j + S'\left(r_k\right) \partial_y r_k S\left(r_j\right) \partial_y \theta_j \\
&= O\left(r^{-4}\right).
\end{aligned}$$

For general $k \neq j \leq 6$, we may have different signs before θ_k, θ_j in the above identity. Hence we have the same estimates. This proves (4.24). ∎

Now we are ready to prove our main theorem in this paper. Since technically the method is quite similar to that of [32], we only sketch the main steps.

Recall that we need to solve

$$\mathbb{L}\left(\eta\right) = -u^{-1} E\left(u\right) + N\left(\eta\right). \tag{4.25}$$

Lemma 4.6 tells us that $\operatorname{Im}\left(e^{-i\tilde{\theta}} E\left(u\right)\right) = O\left(\varepsilon r^{-3}\right)$ for $r > d_1 \varepsilon^{-1}$. We can also estimate $E\left(u\right)$ in terms of r_j, if $r < d_1 \varepsilon^{-1}$. Now if we choose $\sigma > 0$ and $\gamma > 0$ to be sufficiently small. Then the error $E\left(u\right)$ can be estimated in terms of ε as

$$\left\| E\left(u\right) \right\|_{**} \leq C \varepsilon^{1-\beta},$$

where β is a positive constant satisfying $1 - \beta > 2\sigma$. Applying Proposition 4.5 and using contradiction argument, we see that the equation (4.25) can be solved modulo the projection onto the kernels $\partial_x u, \partial_y u$ localized near the vortices(Keep in mind that $\partial_x v_+$ and $\partial_y v_-$ decay like r^{-1} and is not in L^2). More precisely, let $\rho_k \geq 0$ be cutoff functions supported in the region where $\left|z - \varepsilon^{-1} p_k\right| \leq A_0$, where $A_0 > 0$ is a fixed constant. We can find c_k, d_k, η such that

$$\mathbb{L}\eta = -u^{-1} E\left(u\right) + N\left(\eta\right) + \sum_k \left(c_k e^{-i\tilde{\theta}} \partial_x u + d_k e^{-i\tilde{\theta}} \partial_y u \right) \rho_k.$$

Moreover, $\left\|\eta\right\|_* \leq C \varepsilon^{1-\beta-2\sigma}$. Projecting both sides on $\partial_x u, \partial_y u$ and using the estimate of η, we find that if we want all the constants c_k, d_k to be zero, then p_k, q_k should satisfy the system

$$\begin{cases} \sum_{j \neq \alpha} \frac{1}{p_\alpha - p_j} - \sum_j \frac{1}{p_\alpha - q_j} = 1 + O\left(\varepsilon^\delta\right), \text{ for } \alpha = 1, ..., 3, \\ \sum_{j \neq \alpha} \frac{1}{q_\alpha - q_j} - \sum_j \frac{1}{q_\alpha - p_j} = O\left(\varepsilon^\delta\right), \text{ for } \alpha = 1, ..., 3, \end{cases} \tag{4.26}$$

for some small $\delta > 0$. Using the nondegeneracy(Proposition 2.9) of the roots of the Adler-Moser polynomial and the Lipschitz dependence of the $O\left(\varepsilon^\delta\right)$ term on $\{p_k\}, \{q_k\}$, we can solve this system using contraction mapping principle again and get a solution $\left(p_1, p_2, p_3, q_1, q_2, q_3\right)$, close to the roots a, b, of the Adler-Moser polynomials. This gives us the desired traveling wave solutions of the GP equation.

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