LEAPFROGGING VORTEX RINGS FOR THE 3-DIMENSIONAL INCOMPRESSIBLE EULER EQUATIONS

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Abstract. A classical problem in fluid dynamics concerns the interaction of multiple vortex rings sharing a common axis of symmetry in an incompressible, inviscid 3-dimensional fluid. Helmholtz (1858) observed that a pair of similar thin, coaxial vortex rings may pass through each other repeatedly due to the induced flow of the rings acting on each other. This celebrated configuration, known as *leapfrogging*, has not yet been rigorously established. We provide a mathematical justification for this phenomenon by constructing a smooth solution of the 3d Euler equations exhibiting this motion pattern.

1. Introduction

We consider the 3-dimensional Euler equation for an ideal incompressible fluid given by

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p, \\ \operatorname{div}\mathbf{u} = 0, \quad x\in\mathbb{R}^3,\ t\geq 0. \end{cases} \tag{1.1}$$ euler0

For a solution $\mathbf{u}(x,t)$ of (1.1), its vorticity is defined as $\vec{\omega} = \operatorname{curl}\mathbf{u}$. Then $\vec{\omega}$ solves the Euler system (1.1) in its vorticity form,

$$\begin{cases} \vec{\omega}_t + (\mathbf{u}\cdot\nabla)\vec{\omega} = (\vec{\omega}\cdot\nabla)\mathbf{u}, \quad \mathbf{u} = \operatorname{curl}\vec{\psi}, \\ \vec{\psi}(x,t) = (-\Delta)^{-1}\vec{\omega} = \dfrac{1}{4\pi}\displaystyle\int_{\mathbb{R}^3}\frac{1}{|x-y|}\vec{\omega}(y,t)\,dy, \quad x\in\mathbb{R}^3,\ t\geq 0. \end{cases} \tag{1.2}$$ euler

A vortex ring is an axially symmetric solution of (1.2) which does not change shape in time, whose vorticity is mostly concentrated inside a solid torus which moves with constant speed along the symmetry axis. The vortex lines form large circles that fill the torus, whereas fluid particles spin around the vortex core within perpendicular cross sections characterized with a thin torus-shaped region in which the vorticity of the fluid is concentrated. These objects were first described by Helmholtz in his celebrated work [31, 32]. He considered with great attention the situation where the vorticity field is concentrated in a circular vortex-filament of very small section, a thin vortex ring. Helmholtz also analyzed the interaction between two or more similar coaxial vortex rings with thin sections and similar translation speeds. As pointed out by Jerrard and Smets [37], his description reads:

We can now see generally how two ring-formed vortex-filaments having the same axis would mutually affect each other, since each, in addition to its proper motion, has that of its elements of fluid as produced by the other. If they have the same direction of rotation, they travel in the same direction; the foremost widens and travels more slowly, the pursuer shrinks and travels faster till finally, if their velocities are not too different, it overtakes the first and penetrates it. Then the same game goes on in the opposite order, so that the rings pass through each other alternately.

The motion pattern described by Helmholtz is often termed *leapfrogging* in fluid mechanics. Leapfrogging vortex rings are solutions of the Euler equations where several interacting vortex rings sharing a common axis of symmetry move in the same direction along the symmetry axis and pass through each other repeatedly due to the induced flow of the rings acting on each other as depicted in Figure 1.

Even though this phenomenon has been widely studied since Helmholtz, as far as we know it has not yet been mathematically justified. In this paper we present what seems to be the first rigorous construction of a solution to the Euler equations with a leapfrogging motion pattern.

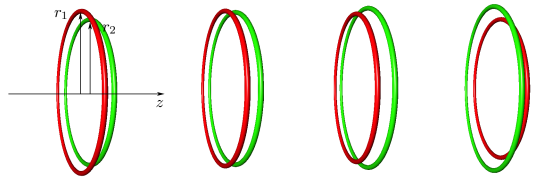

FIGURE 1. Leapfrogging vortex rings.

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The solution we construct belongs to the axisymmetric, no-swirl class, namely with a velocity field $\mathbf{u}(x,t)$ which in standard cylindrical coordinates (r,θ,z) takes the form

$$\mathbf{u}(x,t) = u^r(r,z,t)\,\mathbf{e}_r + u^z(r,z,t)\,\mathbf{e}_z, \quad x = (r\cos\theta, r\sin\theta, z) \in \mathbb{R}^3,$$

where $\mathbf{e}_r = (\cos\theta, \sin\theta, 0)$, $\quad \mathbf{e}_\theta = (-\sin\theta, \cos\theta, 0)$, $\quad \mathbf{e}_z = (0,0,1)$, and

$$(r,z) \in \Sigma = \{(r,z)\,/\,r>0, z\in\mathbb{R}\}. \tag{1.3}$$ defSigma

The corresponding vorticity $\vec{\omega} = \operatorname{curl}\mathbf{u}$ takes the form

$$\vec{\omega}(x,t) = \omega^\theta(r,z,t)\,\mathbf{e}_\theta, \quad \text{where } \omega^\theta = \partial_z u^r - \partial_r u^z.$$

The divergence free condition $\operatorname{div}\mathbf{u} = 0$ becomes

$$\partial_r(ru^r) + \partial_z(ru^z) = 0 \quad \text{in } \Sigma, \quad t>0.$$

This implies the existence of a scalar function $\psi^\theta(r,z,t)$ such that

$$u^r = -\partial_z\psi^\theta, \quad u^z = \frac{1}{r}\partial_r(r\psi^\theta) \quad \text{in } \Sigma, \quad t>0.$$

The relation $\vec{\omega} = \operatorname{curl}\mathbf{u}$ yields

$$-\left(\partial_z^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)\psi^\theta = \omega^\theta.$$

We notice that the vector-valued function $\vec{\psi}(x,t) = \psi^\theta(r,z,t)\,\mathbf{e}_\theta$ satisfies $-\Delta\vec{\psi} = \vec{\omega}$. The Euler equations (1.2) thus become

$$\partial_t\omega^\theta + u^r\partial_r\omega^\theta + u^z\partial_z\omega^\theta = \tfrac{1}{r}u^r\omega^\theta, \quad -[\partial_z^2 + \partial_r^2 + \tfrac{1}{r}\partial_r - \tfrac{1}{r^2}]\psi^\theta = \omega^\theta \quad (r,z)\in\Sigma, \quad t>0. \tag{1.4}$$ eu1

The axisymmetric Euler equation without swirl (1.4) has a formal singularity at $r=0$, which sometimes is inconvenient to work with. To remove the artificial singularity the following change of variable is usually used

$$\omega = \frac{\omega^\theta}{r}, \quad \psi = \frac{\psi^\theta}{r}.$$

The functions ω, ψ are respectively known as the *relative* vorticity and stream function. In terms of these new variables equation (1.4) becomes

$$r\,\partial_t\omega + \nabla^\perp(r^2\psi)\cdot\nabla\omega = 0, \quad -\Delta_5\psi = \omega \quad \text{in } \Sigma, \quad t>0, \tag{1.5}$$ eu2

where $\nabla^\perp = (-\partial_z, \partial_r)$ and

$$\Delta_5\psi = \partial_{rr}\psi + \frac{3}{r}\partial_r\psi + \partial_{zz}\psi, \quad x = (r,z), \tag{1.6}$$ D5

supplemented with the conditions

$$\partial_r\psi(0,z,t) = 0, \quad \lim_{|(r,z)|\to\infty}\psi(r,z,t) = 0.$$

It is well-known that the initial value problem for (1.5) is globally well-posed in $L^1(rdrdz)\cap L^\infty(\Sigma)$, see [50, 43, 12]. A bounded solution ω to (1.5) with sufficient space decay actually satisfies

$$t \mapsto \int_{\mathbb{R}^2} r\omega(r,z,t)\,drdz = constant, \quad t\mapsto \|w(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} = constant. \tag{1.7}$$ time

The same is true if the integral is taken in the time section of connected components of the support of ω. We will use these facts in the formulation of a suitable first approximation for a solution exhibiting the leapfrogging dynamics.

A **vortex ring** moving with constant speed c along the z-axis is a travelling wave solution of (1.5) with the form

$$\omega(r,z,t) = W_0(r, z - c\,t), \quad \psi(r,z,t) = \Psi_0(r, z - c\,t),$$

where W_0 and Ψ_0 solve

$$\nabla^\perp \left(r^2(\Psi_0 - \frac{c}{2})\right) \cdot \nabla W_0 = 0, \quad -[\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2]\Psi_0 = W_0 \qquad \text{in } \Sigma. \tag{1.8}$$ ring0

It is worth mentioning that if $\Psi_0(r,z)$ satisfies a semilinear equation of the form

$$-[\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2]\Psi_0 = f\left(r^2(\Psi_0 - \frac{c}{2})\right) \quad \text{in } \Sigma,$$

for an arbitrary nonlinearity f, then $\Psi_0,\ W_0 = f\left(r^2(\Psi_0 - \frac{c}{2})\right)$ solves (1.8). A first example of a solution with a compactly supported, positive vorticity was exhibited by Hill [40]. In 1970, Fraenkel [26] rigorously found a solution supported inside a torus with a tiny section of radius $\varepsilon > 0$ with a center located at a fixed distance $r_0 > 0$ of the z-axis. See also [45]. To formally derive the correct propagation speed c, let us imagine that we have a family of solutions $(W_\varepsilon, \Psi_\varepsilon)$ of (1.8) with vorticity depending on a small concentration parameter $\varepsilon > 0$ in the form

$$r\,W_\varepsilon(r,z) = \frac{1}{\varepsilon^2} U\left(\frac{x - (r_0, 0)}{\varepsilon}\right)(1 + o(1)), \quad x = (r,z) \tag{1.9}$$ 11

with $o(1) \to 0$ as $\varepsilon \to 0$, for a positive, rapidly decaying radial profile $U(y)$ with a mass that we normalize as $\int_{\mathbb{R}^2} U(y)dy = 8\pi$. To fix ideas, we consider the Kaufmann-Scully vortex

$$U(y) \;=\; \frac{8}{(1 + |y|^2)^2}, \quad y \in \mathbb{R}^2. \tag{1.10}$$ ks

Then we have $r\,W_\varepsilon \rightharpoonup 8\pi\delta_{(r_0,0)}$ as $\varepsilon \to 0$, where $\delta_{(r_0,0)}$ designates a Dirac mass in Σ at the point $(r_0, 0)$. Since

$$-[\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2]\Psi_\varepsilon \approx \frac{1}{r_0\varepsilon^2} U\left(\frac{x - (r_0,0)}{\varepsilon}\right) \quad \text{in } \Sigma,$$

we have that $r_0\Psi_\varepsilon(r,z)$ approaches Green's function $G(r,z)$ where

$$-[\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2]G = 8\pi\delta_{(r_0,0)} \qquad \text{in } \Sigma.$$

It is not hard to show that near $(r_0, 0)$ and up to an additive constant and lower order terms we have

$$G(r,z) = -2\log(|x - (r_0,0)|^2)\left(1 - \frac{3}{2r_0}(r - r_0)\right).$$

Actually, at main order we also have

$$r\Psi_\varepsilon(r,z) = -2\log(|x - (r_0,0)|^2 + \varepsilon^2)(1 - \frac{3}{2r_0}(r - r_0)).$$

Replacing these approximate expressions into the equation $\nabla^\perp\left(r^2(\Psi_\varepsilon - \frac{c}{2})\right)\cdot\nabla W_\varepsilon = 0$ near $(r_0, 0)$ yields

$$[2r_0(\frac{4|\log\varepsilon|}{r_0} - \frac{c}{2}) - \frac{3}{2}4|\log\varepsilon|]\,\mathbf{e_2}\cdot\nabla U \approx 0.$$

Hence, at main order the propagation speed c should satisfy

$$c = \frac{2}{r_0}|\log\varepsilon|(1 + o(1)) \quad \text{as} \quad \varepsilon \to 0. \tag{1.11}$$ massi

This is indeed the speed rigorously derived by Fraenkel [26]. Vortex rings have been analyzed in larger generality in [27, 47, 2, 21, 7]. Expression (1.11) corresponds to a special case of formal asymptotics obtained by Da Rios in 1906 [15, 16], of thin vortex tubes around curves evolving by their binormal flow. See [48, 35, 36]. Helicoidal travelling vortex tubes have been recently found in [19].

Interacting vortex rings. We shall formally derive the dynamics of $k \geq 2$ thin, coaxial vortex rings with comparable speeds given at main order by (1.11) in (1.5). It is convenient to introduce the new unknown $W(r,z,\tau)$ where

$$\omega(r,z,t) \,=\, W(r, z - 2r_0^{-1}\,|\log\varepsilon|\,t, |\log\varepsilon|\,t\,),$$

so that Problem (1.5) in terms of $W(r,z,\tau)$, $\tau = |\log\varepsilon|\,t$, takes the form

$$|\log\varepsilon|\,r\,\partial_\tau W + \nabla^\perp(r^2\,(\Psi - r_0^{-1}|\log\varepsilon|)\,)\cdot\nabla W \,=\, 0, \quad -\Delta_5\Psi = W. \tag{1.12}$$ leap00

Let us assume the presence of a solution $W(r,z,\tau)$ of (1.12) which, in agreement with (1.9), has the approximate form

$$rW(r,z,\tau) \,=\, \sum_{j=1}^{k}\frac{1}{\varepsilon_j^2}U\left(\frac{x-P_j(\tau)}{\varepsilon_j}\right), \quad x=(r,z), \quad P_j(\tau) = (P_j^1(\tau), P_j^2(\tau)),$$

for small numbers $\varepsilon_j \to 0$. This ansatz is consistent with the fact that $\int rW\,drdz$ should be preserved in time on components of the support, see (1.7). On the other hand, the fact that the L^∞-norm of W should be preserved in time on each component suggests that $P_j^1(\tau)\varepsilon_j^2$ should be constant in time, say $= r_0\varepsilon^2$. Thus, we choose

$$\varepsilon_j = \varepsilon\frac{r_0}{\sqrt{P_j^1(\tau)}},$$

so that at main order, near $(r_0, 0)$ we have

$$W(r,z,\tau) \,=\, \sum_{j=1}^{k}\frac{1}{r_0\varepsilon^2}U\left(\frac{x-P_j(\tau)}{\varepsilon_j}\right), \quad x=(r,z),$$

and correspondingly, up to lower order terms

$$\Psi_\varepsilon(r,z,\tau) \,=\, \sum_{j=1}^{k}\frac{2}{P_j^1}\log\left(\frac{1}{|x-P_j(\tau)|^2+\varepsilon_j^2}\right)(1-\frac{3}{2P_j^1}(r-P_j^1)), \quad x=(r,z),$$

for centers $P_j(\tau)$ close to $(r_0, 0)$ to be determined. Substituting this expression into equation (1.12) we obtain that at main order and for each j,

$$\Big[-r|\log\varepsilon|\frac{dP_j}{d\tau} + \nabla_x^\perp\sum_{i\neq j}4r\log\frac{1}{|x-P_i|} - 2\frac{r-r_0}{r_0}|\log\varepsilon|\,\mathbf{e}_2\Big]\cdot\nabla U\left(\frac{x-P_j(\tau)}{\varepsilon_j}\right) \approx 0. \tag{1.13}$$ 12

It is convenient to use the ansatz

$$P_j(\tau) = (r_0, 0) + \frac{1}{\sqrt{|\log\varepsilon|}}q_j(\tau), \quad q_j(\tau) = (q_j^1(\tau), q_j^2(\tau)).$$

Imposing vanishing of the left factor in (1.13) at $x = P_j$, neglecting lower order terms in a fixed interval $\tau\in[0,T]$, we arrive at the limiting system

$$\frac{dq_j}{d\tau} = -4\sum_{\ell\neq j}\frac{(q_j-q_\ell)^\perp}{|q_j-q_\ell|^2} - 2\,\frac{q_j^1}{r_0^2}\begin{pmatrix}0\\1\end{pmatrix}, \quad \tau\in[0,T]. \tag{1.14}$$ car1

This is a Hamiltonian system for the energy

$$H_k(q_1,\dots,q_k) = -2\sum_{i\neq j}\log|q_i-q_j| - \frac{1}{r_0^2}\sum_{j=1}^{k}|q_j^1|^2.$$

For instance, for $k=2$ and restricting ourselves to $q_1 = -q_2 = q = (q^1, q^2)$ we arrive at the system

$$\frac{dq}{d\tau} \,=\, -2\frac{q^\perp}{|q|^2} - 2\,\frac{q^1}{r_0^2}\begin{pmatrix}0\\1\end{pmatrix}, \quad \tau\in[0,T]. \tag{1.15}$$ car2

The level curves for its Hamiltonian $q \mapsto H_2(q,-q)$ are depicted in Figure 2. Any solution of this system is periodic and lies on a closed level curve around the origin.

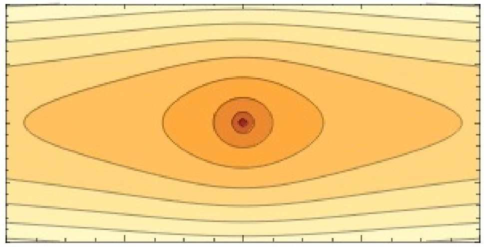

FIGURE 2. Level curves for $q \to H_2(q, -q)$.

⟨fig2⟩

In short, the vorticity of k interacting similar thin vortex rings looks like the superposition of vorticities of k individual rings with small cross section ε, whose centers evolve in time approximately following the *reduced leapfrogging dynamics* (1.14) being located at mutual distances of order $\frac{1}{\sqrt{|\log \varepsilon|}}$.

Several authors investigated the reduced dynamics given by system (1.14), see for instance [22, 23, 14, 5, 33, 39, 40, 44, 1, 6]. Numerical simulations of the leapfrogging, such as in [49, 46, 41, 9], provided theoretical evidence that this phenomenon should actually occur in the Euler equations. The leapfrogging phenomenon was experimentally confirmed in 1978 by Yamada and Matsui [51]. They used vortex rings made of air and used smoke for visualization and successfully created a leapfrogging pair of rings.

Our main result states the existence of a true smooth solution $\omega(r, z, t)$ of Problem (1.5) asymptotically obeying the dynamic law described above, for each given collisionless solution of System (1.14) on the time interval $[0, T]$.

It is more convenient to express, for a small $\varepsilon > 0$, the problem in terms of the equivalent formulation (1.12)

$$|\log \varepsilon|\, r\, \frac{\partial W}{\partial \tau} + \nabla^{\perp}(r^2\,(\Psi - r_0^{-1}|\log \varepsilon|)\,) \cdot \nabla W \;=\; 0, \quad -\Delta_5 \Psi = W, \quad x \in \Sigma, \quad \tau > 0, \tag{1.16}$$

leap01

$$\frac{\partial}{\partial r}\Psi(x,\tau) = 0 \quad \text{on} \quad x \in \partial\Sigma, \quad |\Psi(x,\tau)| \to 0 \quad \text{as } |x| \to \infty,\ \tau > 0.$$

We recall that, if $W(r, z, \tau)$ solves this problem, then

$$\omega(r, z, t) \;=\; W(r, z - 2r_0^{-1}\,|\log \varepsilon|\, t,\ |\log \varepsilon|\, t\,)$$

solves (1.5).

⟨teo⟩ **Theorem 1.** *Let $q(\tau) = (q_1(\tau), \dots q_k(\tau))$ be a solution of System (1.14) without collisions in $[0, T]$ in the sense that $\inf\{|q_i(\tau) - q_j(\tau)|\ /\tau \in [0, T],\ i \neq j\} > 0$. Then there exists a smooth solution $W_\varepsilon(r, z, \tau)$ of Problem (1.16) such that for certain points $P_j^\varepsilon(\tau)$ with the form*

$$P_j^\varepsilon(\tau) = (r_0, 0) + \frac{1}{\sqrt{|\log \varepsilon|}} q_j(\tau) + O\Big(\frac{\log(|\log \varepsilon|)}{|\log \varepsilon|}\Big),$$

we have

$$W_\varepsilon(r, z, \tau) = \frac{1}{\varepsilon^2 r_0} \sum_{j=1}^{k} U\left(\frac{x - P_j^\varepsilon(\tau)}{\varepsilon_j}\right) + \varphi(r, z, \tau), \quad x = (r, z) \tag{1.17}$$

mmx

where $U(y) = \frac{8}{(1+|y|^2)^2}$, $\varepsilon_j^2 = \varepsilon^2 \frac{r_0}{P_j^1(\tau)}$. For some number $C > 0$ we can estimate

$$|\varphi(r, z, \tau)| \;\leq\; C \sum_{j=1}^{k} \frac{\varepsilon^2 |\log \varepsilon|^{60}}{\varepsilon^2 + \left|x - P_j^\varepsilon(\tau)\right|^2} \quad \text{for all} \quad x \in \Sigma,\ \tau \in [0, T],$$

with

$$|\varphi(r, z, \tau)| \;\leq\; C \frac{\varepsilon^2 |\log \varepsilon|^{60}}{1 + |x|^4} \quad \text{if} \quad |x - P_j| > \delta\, \forall j, \qquad \tau \in [0, T].$$

Theorem 1 corresponds to a first mathematically rigorous justification of the leapfrogging motion of vortex rings. Given a solution $q(\tau)$ of the reduced leapfrogging dynamics (1.14) which has no collision in the interval of time $[0,T]$, the function

$$\omega_\varepsilon(r,z,t) \,=\, W_\varepsilon(r, z-2r_0^{-1}\,|\log\varepsilon|\,t,\,|\log\varepsilon|\,t\,),$$

with W_ε given by Theorem 1, is a global in time solution to the 3d Euler equation in the axisymmetric, no-swirl class (1.5). This solution exhibits a leapfrogging motion with k interacting vortex rings in the interval of time $t\in[0,\frac{T}{|\log\varepsilon|}]$, in the sense described by (1.17). Experiments [51] suggest it is not conceivable that the leapfrogging motion persists at all times, as the evolution of the vortices gets deteriorated along the flow. Global-in-time dynamics for other type of vortex configurations in the 2-dimensional Euler equations has been constructed in [52].

Even in the simplified symmetric setting when $k=2$, $q_1=-q_2$ that we discussed before, the Hamiltonian system governing the *reduced leapfrogging dynamics* of the centres (1.15) has periodic orbits, as depicted in Figure 2, but the existence of periodic solutions to (1.16) exhibiting a leapfrogging motion for all times is still an open question.

Motivated by the classical question for the incompressible Euler equations, Jerrard and Smets [37, 38] built up a solution with a vortex-ring leapfrogging pattern in a suitable sense for the Gross-Pitaevskii equation

$$iu_t=\Delta u+\frac{1}{\varepsilon^2}(1-|u|^2)u \quad \text{in } \mathbb{R}^3$$

on the basis of limiting Ginzburg-Landau energy configurations. The method in [37, 38] does not seem applicable to the Euler setting. The Kaufman-Scully vortex (1.10) is a convenient local choice for the building blocks of the rings. The function $\Gamma_0(y)=\log U(y)$ satisfies the Liouville equation

$$-\Delta\Gamma_0=e^{\Gamma_0}=U(y)=\frac{8}{(1+|y|^2)^2} \quad \text{in } \mathbb{R}^2 \tag{1.18}$$ defU

and fine information about the linearized equation is available. Similar statements can be drawn for generic choices of rapidly decaying, radially decreasing positive building blocks $U(|y|)$. Vortices (1.10) have been used to construct solutions with concentrated vorticities in $\mathbb{R}^2$ in [17] and helicoidal travelling waves in [19]. Linearization-stability in $2d$-Euler and with a third dimension added as a vortex filament is a delicate matter, see for instance [4, 25, 28, 29, 30, 34].

In Theorem 1, all individual rings in the vorticity expression (9.9) have similar positive circulations, which is what leads to the leapfrogging phenomenon. Another interesting dynamics of vortex rings is the case of opposite sign circulations, namely dipole dynamics. This is closely related to various scenarios for potential singularities in axi-symmetric Euler flows, see [10, 11, 13, 24, 42]. Our approach may also be applicable to the analysis of dipole dynamics. We expect that this scheme will be a useful tool in studying finite time singularities or the Vortex Filament Conjecture for the incompressible Euler flow.

In the next section we explain the scheme of the proof of Theorem 1 which is carried out in the subsequent sections.

2. Scheme of the proof and organization of the paper.

The proposed strategy for our construction falls into two steps. First we derive an approximate solution, then we solve the full problem setting up an *inner-outer gluing scheme*.

The *inner-outer gluing scheme* has been a very powerful tool in singularity formation problems for nonlinear elliptic and parabolic equations, see for instance [20, 8, 18]. In those applications, the use of maximum principle is essential. In [17] we extend this scheme to the Euler flow in 2-dimensions and we find regular solutions with highly concentrated vorticities around a given number of moving points in the plane, the so-called *desingularized vortex problem.* This paper represents the first attempt to extend this scheme to the three-dimensional Euler flow and find regular solutions with highly concentrated vorticities around a given number of vortex rings interacting in accordance with the Leapfrogging dynamics.

We take advantage of the axi-symmetry and no-swirl assumption to recast the 3-dimensional problem in the 2-dimensional setting (1.16), which allows us to adapt some ideas from [17]. In the construction of solutions with vorticity highly concentrated around points for the 2-dimensional Euler flow in [17], each vortex is well described by a smooth radially symmetric function with fast decay at infinity, the Kaufman-Scully vortex U. The core of each vortex is of size ε and the interaction with other vortices can be easily controlled, as the distance of the centers of two different vortices is of order $O(1)$ as $\varepsilon \to 0$. On the contrary, in order to well approximate vortex rings radially symmetric profiles need an important non radial correction, the reason being the anisotropy in equation (1.8). The core of each ring is still of order ε but the relative distance between the centers of two rings is of size $|\log\varepsilon|^{-\frac{1}{2}}$, which makes their interaction much stronger, and hence delicate to control. These are the features that make the steps in our argument quite involved: the construction of the approximation requires several consecutive adjustments and the inner-outer scheme to find the remainder of the solution has to be designed to properly describe the transition of the problem in different-scaled regions.

Construction of an approximate solution. The basic building block for the construction of the leapfrogging is a single approximate travelling vortex ring with highly ε-concentrated vorticity near a point $P_0 = (r_0, z_0)$, $r_0 > 0$. Here r_0 represents the radius of the ring, and z_0 its vertical displacement. This is achieved finding a constant α and a stream function Ψ_ε *almost* solving the equations for the travelling vortex ring

$$\nabla^\perp(r^2(\Psi_\varepsilon - \alpha|\log\varepsilon|)) \cdot \nabla W_\varepsilon \sim 0, \quad \text{where} \quad W_\varepsilon = -\Delta_5 \Psi_\varepsilon,$$

in a neighbourhood of P_0, with the expectation that the vorticity

$$r\, W_\varepsilon \rightharpoonup 8\pi\delta_{P_0}, \qquad \text{as} \quad \varepsilon \to 0.$$

In accordance with (1.8), (1.11) and the discussion in the introduction, the constant α is expected to satisfy $\alpha \sim \frac{1}{r_0}$ as $\varepsilon \to 0$ and the vorticity $r\,W_\varepsilon$ to have the form $\frac{1}{\varepsilon^2 r_0} U\left(\frac{x-P_0}{\varepsilon}\right)$. More precisely, we find

$$\begin{aligned}
\alpha &= \frac{1}{r_0} + O(|\log\varepsilon|^{-1}) \\
\Psi_\varepsilon(x) &= \frac{1}{r_0}\psi_\varepsilon\left(\frac{x-P_0}{\varepsilon}\right), \quad \psi_\varepsilon(y) = (\Gamma_0(y) - 4\log\varepsilon)\left(1 - \frac{3\varepsilon}{2r_0}y_1\right) + O(1) \\
W_\varepsilon(x) &= \frac{1}{\varepsilon^2 r_0} w_\varepsilon\left(\frac{x-P_0}{\varepsilon}\right), \quad w_\varepsilon(y) = U(y)\left(1 + \frac{\varepsilon y_1}{2r_0}\Gamma_0(y)\right) + U(y)O(\varepsilon y_1)
\end{aligned}$$

in the region around P_0 given by $|x - P_0| < \delta$, for some fixed $\delta > 0$, uniformly as $\varepsilon \to 0$. In the above formula $U(y)$ and $\Gamma_0(y)$ are defined in (1.18). The precise derivation is carried out in Section §3, Proposition 3.1.

The approximate leapfrogging solution for (1.16) will look at main order as the sum of k approximate vortex rings. For simplicity we rename the time-variable τ in (1.16) to be t, so that $t \in [0, T]$. Let $k \geq 2$, and consider k points

$$P_1(t), \quad P_2(t), \quad \ldots \quad P_k(t), \quad P_j = (P_j^1, P_j^2) \in \Sigma$$

and k scaling parameters $\varepsilon_1(t), \ldots, \varepsilon_k(t)$. We allow these points and these parameters to evolve in time, for $t \in [0, T]$ and we assume they have the form

$$P_j = (r_0, 0) + \frac{1}{\sqrt{|\log\varepsilon|}} q_j + \tilde{P}_j + \mathbf{a}_j, \quad P_j^1(t)\,\varepsilon_j^2(t) = r_0\,\varepsilon^2$$

with q_j satisfying the *reduced leapfrogging dynamics* (1.14) and

$$\sqrt{|\log\varepsilon|}\,|\tilde{P}_j| \to 0, \quad |\mathbf{a}_j| \lesssim \varepsilon^3 \qquad \text{as} \quad \varepsilon \to 0,$$

uniformly for $t \in [0, T]$. We refer to Subsection §4.1 for the detailed description of these points and the scaling parameters.

To each point P_j, we associate the approximate vortex ring described before, namely a pair of stream function and vorticity (Ψ_j, W_j), that then we add up together. This gives a good description of the approximate leapfrogging close to the points P_j, but it is far from satisfying the required boundary conditions

$$\frac{\partial}{\partial r}\Psi(x,t) = 0 \quad \text{on} \quad \partial\Sigma \times [0, T), \quad |\Psi(x,t)| \to 0 \quad \text{as } |x| \to \infty.$$

We use a cut-off function and correct the sum of stream functions to have at the same time the boundary conditions and the relation $-\Delta_5\Psi = W$ satisfied.

For $N \in \mathbb{N}$, we set

$$\eta_N(s) = 1, \quad \text{for} \quad s < N, \quad = 0 \quad \text{for} \quad s > 2N \tag{2.1}$$ cutoff

for a smooth cut-off function. Define

$$\eta(x) = \eta_1 \left(\frac{4|x - (r_0, 0)|}{r_0} \right),$$

with $\eta_1$as in (2.3). In Section §4, Subsection §4.2, we prove the existence of a smooth function $H^0(x,t)$, which is uniformly bounded, as $\varepsilon \to 0$, for $(x,t) \in \Sigma \times [0,T]$ so that the pair

$$\Psi^0(x,t) = \eta(x) \sum_{j=1}^{k} \Psi_j + H^0(x,t), \quad \text{and} \quad W^0(x,t) = \sum_{j=1}^{k} W_j,$$

is defined for $(x,t) \in \Sigma \times [0,T]$ and satisfies

$$\frac{\partial}{\partial r}\Psi^0(x,t) = 0 \quad \text{on} \quad \partial\Sigma \times [0,T), \quad |\Psi^0(x,t)| \to 0 \quad \text{as } |x| \to \infty, \quad -\Delta_5 \Psi^0 = W^0 \quad (x,t) \in \Sigma \times [0,T].$$

Let us define the Euler operators

$$\begin{aligned} S_1(W,\Psi) &= |\log\varepsilon| r W_t + \nabla^{\perp}(r^2(\Psi - r_0^{-1}|\log\varepsilon|)) \cdot \nabla W, \\ S_2(W,\Psi) &= \Delta_5 \Psi + W. \end{aligned} \tag{2.2}$$ defS

Hence

$$S_2(W^0, \Psi^0) = 0, \qquad (x,t) \in \Sigma \times [0,T].$$

Besides (Ψ^0, W^0) is a good approximate solution to (1.16) around each point P_j in the following sense. For any $j = 1, \dots, k$, consider the small ball around P_j of radius $|\log\varepsilon|^{-1}$, $B(P_j, |\log\varepsilon|^{-1})$. Under our assumptions on the points P_j and the scaling parameters ε_j, one has that

$$B(P_i, |\log\varepsilon|^{-1}) \cap B(P_j, |\log\varepsilon|^{-1}) = \emptyset, \quad \text{for} \quad i \neq j,$$

for all ε small enough, at any $t \in [0,T]$. In Subsections §4.3 and §4.4 we derive

$$\begin{aligned} \left| S_1(W^0, \Psi^0)\,(x,t) \right| &= O\left(\frac{\varepsilon^{-2}|\log\varepsilon|}{1 + |y|^4} \right), \quad x = (r,z) \\ \text{where} \quad y &= \frac{x - P_j}{\varepsilon}, \quad \text{for} \quad x \in B(P_j, |\log\varepsilon|^{-1}),\, t \in [0,T]. \end{aligned}$$

This estimate can be achieved thanks to the choice of the points P_j at their main order, namely $P_j = P_0 + \frac{1}{\sqrt{|\log\varepsilon|}} q_j$, and the fact that q_j solve (1.14). Observe that on the boundary of the region $B(P_j, |\log\varepsilon|^{-1})$ one has $|S_1(W^0, \Psi^0)| = O(\varepsilon^2 |\log\varepsilon|^5)$, while it is of order $O(\varepsilon^{-2}|\log\varepsilon|)$ close to each P_j.

Next we modify (W^0, Ψ^0) in order to produce a better approximate solution (W^*, Ψ^*). For $N \in \mathbb{N}$, we set

$$\eta_N(s) = 1, \quad \text{for} \quad s < N, \quad = 0 \quad \text{for} \quad s > 2N \tag{2.3}$$ cutoff

for a smooth cut-off function.

In Section §5 and §6 we prove the existence of points $\tilde{P}_j$, $j = 1, \dots, k$ in the decomposition of $P_j = (r_0, 0) + \frac{1}{\sqrt{|\log\varepsilon|}} q_j + \tilde{P}_j + \mathbf{a}_j$, and functions

$$\psi_j^*,\ \phi_j^*, \quad j = 1, \dots, k, \quad \psi^{*,out},\ \phi^{*,out}$$

so that (W^*, Ψ^*) given by

$$\begin{aligned} \Psi^*(x,t) &= \Psi^0 + \sum_{j=1}^{k} \frac{\eta_{j2}}{r_j} \psi_j^*(\frac{x - P_j}{\varepsilon_j}, t) + \psi^{*,out}(x,t) \\ W^*(x,t) &= W^0 + \sum_{j=1}^{k} \frac{\eta_{j1}}{r_j \varepsilon_j^2} \phi_j^*(\frac{x - P_j}{\varepsilon_j}, t) + \phi^{*,out}(x,t) \end{aligned} \tag{2.4}$$ app1

where

$$\eta_{jN}(x,t) = \eta_N\left(|\log\varepsilon|^3\, |x - P_j| \right),$$

is a better approximate leapfrogging. The function η_N is defined in (2.3).

In fact, the boundary conditions are satisfied and we get that

$$
\begin{aligned}
|S_1(W^*,\Psi^*)(x,t)| \ &\leq C\varepsilon^{1-\sigma_*}\sum_{j=1}^k \frac{1}{1+|y_j|^3} + C\frac{\varepsilon^{4-\frac{\sigma_*}{2}}}{1+|x|^4}, \quad y_j = \frac{x-P_j(t)}{\varepsilon},\\
|S_2(W^*,\Psi^*)(x,t)| \ &\leq C\varepsilon^{4-\sigma}\eta_1\left(\frac{4|x-(r_0,0)|}{r_0}\right),
\end{aligned}
$$

for all $x\in\Sigma$ and $t\in[0,T]$, and $|y_j|<\varepsilon^{-1}|\log\varepsilon|^{-3}$. Here η_1 is given by (2.3) and σ^* is a fixed positive number. The functions (W^*,Ψ^*) depend on the points $\mathbf{a}_j$ which are left as parameters to adjust later. Notice that on the boundary of the region $B(P_j,|\log\varepsilon|^{-1})$ one has $|S_1(W^*,\Psi^*)|=O(\varepsilon^{4-\frac{\sigma^*}{2}})$, while it is of order $O(\varepsilon^{1-\sigma^*})$ close to each P_j.

Proposition 5.1 contains a precise description of the error term $S_1(W^*,\Psi^*)$ in the inner regions close to the points P_j, at distance $|x-P_j|\lesssim|\log\varepsilon|^{-3}$, for each $j=1,\ldots,k$, as well as in the complementary outer region. Observe that the inner regions $|x-P_j|\lesssim|\log\varepsilon|^{-3}$ well separate the points P_j, whose relative distance is of order $|\log\varepsilon|^{-\frac{1}{2}}$. The modification to the original stream function Ψ^0 is given by

$$
\sum_{j=1}^k \frac{\eta_{j2}}{r_j}\psi_j^*(\frac{x-P_j}{\varepsilon_j},t)+\psi^{*,out}(x,t).
$$

The terms ψ_j^* are functions of the expanded variables $y=\frac{x-P_j}{\varepsilon_j}$ and encode the local correction needed to improve the approximation near the points P_j. The cut-off functions η_{j2} (as well as η_{j1} for the vorticity) are designed in such a way to guarantee that Ψ^0 is still the main term in the decomposition of Ψ^* in the region where the corresponding cut-off functions are non-zero. The function $\psi^{*,out}$ is a more regular function, expressed in terms of the original variable x and it is responsible of the improvement of the size of the error far from the points P_j. Similar decomposition describes the modification of the initial vorticity W^0.

In Section §5 we describe how we find the improvement (W^*,Ψ^*) and we describe the error of approximation. We make this statement precise in Proposition 5.1. The proof of Proposition 5.1 is contained in Section §6. The construction of the approximation requires 10 subsequent refinements in regions close to the centers, which we call *inner improvements* (to get ψ_j^* and ϕ_j^*), and one global adjustment in the region far from the centers, the *outer improvement* (to get $\psi^{*,out}$ and $\phi^{*,out}$). At the beginning of Section §6, in Subsection §6.1, we describe the general strategy for improvement and then we proceed with the detailed description of the 11 subsequent improvements in Subsections §6.2 to §6.12. For the sixth inner improvement and for the outer improvement we solve two linear transport equations. We study them respectively in Sections §7 and §8.

Solving the full problem. We look for a leapfrogging solution of equations (1.16) of the form

$$
\begin{aligned}
W &= W^*+\varphi\\
\Psi &= \Psi^*+\psi
\end{aligned}
\tag{2.5}
$$

finalform

where φ and ψ are small corrections of the previously found approximation. It is in order to find φ and ψ that we set up an inner-outer gluing scheme. The solution (W,Ψ) will have the form (2.5) with

$$
\begin{aligned}
\varphi &= \sum_{j=1}^k \bar\eta_{j1}\frac{1}{r_j\varepsilon_j^2}\phi_j(\frac{x-P_j}{\varepsilon_j},t)+\phi^{out}(x,t)\\
\psi &= \sum_{j=1}^k \bar\eta_{j2}\frac{1}{r_j}\psi_j(\frac{x-P_j}{\varepsilon_j},t)+\psi^{out}(x,t)
\end{aligned}
$$

where

$$
\bar\eta_{jN}(x)=\eta_N(|\log\varepsilon|^5|x-P_j|),
$$

and η_N as in (2.3).

Comparing with (2.4), you may notice that this ansatz has the same form as the one used for the construction of the improved approximate leapfrogging of vortex rings (W^*,Ψ^*). Observe though that here we are taking cut-offs slightly shorter than the ones taken in the definition of (W^*,Ψ^*).

Let S_1 and S_2 be the Euler operators introduced in (2.2). Then the operator S_1 evaluated at (W,Ψ) becomes

$$S_1(W,\Psi)=\sum_{j=1}^{k}\frac{\bar\eta_{1j}}{\varepsilon_j^4}E_j[\phi_j,\psi_j\,\psi^{out},P]+E^{out}[\phi^{out},\Psi^{out},\phi^{in},\psi^{in},P]$$

where

$$\phi^{in}(y,t)=(\phi_1(y,t),\dots,\phi_k(y,t)),\quad \psi^{in}(y,t)=(\psi_1(y,t),\dots,\psi_k(y,t)).$$

The operators E_j, $j=1,\dots,k$, and E^{out} are defined respectively

$$\begin{aligned}E_j^{in}[\phi_j,\psi_j\,\psi^{out},\mathbf{a}](y,t):=&\,|\log\varepsilon|\varepsilon_j^2(1+\frac{\varepsilon_j}{r_j}y_1)\partial_t\phi_j-|\log\varepsilon|\left(\varepsilon_j\partial_t\varepsilon_j(1+\frac{\varepsilon_j}{r_j}y_1)\nabla\phi_j\cdot y-\frac{\varepsilon_j^2}{r_j}y_1\partial_tP_j\cdot\nabla\phi_j\right)\\&+\nabla^\perp\left((1+\frac{\varepsilon_j}{r_j}y_1)^2(\Psi_j-|\log\varepsilon|+\psi_j^*+\psi_j+r_j\psi^{out})-\varepsilon_j|\log\varepsilon|\partial_t\mathbf{a}_j\right)\cdot\nabla\phi_j\\&+\nabla^\perp\left((1+\frac{\varepsilon_jy_1}{r_j})^2(\psi_j+r_j\psi^{out})\right)\nabla(\varepsilon_j^2r_jW^*)\\&+\varepsilon_j^4S_1(W^*,\Psi^*)(\varepsilon_jy+P_j),\quad |y|<3R,\quad R:=\frac{1}{\varepsilon|\log\varepsilon|^5}\end{aligned}$$

with $y=\frac{x-P_j}{\varepsilon_j}$, and

$$\begin{aligned}E^{out}[\phi^{out},\Psi^{out},\phi^{in},\psi^{in},\mathbf{a}](x,t):=&\,|\log\varepsilon|\,r\,\phi_t^{out}\\&+\nabla_x^\perp(r^2(\Psi^*+\sum_{j=1}^k\frac{\bar\eta_{j2}}{r_j}\psi_j(\frac{x-P_j}{\varepsilon_j})+\psi^{out}-r_0^{-1}|\log\varepsilon|))\nabla_x\phi^{out}\\&+\sum_{j=1}^k\left[r\,|\log\varepsilon|\,\partial_t\bar\eta_{j1}+\nabla_x^\perp(r^2(\Psi^*+\sum_{j=1}^k\frac{\bar\eta_{j2}}{r_j}\psi_j(\frac{x-P_j}{\varepsilon_j})+\psi^{out}-r_0^{-1}|\log\varepsilon|))\nabla\bar\eta_{1j}\right]\frac{\phi_j}{\varepsilon_j^2r_j}\\&+\left[\sum_{j=1}^k(\bar\eta_{2j}-\bar\eta_{1j})\nabla_x^\perp(r^2(\frac{\psi_j}{r_j}+\psi^{out}))+\frac{r^2\psi_j}{r_j}\nabla_x^\perp\bar\eta_{2j}\right]\nabla_xW^*\\&+(1-\sum_{j=1}^k\bar\eta_{2j})\nabla^\perp(r^2\psi^{out})\cdot\nabla W^*+(1-\sum_{j=1}^k\bar\eta_{j1})S_1(W^*,\Psi^*)=0\quad (x,t)\in\Sigma\times[0,T).\end{aligned}$$

With all this set up, we notice that the pair (W,Ψ) is a solution to (1.16) if $(\phi^{in},\psi^{in},\phi^{out},\psi^{out})$ solve the *inner-outer gluing* system given by the inner problems

$$\begin{aligned}E_j^{in}[\phi_j,\psi_j\,\psi^{out},\mathbf{a}](y,t)=0,&\quad (y,t)\in B(0;3R)\times[0,T)\\-\Delta_y\psi_j-\frac{3\varepsilon_j}{r_j+\varepsilon_jy_1}\partial_{y_1}\psi_j=\phi_j,&\quad (y,t)\in B(0;3R)\times[0,T)\end{aligned}$$

for all $j=1,\dots,k$, coupled with the outer problem

$$\begin{aligned}E^{out}[\phi^{out},\psi^{out},\phi^{in},\psi^{in},\mathbf{a}](x,t)=0,&\quad (x,t)\in\Sigma\times[0,T)\\E_1^{out}[\phi^{out},\psi^{out},\phi^{in},\psi^{in},\mathbf{a}](x,t)=0,&\quad (x,t)\in\Sigma\times[0,T)\end{aligned}$$

where

$$E_1^{out}:=\Delta_5\psi^{out}+\phi^{out}+\sum_{j=1}^k(\bar\eta_{j1}-\bar\eta_{j2})\frac{\phi_j}{r_j\varepsilon_j^2}+\sum_{j=1}^k(\frac{\psi_j}{r_j}\Delta_5\bar\eta_{j2}+2\nabla_x\bar\eta_{j2}\nabla_x\frac{\psi_j}{r_j}),\quad (x,t)\in\Sigma\times[0,T)$$

coupled with the boundary and decay conditions on ψ^{out}

$$\frac{\partial}{\partial r}\psi^{out}(x,t)=0,\quad (x,t)\in\partial\Sigma\times[0,T],\quad |\psi^{out}(x,t)|\to0,\qquad \text{as}\quad |x|\to\infty.$$

The solution predicted by Theorem 1 is obtained solving the inner-outer gluing system

$$
\begin{aligned}
&E_j^{in}[\phi_j, \psi_j\, \psi^{out}, \mathbf{a}](y,t) = 0, \quad (y,t) \in B(0;3R)\times[0,T)\\
&-\Delta_y \psi_j - \frac{3\varepsilon_j}{r_j+\varepsilon_j y_1}\partial_{y_1}\psi_j = \phi_j, \quad (y,t)\in B(0;3R)\times[0,T), \quad j=1,\dots,k\\
&E^{out}[\phi^{out},\psi^{out},\phi^{in},\psi^{in},\mathbf{a}](x,t) = 0, \quad (x,t)\in\Sigma\times[0,T)\\
&E_1^{out}[\phi^{out},\psi^{out},\phi^{in},\psi^{in},\mathbf{a}](x,t) = 0, \quad (x,t)\in\Sigma\times[0,T)\\
&\frac{\partial}{\partial r}\psi^{out}(x,t) = 0, \quad (x,t)\in\partial\Sigma\times[0,T], \quad |\psi^{out}(x,t)|\to 0, \qquad \text{as} \quad |x|\to\infty.
\end{aligned}
$$

In order to obtain the desired solution (with initial conditions equal to zero in all the parameter functions) we will formulate the system as a fixed point problem for a compact operator in a ball of a suitable Banach space. We will find a solution by means of a degree theoretical argument. That involves establishing a priori estimates for a homotopical deformation of the problem into a linear one. These arguments are performed in Sections §10 and §11. The rest of this paper is devoted to carrying out in detail the steps outlined above.

3. Approximate travelling vortex ring

⟨sect⟩ In this section we define the basic building block for the construction of the leapfrogging. This object is an approximate travelling vortex ring with highly ε-concentrated vorticity near a point $P\in\Sigma$. It is achieved finding a constant α and a stream function Ψ_ε "almost" solving in a neighbourhood of P the equations for the travelling vortex ring

$$
S_\alpha(W_\varepsilon,\Psi_\varepsilon) := \nabla^\perp(r^2(\Psi_\varepsilon - \alpha|\log\varepsilon|))\cdot\nabla W_\varepsilon \sim 0, \quad \text{where} \quad W_\varepsilon = -\Delta_5\Psi_\varepsilon \tag{3.1}
$$
uu

where Δ_5 is the operator introduced in (1.6). We recall it here

$$
\Delta_5\Psi = \partial_{rr}\Psi + \frac{3}{r}\partial_r\Psi + \partial_{zz}\Psi, \quad x=(r,z).
$$

The point P represents the centre of a travelling ring, and we take it of the form

$$
P = (\bar r,\bar z) = P_0 + q, \quad P_0 = (r_0, 0),
$$

with $r_0>0$ a fixed number, and $|q|\to 0$ as $\varepsilon\to 0$. In accordance with the discussion in the introduction, it is expected that the vorticity rW_ε satisfies

$$
r\,W_\varepsilon(r,z) = \frac{1}{\varepsilon^2}U\left(\frac{x-P}{\varepsilon}\right)(1+o(1)), \quad x=(r,z) \tag{3.2}
$$
star

where U is the rapidly decaying function in (1.10), and that the constant α satisfies $\alpha = \frac{1}{r_0}(1+o(1))$, with $o(1)\to 0$ as $\varepsilon\to 0$. The associated stream function Ψ_ε will correspond to an ε-regularization of the following Green's function

$$
-\Delta_5 G(x;P) = 8\pi\delta_P, \quad \frac{\partial}{\partial r}G(x;P) = 0, \quad \text{on} \quad \partial\Sigma, \quad G(x;P)\to 0 \quad \text{as} \quad |x|\to\infty.
$$

We write the Green's function G as

$$
G(x;P) = \log\frac{1}{|x-P|^4}\big(1 - \frac{3}{2\bar r}(r-\bar r) + H(x;P)\big) + K(x;P),
$$

where $H(\cdot;P)$ and $K(\cdot;P)$ satisfy respectively

$$
\Delta_5\Big(\log\frac{1}{|x-P|^4}H(x;P)\Big) = -30\frac{(r-\bar r)^2}{r\bar r|x-P|^2} + \frac{9}{2r\bar r}\log\frac{1}{|x-P|^4}, \tag{3.3}
$$
eqH

and

$$
\Delta_5 K(x;P) = 0. \tag{3.4}
$$
eqK

Let the ε-regularization of the Green's function G be given by

$$
G_\varepsilon(x;P) := \log\frac{1}{(\varepsilon^2+|x-P|^2)^2}\big(1 - \frac{3}{2\bar r}(r-\bar r) + H(x;P)\big) + K(x;P). \tag{3.5}
$$
greg

Inserting this approximation in the operator defined in (3.1) produces a term of size ε in a neighborhood of P, when expressed in the expanded variable $y=\frac{x-P}{\varepsilon}$. We reduce the size to ε^2 slightly modifying the approximation in a region close to P . For this purpose, we introduce the constant A given by the relation

$$\int_{\mathbb{R}^2} y_1 U(y)\,(\Gamma_0 + A)\,\frac{\partial \Gamma_0}{\partial y_1}\,dy = 0, \tag{3.6}$$ defA

where Γ_0 is given in (1.18), and the function $\Gamma : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$\Gamma(y) := \int_\rho^\infty \frac{(1+\eta^2)^2}{\eta^3} \int_0^\eta \frac{s^3}{1+s^2} U(s)(\Gamma_0(s) + A)\,ds\,d\eta, \quad \rho = |y|. \tag{3.7}$$ defGamma

A direct computation gives that Γ solves

$$\Delta(y_1\Gamma) + U\,y_1\,\Gamma + y_1 U(y)\,(\Gamma_0 + A) = 0 \qquad \text{in} \quad \mathbb{R}^2$$

and satisfies

$$\Gamma(y) = O\left(\frac{\log(1+|y|)}{1+|y|}\right), \qquad \text{as} \quad |y| \to \infty.$$

We make our construction precise in the next Proposition.

⟨f1⟩ **Proposition 3.1.** *Let $P = (\bar r, \bar z) = P_0 + q$, with $P_0 = (r_0, 0)$, $r_0 > 0$ and $|q| \to 0$ as $\varepsilon \to 0$. Define $\alpha = \alpha(\varepsilon, P)$ as*

$$\alpha = \frac{1}{\bar r} - \frac{A + \log 8 - 6 - 2\bar r \partial_r K(P;P) - 4K(P;P)}{4\bar r |\log \varepsilon|}, \tag{3.8}$$ defalpha

where A is given by (3.6), and $\Psi_\varepsilon[P](x)$ as

$$\bar r\,\Psi_\varepsilon[P](x) = G_\varepsilon(x;P) + \frac{r - \bar r}{2\bar r}\Gamma(\frac{x-P}{\varepsilon}), \tag{3.9}$$ f

where $G_\varepsilon(x;P)$ is in (3.5) and Γ in (3.7). Set

$$W_\varepsilon[P](x) = -\Delta_5 \Psi_\varepsilon[P](x).$$

Then for any fixed $\delta > 0$ small and any x with $|x - P| < \delta$, setting

$$y = \frac{x-P}{\varepsilon} = \rho\, e^{i\theta}, \quad |y| < \frac{\delta}{\varepsilon},$$

$S_\alpha(W_\varepsilon, \Psi_\varepsilon)$ as defined in (3.1) has the following expansion

$$\begin{aligned} \varepsilon^4\, S_\alpha(W_\varepsilon, \Psi_\varepsilon)(\varepsilon y + P) &= \frac{\varepsilon^2 |\log \varepsilon|}{1+|y|^4} E_2(\rho,\theta,\varepsilon,P) \\ &+ \frac{\varepsilon^3 |\log\varepsilon|}{1+|y|^3} |\log(\varepsilon(1+|y|))| [E_1(\rho,\theta,\varepsilon,P) + E_3(\rho,\theta,\varepsilon,P)] \\ &+ O\left(\frac{\varepsilon^4 |\log\varepsilon|^2}{1+|y|^2}\right), \qquad \text{as} \quad \varepsilon \to 0. \end{aligned} \tag{3.10}$$ ee1

Here we have written $E_k(\rho,\theta,\varepsilon,P)$ for a function of the form

$$E_k(\rho,\theta,\varepsilon,P) = E_{k,1}(\rho,\varepsilon,P)\cos(k\theta) + E_{k,2}(\rho,\varepsilon,P)\sin(k\theta) \tag{3.11}$$ defEk

where

$$\sum_{j=0}^{2} \left| (1+\rho)^j \frac{\partial^j E_{k,i}}{\partial \rho^j} \right| + \left| \nabla_P E_{k,i} \right| \lesssim 1, \qquad \text{as} \quad \varepsilon \to 0.$$

We next present the proof of Proposition 3.1.

Proof of Proposition 3.1. Let us fix $\delta > 0$ and consider the region of points x with $|x - P| < \delta$. We use the expanded variable

$$y = \frac{x-P}{\varepsilon}, \quad P = (\bar r, \bar z)$$

which we also identify with polar coordinates $y = \rho e^{i\theta}$, $\rho = |y|$.

From the definition of Ψ_ε given in (3.9) we get the following expansions, for $|y|<\frac{\delta}{\varepsilon}$,

$$\begin{aligned}\Psi_\varepsilon[P](x) &= \frac{1}{\bar r}\psi^0(y), \quad \text{where}\\ \psi^0(y) &= \Gamma_0(y) - 4\log\varepsilon - \log 8 + K(P;P)\\ &\quad + \frac{\varepsilon y_1}{2\bar r}\Big(-3\Gamma_0(y) + A_\varepsilon - 4K(P,P) + \Gamma(y)\Big) + \varepsilon^2\theta_1[P](y) + \varepsilon^3\theta_2[P](y),\end{aligned} \tag{3.12}$$ barp00

and

$$\begin{aligned}\frac{r^2}{\bar r}\Big(\Psi_\varepsilon[P](x) - \alpha|\log\varepsilon|\Big) &= \Gamma_0(y) - (4-\alpha\bar r)\log\varepsilon - \log 8 + K(P;P)\\ &\quad + \frac{1}{2\bar r}\Big(\Gamma_0(y) + \bar A + \Gamma(y)\Big)\,\varepsilon y_1 + \varepsilon^2\theta_1[P](y) + \varepsilon^3\theta_2[P](y).\end{aligned} \tag{3.13}$$ barp

In the above expansions, $\bar A$ A_ε are constants given by

$$\bar A = A - 6, \quad A_\varepsilon = 4(4-\alpha\bar r)\log\varepsilon + \bar A + 4\log 8 \tag{3.14}$$ barA

and $\Gamma = \Gamma(y)$ is as in (3.7). Moreover $\theta_1[P](y)$ and $\theta_2[P](y)$ denote generic reminders with the following form

$$\begin{aligned}\theta_1[P](y) = \Bigg[& a(P) + b_1(P)\cos 2\theta + b_2(P)\sin 2\theta\\ & + |\log(\varepsilon(1+|y|))|(a(P) + b_1(P)\cos 2\theta + b_2(P)\sin 2\theta)\Bigg] O(|y|^2)\end{aligned} \tag{3.15}$$ theta1n

and

$$\theta_2[P](y) = a(P) O(|y|^3|\log(\varepsilon(1+|y|)|)) \tag{3.16}$$ theta2n

for a, b_1 and b_2 smooth functions of P, uniformly bounded as $\varepsilon\to 0$. Here $\Gamma_0(y) = \log U(y)$ as in (1.18).

Formula (3.12) gives the asymptotic expansion of the stream function Ψ_ε associated to our approximate vortex ring. In order to describe the asymptotic expansion of the vorticity W_ε, let us introduce

$$f(r^2(\Psi - \alpha|\log\varepsilon|)) = \frac{8}{\bar r} e^{-K(P;P)}\varepsilon^{2-\alpha\bar r} f_0\Big(\frac{r^2}{\bar r}(\Psi - \alpha|\log\varepsilon|)\Big), \quad \text{with} \quad f_0(s) = e^s \tag{3.17}$$ el2

and

$$E[\Psi](x) := \Delta_5\Psi + f(r^2(\Psi - \alpha|\log\varepsilon|)). \tag{3.18}$$ el

The proof of Proposition 3.1 follows from showing that α and Ψ_ε are so that the following expansion for $E[\Psi_\varepsilon]$, given by (3.18)-(3.17), holds true: for all $|y|<\frac{\delta}{\varepsilon}$ we have

$$\begin{aligned}\varepsilon^2\,\bar r\,E[\Psi_\varepsilon](x) = \frac{\varepsilon^2}{1+|y|^2}\Bigg[& a(P) + b_1(P)\cos 2\theta + b_2(P)\sin 2\theta\\ & + |\log(\varepsilon(1+|y|))|(a(P) + b_1(P)\cos 2\theta + b_2(P)\sin 2\theta)\Bigg]\\ & + O(\frac{\varepsilon^3}{1+|y|})|\log(\varepsilon(|y|+1))|,\end{aligned} \tag{3.19}$$ estin1

uniformly as $\varepsilon\to 0$. Here a, b_1 and b_2 denote smooth functions of P, uniformly bounded together with their derivative as $\varepsilon\to 0$, whose definition may change from line to line.

Assume expansion (3.19) is true. Since $W_\varepsilon[P] = -\Delta_5\Psi_\varepsilon[P]$, we have

$$S_\alpha(W_\varepsilon,\Psi_\varepsilon) = \nabla^\perp(r^2(\Psi_\varepsilon - \alpha|\log\varepsilon|))\cdot\nabla(E[\Psi_\varepsilon])$$

and expansion (3.10) readily follows from (3.13) and (3.19).

We also observe that, in the region $|y|<\frac{\delta}{\varepsilon}$, the vorticity of the approximate vortex ring can be described as

$$\begin{aligned}W_\varepsilon[P](x) &= \frac{1}{\varepsilon^2\bar r} w_\varepsilon[P](y)\\ w_\varepsilon[P](y) &= U(y)\Big(1 + \frac{\varepsilon y_1}{2\bar r}(\Gamma_0 + \bar A + \Gamma) + \varepsilon^2\theta_1[P](y) + \varepsilon^3\theta_2[P](y)\Big)\end{aligned} \tag{3.20}$$ Ubar

where $U(y)$ and $\Gamma_0(y)$ are defined in (1.18), $\bar A$ in (3.14), θ_1 and θ_2 in (3.15) and (3.16). Hence $rW_\varepsilon(x)$ approaches, locally around P, a Dirac delta, as $\varepsilon \to 0$, in accordance with the expectation (3.2). We will make use of these estimates in the next Section.

As we explained before, we want to prove the validity of (3.19). We start with the observation that, since the operator in (3.18) is invariant under translations in the z-direction, it is not restrictive to assume that $P = (\bar r, 0)$ and to work in the class of functions that are even in the variable z.

To simplify notation, we drop the dependence on P in the functions G_ε, H and K.

Then for $|x - P| < \delta$ we have

$$G_\varepsilon(x) = (\Gamma_0(y) - 4\log(\varepsilon) - \log 8)\Big(1 - \frac{3}{2\bar r}\varepsilon y_1 + H(P+\varepsilon y)\Big) + K(P+\varepsilon y), \quad y = \frac{x-P}{\varepsilon}$$

where we mean $H(P+\varepsilon y) = H(P+\varepsilon y; P)$, etc. Observe that $\partial_z K(P) = 0$ by symmetry. From (3.3) we get that $H(x;P)$ has the following expansion as $x \to P$

$$H(x) = a_1(P)(r-\bar r)^2 + a_2(P)z^2 + b(P)\frac{|x-P|^2}{\log|x-P|^2} + O(|x-P|^3) \tag{3.21}$$ expH

where a_1, a_2, b are constants whose value depends on P. Using (3.21), we expand

$$\begin{aligned} G_\varepsilon(x) &= \Gamma_0(y) - 4\log\varepsilon - \log 8 + K(P) - \frac{3}{2\bar r}\left[\Gamma_0(y) - 4\log\varepsilon - \log 8 - \frac{2\bar r}{3}\partial_r K(P)\right]\varepsilon y_1 \\ &\quad + \varepsilon^2\theta_1[P](y) + \varepsilon^3\theta_2[P](y) \end{aligned}$$

where θ_1, θ_2 are reminders that can be described as in (3.15) and (3.16).

For $\Psi_P(x) = \frac{1}{\bar r}G_\varepsilon(x)$, we get

$$\begin{aligned} \frac{r^2}{\bar r}\Big(\Psi_P(x) - \alpha|\log\varepsilon|\Big) &= \Gamma_0(y) - (4-\alpha\bar r)\log\varepsilon - \log 8 + K(P) \\ &+ \frac{1}{2\bar r}\Big(\Gamma_0(y) - (4-4\alpha\bar r)\log\varepsilon + 2\bar r\partial_r K(P) + 4K(P) - \log(8)\Big)\varepsilon y_1 \\ &+ \varepsilon^2\theta_1[P](y) + \varepsilon^3\theta_2[P](y), \end{aligned}$$

and

$$\begin{aligned} &\frac{8}{\bar r}e^{-K(P)}\varepsilon^{2-\alpha\bar r} f\Big(\frac{r^2}{\bar r}(\Psi_P(x) - \alpha|\log\varepsilon|)\Big) \\ &= \frac{1}{\varepsilon^2\bar r}U(y)\exp\Big[\frac{1}{2\bar r}\Big(\Gamma_0(y) - (4-4\alpha\bar r)\log\varepsilon + 2\bar r\partial_r K(P) + 4K(P) - \log 8\Big)\varepsilon y_1 \\ &\quad + \varepsilon^2\theta_1[P](y) + \varepsilon^3\theta_2[P](y)\Big], \end{aligned} \tag{3.22}$$ c1

for functions θ_1, θ_2 of the form (3.15) and (3.16).

Next we compute

$$\Delta_5\Psi_P = (1) + (2)$$

where

$$\begin{aligned} (1) &= \frac{1}{\bar r}\Delta_5\Big[(\Gamma_0(\frac{x-P}{\varepsilon}) - 4\log\varepsilon - \log 8)\Big(1 - \frac{3}{2\bar r}(r-\bar r)\Big)\Big] \\ (2) &= \frac{1}{\bar r}\Delta_5[(\Gamma_0(\frac{x-P}{\varepsilon}) - 4\log\varepsilon - \log 8)H], \end{aligned}$$

since by definition $\Delta_5 K = 0$. Setting again $\varepsilon y = x - P$, we get

$$\begin{aligned} (1) &= \frac{1}{\bar r}\Big[\Big(-\frac{1}{\varepsilon^2}U(y) + \frac{3}{\varepsilon(\bar r+\varepsilon y_1)}\Gamma_0'(y)\frac{y_1}{\rho}\Big)\Big(1 - \frac{3}{2\bar r}\varepsilon\, y_1\Big) - \frac{3}{\varepsilon\bar r}\Gamma_0'(\rho)\frac{y_1}{\rho} - \frac{9}{2r\bar r}(\Gamma_0 - 4\log\varepsilon - \log 8)\Big] \\ &= \frac{1}{\bar r}\Big[-\frac{1}{\varepsilon^2}U(y) + \frac{3}{2\varepsilon\bar r}U(y)y_1 - \frac{15}{2}\frac{\Gamma_0'(\rho)}{\rho}\frac{y_1^2}{r\bar r} - \frac{9}{2r\bar r}(\Gamma_0 - 4\log\varepsilon - \log 8)\Big] \\ &= \frac{1}{\varepsilon^2\bar r}\Big[-U(y) + \frac{3}{2\bar r}\varepsilon U(y)y_1 + 30\frac{(r-\bar r)^2}{\bar r r}\frac{\varepsilon^2}{\varepsilon^2+|x-\bar P|^2} - \frac{9}{2r\bar r}\varepsilon^2\log\frac{1}{(\varepsilon^2+|x-\bar P|^2)^2}\Big]. \end{aligned}$$

From (3.3) we get

$$(2) = -30\frac{(r-\bar r)^2}{r\bar r^2|x-P|^2} + \frac{9}{2r\bar r^2}\log\frac{1}{|x-P|^4} + \frac{1}{\bar r}\Delta_5[(\Gamma_0(\frac{x-P}{\varepsilon}) - 4\log\varepsilon - \log 8 - \log\frac{1}{|x-P|^4})H]$$
$$= -30\frac{(r-\bar r)^2}{r\bar r^2|x-P|^2} + \frac{9}{2r\bar r^2}\log\frac{1}{|x-P|^4} + \frac{2}{\bar r}\Delta_5[\log\frac{|\frac{x-P}{\varepsilon}|^2}{1+|\frac{x-P}{\varepsilon}|^2}H]$$

Combining the above expressions we conclude that

$$\begin{aligned} -\Delta_5\Psi_P &= \frac{1}{\bar r}\Big[\frac{1}{\varepsilon^2}U(y) - \frac{3}{\varepsilon\bar r}U(y)y_1 + O(\frac{1}{1+\rho^2})\Big] \\ &= \frac{1}{\bar r\varepsilon^2}U(y)\Big[1 - \frac{3}{\bar r}\varepsilon y_1 + \varepsilon^2\theta_1[P](y) + \varepsilon^3\theta_2[P](y)\Big], \end{aligned} \tag{3.23}$$ c2

where θ_1 and θ_2 are described in (3.15) and (3.16).

Putting together (3.22) and (3.23) we find that

$$\varepsilon^2 E[\Psi_P] = \frac{1}{\bar r}[\varepsilon E_0 + \Theta_\varepsilon(y)] \tag{3.24}$$ err0

where

$$E_0(y) = \frac{y_1}{2\bar r}U(y)\Big(\Gamma_0(y) - 4(1-\alpha\bar r)\log\varepsilon + 2\bar r\partial_r K(P) + 4K(P) - \log 8 + 6\Big),$$

and

$$\begin{aligned} \Theta_\varepsilon(y) &= U(y)\exp\Big[\frac{1}{2\bar r}\Big(\Gamma_0(y) - (4-4\alpha\bar r)\log\varepsilon + 2\bar r\partial_r K(P) + 4K(P) - \log 8\Big)\varepsilon y_1 \\ &\quad + \varepsilon^2\theta_1[P](y) + \varepsilon^3\theta_2[P](y)\Big] \\ &\quad - U(y)\Big[1 + \frac{1}{2\bar r}\Big(\Gamma_0(y) - (4-4\alpha\bar r)\log\varepsilon + 2\bar r\partial_r K(P) + 4K(P) - \log 8\Big)\varepsilon y_1\Big] \\ &\quad + U(y)\left[\varepsilon^2\theta_1[P](y) + \varepsilon^3\theta_2[P](y)\right], \end{aligned}$$

where θ_1, θ_2 denote again generic functions of the form (3.15), (3.16). A closer look at this expression gives that $\Theta_\varepsilon(y)$ can be described as follows

$$\begin{aligned} \Theta_\varepsilon(y) = \frac{\varepsilon^2}{1+|y|^2}\Bigg[&a(P) + b_1(P)\cos 2\theta + b_2(P)\sin 2\theta \\ &+ |\log(\varepsilon(1+|y|))|(a(P) + b_1(P)\cos 2\theta + b_2(P)\sin 2\theta)\Bigg] \\ &+ O(\frac{\varepsilon^3}{1+|y|})\log(\varepsilon(1+|y|))a(P) \end{aligned} \tag{3.25}$$ Thetaex

uniformly as $\varepsilon \to 0$. Also here a, b_1, b_2 stand for generic smooth functions of P, uniformly bounded as $\varepsilon \to 0$.

In order to reduce the size of the error term in (3.24) we solve

$$\Delta\bar\psi + U(y)\bar\psi + \varepsilon E_0(y) = 0 \quad \text{in} \quad \mathbb{R}^2, \quad \lim_{|y|\to\infty}\bar\psi(y) = 0. \tag{3.26}$$ appro

It is here when we introduce the function Γ and we use the explicit definition of α as given in (3.8).

It is known that all bounded solutions to

$$\Delta\bar\psi + U(y)\bar\psi = 0 \quad \text{in} \quad \mathbb{R}^2$$

are given by linear combinations of

$$\frac{\partial\Gamma_0}{\partial y_i}(y), \quad i = 1, 2, \quad 2 + \nabla\Gamma_0\cdot y.$$

This result can be found in [3]. By the standard Fredholm alternative for this problem we need the following solvability condition satisfied

$$\int_{\mathbb{R}^2} E_0(y)\frac{\partial \Gamma_0}{\partial y_1}(y)\,dy \;=\; 0.$$

Infact, the error term E_0 is by definition even with respect to the variable y_2. Hence the remaining solvability conditions

$$\int_{\mathbb{R}^2} E_0(y)\frac{\partial \Gamma_0}{\partial y_2}(y)\,dy \;=\; 0, \qquad \int_{\mathbb{R}^2} E_0(y)\,(2+\nabla\Gamma_0\cdot y)\,dy \;=\; 0$$

are automatically satisfied by symmetry.

Let us then compute

$$\begin{aligned} 2\bar r\int_{\mathbb{R}^2} E_0(y)\frac{\partial \Gamma_0}{\partial y_1}(y)\,dy \;&=\; \Big(\int_{\mathbb{R}^2} U(y)y_1\frac{\partial \Gamma_0}{\partial y_1}(y)\,dy\Big)\,(2\bar r\partial_r K(P)+4K(P)-\log 8+6-4(1-\alpha\bar r)\log\varepsilon)\\ &+\int_{\mathbb{R}^2} U(y)y_1\frac{\partial \Gamma_0}{\partial y_1}(y)\Gamma_0\,dy. \end{aligned}$$

Then $\int_{\mathbb{R}^2} E_0(y)\frac{\partial\Gamma_0}{\partial y_1}(y)\,dy \;=\; 0$ is satisfied if we choose α as in (3.8). In terms of P we have

$$\alpha=\frac{1}{\bar r}+\frac{\beta(P)}{|\log\varepsilon|},\quad \beta(P)=-\frac{A+\log 8-6-2\bar r\partial_r K(P;P)-4K(P;P)}{4\bar r}. \tag{3.27}$$ defbeta

With this choice of α we can solve (3.26). Writing in polar coordinates $y=\rho e^{i\theta}$ we observe that E_0 has the form $E_0(y)=Q(\rho)\cos(\theta)$ with $Q(\rho)=O(\frac{\log\rho}{\rho^3})$ as $\rho\to\infty$. A direct computation yields to

$$\bar\psi(y;P)=\frac{\varepsilon}{2\bar r}\Gamma(\rho)y_1. \tag{3.28}$$ barpsi

Using now the whole expression of Ψ_ε as in (3.9), we decompose

$$\begin{aligned} \varepsilon^2 E[\Psi_\varepsilon]&=\varepsilon^2 E[\Psi_P]+\frac{\varepsilon^3}{\bar r}[(\partial_{rr}+\partial_{zz})\bar\Gamma+f'(\Psi_P)\bar\Gamma]+\varepsilon^3\frac{3}{\bar r r}\partial_r\bar\Gamma\\ &+\varepsilon^2[f(\Psi_P+\varepsilon\bar\Gamma)-f(\Psi_P)-f'(\Psi_P)\varepsilon\bar\Gamma] \end{aligned}$$

where $\bar\Gamma=\frac{r-\bar r}{2\bar r}\Gamma(\frac{x-P}{\varepsilon})$ and f is

$$f(s)=\frac{8}{\bar r}e^{-K(P;P)}\varepsilon^{2-\alpha\bar r}f_0\Big(\frac{r^2}{\bar r}(s-\alpha|\log\varepsilon|)\Big).$$

For $|y|<\frac{\delta}{\varepsilon}$,

$$\begin{aligned} f'(\Psi_P)&=\frac{1}{\varepsilon^2}U(y)\exp\Big[\frac{1}{2\bar r}\Big(\Gamma_0(y)-(4-4\alpha\bar r)\log\varepsilon+2\bar r\partial_r K(P)+4K(P)-\log 8\Big)\,\varepsilon y_1\\ &\quad+\varepsilon^2\theta_1[P](y)+\varepsilon^3\theta_2[P](y)\Big]\times(1+\frac{\varepsilon y_1}{\bar r})^2\\ &=\frac{1}{\varepsilon^2}U(y)\Big[1+\frac{1}{2\bar r}\Big(\Gamma_0(y)-(4-4\alpha\bar r)\log\varepsilon+a(P)\Big)\varepsilon y_1+\varepsilon^2\theta_1[P](y)+\varepsilon^3\theta_2[P](y)\Big] \end{aligned}$$

where θ_1, θ_2 satisfy (3.15), (3.16). We have

$$\begin{aligned} &\frac{\varepsilon^3}{\bar r}[(\partial_{rr}+\partial_{zz})\bar\Gamma+f'(\Psi_P)\bar\Gamma]=\frac{1}{\bar r}[\Delta_y\bar\psi+U(y)\bar\psi]\\ &\quad+U(y)\bar\psi\Big[\frac{1}{2\bar r}\Big(\Gamma_0(y)-(4-4\alpha\bar r)\log\varepsilon+a(P)\Big)\varepsilon y_1+\varepsilon^2\theta_1[P](y)+\varepsilon^3\theta_2[P](y)\Big]. \end{aligned}$$

Recall now that $\varepsilon^2 E[\Psi_P]$ can be written as in (3.24). Since $\bar\psi$ solves (3.26) and has the form (3.28), we get

$$\varepsilon^2 E[\Psi_P]+\frac{\varepsilon^3}{\bar r}[(\partial_{rr}+\partial_{zz})\bar\Gamma+f'(\Psi_P)\bar\Gamma]=\Theta_\varepsilon(y)$$

where Θ_ε can be described as in (3.25). Besides, using the form of the function Γ, as described in (3.9), one sees with a direct inspection that

$$\varepsilon^3\frac{3}{\bar r r}\partial_r\bar\Gamma+\varepsilon^2[f(\Psi_P+\varepsilon\bar\Gamma)-f(\Psi_P)-f'(\Psi_P)\varepsilon\bar\Gamma]=\Theta_\varepsilon(y),$$

with Θ_ε another function of the form (3.25). This concludes the proof of (3.19).

□

4. First approximate leapfrogging

⟨sec3⟩ The rest of the paper is devoted to find a solution to Problem (1.12) with the properties described in Theorem 1. With a little abuse of notation, from now on we will use the variable t instead of τ. Given $r_0>0$, we look for (Ψ, W) solving

$$\begin{cases} |\log\varepsilon|\, r\,\partial_t W + \nabla^\perp (r^2(\Psi - r_0^{-1}|\log\varepsilon|))\cdot\nabla W = 0 \quad \text{in} \quad \Sigma\times[0,T) \\ -\Delta_5\Psi = W, \qquad \text{in} \quad \Sigma\times[0,T) \\ \dfrac{\partial}{\partial r}\Psi(x,t) = 0 \quad \text{on} \quad \partial\Sigma\times[0,T), \quad |\Psi(x,t)|\to 0 \quad \text{as } |x|\to\infty. \end{cases} \tag{4.1}$$ leap0

We recall that $\Sigma = \{x=(r,z) \,:\, r>0,\ z\in\mathbb{R}\}$, see (1.3), and

$$\Delta_5\Psi = \partial_{rr}\Psi + \frac{3}{r}\partial_r\Psi + \partial_{zz}\Psi.$$

This Section is devoted to define a first approximate solution to (4.1), given as a sum of approximate travelling vortex rings, as built in Section 3. These travelling vortex rings are centered at different points, at relative distance $|\log\varepsilon|^{-\frac12}$ one from each other, all of them collapsing to $(r_0,0)$ as $\varepsilon\to 0$. Let us be more precise.

⟨subsec41⟩ 4.1. **The parameter functions.** Fix an integer $k\geq 2$ and consider points $P_j = P_j(t)$, for $j\in\{1,\dots,k\}$, which evolve in time and have the form

$$\begin{aligned} &P_j = P_j(t) = (r_j(t), z_j(t)), \quad t\in[0,T) \quad \text{with} \\ &P_j = \mathbf{P}_j + \mathbf{a}_j(t), \quad \mathbf{P}_j = P_0 + P_j^0(t) + P_j^1(t), \quad P_0 := (r_0, 0). \end{aligned} \tag{4.2}$$ point

Let us describe the different terms in the decomposition of P_j. The points $P_j^0(t) = (r_j^0(t), z_j^0(t))$ are explicit and will be determined towards the end of this section, in the form

$$P_j^0 = \frac{1}{\sqrt{|\log\varepsilon|}} q_j + Q_j, \quad \|Q_j\|_{L^\infty[0,T)} + \|\partial_t Q_j\|_{L^\infty[0,T)} \lesssim \frac{\log|\log\varepsilon|}{|\log\varepsilon|}, \tag{4.3}$$ b0

where q_j are the given solutions to the *leapfrogging dynamics* (1.14).

The points $P_j^1(t) = (r_j^1(t), z_j^1(t))$ in (4.2) will also be determined in the process of the construction of an approximate leapfrogging solution and they will satisfy

$$\|P_j^1\|_{L^\infty[0,T)} + \|\partial_t P_j^1\|_{L^\infty[0,T)} \lesssim \varepsilon^{2-\sigma}, \tag{4.4}$$ b1

for some $\sigma>0$, small and independent of ε.

The points $\mathbf{a}_j(t) = (a_{j1}(t), a_{j2}(t))$ are free parameters to adjust at the end of our proof. For the moment, we ask they are continuous functions in $[0,T]$ for which $\partial_t\mathbf{a}_j$ exists and such that

$$\|\mathbf{a}_j\|_{C^1[0,T)} \lesssim \varepsilon^{3+\sigma}. \tag{4.5}$$ b11

The following notation will be used to identify the different sets of points in the decomposition of P_j given in (4.2)

$$P = (P_1,\dots,P_k), \quad P^b = (P_1^b,\dots,P_k^b), \quad b=0,1, \quad \mathbf{a} = (\mathbf{a}_1,\dots,\mathbf{a}_k).$$

Since the relative distance between two points is of order $|\log\varepsilon|^{-\frac12}$ we have

$$\{x \,:\, |x-P_i| < |\log\varepsilon|^{-1}\}\cap\{x \,:\, |x-P_\ell| < |\log\varepsilon|^{-1}\} = \emptyset$$

for all ε small and $i\neq\ell$. Besides, under assumptions (4.2), (4.3), (4.4) and (4.5), we have that

$$P_j \to (r_0, 0), \qquad \text{as} \quad \varepsilon\to 0, \quad \forall j=1,\dots,k,$$

uniformly for $t\in[0,T]$.

Given the points P_j as described in (4.2), we introduce positive functions $\varepsilon_j = \varepsilon_j(t)>0$ such that, for all $j=1,\dots,k$,

$$t\to r_j(t)\,\varepsilon_j^2(t) \quad \text{is independent of} \quad t,$$

where $r_j(t)$ is the first component of the point $P_j(t)$. For convenience we choose

$$r_j(t)\,\varepsilon_j^2(t) = r_0\,\varepsilon^2 \qquad \text{for all} \qquad t\in[0,T). \tag{4.6}$$ ass11

From (4.2)–(4.5) we get that for all $j=1,\dots,k$,

$$\|\varepsilon_j-\varepsilon\|_{L^\infty[0,T)}+\|\partial_t\varepsilon_j\|_{L^\infty[0,T)}\lesssim \varepsilon|\log\varepsilon|^{-\frac12},\quad \forall \quad j=1,\dots,k, \qquad \text{as} \quad \varepsilon\to 0. \tag{4.7}$$ b2

For such $\varepsilon_j(t)$ and $P_j(t)$, let α_j be given as in (3.8), Proposition 3.1 so that

$$\alpha_j[P_j](t)=\frac{1}{r_j(t)}+\frac{\beta(P_j(t))}{|\log\varepsilon_j(t)|},$$

where $\beta(s)$ is the smooth function defined as in (3.27). Observe that

$$\alpha_j[P_0](t)=\frac{1}{r_0}+\frac{\beta(r_0)}{|\log\varepsilon|}$$

is constant in time. We also have

$$r_0^{-1}-\alpha_j[P_j](t)=\frac{r_j-r_0}{r_0r_j}-\frac{\beta(P_j)}{|\log\varepsilon_j|}, \tag{4.8}$$ alphajalpha0

and $\|r_0^{-1}-\alpha_j\|_{L^\infty[0,T)}=O(|\log\varepsilon|^{-\frac12})$ as $\varepsilon\to0$.

⟨subsec42⟩

4.2. **The function H^0 and definition of the very first approximation.** For any $j=1,\dots,k$, we define

$$\Psi_j^0(x,t):=\Psi_{\varepsilon_j(t)}[P_j(t)](x,t),\quad W_j^0(x,t):=-\Delta_5\Psi_{\varepsilon_j(t)}[P_j(t)](x), \tag{4.9}$$ defWj

where $\Psi_\varepsilon[P](x)$ is the approximate travelling vortex ring introduced in (3.9), Proposition 3.1. Since we are assuming that the point $P_j(t)$ evolves with time, the functions Ψ_j^0 and W_j^0 also depend on the time variable $t\in[0,T]$, and their dependence on time is through P_j (and ε_j). Writing

$$\Psi_j^0(x,t)=\frac{1}{r_j}\psi_j^0(y),\quad y=\frac{x-P_j}{\varepsilon_j} \tag{4.10}$$ fee2

from (4.9) we get

$$W_j^0(x,t)=\frac{1}{\varepsilon_j^2r_j}w_j^0(y),\quad \text{with} \quad -\Delta_{5,j}\psi_j^0=w_j^0 \tag{4.11}$$ fe2

where

$$\Delta_{5,j}=\partial_{y_1}^2+\partial_{y_2}^2+\frac{3\varepsilon_j}{r_j+\varepsilon_jy_1}\,\partial_{y_1}. \tag{4.12}$$ d5

Besides, for a fixed $\delta>0$, we have that, for $y=\frac{x-P_j}{\varepsilon_j}$, in the region $|x-P_j|<\delta$

$$w_j^0(y)=U(y)\left(1+\frac{\varepsilon_jy_1}{2r_j}(\Gamma_0+\bar A+\Gamma)+\varepsilon^2\theta_1[P_j](y)+\varepsilon^3\theta_2[P_j](y)\right) \tag{4.13}$$ Ubar1

where θ_1 and θ_2 are functions also described by (3.15) and (3.16) respectively. This expansion has been obtained in Section 3, formula (3.20).

The starting point of our construction is to assume that the vorticity of a leapfrogging of vortex rings is at main order the sum of the vorticities of vortex rings. We do the same with the stream functions, which we then multiply by a cut off function to make it 0 at infinity.

Having introduced the points $P=(P_1,P_2,\dots,,P_k)$, define

$$\bar\Psi^0(x,t)=\eta(x)\sum_{j=1}^k\Psi_j^0(x,t),\quad W^0(x,t)=\sum_{j=1}^kW_j^0(x,t),$$

where η is the smooth cut-off function given by

$$\eta(x)=\eta_1\left(\frac{4|x-(r_0,0)|}{r_0}\right),$$

with $\eta_1$as in (2.3). We then immediately see that $|\bar\Psi^0(x,t)| \to 0$ as $|x| \to \infty$ and that $\frac{\partial}{\partial r}\bar\Psi^0(x,t) = 0$ on $\partial\Sigma$, for any $t \in [0,T]$. On the other hand we no longer have that $-\Delta_5\bar\Psi^0 = W^0$. We shall then slightly modify $\bar\Psi^0$ by a function H^0 to have, for any $t \in [0,T]$

$$\Delta_5\left(\bar\Psi^0 + H^0\right)(x,t) + W^0(x,t) = 0, \quad x \in \Sigma.$$

For this purpose, consider the linear problem

$$\Delta_5\psi + h = 0, \quad \text{in } \Sigma, \quad \frac{\partial\psi}{\partial r} = 0 \quad \text{on } \partial\Sigma, \quad \psi(x) \to 0, \quad \text{as} \quad |x| \to \infty, \tag{4.14}$$ ext0

for a smooth function h satisfying

$$|h(x)| \le \frac{C}{1+|x|^{2+\nu}}, \tag{4.15}$$ decayh

where $C > 0$. Recall that $\Delta_5 = \frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$, for $x = (r,z) \in \Sigma$.

We have

⟨111⟩ **Lemma 4.1.** *Assume h satisfies (4.15), with $\nu > 0$. Then there exist a solution $\psi = \mathcal{T}(h)$ to (4.14) and a constant $C_1 > 0$ such that*

$$(1+|x|)|\nabla\psi(x)| + |\psi(x)| \le \frac{C_1}{1+|x|^{\min(\nu,3)}}.$$

Proof. Recalling that the Laplacian of a radially symmetric function in $\mathbb{R}^n$ is

$$\Delta_Y = \frac{d^2}{ds^2} + \frac{n-1}{s}\frac{d}{ds}, \quad Y = (Y_1,\dots,Y_n), \quad s = \sqrt{Y_1^2 + \dots Y_n^2}$$

we interpret the differential operator in (4.14) as the Laplacian in $\mathbb{R}^5$ and recast Problem (4.14) in $\mathbb{R}^5$. Define

$$\Psi(Y_1,\dots,Y_4,z) = \psi(\sqrt{Y_1^2+\dots Y_4^2}, z), \quad r = \sqrt{Y_1^2+\dots Y_4^2}, \quad Y_5 = z$$

and

$$H(Y_1,\dots,Y_4,z) = h(\sqrt{Y_1^2+\dots Y_4^2}, z), \quad r = \sqrt{Y_1^2+\dots Y_4^2}, \qquad Y_5 = z.$$

To solve Problem (4.14) we find bounded (non-singular) solutions Ψ to

$$\Delta_{\mathbb{R}^5}\Psi + H = 0, \quad \text{in } \mathbb{R}^5, \quad \Psi(Y) \to 0, \qquad \text{as} \quad |Y| \to \infty.$$

We define Ψ using the Newtonian potential in $\mathbb{R}^5$ as

$$\Psi(Y) = \frac{1}{15\omega_5}\int_{\mathbb{R}^5}\frac{1}{|Z-Y|^3}H(Z)\,dZ,$$

with ω_5 the volume of the unit 5-ball. Moreover $\nabla\Psi(Y) = -\frac{3}{15\omega_5}\int_{\mathbb{R}^5}\frac{(Z-Y)}{|Z-Y|^5}H(Z)\,dZ$. From (4.15) we get that $(1+|Y|^{2+\nu})|H(Y)| \lesssim 1$. Estimating the above integrals splitting them in the region $|Z| < \frac{|Y|}{2}$ and its complement we get

$$(1+|Y|)|\nabla\Psi(Y)| + |\Psi(Y)| \lesssim \frac{1}{1+|Y|^{\min(3,\nu)}}.$$

Going back to the original variables $x = (r,z) \in \Sigma$ we get the required estimates. □

For any $t \in [0,T]$, we denote by $H^0[P] = \mathcal{T}\left(\Delta_5\bar\Psi^0 + W^0\right)$ the solution to

$$\Delta_5 H^0 + \Delta_5\bar\Psi^0 + W^0 = 0 \quad \text{in } \Sigma, \quad \frac{\partial H^0}{\partial r} = 0 \quad \text{on } \partial\Sigma \tag{4.16}$$ defH0

with $H^0[P](x,t) \to 0$ as $|x| \to \infty$, for all $t \in [0,T]$.

A direct computation gives

$$\Delta_5\bar\Psi^0 + W^0 = (1-\eta(x))W^0 + (\Delta_5\eta(x))\sum_{j=1}^k \Psi_j^0 + 2\sum_{j=1}^k \nabla_{r,z}\eta\cdot\nabla_{r,z}\Psi_j^0. \tag{4.17}$$ fe1

The function $\Delta_5\bar\Psi^0+W^0$ is smooth, it is the sum of one term with compact support and size $O(1)$, and a term which is bounded by $O(\frac{\varepsilon^2}{1+|x|^4})$, $(x,t)\in\Sigma\times[0,T]$ uniformly as $\varepsilon\to 0$. From Lemma 4.1 and (4.17) we get that

$$(1+|x|)|\nabla H^0[P]|+|H^0[P]|\lesssim \frac{1}{1+|x|^2},$$

uniformly for $t\in[0,T]$ as $\varepsilon\to 0$.

For $P=(P_1,P_2,\dots,,P_k)$, we can now define the first approximate solution to (4.1) to be

$$\Psi^0[P]=\eta(x)\sum_{j=1}^k\Psi_j^0+H^0[P],\quad W^0[P]=\sum_{j=1}^k W_j^0. \tag{4.18}$$ appro1

⟨subsec43⟩ 4.3. **The very first error.** We recast Problem (4.1) as the problem of finding (W,Ψ) with

$$S_1(W,\Psi)=S_2(W,\Psi)=0\quad\text{in}\quad\Sigma\times[0,T),\quad \frac{\partial\Psi}{\partial r}=0\quad\text{on } \partial\Sigma\times[0,T).$$

Here S_1 and S_2 are the Euler operators introduced in (2.2). The leapfrogging of vortex rings are then solutions (W,Ψ) with W and Ψ close respectively to W^0 and Ψ^0, and $|\Psi(x,t)|\to 0$ as $|x|\to\infty$. By construction what we have so far is that

$$S_2(W^0,\Psi^0)=0\quad\Sigma\times[0,T),\quad \frac{\partial}{\partial r}\Psi^0(r,z,t)=0\quad\partial\Sigma\times[0,T),$$
$$|\Psi^0(x,t)|\to 0,\qquad\text{as}\quad |x|\to\infty.$$

We shall now describe $S_1(W^0,\Psi^0)$. We can write

$$\begin{aligned}S_1(W^0,\Psi^0)&=\sum_{j=1}^k E_j^0,\quad\text{where}\\ E_j^0&=|\log\varepsilon|\,r\,\partial_t W_j^0+\nabla^\perp(r^2(\Psi^0-r_0^{-1}|\log\varepsilon|))\cdot\nabla W_j^0.\end{aligned}\tag{4.19}$$ gru

We start with the following general remark.

⟨r1⟩ **Remark 4.1.** *For a function $W(x,t)$ given in the form*

$$W(x,t)=\frac{1}{r_j\varepsilon_j^2}w\Big(\frac{x-P_j}{\varepsilon_j},t\Big),$$

for some function $w(y,t)$, $y=\frac{x-P_j}{\varepsilon_j}$ we have

$$\varepsilon_j^4|\log\varepsilon|\,r\,\partial_t W=\varepsilon_j^2|\log\varepsilon|(1+\frac{\varepsilon_j}{r_j}y_1)\partial_t w-\varepsilon_j|\log\varepsilon|\nabla w\cdot\partial_t P_j+B_0(w), \tag{4.20}$$ Sexp

where

$$B_0(\phi)=-\varepsilon_j\partial_t\varepsilon_j(1+\frac{\varepsilon_j}{r_j}y_1)\nabla\phi\cdot y-\frac{\varepsilon_j^2}{r_j}y_1\partial_t P_j\cdot\nabla\phi \tag{4.21}$$ defB0

An equivalent expression, which will be useful in the sequel, is

$$\begin{aligned}\varepsilon_j^4|\log\varepsilon|r\partial_t W=|\log\varepsilon|\Bigg[&-\varepsilon_j\nabla w\cdot\partial_t P_j+\varepsilon_j^2\partial_t w+\varepsilon_j\partial_t\varepsilon_j(y_1\partial_1 w-y_2\partial_2 w)\\ &-\frac{\varepsilon_j^2}{r_j}\partial_t z_j y_1\partial_2 w+\frac{\varepsilon_j^3}{r_j}y_1\partial_t w-\frac{\varepsilon_j^2}{r_j}\partial_t\varepsilon_j y_1\nabla w\cdot y\Bigg].\end{aligned}\tag{4.22}$$ Sexp1

Proof of (4.20). *For $x=\varepsilon_j y+P_j$,*

$$\varepsilon_j^4|\log\varepsilon|\,r\,\partial_t W=|\log\varepsilon|(1+\frac{\varepsilon_j}{r_j}y_1)\Bigg[\varepsilon_j^2 w_t-(\frac{\partial_t r_j}{r_j}+\frac{2\partial_t\varepsilon_j}{\varepsilon_j})\varepsilon_j^2 w-\varepsilon_j\partial_t\varepsilon_j\nabla w\cdot y-\varepsilon_j\nabla w\cdot\partial_t P_j\Bigg].$$

From our first assumption (4.6) on the parameters r_j and ε_j we get

$$\frac{\partial_t r_j}{r_j}=-2\frac{\partial_t\varepsilon_j}{\varepsilon_j},\qquad\text{for all}\quad t \tag{4.23}$$ ass1

which gives (4.20).

Proof of (4.22). *This expression follows from observing that*

$$\varepsilon_j^4|\log\varepsilon| r\partial_t W = |\log\varepsilon|\Bigg[-\varepsilon_j\nabla w\cdot\partial_t P_j + \varepsilon_j^2\partial_t w - \varepsilon_j\partial_t\varepsilon_j\nabla w\cdot y - \frac{\varepsilon_j^2}{r_j}y_1\partial_t P_j\cdot\nabla w + \frac{\varepsilon_j^3}{r_j}y_1\partial_t w - \frac{\varepsilon_j^2}{r_j}\partial_t\varepsilon_j y_1\nabla w\cdot y\Bigg]$$

and (again (4.23) *for* $P_j(t) = (r_j(t), z_j(t))$*)*

$$-\varepsilon_j\partial_t\varepsilon_j\nabla w\cdot y - \frac{\varepsilon_j^2}{r_j}y_1\partial_t P_j\cdot\nabla w = \varepsilon_j\partial_t\varepsilon_j(y_1\partial_1 w - y_2\partial_2 w) - \frac{\varepsilon_j^2}{r_j}\partial_t z_j y_1\partial_2 w.$$

Recalling (4.13) and using (4.20) and (4.11) we easily get that the first term in E_j^0 in (4.19) is given by

$$\varepsilon_j^4|\log\varepsilon|\, r\,\partial_t W_j^0 = -\varepsilon_j|\log\varepsilon|(1+\frac{\varepsilon_j}{r_j}y_1)\,\partial_t P_j\cdot\nabla w_j^0 + \varepsilon_j|\log\varepsilon|(1+\frac{\varepsilon_j}{r_j}y_1)\left[\varepsilon_j\partial_t w_j^0 - \partial_t\varepsilon_j\nabla w_j^0\cdot y\right].$$

We now pass to the second term in E_j^0 given in (4.19). It is convenient to use the decomposition

$$r^2\left(\Psi^0 - r_0^{-1}|\log\varepsilon|\right) = r^2\left(\Psi_j^0 - \alpha_j|\log\varepsilon_j|\right) + r^2\left(\alpha_j|\log\varepsilon_j| - r_0^{-1}|\log\varepsilon|\right) + r^2\left(H^0 + \sum_{\ell\neq j}\Psi_\ell^0\right).$$

In the region $\{x : |x-P_j| < |\log\varepsilon|^{-1}\}$ we write, for $y = \frac{x-P_j}{\varepsilon_j}$, $|y| < \varepsilon_j^{-1}|\log\varepsilon|^{-1}$,

$$\varepsilon_j^4\nabla^\perp(r^2(\Psi^0 - r_0^{-1}|\log\varepsilon|))\cdot\nabla W_j^0 = \nabla^\perp\left((1+\frac{\varepsilon_j}{r_j}y_1)^2(\psi_j^0 - r_j\alpha_j|\log\varepsilon_j|)\right)\cdot\nabla w_j^0 + \nabla^\perp(\tilde\varphi_j(\varepsilon_j y + P_j;P)\,)\cdot\nabla w_j^0,$$

where

$$\tilde\varphi_j(x;P) = \tilde\varphi_j(\varepsilon_j y + P_j;P) = r_j(1+\frac{\varepsilon_j}{r_j}y_1)^2\times \left(\alpha_j|\log\varepsilon_j| - r_0^{-1}|\log\varepsilon| \;+ H^0[P](\varepsilon_j y + P_j) + \sum_{\ell\neq j}\frac{1}{r_\ell}\;\psi_\ell^0(\frac{\varepsilon_j}{\varepsilon_\ell}y + \frac{P_j - P_\ell}{\varepsilon_\ell})\right), \tag{4.24}$$

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with H^0 the correction introduced in (4.16). Referring to (4.19), in the region $|x-P_j| < |\log\varepsilon|^{-1}$, we have, for $y = \frac{x-P_j}{\varepsilon_j}$,

$$\varepsilon_j^4\,E_j^0(P_j+\varepsilon_j y) = \nabla^\perp\mathcal{R}_j(y,t)\cdot\nabla w_j^0 + \varepsilon_j^2|\log\varepsilon|(1+\frac{\varepsilon_j}{r_j}y_1)\partial_t w_j^0 + |\log\varepsilon|B_0(w_j^0) + \nabla^\perp\left((1+\frac{\varepsilon_j}{r_j}y_1)^2(\psi_j^0 - r_j\alpha_j|\log\varepsilon_j|)\right)\cdot\nabla w_j^0,$$

where B_0 is defined in (4.21) and

$$\mathcal{R}_j(y,t) = \varepsilon_j|\log\varepsilon|\,\partial_t P_j^\perp\cdot y + \tilde\varphi_j(\varepsilon_j y + P_j;P) - \tilde\varphi_j(P_j;P). \tag{4.25}$$

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In the same region, for $i\neq j$,

$$\varepsilon_j^4\,E_i^0(P_j+\varepsilon_j y) = \frac{\varepsilon_j^4}{\varepsilon_i^4}\,\varepsilon_i^4\,E_i^0((\frac{\varepsilon_j}{\varepsilon_i}\frac{P_j-P_i}{\varepsilon_j} + \frac{\varepsilon_j}{\varepsilon_i}y) = O(\frac{\varepsilon^5|\log\varepsilon|^2}{1+|y|}).$$

From the result in Proposition 3.1 we recognize that

$$\begin{aligned}(1+\frac{\varepsilon_j}{r_j}y_1)^2(\psi_j^0 - r_j\alpha_j|\log\varepsilon_j|) &= \Gamma_0(y) - (4-\alpha_j r_j)\log\varepsilon_j - \log 8 + K(P_j;P_j)\\ &+ \frac{\varepsilon_j\, y_1}{2r_j}\Big(\Gamma_0(y) + \bar{A} + \Gamma(y)\Big) + \varepsilon^2\theta_1[P_j](y) + \varepsilon^3\theta_2[P_j](y),\end{aligned} \tag{4.26}$$

fee22

where $\bar{A}$ is the constant defined in (3.14), $\Gamma = \Gamma(y)$ as in (3.7), θ_1 and θ_2 have the form described in (3.15) and (3.16). Moreover

$$\nabla^\perp\left((1+\frac{\varepsilon_j}{r_j}y_1)^2(\psi_j^0 - r_j\alpha_j|\log\varepsilon_j|)\right)\cdot\nabla w_j^0 = r_j^2\,\varepsilon_j^4\,S_{\alpha_j}(W_j^0;\Psi_j^0)(\varepsilon_j y + P_j)$$

where the operator S_α is given by (3.1). From estimates (3.10) on this term, we conclude that, for $\rho = |y| = |\frac{x-P_j}{\varepsilon_j}| < \varepsilon_j^{-1}|\log\varepsilon|^{-1}$,

$$\begin{aligned}\varepsilon_j^4\,S_1(W^0,\Psi^0)(P_j+\varepsilon_j y) &= \varepsilon_j^4\sum_{i=1}^k E_i^0(P_j+\varepsilon_j y)\\ &= \nabla^\perp\mathcal{R}_j(y,t)\cdot\nabla w_j^0 + \varepsilon_j^2|\log\varepsilon|(1+\frac{\varepsilon_j}{r_j}y_1)\partial_t w_j^0 + |\log\varepsilon|B_0(w_j^0)\\ &+ \frac{\varepsilon^2|\log\varepsilon|}{1+|y|^4}E_2(\rho,\theta,t,\varepsilon) + \frac{\varepsilon^3|\log\varepsilon|^2}{1+|y|^3}[E_1(\rho,\theta,t,\varepsilon)+E_3(\rho,\theta,t,\varepsilon)] + O\left(\frac{\varepsilon^4|\log\varepsilon|^2}{1+|y|^2}\right)\end{aligned} \tag{4.27}$$

error1

as $\varepsilon\to 0$. To get this estimate we have used (3.10). The functions $E_j(\rho,\theta,t,\varepsilon)$ have the following form

$$E_j(\rho,\theta,t,\varepsilon) = E_{j,1}(\rho,\varepsilon,P(t))\cos(j\theta) + E_{j,2}(\rho,\varepsilon,P(t))\sin(j\theta)$$

where

$$\sum_{i=0}^2\Big|(1+\rho)^i\frac{\partial^i E_{j,\ell}}{\partial\rho^i}\Big| + \Big|\nabla_P E_{j,\ell}\Big| \lesssim 1 \quad \ell=1,2, \qquad \text{as} \quad \varepsilon\to 0,$$

uniformly for $t\in[0,T)$

⟨subsec44⟩ 4.4. **Dynamics for P_j^0 and reduction of the very first error.** We shall see that it is possible to reduce the size of the error term (4.27) in each of the regions $|\frac{x-P_j}{\varepsilon_j}| < \varepsilon_j^{-1}|\log\varepsilon|^{-1}$, $j=1,\dots,k$, by choosing properly the points P_j^0 as in (4.3). Write the function $\tilde\varphi_j$ in (4.24) as

$$\tilde\varphi_j(x;P) = \varphi_j(x;P) + \varepsilon^2\theta_{j\varepsilon_j}(x;P)$$

with

$$\varphi_j(x;P) = \frac{r^2}{r_j}\left(\alpha_j|\log\varepsilon_j| - r_0^{-1}|\log\varepsilon| \;+ H^0(x;P) + \sum_{\ell\neq j}\frac{1}{r_\ell}\;\bar\psi_\ell^0(x;P_\ell)\right) \tag{4.28}$$

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$$\bar\psi_\ell^0(x;P_\ell) = \log\frac{1}{|x-P_\ell|^4}\big(1-\frac{3}{2r_\ell}(r-r_\ell) + H(x;P_\ell)\big) + K(x;P_\ell) + \varepsilon_\ell\Gamma(x;P_\ell)$$

$$\text{where}\quad \varepsilon_\ell\Gamma(x;P_\ell) = \frac{r-r_\ell}{2r_\ell}\Gamma(\frac{x-P_\ell}{\varepsilon_\ell})\quad\text{and}$$

$$\theta_{j\varepsilon_j}(x;P) = \frac{1}{\varepsilon^2}\sum_{\ell\neq j}\frac{r^2}{r_j r_\ell}\log\frac{|x-P_\ell|^4}{(\varepsilon_\ell^2+|x-P_\ell|^2)^2}\big(1-\frac{3}{2r_\ell}(r-r_\ell)+H(x;P_\ell)\big).$$

We refer to (3.3), (3.4) and (3.9) for the definition of H, K and Γ. We Taylor expand

$$\begin{aligned}\tilde\varphi_j(P_j+\varepsilon_j y;P) &= \tilde\varphi_j(P_j;P) + \varepsilon_j\nabla_x\varphi_j(P_j;P)\cdot y + \bar\varphi_j\\ \bar\varphi_j &= \sum_{k=2}^4\frac{\varepsilon_j^k}{k!}D_x^k\varphi_j(P_j;P)[y]^k \;+ \varepsilon_j^3 D_x\theta_{j\varepsilon_j}(P_j;P)[y] + \frac{\varepsilon_j^4}{2}D_x^2\theta_{j\varepsilon_j}(P_j;P)[y]^2 + Q(\varepsilon_j y,P)\end{aligned} \tag{4.29}$$

ta

where $Q(z,\zeta)$ is a function smooth in its arguments that satisfies

$$|\nabla_z Q(z,\zeta)| \;\le\; C\varepsilon|z|^4.$$

Recalling the definition of $\mathcal{R}_j$ in (4.25) we can write

$$\nabla^\perp \mathcal{R}_j(y,t) = -\varepsilon_j |\log\varepsilon|\,\partial_t P_j + \varepsilon_j \nabla_x^\perp \varphi_j(P_j;P) + \nabla^\perp \bar\varphi_j,$$

and we choose the points P_j^0 in (4.2)-(4.3) to satisfy the ODEs system

$$-\,|\log\varepsilon|\,\partial_t P_j^0 + \nabla_x^\perp \varphi_j(P_0+P_j^0;P_0+P^0) = 0, \quad j=1,\dots k. \tag{4.30}$$ dinl

Here $P_0=(r_0,0)$ and $P^0=(P_1^0,\dots,P_k^0)$. We use (4.8) to write

$$\nabla_x \varphi_j^\perp(P_j;P) = \Theta_j(P) = \Theta_j^0(P) + \Theta_j^1(P)$$

where for $P_j=(r_j,z_j)$,

$$\Theta_j^0(P) = -4\sum_{\ell\neq j} \frac{(P_j-P_\ell)^\perp}{|P_j-P_\ell|^2} - 2\,\frac{r_j-r_0}{r_0^2}\,|\log\varepsilon|\,\mathbf{e_2}.$$

Recall now the form of the points P_j^0 from (4.3)

$$P_j^0 = \frac{1}{\sqrt{|\log\varepsilon|}} q_j + Q_j.$$

Here q_j are the given solutions to the *leapfrogging dynamics* (1.14), which gets rewritten as

$$-|\log\varepsilon|^{\frac12}\partial_t q_j + \Theta_j^0(P_0 + \frac{1}{\sqrt{|\log\varepsilon|}} q) = 0 \quad t\in[0,T].$$

We choose Q_j to solve the initial value problem

$$\begin{aligned}
&-\partial_t Q_j + \tilde\Theta_j(t,Q) = 0 \quad t\in[0,T]\\
&\tilde\Theta_j(t,Q) := \frac{1}{|\log\varepsilon|}\left[\Theta_j^0(P_0+P^0) - \Theta_j^0(P_0+\frac{1}{\sqrt{|\log\varepsilon|}}q)\right] + \frac{1}{|\log\varepsilon|}\Theta_j^1(P_0+P^0),\\
&Q_j(0)=0.
\end{aligned}$$

A direct computation gives that

$$\|\frac{1}{|\log\varepsilon|}\Theta_j^1(P_0+\frac{1}{\sqrt{|\log\varepsilon|}}q)\|_{L^\infty([0,T])} \lesssim \frac{\log|\log\varepsilon|}{|\log\varepsilon|}$$

The functions $\tilde\Theta_j$ are Lipschtz continuous in Q in the set

$$\|Q_j\|_{L^\infty[0,T)} \lesssim \frac{\log|\log\varepsilon|}{|\log\varepsilon|},$$

and continuous in t, for $t\in[0,T]$. Standard ODEs theory ensures the existence of a solution $(Q_1,\dots,Q_k)$ satisfying the bounds (4.3).

This choice for P^0 automatically reduces from ε to ε^2 the size of the term $\nabla^\perp\mathcal{R}_j(y,t)\cdot\nabla w_j^0$ in the first line of the error of approximation described by (4.27). Let us now analyze the terms in (4.27) given by

$$\varepsilon_j|\log\varepsilon|(1+\frac{\varepsilon_j}{r_j}y_1)\left[\varepsilon_j\partial_t w_j^0 - \partial_t\varepsilon_j\nabla w_j^0\cdot y\right] - \varepsilon_j^2|\log\varepsilon|y_1\partial_t P_j\cdot\nabla w_j^0.$$

They have the same form as the terms in the following line in (4.27), with the only difference that in this case the functions $E_j(\rho,\theta,t,\varepsilon)$ do also depend on $\partial_t P$ and not only on P. This fact is consequence of (4.22) and of our assumptions on the points P_j.

We are thus in a position to conclude that if we choose the points P^0 of the form (4.3) to satisfy (4.30), the error of approximation (4.27) in the region $|x-P_j|<|\log\varepsilon|^{-1}$ can be described as, for $y=\frac{x-P_j}{\varepsilon_j}$, $\rho=|y|$

$$\begin{aligned}
\varepsilon_j^4 S_1(W^0,\Psi^0)(P_j+\varepsilon_j y) &= \nabla^\perp\mathcal{R}_j(y,t;P)\cdot\nabla w_j^0 + \frac{\varepsilon^2|\log\varepsilon|}{1+|y|^4}E_2(\rho,\theta,t,\varepsilon)\\
&+\frac{\varepsilon^3|\log\varepsilon|^2}{1+|y|^3}\left[E_1(\rho,\theta,t,\varepsilon)+E_3(\rho,\theta,t,\varepsilon)\right] + O\left(\frac{\varepsilon^4|\log\varepsilon|^2}{1+|y|^2}\right)
\end{aligned} \tag{4.31}$$ error11

as $\varepsilon \to 0$, where the functions $E_j(\rho,\theta,t,\varepsilon)$ now depends also on $\partial_t P$, and have the form

$$E_j[P,\partial_t P](\rho,\theta,t,\varepsilon) = E_{j,1}(\rho,\varepsilon,P(t),\partial_t P(t))\cos(j\theta) + E_{j,2}(\rho,\varepsilon,P(t),\partial_t P(t))\sin(j\theta), \tag{4.32}$$ Ekkn

where

$$\sum_{i=0}^{2}\left|(1+\rho)^i\frac{\partial^i E_{j,\ell}}{\partial\rho^i}\right| + \left|\nabla_P E_{j,\ell}\right| + \left|\nabla_{\partial_t P}E_{j,\ell}\right| \lesssim 1 \quad \ell=1,2, \qquad \text{as} \quad \varepsilon\to 0.$$

Besides,

$$\mathcal{R}_j(y,t;P) = \varepsilon_j|\log\varepsilon|\partial_t(P_j-P_j^0)^\perp\cdot y + \tilde\varphi_j(\varepsilon_j y+P_j;P) - \tilde\varphi_j(P_j;P) - \nabla_x\varphi_j(\bar P_j^0;\bar P^0)\,\varepsilon_j y. \tag{4.33}$$ defRj

where $\bar P_j^0(t) = P_0 + P_j^0(t)$.

5. Improvement of the approximation

⟨sec4⟩ In the previous section we introduced the functions $\Psi^0(x,t)$ and $W^0(x,t)$ defined in (4.18), where P is a collection of k points of the form

$$P=(P_1,\dots,P_k),\quad P_j(t) = P_0+P_j^0(t)+P_j^1(t)+\mathbf{a}_j(t),\quad P_0=(r_0,0).$$

So far we have defined P_j^0 explicitly with the form (4.3) to solve (4.30), while P_j^1 and $\mathbf{a}_j$ are still parameters that are assumed to satisfy (4.4) and (4.5) respectively.

The next step in our argument is to modify (W^0,Ψ^0) in order to produce a better approximate solution (W^*,Ψ^*). We do it taking (W^*,Ψ^*) of the form

$$\Psi^*(x,t;P) = \Psi^0 + \sum_{j=1}^k \frac{\eta_{j2}}{r_j}\psi_j^*(\frac{x-P_j}{\varepsilon_j},t) + \psi^{*,out}(x,t) \tag{5.1}$$ f110

$$W^*(x,t;P) = W^0 + \sum_{j=1}^k \frac{\eta_{j1}}{r_j\varepsilon_j^2}\phi_j^*(\frac{x-P_j}{\varepsilon_j},t) + \phi^{*,out}(x,t) \tag{5.2}$$ f220

where

$$\eta_{jN}(x,t) = \eta_N\left(|\log\varepsilon|^\zeta\,|x-P_j|\right), \tag{5.3}$$ zeta

and η_N is given by (2.3). Here $\zeta>1$ is a positive number, independent of ε. Since the relative distance between P_j and P_i, $j\neq i$, is of the order $|\log\varepsilon|^{-\frac12}$, we have

$$P_i\in \text{support}\,(\eta_{i2}),\quad \text{support}\,(\eta_{i2})\cap\text{support}\,(\eta_{j2})=\emptyset,\quad i\neq j.$$

It will be convenient to choose

$$\zeta = 3.$$

This will guarantee that Ψ^0 and W^0 are the main terms in the decomposition of Ψ^* and W^* in the region where the corresponding cut-off functions η_{j2}, η_{j1} are non-zero.

If we insert the expressions of Ψ^* and W^* given by (5.1) and (5.2) in the Euler operator S_1 (see (2.2)) we get

$$S_1(W^*,\Psi^*) = \sum_{j=1}^k \frac{\eta_{1j}}{\varepsilon_j^4}E_j^{in}[\phi_j^*,\psi_j^*\,\psi^{*,out},P] + E^{out}[\phi^{*,out},\psi^{*,out},\phi^{*,in},\psi^{*,in},P], \tag{5.4}$$ fullS10

where

$$\phi^{*,in} = (\phi_1^*,\dots,\phi_k^*),\quad \psi^{*,in}=(\psi_1^*,\dots,\psi_k^*).$$

The inner-operators E_j^{in}, $j = 1, \dots, k$, are defined as

$$
\begin{aligned}
E_j^{in}[\phi_j, \psi_j\, \psi^{out}, P](y,t) &:= |\log\varepsilon|\varepsilon_j^2(1+\frac{\varepsilon_j}{r_j}y_1)\partial_t\phi_j + |\log\varepsilon| B_0(\phi_j)\\
&+\nabla^\perp\left((1+\frac{\varepsilon_j}{r_j}y_1)^2(\psi_j^0 - r_j\alpha_j|\log\varepsilon_j| + \psi_j + r_j\psi^{out}) + \mathcal{R}_j(y,t;P)\right)\cdot\nabla\phi_j\\
&+\nabla^\perp\left((1+\frac{\varepsilon_j y_1}{r_1})^2(\psi_j + r_j\psi^{out})\right)\nabla(w_j^0 + \sum_{i\neq j}\frac{r_j\varepsilon_j^2}{r_i\varepsilon_i^2}w_i^0)\\
&+\varepsilon_j^4 S_1(W^0,\Psi^0)(\varepsilon_j y + P_j), \quad |y| < 3R_j, \quad R_j := \frac{1}{\varepsilon_j|\log\varepsilon|^\varsigma}
\end{aligned}
\tag{5.5}
$$

with $y = \frac{x-P_j}{\varepsilon_j}$, $t\in[0,T)$, and $\mathcal{R}_j$ defined in (4.33). Moreover B_0 is the operator defined in (4.21), which can be equivalently written as

$$
B_0(\phi) = \varepsilon_j\partial_t\varepsilon_j(y_1\partial_1\phi - y_2\partial_2\phi) - \frac{\varepsilon_j^2}{r_j}\partial_t\varepsilon_j y_1 y\cdot\nabla\phi - \frac{\varepsilon_j^2}{r_j}\partial_t z_j y_1\partial_2\phi
$$

see (4.20) and (4.22) for the derivation of B_0, and its equivalent form. The outer-operator E^{out} is given by

$$
\begin{aligned}
&E^{out}[\phi^{out},\psi^{out},\phi^{in},\psi^{in},P](x,t) := |\log\varepsilon|\, r\,\phi_t^{out} \\
&+\nabla_x^\perp(r^2(\Psi^0 + \sum_{j=1}^k\frac{\eta_{j2}}{r_j}\psi_j(\frac{x-P_j}{\varepsilon_j}) + \psi^{out} - r_0^{-1}|\log\varepsilon|))\cdot\nabla_x\phi^{out}\\
&+\sum_{j=1}^k\left[r\,|\log\varepsilon|\,\partial_t\bar\eta_{j1} + \nabla_x^\perp(r^2(\Psi^0 + \sum_{j=1}^k\frac{\eta_{j2}}{r_j}\psi_j(\frac{x-P_j}{\varepsilon_j}) + \psi^{out} - r_0^{-1}|\log\varepsilon|))\nabla\bar\eta_{1j}\right]\frac{\phi_j}{\varepsilon_j^2 r_j}\\
&+\left[\sum_{j=1}^k(\eta_{2j}-\eta_{1j})\nabla_x^\perp(r^2(\frac{\psi_j}{r_j}+\psi^{out})) + \frac{r^2\psi_j}{r_j}\nabla_x^\perp\eta_{2j}\right]\cdot\nabla_x W^0\\
&+(1-\sum_{j=1}^k\eta_{2j})\nabla^\perp(r^2\psi^{out})\cdot\nabla W^0 + (1-\sum_{j=1}^k\eta_{j1})S_1(W^0,\Psi^0) = 0 \quad (x,t)\in\Sigma\times[0,T).
\end{aligned}
\tag{5.6}
$$

If we now insert the expressions of Ψ^* and W^* given by (5.1) and (5.2) in the Euler operator S_2 (see (2.2)) we get

$$
\begin{aligned}
&S_2[\Psi^*,W^*] = S_2[\phi^{*,in},\psi^{*,in},\phi^{*,out},\psi^{*,out},P], \quad \text{with}\\
&S_2[\phi^{in},\psi^{in},\phi^{out},\psi^{out},P] = \sum_{j=1}^k\frac{\eta_{1j}}{r_j\varepsilon_j^2}[\Delta_{5,j}\psi_j + \phi_j] + \Delta_5\psi^{out} + \phi^{out}\\
&\qquad + \sum_{j=1}^k(\eta_{j1}-\eta_{j2})\frac{\Delta_{5,j}\psi_j}{r_j\varepsilon_j^2} + \sum_{j=1}^k(\frac{\psi_j}{r_j}\Delta_5\eta_{j2} + 2\nabla_x\eta_{j2}\nabla_x\frac{\psi_j}{r_j}).
\end{aligned}
\tag{5.7}
$$

Here $\Delta_{5,j}$ is the differential operator in the expanded y-variable defined in (4.12), while Δ_5 is the differential operator defined in the original x-variable as given in (1.6). We require conditions on the boundary and at infinity on $\psi^{*,out}$: for all $t\in[0,T]$

$$
\frac{\partial\psi^{*,out}}{\partial r}(x,t) = 0, \quad \text{on} \quad \partial\Sigma\times[0,T], \quad |\psi^{*,out}(x,t)|\to 0, \qquad \text{as} \quad |x|\to\infty.
$$

We are able to prove that there exist points $P_1^1,\dots,P_k^1$ in (4.2) such that, for any choice of points $\mathbf{a}_1,\dots,\mathbf{a}_k$ in (4.2) satisfying (4.5), it is possible to construct a good approximate leapfrogging of vortex rings (W^*,Ψ^*) with the form (5.1)-(5.2). This is the content of next Proposition.

Proposition 5.1. *There exist constants $\sigma>0$ in (4.4)-(4.5), $\sigma_*>0$, $b>0$, points*

$$
\mathbf{P} = (\mathbf{P}_1,\dots,\mathbf{P}_k), \quad \mathbf{P}_j = P_0 + P_j^0 + P_j^1, \quad P_0 = (r_0, 0)
$$

satisfying (4.4)*, and functions*

$$\phi^{*,in} := (\phi_1^*(y,t),\dots,\phi_k^*(y,t)), \quad \psi^{*,in} := (\psi_1^*(y,t),\dots,\psi_k^*(y,t)), \quad \phi^{*,out}, \quad \psi^{*,out}$$

in (5.1), (5.2) *such that, for any points* $\mathbf{a}$ *satisfying* (4.5)*, the following facts hold. For*

$$P = \mathbf{P} + \mathbf{a},$$

we have

$$E_j^{in}[\phi_j^*, \psi_j^* \, \psi^{*,out}, P](y,t) = \varepsilon_j \nabla^\perp \left[\left(|\log\varepsilon| \partial_t \mathbf{a}_j^\perp + D_x \nabla_x \varphi_j(\mathbf{P}_j; \mathbf{P})[\mathbf{a}] \right) \cdot y \right] \nabla U + \mathcal{E}_j[\phi_j^*, \psi_j^* \, \psi^{*,out}, \mathbf{a}](y,t)$$

where φ_j *is given in* (4.28) *and*

$$\left| \mathcal{E}_j[\phi_j^*, \psi_j^* \, \psi^{*,out}, \mathbf{a}](y,t) \right| = O\left(\frac{\varepsilon^{5-\sigma_*}}{1+|y|^3} \right), \quad |y| < 3R_j, \quad R_j := \frac{1}{\varepsilon_j |\log\varepsilon|^3}$$

for all $j = 1,\dots,k$*, and*

$$\left| E^{out}[\phi^{*,out}, \psi^{*,out}, \phi^{*,in}, \psi^{*,in}, P](x,t) \right| \lesssim \frac{\varepsilon^{4-\frac{\sigma_*}{2}}}{1+|x|^4}, \quad (x,t) \in \Sigma \times [0,T].$$

We refer to (5.5) *and* (5.6) *for the explicit definitions of* E_j^{in} *and* E^{out}*. Besides,*

$$\begin{aligned} (1+|y|)|\nabla\psi_j^*(y,t)| + |\psi_j^*(y,t)| &\lesssim \varepsilon^2 |\log\varepsilon|^b, \quad |y| < 3R_j,\ t \in [0,T] \\ (1+|y|^2)|\phi_j^*(y,t)| &\lesssim \varepsilon^2 |\log\varepsilon|^b, \quad |y| < 3R_j,\ t \in [0,T] \\ (1+|x|^4)|\phi^{*,out}(x,t)| + (1+|x|^2)|\psi^{*,out}(x,t)| &\lesssim \varepsilon^2 |\log\varepsilon|^b, \quad (x,t) \in \Sigma \times [0,T]. \end{aligned}$$

This result is telling that the new approximate solution (W^*, Ψ^*) in (5.1)-(5.2) with the parameter functions considered above produces total errors in (5.4) and (5.7) which can be estimated as follows

$$\begin{aligned} |S_1(W^*, \Psi^*)(x,t)| &\le C\varepsilon^{1-\sigma_*} \sum_{j=1}^k \frac{1}{1+|y_j|^3} + C\frac{\varepsilon^{4-\frac{\sigma_*}{2}}}{1+|x|^4}, \quad y_j = \frac{x - P_j(t)}{\varepsilon}, \\ |S_2(W^*, \Psi^*)(x,t)| &\le C\varepsilon^{4-\sigma} \eta_1 \left(\frac{4|x - (r_0, 0)|}{r_0} \right), \end{aligned}$$

for all $x \in \Sigma$ and $t \in [0,T]$. By construction these errors are also uniformly Lipschitz in the parameter points $\mathbf{a}$. Besides, in combination with (3.17)-(4.13) we get that, for all $j = 1,\dots,k$,

$$\begin{aligned} W^*(x,t;P) &= \frac{1}{r_j \varepsilon_j^2} f\left((1 + \frac{\varepsilon_j}{r_j} y_1)^2 (\psi_j^0 - r_j \alpha_j |\log\varepsilon_j| + \psi_j^*) \right) \left(1 + O(\varepsilon^2)\right), \\ \text{where} \quad f(s) &= 8\, e^{-K(P_j,P_j)}\, \varepsilon_j^{2-\alpha_j r_j}\, e^s, \quad \text{and} \quad \Psi^*(x,t;P) = \psi_j^0 + O(\varepsilon^2 |\log\varepsilon|) \end{aligned} \tag{5.8}$$ d

uniformly in the region $|x - P_j| < |\log\varepsilon|^{-3}$, with $x = P_j + \varepsilon_j y$. The definition of ψ_j^0 is given in (4.10), see also (4.26). We also have

$$\begin{aligned} (1 + \frac{\varepsilon_j}{r_j} y_1)^2 (\psi_j^0 - r_j \alpha_j |\log\varepsilon_j| + \psi_j^*) &= \Gamma_0(y) - (4 - \alpha_j r_j)\log\varepsilon_j - \log 8 + K(P_j; P_j) \\ &\quad + \frac{\varepsilon_j y_1}{2r_j} \left(\Gamma_0(y) + \bar{A} + \Gamma(y) \right) + b_j^{**}(y,t) \end{aligned} \tag{5.9}$$ barp00n

where

$$|\log\varepsilon|^{\frac12} |\partial_t b_j^{**}(y,t)| + (1+|y|)\,|\nabla_y b_j^{**}(y,t)| + |b_j^{**}(y,t)| \le C\varepsilon^2 \,(1+|y|^2)\, \log(2+|y|)$$

for $y = \frac{x-P_j}{\varepsilon_j}$, $|y| < |\log\varepsilon|^{-3}$. Next section will be devoted to build this approximate solution and to prove Proposition 5.1.

6. Improvement of the approximation: Proof of Proposition 5.1

⟨sec5⟩ The approximate solution (W^*, Ψ^*) predicted by Proposition 5.1 is constructed improving the inner errors E_j^{in} defined in (5.5) ten successive times, and improving the outer error E^{out} defined in (5.6) once.

It is useful to have at hand a more explicit expression of the inner-operators E_j^{in} in (5.5). The key observation is that in the region $|x - P_j| < 3|\log\varepsilon|^{-\zeta}$, we have, for $y = \frac{x-P_j}{\varepsilon_j}$,

$$
\begin{aligned}
(1 + \frac{\varepsilon_j}{r_j} y_1)^2 (\psi_j^0 - r_j\alpha_j |\log\varepsilon_j|) + \mathcal{R}_j(y,t;P)& \\
= \Gamma_0(y) - (4 - \alpha r_0)\log\varepsilon_j - \log 8 + K(P_j;P_j)& \\
+ \frac{\varepsilon_j y_1}{2r_j}\Big(\Gamma_0(y) + \bar{A} + \Gamma(y)\Big) + \mathcal{R}_j^0(y,t;P), \quad \text{where}& \qquad (6.1)\ \texttt{newR}\\
\mathcal{R}_j^0(y,t;P) = \varepsilon_j |\log\varepsilon| \partial_t (P_j - P_j^0)^\perp \cdot y + \varepsilon_j [\nabla_x \varphi_j(P_j;P) - \nabla_x\varphi_j(\bar{P}_j^0;\bar{P}^0)]\cdot y& \\
+ \varepsilon_j^3 D_x \theta_{j\varepsilon_j}(P_j;P)[y] + \varepsilon^2 |\log\varepsilon| \mathcal{Q}_1[P](y) + \varepsilon^3 |\log\varepsilon|^{\frac{3}{2}} \mathcal{Q}_2[P](y)&
\end{aligned}
$$

with $\bar{P}_j^0 = P_0 + P_j^0$, $\bar{P}^0 = (\bar{P}_1^0, \dots, \bar{P}_k^0)$, $\mathcal{Q}_1$, $\mathcal{Q}_2$ denote functions of the form, for $y = \rho e^{i\theta}$,

$$
\begin{aligned}
\mathcal{Q}_1[P](y) &= \Big[a(P) + b_1(P)\cos 2\theta + b_2(P)\sin 2\theta\Big] O(|y|^2) \\
\mathcal{Q}_2[P_0](y) &= a(P) O(|y|^3),
\end{aligned}
\qquad (6.2)\ \texttt{calQ}
$$

for a, b_1 and b_2 smooth functions of the points P, which are uniformly bounded as $\varepsilon \to 0$. Also: $\bar{A}$ is the explicit constant defined by (3.14) and $\Gamma = \Gamma(y)$ is the function introduced in (3.7).

Proof of (6.1). In order to prove (6.1), we combine an expansion of the term $(1+\frac{\varepsilon_j}{r_j}y_1)^2(\psi_j^0 - r_j\alpha_j|\log\varepsilon_j|)$ following the lines to get (3.13). From (4.29) and (4.33) we get

$$
\begin{aligned}
\mathcal{R}_j(y,t;P) &= \varepsilon_j |\log\varepsilon| \partial_t (P_j - P_j^0)^\perp \cdot y + \varepsilon_j [\nabla_x \varphi_j(P_j;P) - \nabla_x\varphi_j(\bar{P}_j^0;\bar{P}^0)]\cdot y \\
&+ \varepsilon_j^3 D_x \theta_{j\varepsilon_j}(P_j;P)[y] + \sum_{k=2}^{4} \frac{\varepsilon_j^k}{k!} D_x^k \varphi_j(P_j;P)[y]^k \\
&+ \frac{\varepsilon_j^4}{2} D_x^2 \theta_{j\varepsilon_j}(P_j;P)[y]^2 + Q(\varepsilon_j y, P)
\end{aligned}
$$

Under our assumptions (4.2), (4.4) and (4.5) on the form and size of the points P_j, we have that

$$
\begin{aligned}
\mathcal{R}_j(y,t;P) &= \varepsilon_j |\log\varepsilon| \partial_t (P_j - P_j^0)^\perp \cdot y + \varepsilon_j [\nabla_x \varphi_j(P_j;P) - \nabla_x\varphi_j(\bar{P}_j^0;\bar{P}^0)]\cdot y \\
&+ \varepsilon_j^3 D_x \theta_{j\varepsilon_j}(P_j;P)[y] + \varepsilon^2 |\log\varepsilon| \mathcal{Q}_1[P](y) + \varepsilon^3 |\log\varepsilon|^{\frac{3}{2}} \mathcal{Q}_2[P](y)
\end{aligned}
\qquad (6.3)\ \texttt{expRj}
$$

where $\mathcal{Q}_1$, $\mathcal{Q}_2$ satisfy (6.2). This fact gives (6.1).

□

⟨strategy⟩ 6.1. **Strategy for the improvement.** Replacing (6.1) in (5.5) and using (4.6) we re-write the inner operator E_j^{in} (see (5.5)) as

$$
\begin{aligned}
E_j^{in}[\phi_j, \psi_j, \psi^{out}, P](y,t) := L_1(\phi_j) + L_2(\phi_j) + L_3(\psi_j) + Q(\psi_j,\phi_j) + L_3(r_j\psi^{out}) + Q(\psi^{out},\phi_j)& \\
+ \varepsilon_j^4 S_1(W^0,\Psi^0)(\varepsilon_j y + P_j),& \qquad (6.4)\ \texttt{Ej1}
\end{aligned}
$$

where

$$
\begin{aligned}
L_1(\phi) &:= |\log\varepsilon|\, \varepsilon_j^2 (1 + \frac{\varepsilon_j}{r_j} y_1)\partial_t\phi + |\log\varepsilon| B_0(\phi) \\
L_2(\phi) &:= \nabla^\perp \left(\Gamma_0 + \frac{\varepsilon_j}{2r_j} y_1 \Big(\Gamma_0(y) + \bar{A} + \Gamma(y)\Big) + \mathcal{R}_j^0(y,t;P)\right)\cdot\nabla\phi \\
L_3(\psi) &:= \nabla^\perp ((1 + \frac{\varepsilon_j y_1}{r_1})^2\psi)\cdot\nabla(w_j^0 + \sum_{i\neq j} w_i^0) \\
Q(\psi,\phi) &:= \nabla^\perp \left((1 + \frac{\varepsilon_j}{r_j} y_1)^2\psi\right)\cdot\nabla\phi.
\end{aligned}
\qquad (6.5)\ \texttt{l1}
$$

We recall the definition of the operator B_0 in (4.21).

The strategy to improve the inner error requires ten consecutive adjustments, which are built upon three different types of mechanisms. The choice of the mechanism to use is dictated by the form of the part of the error we aim at removing.

Suppose the error $E(y,t)$, with $y=\rho e^{i\theta}$, has a Fourier decomposition in the θ-variable given by

$$E(y,t)=E(\rho e^{i\theta},t)=\sum_{n\in\mathbb{Z}}E_n(\rho,t)e^{i\,n\,\theta},\quad E_n(\rho,t)=\int_0^{2\pi}E(\rho e^{in\theta})e^{in\theta}\,d\theta.$$

We call E_n the $n-$th mode in the Fourier decomposition of E.

The elliptic improvement. If the part of the error we want to remove has no 0-th mode in its Fourier decomposition then we will solve using the elliptic operator

$$L[\psi]\,:=\,\nabla^{\perp}\Gamma_0\cdot\nabla\phi+\nabla^{\perp}\psi\cdot\nabla U,\quad \phi=-\Delta_{5,j}\psi.$$

You recover $L[\psi]$ from $L_2(\phi_j)+L_3(\psi_j)$ in (6.5) just formally taking $\varepsilon_j=0$, $\omega_j^0=U$ and $\omega_i^0=0$ for $i\neq j$. In other words, $L[\psi]$ selects the main terms in $L_2(\phi)$ and $L_3(\psi)$. Using the fact that $-\Delta\Gamma_0=f_0(\Gamma_0)=U$ where $f_0(u)=e^u$ and $-\Delta_{5,j}\psi=\phi$ we see that

$$L[\psi]\;=-\,\nabla^{\perp}\Gamma_0\cdot\nabla[\Delta_y\psi+\frac{3\varepsilon_j}{r_j+\varepsilon_jy_1}\partial_{y_1}\psi+f_0'(\Gamma_0)\psi].$$

In polar coordinates $y=\rho e^{i\theta}$ in $\mathbb{R}^2$, we check that

$$L[\psi]=-\frac{4}{\rho^2+1}\frac{\partial}{\partial\theta}[\Delta_y\psi+\frac{3\varepsilon_j}{r_j+\varepsilon_jy_1}\partial_{y_1}\psi+f_0'(\Gamma_0)\psi].$$

It is enough to use the simplified version of this operator given by

$$L_0[\psi]:=-\frac{4}{\rho^2+1}\frac{\partial}{\partial\theta}[\Delta_y\psi+f_0'(\Gamma_0)\psi],\quad \phi=-\Delta_{5,j}\psi. \tag{6.6}$$ gadd

For terms in the error of Fourier mode 1 or higher, we will solve with the elliptic operator in (6.6). When the error has mode 1, it will be possible to solve only under certain orthogonality conditions, which requires a proper adjustment of the points P_j^1. We call this procedure the *elliptic improvement*. We will discuss the solvability and a-priori bounds for problems of the form $L_0(\psi)=E$ in Lemma 6.1.

The ODEs improvement. If the part of inner error $E(y,t)$ we want to remove has 0-th mode in its Fourier decomposition, we will solve

$$|\log\varepsilon|\,\varepsilon_j^2\,\partial_t\phi=E,\quad \phi(0,\cdot)=0.$$

This is the main part of $L_1(\phi)$ in (6.5). We can solve the above ODEs at the expenses of losing two power of ε_j (and gaining one power of $|\log\varepsilon|$) in the estimates for the solution. We will see that the construction is decided by powers of ε, in the sense that the exact number of powers of $|\log\varepsilon|$ we gain will be easily absorbed in a slightly smaller power of ε. We call this procedure the *ODEs improvement*. It has the advantage of keeping track of the Fourier modes of the solutions.

The transport improvement. After the fifth inner improvement, we will need to reduce not only the size of the error, expressed in terms of powers of ε, but also its decay rate in the $\rho=|y|$-variable. In this case we will solve the transport-type equation

$$L_1(\phi)+L_2(\phi)=E.$$

We call this procedure the *transport improvement*. A-priori estimates for solutions of this transport-type equation are contained in Lemma 6.2. After solving this linear equation, we will have lost control on the Fourier modes of the solution ϕ, but the structure of the problem will automatically give a new error whose main term has no 0-th mode in its Fourier decomposition. This will be crucial for the scheme of improvement to work.

We summarize the complete process of improvement in the following diagram: for $\rho=\frac{|x-P_j|}{\varepsilon_j}$

$$e_1=\frac{\varepsilon^2|\log\varepsilon|}{1+\rho^4}E_2\underset{\mathcal{E}}{\Rightarrow}e_2=\frac{\varepsilon^3|\log\varepsilon|^2}{1+\rho^3}E_1\underset{\mathcal{E}\&P^1}{\Rightarrow}e_3=\frac{\varepsilon^4|\log\varepsilon|^4}{1+\rho^2}E_0\underset{\mathcal{O}DEs}{\Rightarrow}e_4=\frac{\varepsilon^3|\log\varepsilon|^6}{1+\rho^3}E_{12}\sin\theta\underset{\mathcal{E}\&P}{\Rightarrow}$$

$$e_5=\frac{\varepsilon^4|\log\varepsilon|^8}{1+\rho^2}E_2 \underset{\mathcal{E}}{\Rightarrow} e_6=\frac{\varepsilon^5|\log\varepsilon|^8}{1+\rho}E_0 \underset{\mathcal{T}}{\Rightarrow} e_7=\frac{\varepsilon^3|\log\varepsilon|^8}{1+\rho^5}E_1 \underset{\mathcal{O}ut}{\Rightarrow} e_8=\frac{\varepsilon^3|\log\varepsilon|^b}{1+\rho^5}E_1 \underset{\mathcal{E}\&P^1}{\Rightarrow}$$

$$e_9=\frac{\varepsilon^4|\log\varepsilon|^b}{1+\rho^4}E_0 \underset{\mathcal{O}DEs}{\Rightarrow} e_{10}=\frac{\varepsilon^3|\log\varepsilon|^b}{1+\rho^5}E_{12}\sin\theta \underset{\mathcal{E}\&P^1}{\Rightarrow} e_{11}=\frac{\varepsilon^4|\log\varepsilon|^b}{1+\rho^4}E_2 \underset{\mathcal{E}}{\Rightarrow} e_{12}=\frac{\varepsilon^5|\log\varepsilon|^b}{1+\rho^3}E_0.$$

Here $\mathcal{E}$ stands for *inner elliptic improvement*, $\mathcal{E}\&P^1$ stands for *inner elliptic improvement with adjustment of the points* P^1, $\mathcal{T}$ stands for *inner transport improvement*, $\mathcal{O}DEs$ stands for *inner ODEs improvement*, and $\mathcal{O}ut$ stands for *outer improvement*. Besides b is a positive number whose value may change from line to line and within the same line.

Let us explain how to interpret the diagram: we start with an initial error of which we aim at eliminating the part of size $\varepsilon^2|\log\varepsilon|$, decay in space $\frac{1}{1+\rho^4}$ and Fourier mode 2: we write it as $e_1=\frac{\varepsilon^2|\log\varepsilon|}{1+\rho^4}E_2$. To do so, we proceed with the inner elliptic improvement: we write $\underset{\mathcal{E}}{\Rightarrow}$. After this correction is done, we have a new error. Of the new error we aim now at eliminating the main term, which has size $\varepsilon^3|\log\varepsilon|^2$, decay in space $\frac{1}{1+\rho^3}$ and Fourier mode 1: we write $e_2=\frac{\varepsilon^3|\log\varepsilon|^2}{1+\rho^3}E_1$. And so on. Notice that the errors e_4 and e_{10} has mode 1, but only with $\sin\theta$ (mode 1, odd in y_2).

Before starting the process of improvement of the approximation, we observe that the expression for the initial error $\varepsilon_j^4 S_1(W^0,\Psi^0)(P_j+\varepsilon_j y)$ in (4.31) gets a simpler form if we use (6.1). Using (6.3) and (3.20), we write

$$\begin{aligned}
&\mathcal{R}_j(y,t;P)=\mathcal{R}_j^{00}(y,t;P)+\varepsilon^2|\log\varepsilon|\mathcal{Q}_1[P](y)+\varepsilon^3|\log\varepsilon|^{\frac32}\mathcal{Q}_2[P](y)\\
&\mathcal{R}_j^{00}(y,t;P):=\varepsilon_j|\log\varepsilon|\partial_t(P_j-P_j^0)\cdot y+\varepsilon_j[\nabla_x\varphi_j(P_j;P)-\nabla_x\varphi_j(P_j^0;P^0)]\cdot y\\
&\qquad\qquad +\varepsilon_j^3 D_x\theta_{j\varepsilon_j}(P_j;P)[y],
\end{aligned} \tag{6.7}$$
R00

and

$$w_j^0=U(y)\left(1+\frac{\varepsilon_j y_1}{2r_j}(\Gamma_0+\bar A+\Gamma)+\varepsilon^2|\log\varepsilon|\mathcal{Q}_1[P](y)+\varepsilon^3|\log\varepsilon|^{\frac32}\mathcal{Q}_2[P](y)\right),$$

with $\mathcal{Q}_i$, $i=1,2$ as in (6.2). Under the constraints (4.2)-(4.3)-(4.4)-(4.5) on the points P_j, we get

$$\begin{aligned}
\mathcal{R}_j(y,t;P)\cdot\nabla w_j^0&=\mathcal{R}_j^{00}(y,t;P)\cdot\nabla U+\frac{\varepsilon^2|\log\varepsilon|}{1+|y|^4}E_2(\rho,\theta,t,\varepsilon)\\
&\quad+\frac{\varepsilon^3|\log\varepsilon|^2}{1+|y|^3}(E_1+E_3)(\rho,\theta,t,\varepsilon)+O\left(\frac{\varepsilon^4|\log\varepsilon|^2}{1+|y|^2}\right);
\end{aligned}$$

hence $\varepsilon_j^4 S_1(W^0,\Psi^0)(P_j+\varepsilon_j y)$ in (4.31) becomes

$$\begin{aligned}
\varepsilon_j^4 S_1(W^0,\Psi^0)(P_j+\varepsilon_j y)&=\nabla^\perp\mathcal{R}_j^{00}(y,t;P)\cdot\nabla U+\frac{\varepsilon^2|\log\varepsilon|}{1+|y|^4}E_2(\rho,\theta,t,\varepsilon)\\
&\quad+\frac{\varepsilon^3|\log\varepsilon|^2}{1+|y|^3}(E_1+E_3)(\rho,\theta,t,\varepsilon)+O\left(\frac{\varepsilon^4|\log\varepsilon|^2}{1+|y|^2}\right)
\end{aligned} \tag{6.8}$$
error111

as $\varepsilon\to0$, where the functions $E_i(\rho,\theta,t,\varepsilon)$ now depends also on $\partial_t P$, and have the form (4.32). These estimates are valid for $|y|<3\varepsilon_j^{-1}|\log\varepsilon|^{-\zeta}$.

We are now ready to start improving. At each step of improvement we analyze the term of the error we want to remove, we describe the strategy to do it and we compute the new error produced by the correction.

⟨prima⟩ 6.2. **First inner improvement.** The first improvement of the error will remove part of the mode-2 term of size $\varepsilon^2|\log\varepsilon|$ given by

$$\frac{\varepsilon^2|\log\varepsilon|}{1+|y|^4}E_2(\rho,\theta,t,\varepsilon)$$

in the error $\varepsilon_j^4 S_1(W^0,\Psi^0)(\varepsilon_j y+P_j)$ computed in (6.8). This term can be decomposed as the sum of two parts, one depending on $\partial_t P$ and one not:

$$\frac{\varepsilon^2|\log\varepsilon|}{1+|y|^4}E_2 = \frac{\varepsilon^2|\log\varepsilon|}{1+|y|^4}E_2^*[P] + \frac{\varepsilon^2|\log\varepsilon|}{1+|y|^4}E_2^{**}[P,\partial_t P].$$

In fact the origin of $\frac{\varepsilon^2|\log\varepsilon|}{1+|y|^4}E_2^{**}[P,\partial_t P]$ is $\varepsilon_j^2|\log\varepsilon|(1+\frac{\varepsilon_j}{r_j}y_1)\partial_t w_j^0$, $|\log\varepsilon|B_0(w_j^0)$ and $\mathcal{R}_j(y,t;P)\cdot\nabla w_j^0$, and the dependence on $\partial_t P$ is linear. We check that

$$\frac{\varepsilon^2|\log\varepsilon|}{1+|y|^4}E_2^{**}[P,\partial_t P] = \frac{\varepsilon^2|\log\varepsilon|}{1+|y|^4}E_2^{**}[r_0\mathbf{e}_1+P^0+P^1,\partial_t(P^0+P^1)] + \frac{\varepsilon^5|\log\varepsilon|}{1+|y|^4}E_2$$

with E_2 satisfying (4.32). See (4.2) for the assumptions on P, P^0 and P^1. We will remove the part of the error $\frac{\varepsilon^2|\log\varepsilon|}{1+|y|^4}E_2$ given by

$$e_1 := \frac{\varepsilon^2|\log\varepsilon|}{1+|y|^4}E_2^*[P] + \frac{\varepsilon^2|\log\varepsilon|}{1+|y|^4}E_2^{**}[(r_0,0)+P^0+P^1,\partial_t(P^0+P^1)], \tag{6.9}$$ e2

thus leaving out what depends on $\partial_t\mathbf{a}$. We will do it solving the simplified linear elliptic operator

$$L_0[\psi] := -\frac{4}{\rho^2+1}\frac{\partial}{\partial\theta}[\Delta_y\psi + f_0'(\Gamma_0)\psi].$$

introduced in (6.6). We freeze the time variable and consider the problem

$$\begin{aligned} &\frac{4}{1+\rho^2}\frac{\partial}{\partial\theta}\left[\Delta\psi + e^{\Gamma_0(y)}\psi\right] + g(y) = 0 \quad \text{in } B(0,\frac{8}{\varepsilon|\log\varepsilon|^\varsigma}),\\ &\psi = 0 \quad \text{in } \partial B(0,\frac{8}{\varepsilon|\log\varepsilon|^\varsigma}), \quad \phi = -\Delta_{5,j}\psi, \end{aligned} \tag{6.10}$$ linear1

for a bounded function $g: B(0,\frac{8}{\varepsilon|\log\varepsilon|^\varsigma})\subset\mathbb{R}^2\to\mathbb{R}$. A necessary condition for the solvability of (6.10) is that

$$\int_0^{2\pi} g(\rho e^{i\theta})\,d\theta = 0 \quad \text{for all} \quad \rho\in(0,8R_\varepsilon), \quad R_\varepsilon = \frac{1}{\varepsilon|\log\varepsilon|^\varsigma}. \tag{6.11}$$ ort0

As

$$L_0[Z_\ell]=0, \quad Z_\ell(y)=\partial_{y_\ell}\Gamma_0(y).$$

we also assume the orthogonality conditions

$$\int_{B(0,8R_\varepsilon)}(1+|y|^2)\,g(y)\,Z_\ell(y)\,dy \;=\; 0, \quad \ell=1,2. \tag{6.12}$$ ort2

We have the validity of the following result.

⟨alpha⟩ **Lemma 6.1.** *Assume that* $3\le m\le 5$ *and*

$$|g(y)| \;\le\; (1+|y|)^{-m} \tag{6.13}$$ decay

There exists a constant $C>0$ *such that for all* $\varepsilon>0$ *sufficiently small and* $g\in L^\infty(B(0,8R_\varepsilon))$ *that satisfies conditions* (6.11), (6.12) *and* (6.13), *there exists a unique solution* (ψ,ϕ) *of equation* (6.10) *that satisfies*

$$\int_0^{2\pi}\psi(\rho e^{i\theta})\,d\theta = 0 \quad \textit{for all} \quad \rho\in(0,8R_\varepsilon)$$

and the estimate

$$\begin{aligned} &|\psi(y)| \;+\; (1+|y|)\,|\nabla\psi(y)| \;+\; (1+|y|^2)\,|\phi(y)| \\ &\le\; C(1+|y|)^{4-m}\begin{cases}\log\left(\frac{16\varepsilon^{-1}}{|y|+1}\right) & \textit{if} \quad m=5\\ 1 & \textit{if} \quad 3<m<5\\ \log\left(\frac{16\varepsilon^{-1}}{|y|+1}\right) & \textit{if} \quad m=3\end{cases}\end{aligned}$$

Proof. Let $y = \rho\, e^{i\theta}$ and decompose ψ and g in Fourier series in the θ-variable

$$g(\rho e^{i\theta}) = \sum_{n\in\mathbb{Z}} g_n(\rho) e^{i\,n\,\theta}, \quad \psi(\rho e^{i\theta}) = \sum_{n\in\mathbb{Z}} p_n(\rho) e^{i\,n\,\theta}.$$

Condition (6.11) amounts to $g_0 \equiv 0$. Imposing $p_0 \equiv 0$, equation (6.10) decouples into the infinitely many problems.

$$\mathcal{L}_n[p_n] := \partial_\rho^2 p_n + \frac{1}{\rho}\partial_\rho p_n - \frac{n^2}{\rho^2} p_n + \frac{8p_n}{(1+\rho^2)^2} = \frac{i(1+\rho^2)}{4n} g_n(\rho), \quad p_n(8R_\varepsilon) = 0. \tag{6.14}$$ pk

For each $n \neq 0$, there exists a positive function $\zeta_n(\rho)$ such that $\mathcal{L}_n[\zeta_n] = 0$ and

$$\zeta_n(\rho) = \rho^{|n|}(1 + o(1)) \quad \text{as } \rho \to 0$$

For $n = \pm 1$ we explicitly have $\zeta_n(\rho) = \frac{\rho}{1+\rho^2}$, while for $|n| \geq 2$ we have

$$\zeta_n(\rho) = \rho^{|n|}(1 + o(1)) \quad \text{as } \rho \to +\infty.$$

Problem (6.14) is uniquely solved by the formula

$$p_n(\rho) = \mathcal{L}_n^{-1}[g_n] := \frac{i}{4n}\zeta_n(\rho) \int_\rho^{8R_\varepsilon} \frac{dr}{r\zeta_n(r)^2} \int_0^r (1+s^2) g_n(s)\zeta_n(s)\, s\, ds.$$

Let us consider the case $|n| \geq 2$. We have

$$|p_n(\rho)| \ \leq\ \mathcal{L}_n^{-1}[(1+\rho)^{-\alpha}] \leq c_n (1+\rho)^{|n|} \int_\rho^{8R_\varepsilon} (1+r)^{-|n|+3-\alpha} dr$$

so that

$$|p_n(\rho)| \ \leq\ c_n (1+\rho)^{4-\alpha}$$

since $m + |n| > 4$. Let us denote $\bar{P}(\rho) = \mathcal{L}_2^{-1}[(1+\rho)^{\alpha}]$ we claim that for some $\gamma > 0$ and all $|n| \geq 2$ we have the validity of the estimate

$$|p_n(\rho)| \ \leq\ \frac{\gamma}{n^3}\bar{P}(\rho).$$

That follows from the fact that the right hand side defines a positive supersolution for the real and imaginary parts of (6.14). Indeed, if γ is taken sufficiently large we get

$$\mathcal{L}_n\big[\frac{\gamma}{n^3}\bar{P}(\rho)\big] + \frac{1+\rho^2}{4n}|g_n(\rho)| \leq \gamma \frac{4-n^2}{10\rho^2}(1+\rho)^{4+\alpha}$$

Fourier modes ± 1 need to be separately treated because $\zeta_1(\rho)$ decays at infinity. At this point we observe that

$$Z_1(y) = \zeta_1(\rho)\cos\theta, \quad Z_2(y) = \zeta_1(\rho)\sin\theta$$

and that the orthogonality conditions (6.12) assumed are equivalent to

$$\int_0^{8R_\varepsilon} (1+\rho^2)\, g_{\pm 1}(\rho)\zeta_1(\rho)\,\rho\, d\rho = 0.$$

Therefore we can write

$$p_{\pm 1}(\rho) = \mp\frac{i}{4}\zeta_1(\rho) \int_\rho^{8R_\varepsilon} \frac{dr}{r\zeta_1(r)^2} \int_r^{8R_\varepsilon} (1+s^2) g_{\pm 1}(s)\zeta_1(s)\, s\, ds.$$

and we obtain, if we now assume $3 < m \leq 5$,

$$|p_{\pm 1}(\rho)| \ \lesssim\ \begin{cases} (1+\rho)^{-1}\log\frac{16R_\varepsilon}{\rho+1} & \text{if } m = 5 \\ (1+\rho)^{4-m} & \text{if } 3 < m < 5 \\ (1+\rho)\log\frac{16R_\varepsilon}{\rho+1} & \text{if } m = 3. \end{cases}$$

The desired result then follows from addition of the above estimates since

$$|\psi(y)| \ \leq\ \sum_{n\in\mathbb{Z}} |p_n(|y|)|.$$

Finally, since ψ satisfies the equation

$$\Delta\psi + e_0^\Gamma \psi = -\frac{1}{4}(1+\rho^2)\int_0^\theta g(\rho,\theta)\, d\theta, \quad \text{in } B(0, 8R_\varepsilon), \quad \psi = 0 \quad \text{on } \partial B(0, 8R_\varepsilon),$$

the bounds for ϕ and $\nabla\psi$ follow from standard elliptic estimates.

□

We extend the function e_1 in (6.9) to be equal to 0 outside $B(0,3R_j)$. Under the assumptions (4.2) on P, P^0 and P^1, we have that

$$(1+|y|^4)\left(|e_1(y,t)|+|\log\varepsilon|^{\frac12}|\partial_t e_1(y,t)|\right)\lesssim \varepsilon^2|\log\varepsilon|.$$

Moreover the function e_1 satisfies authomatically the orthogonality conditions (6.11) and (6.12). Let $\psi_j^1=\psi_j^1(y,t)$ and $\phi_j^1(y,t)=-\Delta_{5,j}\psi_j^1$ be the solution to (6.10) when $g=e_1$, as predicted by Lemma 6.1. We have

$$|\psi_j^1(y,t)|\,+\,(1+|y|)\,|\nabla\psi_j^1(y,t)|\,\lesssim\varepsilon^2|\log\varepsilon|,\qquad \text{in } B(0,3R_j)\times[0,T].$$

and

$$(1+|y|^3)\,|\nabla_y\phi_j^1(y,t)|+\,(1+|y|^2)\,|\phi_j^1(y,t)|\lesssim\varepsilon^2|\log\varepsilon|\qquad \text{in } B(0,3R_j)\times[0,T].$$

Having left out what depends on $\partial_t\mathbf{a}$ in the part of the error e_1 we are considering (see (6.9)), we can harmlessly differentiate in time equations (6.10), to also get

$$(1+|y|^2)\,|\partial_t\phi_j^1(y,t)|\lesssim\varepsilon^2|\log\varepsilon|^{\frac12}\qquad \text{in } B(0,3R_j)\times[0,T].$$

Besides, a consequence of the proof of Lemma 6.1 is that the Fourier decomposition of ψ_j^1 only contains mode-2 terms. Hence the Fourier decomposition of the function $\phi_j^1(y,t)$ has a mode-2 term of size $\varepsilon_j^2|\log\varepsilon|$, mode-1 and mode-3 terms of size $\varepsilon_j^3|\log\varepsilon|$ or smaller.

Using these properties of the functions ψ_j^1, ϕ_j^1, and the notations introduced in (6.5) we get

$$L_1(\phi_j^1)=\frac{\varepsilon_j^4|\log\varepsilon|^{\frac32}}{1+|y|^2}E_2+\frac{\varepsilon_j^5|\log\varepsilon|^{\frac32}}{1+|y|}E_1,$$

$$\nabla^\perp\Gamma_0\cdot\nabla[\frac{3\varepsilon_j}{r_j+\varepsilon_j y_1}\partial_{y_1}\psi_j^1]=\frac{\varepsilon_j^3|\log\varepsilon|}{1+|y|^3}(E_1+E_3)+\frac{\varepsilon_j^4|\log\varepsilon|}{1+|y|^2}(E_0+E_2),$$

and

$$L_2[\phi_j^1]+L_3[\psi_j^1]-L[\psi_j^1]+Q(\psi_j^1,\phi_j^1)=\frac{\varepsilon_j^3|\log\varepsilon|^2}{1+|y|^3}(E_1+E_3)+\frac{\varepsilon_j^4|\log\varepsilon|^2}{1+|y|^2}E_{i\geq0}.$$

In the above expression E_j denotes the j-th mode in the Fourier decomposition of $E(y,t)$, in accordance with the notation introduced in (3.11). Inserting this information in (6.4) we get the description of the new error, in the region $|x-P_j|<3|\log\varepsilon|^{-\zeta}$, for $y=\frac{x-P_j}{\varepsilon_j}$,

$$\begin{aligned}E_j^{in}[\phi_j^1,\psi_j^1,0,P](y,t):=&\nabla^\perp\mathcal{R}_j^{00}(y,t;P)\cdot\nabla U+\frac{\varepsilon_j^3|\log\varepsilon|^2}{1+|y|^3}(E_1+E_3)\\&+\frac{\varepsilon_j^4|\log\varepsilon|^2}{1+|y|^2}(E_0+E_2)+\frac{\varepsilon_j^5|\log\varepsilon|^2}{1+|y|}E_{i\geq0},\end{aligned}$$

as $\varepsilon\to0$, where the functions $E_j(\rho,\theta,t,\varepsilon)$ have the form described in (4.32). For our next improvement it is convenient to write the error as a term of size $\varepsilon_j^3|\log\varepsilon|^2$-size in Fourier mode 1 or higher, with spacial decay bounded by $\frac{1}{1+|y|^3}$, and that does depend on P, $\partial_t P^0$, but not on $\partial_t P^1+\partial_t\mathbf{a}$. We use the fact that the error depends in a linear way on $\partial_t P$ and assumptions (4.4)-(4.5) to conclude that

$$E_j^{in}[\phi_j^1,\psi_j^1,0,P](y,t):=\nabla^\perp\mathcal{R}_j^{00}(y,t;P)\cdot\nabla U+\frac{\varepsilon^3|\log\varepsilon|^2}{1+|y|^3}(E_1+E_3)[P,\partial_tP^0]++O\left(\frac{\varepsilon^4|\log\varepsilon|^2}{1+|y|^2}\right)\quad(6.15)\ \text{Ej2}$$

6.3. **Second inner improvement.** Our next step is to eliminate the terms in the error (6.15) of size $\varepsilon^3|\log\varepsilon|^2$. A difference with the first improvement is that this time these terms posses a mode 1. We will solve again an elliptic problem of the form (6.10), but only at the expenses of asking that the orthogonality conditions (6.12) for the right-hand side are satisfied. We shall see that this is possible with an adjustment of the points P.

In the process of the construction of the approximation (Ψ^*, W^*), we will need to correct these points several times. All these corrections are encoded in the point we called P^1 in (4.2)-(4.4). The final definition of P^1 is given by

$$P^1(t) = \sum_{\ell=1}^{4} P^{1\ell}(t). \tag{6.16}$$ P1

where $P^{1\ell}$ correspond to successive explicit adjustments of P^1. Take $\ell = 1$, for the first adjustment, and for

$$\bar P^1 = (\bar P_1^1, \dots, \bar P_k^1), \quad \bar P_j^1 = (r_0, 0) + P_j^0 + P_j^{11},$$

we take

$$e_2(y,t) := \nabla^{\perp}\mathcal{R}_j^{00}(y,t;\bar P^1)\cdot\nabla U + \frac{\varepsilon_j^3|\log\varepsilon|^2}{1+|y|^3}(E_1+E_3)[\bar P^1, \partial_t P^0](\rho,\theta,t,\varepsilon).$$

We directly check that $\int_0^{2\pi} e_2(\rho e^{i\theta},t)\,d\theta = 0$, and e_2 satisfies the decay (6.13) with $m=3$. In fact, under the assumptions (4.2) on P, we have

$$(1+|y|^3)(|e_2(y,t)| + |\log\varepsilon|^{\frac12}|\partial_t e_2(y,t)|) \lesssim \varepsilon^3|\log\varepsilon|^2, \quad |y| < 3R_j.$$

The orthogonality conditions

$$\int_{B_{8R_j}} (1+|y|^2)e_2(y,y)Z_\ell(y)\,dy = 0, \quad \ell = 1,2$$

become a system of ODEs for the point $P^{11} = (P_1^{11}, \dots, P_k^{11})$ that has the form

$$\varepsilon_j|\log\varepsilon|\left[\partial_t P_j^{11} + AP^{11} + |\log\varepsilon|^{-\frac12}B(P^{11})\right] = \varepsilon^3|\log\varepsilon|^3 f(\bar P^1) \tag{6.17}$$ reduced

where A is the 2×2 matrix defined by

$$A = |\log\varepsilon|^{-1}D_x^2\varphi_j(\bar P_j^0; P^0), \quad t\in[0,T),$$

and B

$$B(P^{11}) = |\log\varepsilon|^{-1}[\nabla_x\varphi_j(\bar P_j^1;\bar P^1) - \nabla_x\varphi_j(P_j^0;P^0) - D_x^2\varphi_j(\bar P_j^0;P^0)P^{11}].$$

See (4.28) for the definition of φ_j. Under our assumptions (4.3) on the points P^0, we have

$$A = O(1), \quad B(P^{11}) = (|\log\varepsilon|^{-\frac12}), \quad t\in[0,T),$$

uniformly as $\varepsilon\to 0$. Besides, $f(\bar P^1)$ is a smooth function of P^{11}, which is uniformly bounded, together with its derivative, for $t\in[0,T)$, uniformly as $\varepsilon\to 0$. Standard ODEs theory gives that, for all $\varepsilon > 0$ small enough there exists a unique solution P^{11} to (6.17) with initial condition $P^{11}(0) = 0$, which satisfy the bounds

$$\|P_j^{11}\|_{L^\infty[0,T)} + \|\partial_t P_j^{11}\|_{L^\infty[0,T)} \lesssim \varepsilon^2|\log\varepsilon|^2.$$

We denote by $\psi_j^2 = \psi_j^2(y,t)$ and $\phi_j^2(y,t) = -\Delta_{5,j}\psi_j^2$ the solution to (6.10), with $g = e_2$, whose existence and estimates are given by Lemma 6.1. We have that $\int_0^{2\pi}\psi_j^2(\rho e^{i\theta},t)\,d\theta = 0$ and

$$|\psi_j^2(y,t)| + (1+|y|)\,|\nabla\psi_j^2(y,t)| \lesssim \varepsilon^3|\log\varepsilon|^3\,(1+|y|), \qquad \text{in } B(0,3R_j)\times[0,T].$$

We also have

$$(1+|y|^2)\,|\nabla_y\phi_j^2(y,t)| + (1+|y|)\,|\phi_j^2(y,t)| \lesssim \varepsilon^3|\log\varepsilon|^3 \qquad \text{in } B(0,3R_j)\times[0,T].$$

For the same reason we did it before, we can differentiate in time the equation, to also get

$$(1+|y|)\,|\log\varepsilon|^{\frac12}\,|\partial_t\phi_j^2(y,t)| \lesssim \varepsilon^3|\log\varepsilon|^3 \qquad \text{in } B(0,3R_j)\times[0,T].$$

From Lemma 6.1 we also get that the Fourier decomposition of ψ_j^2 only contains Fourier mode-1 and mode-3 terms. Hence the Fourier decomposition of the function $\phi_j^2(y,t)$ has mode-1 and mode-3 terms of size $\varepsilon^3|\log\varepsilon|^3$, terms of mode-0, mode-2 or higher of size $\varepsilon^4|\log\varepsilon|^3$ or smaller. Using these properties and the notations introduced in (6.5) we get

$$L_1(\phi_j^2) = \frac{\varepsilon^5|\log\varepsilon|^{\frac{7}{2}}}{1+|y|}E_{i\geq 0},$$

$$\nabla^\perp\Gamma_0\cdot\nabla[\frac{3\varepsilon_j}{r_j+\varepsilon_j y_1}\partial_{y_1}\psi_j^2] = \frac{\varepsilon^4|\log\varepsilon|^3}{1+|y|^2}(E_0+E_{i\geq 2}) + \frac{\varepsilon^5|\log\varepsilon|^3}{1+|y|}E_{i\geq 0},$$

$$L_2[\phi_j^2] + L_3[\psi_j^2] - L[\psi_j^2] + Q(\psi_j^2,\phi_j^2) = \frac{\varepsilon^4|\log\varepsilon|^4}{1+|y|^2}(E_0+E_{i\geq 2}) + \frac{\varepsilon^5|\log\varepsilon|^4}{1+|y|}E_{i\geq 0}$$

and

$$Q(\psi_j^1,\phi_j^2) + Q(\psi_j^2,\phi_j^1) = \frac{\varepsilon^5|\log\varepsilon|^4}{1+|y|}E_{i\geq 0}.$$

Define

$$\tilde\phi_j^2 = \sum_{i=1}^2 \phi_j^i, \quad \tilde\psi_j^2 = \sum_{i=1}^2 \psi_j^i.$$

We get the new error

$$\begin{aligned} E_j^{in}[\tilde\phi_j^2,\tilde\psi_j^2,0,P](y,t) &:= [\nabla^\perp\mathcal{R}_j^{00}(y,t;P) - \nabla^\perp\mathcal{R}_j^{00}(y,t;\bar P^1)]\cdot\nabla U \\ &+ \frac{\varepsilon^4|\log\varepsilon|^4}{1+|y|^2}(E_0+E_{i\geq 2})[P,\partial_t P] + \frac{\varepsilon^5|\log\varepsilon|^4}{1+|y|}E_{i\geq 0}[P,\partial_t P]. \end{aligned} \tag{6.18}$$ Ej3

Using the definition of $\mathcal{R}_j^{00}$ in (6.7), we check that

$$\begin{aligned} \mathcal{R}_j^{00}(y,t;P) - \mathcal{R}_j^{00}(y,t;\bar P^1) &= \varepsilon_j|\log\varepsilon|\partial_t(P_j-\bar P_j^1)^\perp\cdot y \\ &+ \varepsilon_j[\nabla_x\varphi_j(P_j;P) - \nabla_x\varphi_j(\bar P_j^1;\bar P^1)]\cdot y \\ &+ \varepsilon_j^3 D_x\theta_{j\varepsilon_j}(P_j;P)[y] - \varepsilon_j^3 D_x\theta_{j\varepsilon_j}(\bar P_j^1;\bar P^1)[y]. \end{aligned}$$

6.4. **Third inner improvement.** Our next step is the elimination of the Fourier mode-0 term of size $\varepsilon^4|\log\varepsilon|^4$ in formula (6.18). We define ϕ_j^3 as follows

$$\phi_j^3(y,t) = -\frac{\varepsilon^4\,|\log\varepsilon|^3}{1+|y|^2}\int_0^t \varepsilon_j^{-2}(s)E_0(\rho,\theta,s,\varepsilon)\,ds.$$

It solves

$$\begin{aligned} |\log\varepsilon|\varepsilon_j^2\partial_t\phi_j^3 + \frac{\varepsilon^4|\log\varepsilon|^4}{1+|y|^2}E_0 = 0 \quad (y,t)\in B(0,3R_j)\times[0,T) \\ \phi_j^3(y,0) = 0, \end{aligned}$$

it satisfies

$$(1+|y|)|\nabla_y\phi_j^3(y,t)| + |\phi_j^3(y,t)| \lesssim \frac{\varepsilon_j^2|\log\varepsilon|^3}{1+|y|^2},$$

and its Fourier decomposition only has mode 0. Let ψ_j^3 be the solution to

$$-\Delta_{5,j}\psi_j^3 = \phi_j^3 \quad (y,t)\in B(0,4R_j)\times[0,T), \quad \psi_j^3 = 0 \quad (y,t)\in\partial B(0,4R_j)\times[0,T).$$

Let $p(y,t)$, with $p=0$ on $\partial B(0,4R_j)$ be the positive radial solution to

$$\Delta_y p + \frac{4}{1+|y|^2} = 0, \quad y\in B(0,4R_j), \quad p(\rho,t) = 4\int_\rho^{4R_j}\frac{\log(1+s^2)}{s}\,ds.$$

Then

$$\Delta_y p + \frac{3\varepsilon_j}{r_j+\varepsilon_j y_1}\partial_1 p + \frac{2}{1+|y|^2} \leq 0,$$

thanks to the fact that $R_j = \frac{\delta}{\varepsilon_j|\log\varepsilon|^\zeta}$ with $\zeta>1$. Take $\bar\psi(y,t) = M\,\varepsilon^2|\log\varepsilon|^3\,p$, for some $M>0$. This is a super solution for

$$\Delta_{5,j}\psi + \phi_j^3 \leq 0,$$

and gives that

$$|\psi_j^3(y,t)| + |(1+|y|)\nabla\psi_j^3(y,t)| \lesssim \varepsilon_j^2\,|\log\varepsilon|^5.$$

Besides the Fourier decomposition of the function ψ_j^3 has mode-0 terms of size $\varepsilon^2|\log\varepsilon|^5$, terms of mode-1 or higher of size $\varepsilon^3|\log\varepsilon|^5$ or smaller.

It is important to notice that $B_0(g)$ only contains Fourier mode-2 or mode-1 terms if g is a Fourier mode-0 function. We refer to (4.21) and Remark 4.1 for the definition of B_0 and equivalent formulations. This observation yields that

$$L_1(\phi_j^3) - |\log\varepsilon|\varepsilon_j^2\partial_t\phi_j^3 = \frac{\varepsilon^4|\log\varepsilon|^4}{1+|y|^2}E_2 + \frac{\varepsilon^5|\log\varepsilon|^4}{1+|y|}E_1.$$

If f is a Fourier mode-2 function and g is a Fourier mode-0 function, then $\nabla^\perp f\cdot\nabla g$ is a Fourier mode-2 function. We use this, together with the explicity expression of $\mathcal{R}_j^0$ in (6.1) and the explicit form of the operator L_2 in (6.5) to get

$$L_2(\phi_j^3) = \frac{\varepsilon^3|\log\varepsilon|^4}{1+|y|^3}E_{1,2}\sin\theta + \frac{\varepsilon^4|\log\varepsilon|^4}{1+|y|^2}E_2 + \frac{\varepsilon^5|\log\varepsilon|^4}{1+|y|}E_{i\geq 0}.$$

The key fact is that the main error of size $\varepsilon^3|\log\varepsilon|^4$ has a Fourier mode-1 term, but only containing $\sin\theta$, not $\cos\theta$: it is of the form described in (4.32) for $j=1$ and $E_{j,1}=0$. A similar expression is valid for $L_3(\psi_j^3)$, with two more powers of $|\log\varepsilon|$:

$$L_3(\psi_j^3) = \frac{\varepsilon^3|\log\varepsilon|^6}{1+|y|^3}E_{1,2}\sin\theta + \frac{\varepsilon^4|\log\varepsilon|^6}{1+|y|^2}E_2 + \frac{\varepsilon^5|\log\varepsilon|^6}{1+|y|}E_{i\geq 0}.$$

We also have

$$Q(\psi_j^3,\phi_j^3) = \frac{\varepsilon^5|\log\varepsilon|^8}{1+|y|}E_{i\geq 0}$$

$$Q(\psi_j^1,\phi_j^2) + Q(\psi_j^2,\phi_j^1) = \frac{\varepsilon^4|\log\varepsilon|^6}{1+|y|^4}E_2 + \frac{\varepsilon^5|\log\varepsilon|^6}{1+|y|}E_{i\geq 0}$$

With this in mind, calling

$$\tilde\phi_j^3 = \sum_{i=1}^3\phi_j^i,\quad \tilde\psi_j^3 = \sum_{i=1}^3\psi_j^i$$

we get the new error

$$\begin{aligned} E_j^{in}[\tilde\phi_j^3,\tilde\psi_j^3,0,P](y,t) &:= [\nabla^\perp\mathcal{R}_j^{00}(y,t;P) - \nabla^\perp\mathcal{R}_j^{00}(y,t;\bar P^1)]\cdot\nabla U \\ &+ \frac{\varepsilon_j^3|\log\varepsilon|^6}{1+|y|^3}\,E_{1,1}\sin\theta + \frac{\varepsilon_j^4|\log\varepsilon|^6}{1+|y|^2}\,E_2[P,\partial_tP] + \frac{\varepsilon_j^5|\log\varepsilon|^8}{1+|y|}\,E_{i\geq 0}[P,\partial_tP]. \end{aligned} \tag{6.19}$$ Ej4

If we compare this error $E_j^{in}[\tilde\phi_j^3,\tilde\psi_j^3,0,P]$ with the error $E_j^{in}[\phi_j^1,\psi_j^1,0,P]$ in (6.15), we observe a crucial difference: even though their main term have size ε^3 (multiplied by powers of $|\log\varepsilon|$) and are in Fourier mode-1, in $E_j^{in}[\tilde\phi_j^3,\tilde\psi_j^3,0,P]$ Fourier mode-1 enters only with a $\sin\theta$. We shall proceed as in the second improvement of the approximation, with adjusting the points P^1 and solving the same elliptic linear problem, but this time the new error will not have Fourier mode-0 of size ε^4. This subtle fact allows us to proceed with the construction.

6.5. **Fourth inner improvement.** Our next step is to eliminate the term $\frac{\varepsilon_j^3|\log\varepsilon|^6}{1+|y|^3}\,E_{1,1}\sin\theta$ in the error (6.19). Take $\ell=2$ in the decomposition (6.16) of $P^1(t)$ and for $\bar P^2 = (\bar P_1^2,\ldots,\bar P_k^2)$, $\bar P_j^2 = P_0 + P_j^0 + P_j^{11} + P_j^{12}$, we take

$$e_4(y,t) := \nabla^\perp\mathcal{R}_j^{00}(y,t;\bar P^2)\cdot\nabla U + \frac{\varepsilon_j^3|\log\varepsilon|^6}{1+|y|^3}\,E_{1,2}[\bar P^2,\partial_t(P^0+P^{11})]\sin\theta.$$

We have $\int_0^{2\pi} e_4(\rho e^{i\theta},t)\,d\theta = 0$, and e_4 satisfies the decay (6.13) with $m=3$. In fact, under the assumptions (4.2) on P, we have

$$(1+|y|^3)(|e_4(y,t)| + |\log\varepsilon|^{\frac12}|\partial_t e_4(y,t)|) \lesssim \varepsilon^3|\log\varepsilon|^5,\quad |y|<3R_j.$$

The orthogonality conditions

$$\int_{B_{8R_j}} (1+|y|^2)e_4(y,y)Z_\ell(y)\,dy = 0, \quad \ell = 1,2$$

become a system of ODEs for the point $P^{12} = (P_1^{12}, \dots, P_k^{12})$ of the same form as (6.17). Standard ODEs theory gives that, for all $\varepsilon > 0$ small enough there exists a unique solution P^{12} to (6.17) with initial condition $P^{12}(0) = 0$, which satisfy the bounds

$$\|P_j^{12}\|_{L^\infty[0,T)} + \|\partial_t P_j^{12}\|_{L^\infty[0,T)} \lesssim \varepsilon^2 |\log \varepsilon|^6.$$

We denote by $\psi_j^4 = \psi_j^4(y,t)$ and $\phi_j^4(y,t) = -\Delta_{5,j}\psi_j^4$ the solution to (6.10), with $g = e_4$, whose existence and estimates are given by Lemma 6.1. We have that $\int_0^{2\pi} \psi_j^4(\rho e^{i\theta}, t)\,d\theta = 0$ and

$$|\psi_j^4(y,t)| + (1+|y|)\,|\nabla \psi_j^4(y,t)| \lesssim \varepsilon^3 |\log\varepsilon|^7 \,(1+|y|), \qquad \text{in } B(0,3R_j)\times[0,T].$$

We also have

$$(1+|y|^2)\,|\nabla_y \phi_j^4(y,t)| + (1+|y|)\,|\phi_j^4(y,t)| \lesssim \varepsilon^3 |\log\varepsilon|^7 \qquad \text{in } B(0,3R_j)\times[0,T].$$

We can differentiate in time the equation, to also get

$$(1+|y|)\,|\log\varepsilon|^{\frac12}\,|\partial_t \phi_j^4(y,t)| \lesssim \varepsilon^3|\log\varepsilon|^7 \qquad \text{in } B(0,3R_j)\times[0,T].$$

From Lemma 6.1 we infer that is that the Fourier decomposition of ψ_j^4 contains Fourier mode-1 terms, but only with $\cos\theta$. Hence the Fourier decomposition of the function $\phi_j^4(y,t)$ has mode-1 terms with $\cos\theta$ of size $\varepsilon^2|\log\varepsilon|^7$, terms of mode-0, mode-2 or higher of size $\varepsilon^4|\log\varepsilon|^7$ or smaller. Using these properties and the notations introduced in (6.5) we get

$$
\begin{aligned}
&L_1(\phi_j^4) = \frac{\varepsilon^5|\log\varepsilon|^7}{1+|y|} E_{i\ge 0},\\
&\nabla^\perp\Gamma_0\cdot\nabla\Big[\frac{3\varepsilon_j}{r_j+\varepsilon_j y_1}\partial_{y_1}\psi_j^4\Big] = \frac{\varepsilon^4|\log\varepsilon|^7}{1+|y|^2}E_{i\ge2} + \frac{\varepsilon^5|\log\varepsilon|^7}{1+|y|}E_{i\ge0},\\
&L_2[\phi_j^4] + L_3[\psi_j^4] - L[\psi_j^4] + Q(\psi_j^4,\phi_j^4) = \frac{\varepsilon^4|\log\varepsilon|^8}{1+|y|^2}E_{i\ge2} + \frac{\varepsilon^5|\log\varepsilon|^4}{1+|y|}E_{i\ge0}\\
&\quad\text{and}\\
&Q(\tilde\psi_j^3,\phi_j^4) + Q(\psi_j^4,\tilde\phi_j^3) = \frac{\varepsilon^5|\log\varepsilon|^8}{1+|y|}E_{i\ge0}.
\end{aligned}
$$

Define

$$\tilde\phi_j^4 = \sum_{i=1}^4 \phi_j^i, \quad \tilde\psi_j^4 = \sum_{i=1}^4 \psi_j^i.$$

We get the new error, for $y\in B(0,3R_j)$,

$$
\begin{aligned}
E_j^{in}[\tilde\phi_j^4,\tilde\psi_j^4,0,P](y,t) &:= [\nabla^\perp\mathcal{R}_j^{00}(y,t;P) - \nabla^\perp\mathcal{R}_j^{00}(y,t;\bar P^2)]\cdot\nabla U\\
&\quad + \frac{\varepsilon^4|\log\varepsilon|^8}{1+|y|^2}\,E_{i\ge2} + \frac{\varepsilon^5|\log\varepsilon|^8}{1+|y|}\,E_{i\ge0}[P,\partial_t P].
\end{aligned}
\tag{6.20}
$$

Ej5-1

The 0-th Fourier mode of the new error comes with size ε^5 (and powers of $|\log\varepsilon|$).

6.6. **Fifth inner improvement.** We now remove part of the Fourier mode-2 or higher terms of size $\varepsilon^4|\log\varepsilon|^8$ in (6.20). As in the first inner improvement, we take just the parts that depend on P and on $\partial_t P^0$, $\partial_t P^1$, but not the ones depending on $\partial_t \mathbf{a}$, and we call it e_5. We extend the function e_5 in (6.9) to be equal to 0 outside $B_j = B(0,3R_j)$. Under the assumptions (4.2) on P, P^0 and P^1, we have that

$$(1+|y|^2)(|e_5(y,t)| + |\log\varepsilon|^{\frac12}|\partial_t e_5(y,t)|) \lesssim \varepsilon^4|\log\varepsilon|^8.$$

Moreover the function e_5 satisfies authomatically the orthogonality conditions (6.11) and (6.12). Let ψ_j^5 and $\phi_j^5(y,t) = -\Delta_{5,j}\psi_j^5$ be the solution to (6.10) when $g = e_5$, as predicted by Lemma 6.1. We have

$$|\psi_j^5(y,t)| + (1+|y|)\,|\nabla\psi_j^5(y,t)| \lesssim \varepsilon^4|\log\varepsilon|^8(1+|y|^2), \qquad \text{in } B(0,8R_j)\times[0,T].$$

and

$$(1+|y|)\,|\nabla_y \phi_j^5(y,t)| + \ |\phi_j^5(y,t)| \lesssim \varepsilon^4 |\log\varepsilon|^8 \qquad \text{in } B(0,8R_j)\times[0,T].$$

We can differentiate in time equations (6.10), to also get

$$|\partial_t \phi_j^5(y,t)| \lesssim \varepsilon^4|\log\varepsilon|^8 \qquad \text{in } B(0,8R_j)\times[0,T].$$

Define

$$\tilde\phi_j^5 = \sum_{i=1}^5 \phi_j^i, \quad \tilde\psi_j^5 = \sum_{i=1}^5 \psi_j^i.$$

We get the new error, for $y\in B(0,3R_j)$

$$E_j^{in}[\tilde\phi_j^5,\tilde\psi_j^5,0,P](y,t) := [\nabla^\perp \mathcal{R}_j^{00}(y,t;P) - \nabla^\perp \mathcal{R}_j^{00}(y,t;\bar P^2)]\cdot\nabla U + \frac{\varepsilon^5|\log\varepsilon|^8}{1+|y|}E_{i\geq 0}. \tag{6.21}$$ Ej5-2

6.7. **Sixth inner improvement.** Our next step is the elimination of the $\varepsilon_j^5|\log\varepsilon|^8$-term in (6.21), $\frac{\varepsilon_j^5|\log\varepsilon|^8}{1+|y|}E_{i\geq 0}$, to get faster decay in the y variable. To do so, rather than solving an elliptic problem or an ODEs, we solve the transport-type equation

$$\begin{aligned} &L_1(\phi_j)+L_2(\phi_j)\\ &= |\log\varepsilon|\,\varepsilon_j^2(1+\frac{\varepsilon_j}{r_j}y_1)\partial_t\phi_j + |\log\varepsilon|B_0(\phi_j)\\ &+\nabla^\perp\left(\Gamma_0+\frac{\varepsilon_j}{2r_j}\,y_1\left(\Gamma_0(y)+\bar A+\Gamma(y)\right)+\mathcal{R}_j^0(y,t;P)\right)\cdot\nabla\phi_j = E(y,t),\end{aligned}$$

in $B_{3R_j}\times[0,T]$, with initial condition $\phi(y,0)=0$ in $B(0,3R_j)$. We write the operator B_0 defined in (4.21) as follows

$$\begin{aligned} |\log\varepsilon|B_0(\phi) &= \mathcal{B}_j^0(y,t;P)\cdot\nabla\phi\\ \mathcal{B}_j^0(y,t;P) &= -\frac{|\log\varepsilon|}{2}\partial_t\varepsilon_j^2(1+\frac{\varepsilon_j}{r_j}y_1)\,y - |\log\varepsilon|\frac{\varepsilon_j^2}{r_j}y_1\partial_t P_j.\end{aligned}$$

We will need uniform differentiability in t of the coefficients $\mathcal{R}_j^0$ and $\mathcal{B}_j^0$. Thus we consider the following slightly-modified transport equation

$$\begin{cases} \varepsilon_j^2|\log\varepsilon|(1+\frac{\varepsilon_j}{r_j}y_1)\partial_t\phi + \nabla_y^\perp(\Gamma_0(y)+\tilde{\mathcal{R}}_j(y,t;P))\cdot\nabla_y\phi + \mathcal{B}_j(y,t;P)\cdot\nabla_y\phi \\ \qquad\qquad = E(y,t), \quad \text{in } B(0,3R_j)\times[0,T]\\ \phi(y,0)=0, \quad \text{in } B(0,3R_j)\end{cases} \tag{6.22}$$ pico

Here

$$\tilde{\mathcal{R}}_j(y,t) = \frac{\varepsilon_j}{2r_j}\,y_1\left(\Gamma_0(y)+\bar A+\Gamma(y)\right)+\mathcal{R}_j^1(y,t;P)$$

where

$$\mathcal{R}_j^0(y,t;P) = \mathcal{R}_j^1(y,t;P)+\varepsilon_j|\log\varepsilon|\,\partial_t\mathbf{a}_j\cdot y$$

see (6.1). In other words, we leave out the term involving $\partial_t\mathbf{a}$. We do the same to $\mathcal{B}_j^0$: using the fact that $\partial_t\varepsilon_j^2 = -r_0\varepsilon^2\frac{\partial_t r_j}{r_j^2}$, we write

$$\mathcal{B}_j^0(y,t;P) = \mathcal{B}_j(y,t;P) + \frac{|\log\varepsilon|\varepsilon^2 r_0}{2}\frac{\partial_t\mathbf{a}_{j1}}{r_j^2}(1+\frac{\varepsilon_j}{r_j}y_1)y - |\log\varepsilon|\frac{\varepsilon_j^2}{r_j}y_1\partial_t\mathbf{a}_j.$$

It is straightforward to check that, under our assumptions (4.2), (4.3) and (4.4), for $y\in B(0,3R_j)$ we have

$$\begin{aligned} |\mathcal{R}_j^1(y,t)|+|\partial_t\mathcal{R}_j^1(y,t)| &\leq M\varepsilon^2|y|^2,\\ |\nabla_y\mathcal{R}_j^1(y,t)|+|\partial_t\nabla_y\mathcal{R}_j^1(y,t)| &\leq M\varepsilon^2|y|,\\ |D_y^2\mathcal{R}_j^1(y,t)| &\leq M\varepsilon^2,\\ |\mathcal{B}_j(y,t)|+|\partial_t\mathcal{B}_j(y,t)| &\leq M\varepsilon^2|\log\varepsilon|^{\frac12}|y|,\\ |\nabla_y\mathcal{B}_j(y,t)| &\leq M\varepsilon^2|\log\varepsilon|^{\frac12},\end{aligned} \tag{6.23}$$ papa-new

for some positive constant M independent of ε.

We consider a smooth cut-off function $\eta_4(s)$ as in (2.3), and take

$$\eta_{4\varepsilon}(y)=\eta_4(|\log\varepsilon|^{\zeta}\,\varepsilon\,|y|). \tag{6.24}$$ eta2epsilon

We will then have a solution to (6.22) by restricting to $B(0,3R_j)$ the solution of the Cauchy problem

$$\begin{cases} \varepsilon_j^2|\log\varepsilon|(1+\eta_{4\varepsilon}\frac{\varepsilon_j}{r_j}y_1)\partial_t\phi+\nabla_y^{\perp}(\Gamma_0(y)+\eta_{4\varepsilon}\tilde{\mathcal{R}}_j(y,t;\xi))\cdot\nabla_y\phi+\eta_{4\varepsilon}\mathcal{B}_j(y,t;P)\cdot\nabla_y\phi \\ \qquad =\eta_{4\varepsilon}E(y,t),\quad \text{in } \mathbb{R}^2\times[0,T] \\ \phi(y,0)=0,\quad \text{in } \mathbb{R}^2 \end{cases} \tag{6.25}$$ pico1

In § 7 we prove the following result

⟨transport3-new⟩ **Lemma 6.2.** *Let us assume that $\tilde{\mathcal{R}}_j$ and $\mathcal{B}_j$ satisfy (6.23). Then there exist numbers $C,\delta>0$ such that for all sufficiently small ε and any function $E(y,t)$ that satisfies for some $C,\alpha\in\mathbb{R}$*

$$(1+|y|^2)|D_y^2E(y,t)|+(1+|y|)\left(|\nabla_yE(y,t)|+|\partial_t\nabla_yE(y,t)|\right)+|E(y,t)|+|\partial_tE(y,t)|\ \le\ C\,(1+|y|)^{\alpha},$$

the solution of (6.25) satisfies

$$(1+|y|)|\nabla_y\phi(y,t)|+|\phi(y,t)|\ \le\ C\varepsilon^{-2}|\log\varepsilon|^{-1}C(1+|y|)^{\alpha}$$

for all $y\in\mathbb{R}^2$, with $|y|<4\varepsilon^{-1}|\log\varepsilon|^{-\zeta}$, $t\in[0,T)$.

We want to apply Lemma 6.2 for the right hand side

$$E(y,t)=e_6,\quad e_6=\frac{\varepsilon^5|\log\varepsilon|^8}{1+|y|}\,E_{i\ge0}(\rho,\theta,t,\varepsilon)$$

where we freeze this term at $P_j=P_0+P_j^0+P_j^1$. We define ϕ_j^6 to be the solution to problem (6.22), with E as above, predicted by Lemma 6.2. It satisfies

$$(1+|y|)|\nabla_y\phi_j^6(y,t)|+|\phi_j^6(y,t)|\ \le\ C\frac{\varepsilon^3|\log\varepsilon|^7}{1+|y|}.$$

Let ψ_j^6 be the solution to

$$-\Delta_{5,j}\psi_j^6=\phi_j^6\quad (y,t)\in B(0,4R_j)\times[0,T),\qquad \psi_j^6=0\quad (y,t)\in\partial B(0,4R_j)\times[0,T).$$

Let $p(y,t)$, with $p=0$ on $\partial B(0,4R_j)$ be the positive smooth radial solution to

$$\Delta_yp+\frac{4}{1+|y|}=0,\quad y\in B(0,4R_j).$$

Then $p(y)=16(R_j-|y|)-4\int_{|y|}^{4R_j}\frac{\log(1+s)}{s}\,ds$, and $|p(y)|\lesssim(1+|y|)$ and

$$\Delta_yp+\frac{3\varepsilon_j}{r_j+\varepsilon_jy_1}\partial_1p+\frac{2}{1+|y|}\le0.$$

Take $\bar\psi(y,t)=M\,\varepsilon^3|\log\varepsilon|^7\,p$, for some $M>0$. It is a super solution for

$$\Delta_{5,j}\psi+\phi_j^6\le0,$$

and gives that

$$(1+|y|^2)|D_y^2\psi_j^6(y,t)|+(1+|y|)|\nabla_y\psi_j^6(y,t)|+|\psi_j^6(y,t)|\ \le\ C\varepsilon^3|\log\varepsilon|^8(1+|y|).$$

We use this information to compute the following terms

$$L_3(\psi_j^6)+Q(\psi_j^6,\phi_j^6)=\frac{\varepsilon^3|\log\varepsilon|^8}{1+|y|^5}E_{i\ge1},\quad Q(\psi_j^6,\tilde\phi_j^5)+Q(\tilde\psi_j^5,\phi_j^6)=\frac{\varepsilon_j^5|\log\varepsilon|^8}{1+|y|^3}E_{i\ge0}$$

Define

$$\tilde\phi_j^6=\sum_{i=1}^{6}\phi_j^i,\quad \tilde\psi_j^6=\sum_{i=1}^{6}\psi_j^i.$$

We get the new error

$$E_j^{in}[\tilde{\phi}_j^6, \tilde{\psi}_j^6, 0, P](y,t) := [\nabla^\perp \mathcal{R}_j^{00}(y,t;P) - \nabla^\perp \mathcal{R}_j^{00}(y,t;\bar{P}^2)] \cdot \nabla U + \frac{\varepsilon^3 |\log\varepsilon|^8}{1+|y|^5} E_{i\geq 1} + \frac{\varepsilon^5 |\log\varepsilon|^8}{1+|y|^3} E_{i\geq 0}. \tag{6.26}$$

Ej6

If we compare this error with the one in (6.15), we see that both their main terms have size ε^3 multiplied by a power of $|\log\varepsilon|$, and have Fourier mode-1. The difference is in their decay in the y-variable: the error in (6.26) has a much faster decay, which will be crucial to make the final argument of our construction work. We explain this in Section 9 where the *inner-outer* scheme is described in detail.

At this point of our construction we choose ζ in (5.3). Take

$$\zeta = 3.$$

Then

$$W^0(x) + \frac{\eta_{j1}}{r_j \varepsilon_j^2} \tilde{\phi}_j^6 (\frac{x-P_j}{\varepsilon_j}, t) = \frac{1}{r_j\varepsilon_j^2} U(\frac{x-P_j}{\varepsilon_j})(1+o(1)), \quad \text{as} \quad \varepsilon \to 0$$

for $|x-P_j| < |\log\varepsilon|^{-3}$.

6.8. **The outer improvement.** So far we have modified the approximate solution in the inner regions, namely at a small distance from the points P_j. We will now improve the outer error given by

$$\begin{aligned} E_0(x,t;P) &= E^{out}[0,0,\tilde{\phi}^6, \tilde{\psi}^6, P](x,t) \\ &:= \sum_{j=1}^k \left[r\,|\log\varepsilon|\,\partial_t \bar{\eta}_{j1} + \nabla_x^\perp (r^2(\Psi^0 + \sum_{j=1}^k \frac{\eta_{j2}}{r_j} \tilde{\psi}_j^6 (\frac{x-P_j}{\varepsilon_j}) - r_0^{-1} |\log\varepsilon|)) \nabla \bar{\eta}_{1j} \right] \frac{\tilde{\phi}_j^6}{\varepsilon_j^2 r_j} \\ &+ \left[\sum_{j=1}^k (\eta_{2j} - \eta_{1j}) \nabla_x^\perp (r^2 \frac{\tilde{\psi}_j^6}{r_j}) + \frac{r^2 \tilde{\psi}_j^6}{r_j} \nabla_x^\perp \eta_{2j} \right] \nabla_x W^0 \\ &+ (1 - \sum_{j=1}^k \eta_{j1}) S_1(W^0, \Psi^0) \quad (x,t) \in \Sigma \times [0,T) \end{aligned}$$

We also want to reduce the size of $S_2[\tilde{\phi}^6, \tilde{\psi}^6, 0, 0, P]$. We refer to (2.2) and also (5.7). Observe that the function $\Psi^0 + \sum_{j=1}^k \frac{\eta_{j2}}{r_j} \tilde{\psi}_j^6 (\frac{x-P_j}{\varepsilon_j})$ satisfies the conditions on the boundary and at infinity, see (4.18): for all $t \in [0,T]$

$$\frac{\partial\psi}{\partial r}(x,t) = 0, \quad \text{on} \quad \partial\Sigma, \quad |\psi(x,t)| \to 0, \qquad \text{as} \quad |x| \to \infty.$$

To reduce the outer error, we first solve in ϕ^{out} the *outer transport equation*

$$\begin{aligned} |\log\varepsilon|\, r\, \partial_t \phi^{out} + \nabla_x^\perp (r^2(\Psi^0 - r_0^{-1} |\log\varepsilon|)) \nabla_x \phi^{out} &= E_0, \quad \text{in} \quad \Sigma \times [0,T) \\ \phi^{out}(x,0) &= 0, \qquad \text{in} \quad \Sigma. \end{aligned} \tag{6.27}$$

transport-outer

We define

$$\tilde{E}_0(x,t) = \frac{E_0(x,t)}{r}, \quad |\log\varepsilon| B(x,t) = \frac{1}{r} \nabla_x^\perp (r^2(\Psi^0 - \alpha_0 |\log\varepsilon|)) \tag{6.28}$$

ou1

and re-write (6.27) as

$$\begin{aligned} |\log\varepsilon|\, \partial_t \phi^{out} + |\log\varepsilon| B \cdot \nabla_x \phi^{out} &= \tilde{E}_0, \quad \text{in} \quad \Sigma \times [0,T) \\ \phi^{out}(x,0) &= 0, \qquad \text{in} \quad \Sigma. \end{aligned} \tag{6.29}$$

to1

From the very definition of Ψ^0 the following properties for $B(x,t)$ follow: if $B(x,t) = (B_1(x,t), B_2(x,t))$, we have

$$B_1(x,t) = 0 \quad \text{in } (x,t) \in \partial\Sigma \times [0,T] \tag{6.30}$$

estB

and

$$B(x,t) = O(1) \quad \text{in } (x,t) \in \Sigma \times [0,T], \qquad \text{as} \quad \varepsilon \to 0. \tag{6.31}$$

estB1

Since the function B is continuous and log-Lipschitz in x uniformly in t, for $(x,t)\in\Sigma\times[0,T)$, we can represent the solution to (6.29) using the Duhamel's representation formula

$$\phi^{out}(x,t)=|\log\varepsilon|^{-1}\int_0^t \tilde E_0(\bar x(s;x,t),s)\,ds \tag{6.32}$$ phi3

where $\bar x(s;x,t)$ are the characteristic curves defined as

$$\frac{d}{ds}\bar x(s;x,t)=B(\bar x(s;x,t),s)\quad s\in(0,t),\quad \bar x(t;x,t)=x, \tag{6.33}$$ char

which exist and are unique, mainly thanks to (6.30) as we shall prove later.

The right-hand side of our equation (6.27) is supported away from the vortices $P_j(t)$ at a distance proportional to $|\log\varepsilon|^{-3}$. We assume that for some fixed number δ' we have

$$\tilde E_0,\, E_0\equiv 0\quad \text{in } \Big\{(x,t)\in\Sigma\times[0,T]\ /\ x\in\bigcup_{j=1}^k B(P_j(t),\delta'|\log\varepsilon|^{-3})\Big\}. \tag{6.34}$$ assE

We have the validity of the following

⟨int1⟩ **Lemma 6.3.** *For $x\in\Sigma$ the characteristic curves (6.33) satisfy $\bar x(s;t,x)\in\Sigma$ for all $0\le s\le t$. The solution of (6.27) given by (6.32) satisfies for any $1\le p\le+\infty$,*

$$\|\phi(\cdot,t)\|_{L^p(\Omega)}\le|\log\varepsilon|^{-1}t\sup_{0\le s\le t}\|\tilde E_0(\cdot,s)\|_{L^p(\Omega)}.$$

If E_0 satisfies (6.34), there exists a number $\delta^*>0$ independent of δ' and ε such that the solution of (6.27) satisfies

$$\phi\equiv0\quad in\ \Big\{(x,t)\in\Sigma\times[0,T]\ /\ x\in\bigcup_{j=1}^k B(P_j(t),\delta^*|\log\varepsilon|^{-3})\Big\}.$$

Moreover, if $\tilde E_0$ satisfies (6.34) and

$$(1+|x|)|\nabla_x\tilde E_0(x,t)|+|\tilde E_0(x,t)|\le\frac{A}{1+|x|^4}\quad for\ all\quad (x,t)\in\Sigma\times[0,T],$$

then there exists $C>0$ such that the solution of (6.27) satisfies the estimate

$$(1+|x|)|\nabla_x\phi(x,t)|+|\phi_t(x,t)|+|\phi(x,t)|\ \le\ C|\log\varepsilon|^{-1}\frac{A}{1+|x|^4}. \tag{6.35}$$ poto1

For the proof of Lemmas 6.3 see § 8.

From (4.19), we explicitly compute

$$\frac{(1-\sum_{j=1}^k\eta_{j1})S_1(W^0,\Psi^0)}{r}=|\log\varepsilon|(1-\sum_{j=1}^k\eta_{j1})\sum_{j=1}^k\Big[\partial_tW_j^0+\frac{r}{|\log\varepsilon|}\nabla^\perp(\Psi^0-\alpha_0|\log\varepsilon|)\cdot\nabla W_j^0$$
$$-\frac{2}{|\log\varepsilon|}(\Psi^0-\alpha_0|\log\varepsilon|)\,\mathbf{e_2}\cdot\nabla W_j^0\Big].$$

This fact, together with (6.27) and the estimates on $\tilde\phi^6$, $\tilde\psi^6$ give that $\tilde E_0(x,t)$ has no singularity at $r=0$ and satisfies

$$\begin{aligned}\tilde E_0(x,t)&=\sum_{j=1}^k(1-\eta_{j1})\frac{O(\varepsilon^2|\log\varepsilon|)}{(|\log\varepsilon|^{-3}+|x-P_j|)^4},\\ \nabla_x\tilde E_0(x,t)&=\sum_{j=1}^k(1-\eta_{j1})\frac{O(\varepsilon^2|\log\varepsilon|)}{(|\log\varepsilon|^{-3}+|x-P_j|)^5}.\end{aligned} \tag{6.36}$$ est-tilde-E0

We define ϕ_1^{out} to solve (6.27). Since (6.36), from Lemma 6.3 we obtain that

$$(1+|x|)|\nabla_x\phi_1^{out}(x,t)|+|\partial_t\phi_1^{out}(x,t)|+|\phi_1^{out}(x,t)|\ \le\ C\,\frac{\varepsilon^2|\log\varepsilon|^{13}}{1+|x|^4}.$$

We next introduce ψ_1^{out} to solve

$$-\Delta_5\psi_1^{out} = \phi_1^{out} + S_2[\tilde\phi^6, \tilde\psi^6, 0, 0, P], \quad \text{on} \quad \Sigma\times[0,T),$$
$$\frac{\partial\psi_1^{out}}{\partial r} = 0 \quad \text{on} \quad \partial\Sigma\times[0,T), \quad |\psi_1^{out}(x,t)|\to 0 \quad \text{as } |x|\to\infty.$$

From the estimates on $\tilde\phi^6$ and $\tilde\psi^6$ we get

$$S_2[\tilde\phi^6, \tilde\psi^6, 0, 0, P] = \sum_{j=1}^k (\eta_{j1}-\eta_{j2})\frac{\Delta_{5,j}\tilde\psi_j^6}{r_j\varepsilon_j^2}$$
$$+\sum_{j=1}^k(\frac{\tilde\psi_j^6}{r_j}\Delta_5\eta_{j2} + 2\nabla_x\eta_{j2}\nabla_x\frac{\tilde\psi_j^6}{r_j}) = O(\varepsilon^2|\log\varepsilon|^{11})\mathbf{1}_{\{|\log\varepsilon|^{-3}<|x-P_j|<4|\log\varepsilon|^{-3}\}}.$$

With the aid of Lemma 4.1, we have

$$(1+|x|)|\nabla\psi_1^{out}(x,t)| + |\psi_1^{out}(x,t)| \lesssim \frac{\varepsilon^2|\log\varepsilon|^{13}}{1+|x|^2}, \tag{6.37}$$ psiout1

and hence

$$S_2[\tilde\phi^6, \tilde\psi^6, \phi_1^{out}, \psi_1^{out}, P] = 0 \quad (x,t)\in\Sigma\times[0,T). \tag{6.38}$$ S2000

Having introduced the outer corrections ϕ_1^{out} and ψ_1^{out}, we compute the new outer error $E^{out}[\phi_1^{out}, \psi_1^{out}, \tilde\phi^6, \tilde\psi^6, P]$ and the new inner error $E_j^{in}[\tilde\phi_j^6, \tilde\psi_j^6, \psi_1^{out}, P]$.

The new outer error is given by

$$E^{out}[\phi_1^{out}, \psi_1^{out}, \tilde\phi^6, \tilde\psi^6, P](x,t) := \nabla_x^\perp(r^2(\sum_{j=1}^k\frac{\eta_{j2}}{r_j}\tilde\psi_j^6 + \psi_1^{out}))\cdot\nabla_x\phi_1^{out}$$
$$+\sum_{j=1}^k\nabla_x^\perp(r^2\psi_1^{out})\nabla\bar\eta_{1j}\frac{\tilde\phi_j^6}{\varepsilon_j^2 r_j} + \sum_{j=1}^k(\eta_{2j}-\eta_{1j})\nabla_x^\perp(r^2\psi_1^{out})\cdot\nabla_x W^0$$
$$+(1-\sum_{j=1}^k\eta_{2j})\nabla^\perp(r^2\psi_1^{out})\cdot\nabla W^0 \quad (x,t)\in\Sigma\times[0,T)$$

We see that $E^{out}[\phi_1^{out}, \psi_1^{out}, \tilde\phi^6, \tilde\psi^6, P](x,t)$ is a regular function, which is $\equiv 0$ in the region

$$\Big\{(x,t)\in\Sigma\times[0,T]\ /\ x\in\bigcup_{j=1}^k B(P_j(t),\bar\delta|\log\varepsilon|^{-3})\Big\}$$

for some $\bar\delta>0$, and satisfies the bounds

$$|E^{out}[\phi_1^{out}, \psi_1^{out}, \tilde\phi^6, \tilde\psi^6, P](x,t)| \lesssim \frac{\varepsilon^4|\log\varepsilon|^{26}}{1+|x|^4}. \tag{6.39}$$ Eout1-2

The new inner error is given by

$$E_j^{in}[\tilde\phi_j^6, \tilde\psi_j^6, \psi_1^{out}, P](y,t) := \nabla^\perp((1+\frac{\varepsilon_j y_1}{r_1})^2 r_j\psi_1^{out})\cdot\nabla(w_j^0 + \sum_{i\neq j}w_i^0)$$
$$+\nabla^\perp\left((1+\frac{\varepsilon_j}{r_j}y_1)^2 r_j\psi_1^{out}\right)\cdot\nabla\tilde\phi_j^6 + E_j^{in}[\tilde\phi_j^6, \tilde\psi_j^6, 0, P](y,t).$$

The biggest term in this expression comes from $\nabla^\perp((1+\frac{\varepsilon_j y_1}{r_1})^2 r_j\psi_1^{out})\cdot\nabla(w_j^0+\sum_{i\neq j}w_i^0)$. Using (6.37) we see that the size of this term is $\frac{\varepsilon^3|\log\varepsilon|^{13}}{1+|y|^5}$. Moreover, since w_j^0 satisfies (4.13), its Fourier modes-0 come at order $\varepsilon^4|\log\varepsilon|^{26}$ or smaller. In combination with (6.26), we get

$$E_j^{in}[\tilde\phi_j^6, \tilde\psi_j^6, \psi_1^{out}, P](y,t) := [\nabla^\perp\mathcal{R}_j^{00}(y,t;P) - \nabla^\perp\mathcal{R}_j^{00}(y,t;\bar P^2)]\cdot\nabla U$$
$$+\frac{\varepsilon^3|\log\varepsilon|^{14}}{1+|y|^5}E_{i\geq 1} + \frac{\varepsilon^4|\log\varepsilon|^{26}}{1+|y|^4}E_0 + \frac{\varepsilon^5|\log\varepsilon|^{20}}{1+|y|^3}E_{i\geq 0}. \tag{6.40}$$ Ej7

6.9. **Seventh inner improvement.** Our next step is to eliminate the terms in the error (6.40) given by $\frac{\varepsilon^3|\log\varepsilon|^{13}}{1+|y|^5}E_{i\geq 1}$. We proceed as in the second and fourth improvements. Take $\ell = 3$ in the decomposition (6.16) of $P^1(t)$ and for $\bar P^3 = (\bar P^3_1, \dots, \bar P^3_k)$, $\bar P^3_j = P_0 + P^0_j + P^{11}_j + P^{12}_j + P^{13}_j$, we take

$$e_7(y,t) := \nabla^\perp \mathcal{R}^{00}_j(y,t;\bar P^3)\cdot\nabla U + \frac{\varepsilon_j^3|\log\varepsilon|^{14}}{1+|y|^5}\, E_{i\geq 1}[\bar P^2, \partial_t(P^0+P^{11}+P^{12})].$$

We have $\int_0^{2\pi} e_7(\rho e^{i\theta},t)\,d\theta = 0$, and e_7 satisfies the decay (6.13) with $m=5$. In fact, under the assumptions (4.2) on P, we have

$$(1+|y|^5)(|e_7(y,t)| + |\log\varepsilon|^{\frac12}|\partial_t e_7(y,t)|) \lesssim \varepsilon^3|\log\varepsilon|^{14}, \quad |y|<3R_j.$$

The orthogonality conditions

$$\int_{B_{8R_j}} (1+|y|^2)e_7(y,y)Z_\ell(y)\,dy = 0, \quad \ell = 1,2$$

become a system of ODEs for the point $P^{13} = (P^{13}_1,\dots,P^{13}_k)$ of the same form as (6.17). Standard ODEs theory gives that, for all $\varepsilon>0$ small enough there exists a unique solution P^{13} to (6.17) with initial condition $P^{13}(0)=0$, which satisfy the bounds

$$\|P^{13}_j\|_{L^\infty[0,T)} + \|\partial_t P^{13}_j\|_{L^\infty[0,T)} \lesssim \varepsilon^2|\log\varepsilon|^c,$$

for some $c>0$. We denote by $\psi^7_j = \psi^7_j(y,t)$ and $\phi^7_j(y,t) = -\Delta_{5,j}\psi^7_j$ the solution to (6.10), with $g = e_7$, whose existence and estimates are given by Lemma 6.1. We have that $\int_0^{2\pi}\psi^7_j(\rho e^{i\theta},t)\,d\theta = 0$ and

$$|\psi^7_j(y,t)| + (1+|y|)\,|\nabla\psi^7_j(y,t)| \lesssim \frac{\varepsilon^3|\log\varepsilon|^{15}}{1+|y|}, \qquad \text{in } B(0,3R_j)\times[0,T].$$

We also have

$$(1+|y|^4)\,|\nabla_y\phi^7_j(y,t)| + (1+|y|^3)\,|\phi^7_j(y,t)| \lesssim \varepsilon^3|\log\varepsilon|^{15} \qquad \text{in } B(0,3R_j)\times[0,T].$$

We can differentiate in time the equation, to also get

$$(1+|y|^3)\,|\log\varepsilon|^{\frac12}\,|\partial_t\phi^7_j(y,t)| \lesssim \varepsilon^3|\log\varepsilon|^{15} \qquad \text{in } B(0,3R_j)\times[0,T].$$

Define

$$\tilde\phi^7_j = \sum_{i=1}^7 \phi^i_j, \quad \tilde\psi^7_j = \sum_{i=1}^7 \psi^i_j.$$

Arguing as in the second improvement we get

$$\begin{aligned} E^{in}_j[\tilde\phi^7_j, \tilde\psi^7_j, \psi^{out}_1, P](y,t) &:= [\nabla^\perp\mathcal{R}^{00}_j(y,t;P) - \nabla^\perp\mathcal{R}^{00}_j(y,t;\bar P^3)]\cdot\nabla U \\ &+ \frac{\varepsilon^4|\log\varepsilon|^{26}}{1+|y|^4}\,E_{i\geq0} + \frac{\varepsilon^5|\log\varepsilon|^{28}}{1+|y|^3}\,E_{i\geq0}[P,\partial_t P]. \end{aligned} \tag{6.41}$$

Ej8

6.10. **Eighth inner improvement.** Our next step is the elimination of the Fourier mode-0 term of size $\varepsilon^4|\log\varepsilon|^{26}$ in formula (6.41). We proceed as in the third inner improvement and define ϕ^8_j as follows

$$\phi^8_j(y,t) = -\frac{\varepsilon^4\,|\log\varepsilon|^{25}}{1+|y|^4}\int_0^t \varepsilon_j^{-2}(s)E_0(\rho,\theta,s,\varepsilon)\,ds.$$

It solves

$$\begin{aligned} |\log\varepsilon|\varepsilon_j^2\partial_t\phi^8_j + \frac{\varepsilon^4|\log\varepsilon|^{21}}{1+|y|^4}\,E_0 = 0 \quad (y,t)\in B(0,3R_j)\times[0,T) \\ \phi^8_j(y,0) = 0 \quad \text{in } B(0,3R_j), \end{aligned}$$

it satisfies

$$(1+|y|)|\nabla_y\phi^8_j(y,t)| + |\phi^8_j(y,t)| \lesssim \frac{\varepsilon^2|\log\varepsilon|^{25}}{1+|y|^4}\,,$$

and its Fourier decomposition only has mode 0. Let ψ^8_j be the solution to

$$-\Delta_{5,j}\psi^8_j = \phi^8_j \quad (y,t)\in B_{4R_j}\times[0,T), \quad \psi^8_j = 0 \quad (y,t)\in\partial B_{4R_j}\times[0,T).$$

Thanks to the space decay in ϕ_j^8 we get

$$|\psi_j^8(y,t)| + |(1+|y|)\nabla\psi_j^8(y,t)| \lesssim \frac{\varepsilon^2\,|\log\varepsilon|^{25}}{1+|y|^2}.$$

As in the third inner improvement we get, for

$$\tilde\phi_j^8 = \sum_{i=1}^{8}\phi_j^i, \quad \tilde\psi_j^8 = \sum_{i=1}^{8}\psi_j^i,$$

$$E_j^{in}[\tilde\phi_j^8, \tilde\psi_j^8, \psi_1^{out}, P](y,t) := [\nabla^\perp\mathcal{R}_j^{00}(y,t;P) - \nabla^\perp\mathcal{R}_j^{00}(y,t;\bar P^3)]\cdot\nabla U$$
$$+\frac{\varepsilon^3|\log\varepsilon|^{26}}{1+|y|^5}E_{1,1}\sin\theta + \frac{\varepsilon^4|\log\varepsilon|^{50}}{1+|y|^4}E_2[P,\partial_t P] + \frac{\varepsilon^5|\log\varepsilon|^{40}}{1+|y|^3}E_{i\ge 0}[P,\partial_t P]. \tag{6.42}$$ Ej9

In comparison with (6.41), we loose one power of ε and produce a new error in mode 1, in the form of $\sin\theta$. We have already seen how to deal with such a situation in the fourth inner improvement and in this way we proceed next.

6.11. **Nineth inner improvement.** We eliminate the terms in the error (6.42) $\frac{\varepsilon^3|\log\varepsilon|^{26}}{1+|y|^5}E_{1,1}\sin\theta$. Take $\ell=4$ in the decomposition (6.16) of $P^1(t)$ and for $\bar P^4 = (\bar P_1^4,\dots,\bar P_k^4)$, $\bar P_j^4 = P_0+P_j^0+P_j^{11}+P_j^{12}+P_j^{13}+P_j^{14}$, we take

$$e_9(y,t) := \nabla^\perp\mathcal{R}_j^{00}(y,t;\bar P^4)\cdot\nabla U + \frac{\varepsilon^3|\log\varepsilon|^{26}}{1+|y|^5}E_{1,1}[\bar P^4, \partial_t(P^4-P^{14})]\sin\theta.$$

We have $\int_0^{2\pi} e_9(\rho e^{i\theta},t)\,d\theta = 0$, and e_9 satisfies the decay (6.13) with $m=5$. In fact, under the assumptions (4.2) on P, we have

$$(1+|y|^5)(|e_9(y,t)| + |\log\varepsilon|^{\frac12}|\partial_t e_9(y,t)|) \lesssim \varepsilon^3|\log\varepsilon|^{21}, \quad |y|<3R_j.$$

The orthogonality conditions

$$\int_{B_{8R_j}}(1+|y|^2)e_8(y,y)Z_\ell(y)\,dy = 0, \quad \ell=1,2$$

become a system of ODEs for the point P^{14} of the same form as (6.17). Standard ODEs theory gives that, for all $\varepsilon>0$ small enough there exists a unique solution P^{14} to (6.17) with initial condition $P^{14}(0)=0$, which satisfy the bounds

$$\|P_j^{14}\|_{L^\infty[0,T)} + \|\partial_t P_j^{14}\|_{L^\infty[0,T)} \lesssim \varepsilon^2|\log\varepsilon|^c$$

for some $c>0$. We denote by $\psi_j^9 = \psi_j^9(y,t)$ and $\phi_j^9(y,t) = -\Delta_{5,j}\psi_j^9$ the solution to (6.10), with $g=e_9$, whose existence and estimates are given by Lemma 6.1. We have that $\int_0^{2\pi}\psi_j^9(\rho e^{i\theta},t)\,d\theta = 0$ and

$$|\psi_j^9(y,t)| + (1+|y|)\,|\nabla\psi_j^9(y,t)| \lesssim \frac{\varepsilon^3|\log\varepsilon|^{27}}{1+|y|}, \qquad \text{in } B(0,3R_j)\times[0,T].$$

We also have

$$(1+|y|^4)\,|\nabla_y\phi_j^9(y,t)| + (1+|y|^3)\,|\phi_j^9(y,t)| \lesssim \varepsilon^3|\log\varepsilon|^{27} \qquad \text{in } B(0,3R_j)\times[0,T].$$

We can differentiate in time the equation, to also get

$$(1+|y|^3)\,|\log\varepsilon|^{\frac12}\,|\partial_t\phi_j^9(y,t)| \lesssim \varepsilon^3|\log\varepsilon|^{27} \qquad \text{in } B(0,3R_j)\times[0,T].$$

Arguing as in the fourth improvement we get, for

$$\tilde\phi_j^9 = \sum_{i=1}^{9}\phi_j^i, \quad \tilde\psi_j^9 = \sum_{i=1}^{9}\psi_j^i.$$

We get the new error

$$E_j^{in}[\tilde\phi_j^9, \tilde\psi_j^9, \psi_1^{out}, P](y,t) := [\nabla^\perp\mathcal{R}_j^{00}(y,t;P) - \nabla^\perp\mathcal{R}_j^{00}(y,t;\bar P^4)]\cdot\nabla U$$
$$+\frac{\varepsilon^4|\log\varepsilon|^{50}}{1+|y|^4}E_{i\ge 2} + \frac{\varepsilon^5|\log\varepsilon|^{52}}{1+|y|^3}E_{i\ge 0}[P,\partial_t P].$$

⟨ultima⟩ 6.12. **Tenth (and final) inner improvement.** Our last step consists in removing the term $\frac{\varepsilon^4|\log\varepsilon|^{50}}{1+|y|^4}E_{i\geq 2}$ and we argue exactly as in the first inner improvement, but with two more power of ε. We find ψ_j^{10}, ϕ_j^{10} satisfying

$$|\psi_j^{10}(y,t)| + (1+|y|)\,|\nabla\psi_j^{10}(y,t)| \lesssim \varepsilon^4|\log\varepsilon|^{50}, \qquad \text{in } B(0,8R_j)\times[0,T].$$

and

$$(1+|y|^3)\,|\nabla_y\phi_j^{10}(y,t)| + (1+|y|^2)\,|\phi_j^{10}(y,t)| \lesssim \varepsilon^4|\log\varepsilon|^{50} \qquad \text{in } B(0,8R_j)\times[0,T],$$

such that, for

$$\tilde\phi_j^{10} = \sum_{i=1}^{10}\phi_j^i, \quad \tilde\psi_j^{10} = \sum_{i=1}^{10}\psi_j^i.$$

we get

$$E_j^{in}[\phi_j^{10},\psi_j^{10},\psi_1^{out},P](y,t) := [\nabla^\perp\mathcal{R}_j^{00}(y,t;P) - \nabla^\perp\mathcal{R}_j^{00}(y,t;\bar P^4)]\cdot\nabla U + \frac{\varepsilon^5|\log\varepsilon|^{52}}{1+|y|^3}E_{i\geq 0}[P,\partial_t P]$$

as $\varepsilon\to 0$, where the functions $E_j(\rho,\theta,t,\varepsilon)$ have the form described in (4.32). Here b can be taken to be 50. Using the definition of $\mathcal{R}_j^{00}$ in (6.7), we check that

$$\begin{aligned}\mathcal{R}_j^{00}(y,t;P) - \mathcal{R}_j^{00}(y,t;\bar P^4) &= \varepsilon_j|\log\varepsilon|\partial_t\mathbf{a}_j\cdot y\\ &+ \varepsilon_j[\nabla_x\varphi_j(P_j;P) - \nabla_x\varphi_j(\bar P_j^4;\bar P^4)]\cdot y\\ &+ \varepsilon_j^3 D_x\theta_{j\varepsilon_j}(P_j;P)[y] - \varepsilon_j^3 D_x\theta_{j\varepsilon_j}(\bar P_j^4;\bar P^4)[y].\end{aligned}$$

We take

$$\phi_j^* = \bar\phi_j^{10}, \quad \psi_j^* = \bar\psi_j^{10}, \quad \psi^{*,out} = \psi_1^{out}, \quad \phi^{*,out} = \phi_1^{out}.$$

Recall from (6.38) that

$$S_2[\tilde\phi^6,\tilde\psi^6,\phi_1^{out},\psi_1^{out},P] = 0 \quad (x,t)\in\Sigma\times[0,T).$$

Hence

$$|S_2[\phi^{*,in},\psi^{*,in},\phi^{*,out},\psi^{*,out},P]| \;\leq\; C\varepsilon^{4-\sigma}\eta_1\left(\frac{4|x-(r_0,0)|}{r_0}\right).$$

Besides, from (6.39) we get

$$\left|E^{out}[\phi^{*,out},\psi^{*,out},\phi^{*,in},\psi^{*,in},\mathbf{a}](x,t)\right| \lesssim \frac{\varepsilon^4|\log\varepsilon|^{52}}{1+|x|^4}, \quad (x,t)\in\Sigma\times[0,T].$$

This concludes the proof of Proposition 5.1.

7. The inner modified transport equation

⟨proofTransportInner⟩ This section is devoted to discuss the result of Lemma 6.2, on the solution of the Cauchy Problem (6.25). We write it as

$$\begin{aligned}&\varepsilon_j^2|\log\varepsilon|\partial_t\phi + \frac{\nabla_y^\perp(\Gamma_0(y)+\eta_{4\varepsilon}\tilde{\mathcal{R}}_j(y,t)) + \eta_{4\varepsilon}\mathcal{B}_j(y,t)}{(1+\frac{\varepsilon_j}{r_j}\eta_{4\varepsilon}y_1)}\cdot\nabla_y\phi = \frac{\eta_{4\varepsilon}E(y,t)}{(1+\frac{\varepsilon_j}{r_j}\eta_{4\varepsilon}y_1)}, \quad \text{in } \mathbb{R}^2\times[0,T],\\ &\phi(y,0) = 0, \quad \text{in } \mathbb{R}^2.\end{aligned} \tag{7.1}$$ pico1-new

We refer to (6.24) for the definition of $\eta_{4\varepsilon}$. Problem (7.1) is a transport equation. It is convenient to introduce the change of variable in time and let $t = t(\tau)$,

$$t = |\log\varepsilon|\int_0^\tau \varepsilon_j^2(s)\,ds, \quad T = |\log\varepsilon|\int_0^{\tau_T}\varepsilon_j^2(s)\,ds.$$

Problem (7.1) gets transformed into a problem for $\phi = \phi(y,t(\tau)) = \phi(y,\tau)$ as

$$\begin{aligned}&\phi_\tau + \frac{\nabla_y^\perp(\Gamma_0(y)+\eta_{4\varepsilon}\tilde{\mathcal{R}}_j(y,t)) + \eta_{4\varepsilon}\mathcal{B}_j(y,t)}{(1+\frac{\varepsilon_j}{r_j}\eta_{4\varepsilon}y_1)}\cdot\nabla_y\phi = \frac{\eta_{4\varepsilon}E(y,t)}{(1+\frac{\varepsilon_j}{r_j}\eta_{4\varepsilon}y_1)}, \quad \text{in } \mathbb{R}^2\times[0,\tau_T],\\ &\phi(y,0) = 0, \quad \text{in } \mathbb{R}^2\end{aligned} \tag{7.2}$$ pico2

We solve (7.2) by the method of characteristics.

The characteristic curve $\bar{y}(s;\tau,y)$ is by definition the solution $\bar{y}(s)$ of the ODE system

$$\begin{cases} \dfrac{d\bar{y}}{ds}(s) = \mathcal{B}(\bar{y}(s), t(s)) \\ \ \bar{y}(\tau) = y. \end{cases} \tag{7.3}$$

characteristic1-new

where

$$\mathcal{B}(y,t(\tau)) = \frac{\nabla_y^{\perp}(\Gamma_0(y) + \eta_{4\varepsilon}\tilde{\mathcal{R}}_j(y,t)) + \eta_{4\varepsilon}\mathcal{B}_j(y,t)}{(1+\frac{\varepsilon_j}{r_j}\eta_{4\varepsilon}y_1)}.$$

Observe that $(1+\frac{\varepsilon_j}{r_j}\eta_{4\varepsilon}y_1) = 1 + O(|\log\varepsilon|^{-3})$, uniformly in ε and that $\mathcal{B}$ is log-Lipschitz in y uniformly in t, that is, for some $L>0$,

$$|\mathcal{B}(y_1,t) - \mathcal{B}(y_2,t)| \le L|y_1-y_2|(1+|\log|y_1-y_2||)$$

and continuous in its two variables. Hence system (7.3) has a unique solution. For a locally bounded function E, the unique solution of (7.2) is then represented by the formula

$$\phi(y,\tau) = \int_0^{\tau} \mathcal{E}(\bar{y}(s;\tau,y),t(s))\,ds, \quad \mathcal{E} = \frac{\eta_{4\varepsilon}E(y,t)}{(1+\frac{\varepsilon_j}{r_j}\eta_{4\varepsilon}y_1)}. \tag{7.4}$$

phi

Under the assumptions in Lemma 6.2, we get that

$$(1+|y|^2)|D_y^2E(y,t)| + (1+|y|)|\nabla_y\partial_t\mathcal{E}(y,t)| + (1+|y|)|\nabla_y\mathcal{E}(y,t)| + |\mathcal{E}(y,t)| + |\partial_t\mathcal{E}(y,t)| \ \le\ C\,(1+|y|)^{\alpha}, \tag{7.5}$$

cotita

Our first result is

⟨transport1-new⟩ **Lemma 7.1.** *Let $1\le p\le +\infty$ and $m\in\mathbb{R}$. There exists a number $C>0$ such that for any function $E(y,t)$ that satisfies*

$$\sup_{t\in[0,T]} \|(1+|\cdot|)^{-m}E(\cdot,t)\|_{L^p(\mathbb{R}^2)} \ <\ +\infty$$

we have that for all sufficiently small ε, the solution of (7.1) satisfies

$$\sup_{t\in[0,T]} \|(1+|\cdot|)^{-m}\phi(\cdot,t)\|_{L^p(\mathbb{R}^2)} \ \le\ C\varepsilon^{-2}|\log\varepsilon|^{-1} \sup_{t\in[0,T]} \|(1+|\cdot|)^{-m}E(\cdot,t)\|_{L^p(\mathbb{R}^2)}.$$

Proof. Let

$$H(y,s) = \Gamma_0(y) + \eta_{4\varepsilon}\tilde{\mathcal{R}}_j(y,t(s)).$$

We use the explicit expression for $\mathcal{R}_j(y,t(s))$ to get that along the characteristic curves defined in (7.3) one has

$$\begin{aligned} \frac{d}{ds}H(\bar{y}(s),t(s)) &= \nabla_y(\Gamma_0(\bar{y}) + \eta_{4\varepsilon}\tilde{\mathcal{R}}_j(\bar{y},t(s)))\cdot\frac{d\bar{y}}{ds}(s) + \partial_s(\eta_{4\varepsilon}\tilde{\mathcal{R}}_j(\bar{y},t(s))) \\ &= \nabla_y(\Gamma_0(\bar{y}) + \eta_{4\varepsilon}\tilde{\mathcal{R}}_j(\bar{y},t(s)))\cdot\frac{\eta_{4\varepsilon}\mathcal{B}_j(\bar{y},t(s))}{(1+\frac{\varepsilon_j}{r_j}\eta_{4\varepsilon}y_1)} + \partial_s(\eta_{4\varepsilon}\tilde{\mathcal{R}}_j(\bar{y},t(s))). \end{aligned}$$

Observe that

$$\begin{aligned} \partial_s(\eta_{4\varepsilon}\tilde{\mathcal{R}}_j(\bar{y},t(s))) &= \eta_{4\varepsilon}\frac{d}{dt}(\frac{\varepsilon_j}{2r_j})\frac{dt}{ds}\,\bar{y}_1\left(\Gamma_0(\bar{y}) + \bar{A} + \Gamma(\bar{y})\right) + \eta_{4\varepsilon}\partial_t\tilde{\mathcal{R}}_j^1(\bar{y},t;P)\frac{dt}{ds} \\ &= \eta_{4\varepsilon}\frac{d}{dt}(\frac{\varepsilon_j}{2r_j})\varepsilon_j^2|\log\varepsilon|\,\bar{y}_1\left(\Gamma_0(\bar{y}) + \bar{A} + \Gamma(\bar{y})\right) + \eta_{4\varepsilon}\partial_t\tilde{\mathcal{R}}_j^1(\bar{y},t;P)\varepsilon_j^2|\log\varepsilon|, \end{aligned}$$

as $\frac{dt}{ds} = \varepsilon_j^2|\log\varepsilon|$. This yields

$$\frac{d}{ds}H(\bar{y}(s),t(s)) = O(\varepsilon^2|\log\varepsilon|).$$

We integrate both sides from τ to s and get

$$\begin{aligned} H(\bar{y}(s),t(s)) - H(y,t(\tau)) &= \int_{\tau}^{s} \nabla_y(\Gamma_0(\bar{y}(\xi)) + \eta_{4\varepsilon}\tilde{\mathcal{R}}_j(\bar{y}(\xi),t(\xi)))\cdot\frac{\eta_{4\varepsilon}\mathcal{B}_j(\bar{y}(\xi),t(\xi))}{(1+\frac{\varepsilon_j}{r_j}\eta_{4\varepsilon}\bar{y}_1(\xi))} \\ &+ |\log\varepsilon|\int_{\tau}^{s}\eta_{4\varepsilon}\frac{d}{dt}(\frac{\varepsilon_j}{2r_j})\varepsilon_j^2\,\bar{y}_1(\xi)\left(\Gamma_0(\bar{y}(\xi)) + \bar{A} + \Gamma(\bar{y}(\xi))\right)d\xi \\ &+ |\log\varepsilon|\int_{\tau}^{s}\eta_{4\varepsilon}\partial_t\tilde{\mathcal{R}}_j^1(\bar{y}(\xi),t;P)\varepsilon_j^2(t(\xi))\,d\xi. \end{aligned}$$

By assumptions (4.2) and (4.7), we have that $\frac{d}{dt}(\frac{\varepsilon_j}{2r_j}) = O(\varepsilon|\log\varepsilon|^{-\frac{1}{2}})$. Furthermore, $s \in [0,\tau)$, $\tau \in [0, \varepsilon_j^{-2}|\log\varepsilon|^{-1}]$, and using (6.23) we get

$$H(\bar{y}(s), t(s)) - H(y, t(\tau)) = |\log\varepsilon|^{-\frac{1}{2}} g(s)$$

for $g(s) = O(1)$ as $s \in [0,\tau)$, uniformly as $\varepsilon \to 0$, such that $g(\tau) = 0$. This relation is equivalent to

$$\begin{aligned}(1 + \eta_{4\varepsilon}\frac{\varepsilon_j}{2r_j}\bar{y}_1(s))\Gamma_0(\bar{y}(s)) + \eta_{4\varepsilon}\frac{\varepsilon_j}{2r_j}\,\bar{y}_1(s)\left(\bar{A} + \Gamma(\bar{y}(s))\right) + \eta_{4\varepsilon}\tilde{\mathcal{R}}_j^1(\bar{y}(s), t(s); P)\\ = (1 + \eta_{4\varepsilon}\frac{\varepsilon_j}{2r_j}y_1)\Gamma_0(y) + \eta_{4\varepsilon}\frac{\varepsilon_j}{2r_j}\,y_1\left(\bar{A} + \Gamma(y)\right) + \eta_{4\varepsilon}\tilde{\mathcal{R}}_j^1(y, t(\tau); P) + O(|\log\varepsilon|^{-\frac{1}{2}}).\end{aligned}$$

A closer look at this identity gives

$$\log(1 + |\bar{y}(s)|^2) = (1 + O(\eta_{4\varepsilon}|\log\varepsilon|^{-1}))\log(1 + |y|^2) + O(|\log\varepsilon|^{-\frac{1}{2}}),$$

and hence, taking the exponential on both sides and using the fact that $(\varepsilon|\log\varepsilon|)^{-|\log\varepsilon|^{-1}} = O(1)$ as $\varepsilon \to 0$,

$$\begin{aligned}(1 + |\bar{y}(s)|^2) &= (1 + |y|^2)\,[2 + O(\frac{\log|\log\varepsilon|}{|\log\varepsilon|}) + O(|\log\varepsilon|^{-\frac{1}{2}})]\\ &= (1 + |y|^2)\,[2 + O(|\log\varepsilon|^{-\frac{1}{2}})].\end{aligned}$$

Therefore there are positive constants a, b independent of $\tau \in (0, \tau_T)$ and ε such that

$$a(1 + |y(s)|^2) \le 1 + |\bar{y}(s)|^2 \le b(1 + |y(s)|^2) \quad \forall s \in [0, \tau_T). \tag{7.6}$$ bounds1

We recall the representation formula (7.4)

$$\phi(y,\tau) = \int_0^\tau \mathcal{E}(\bar{y}(s;\tau,y), t(s))\,ds, \quad \mathcal{E} = \frac{\eta_{4\varepsilon}E(y,t)}{(1 + \frac{\varepsilon_j}{r_j}\eta_{4\varepsilon}y_1)}$$

Using (7.6) we readily get

$$\sup_{t\in[0,T]}\|(1 + |\cdot|)^{-m}\phi(\cdot,t)\|_{L^p(\mathbb{R}^2)} \;\le\; C\varepsilon^{-2}|\log\varepsilon|^{-1}\sup_{t\in[0,T]}\|(1 + |\cdot|)^{-m}E(\cdot,t)\|_{L^p(\mathbb{R}^2)}.$$

for any $1 \le p \le +\infty$, as desired. □

A consequence of the bounds (7.6) on the characteristic curves is the fact if the spacial support of the function E stays at a uniform large distance of the origin then so does the solution of (7.1). This property that will be useful for the analysis of the outer problem.

⟨transport2⟩ **Lemma 7.2.** *There exist numbers $\varepsilon_0 > 0$, $\beta > 0$ such that for any sufficiently small $0 < \varepsilon < \varepsilon_0$ and any locally bounded function E such that*

$$E(y,t) = 0 \quad \text{for all} \quad (y,t) \in B_{\varepsilon^{-1}|\log\varepsilon|^{-\varsigma}}(0) \times [0,T]$$

we have that the solution of (7.1) satisfies

$$\phi(y,t) = 0 \quad \text{for all} \quad (y,t) \in B_{\beta\varepsilon^{-1}|\log\varepsilon|^{-\varsigma}}(0) \times [0,T].$$

Proof. This is an immediate consequence of estimates (7.6) for the characteristics and the representation formula (7.4). □

We can now prove Lemma 6.2.

Proof of Lemma 6.2.

The bound on the function $|\phi(y,t)|$ follows directly from the control on the characteristic curves we obtained in (7.6) and the representation formula (7.4).

To estimate $\partial_{y_\ell}\phi(y,\tau)$, we formally differentiate (7.4) with respect to y_ℓ, $\ell = 1, 2$ and obtain

$$\partial_{y_\ell}\phi(y,\tau) = \int_0^\tau \nabla_{\bar{y}}\mathcal{E}(\bar{y}(s;\tau,y), t(s)) \cdot \bar{y}_{y_\ell}(s;\tau,y)\,ds. \tag{7.7}$$ phi1

Recall that

$$\left(1 + \frac{\varepsilon_j}{r_j}\eta_{4\varepsilon}(\bar{y})\bar{y}_1\right)\frac{d\bar{y}}{ds}(s) = \nabla_y^\perp(\Gamma_0(\bar{y}) + \eta_{4\varepsilon}\tilde{\mathcal{R}}_j(\bar{y}, t(s))) + \eta_{4\varepsilon}\mathcal{B}_j(\bar{y}, t), \quad \bar{y}(\tau) = y,$$

(see (7.3)). Under assumptions (6.23), the above system has the form

$$\left(1+\frac{\varepsilon_j}{r_j}\eta_{4\varepsilon}(\bar y)\bar y_1\right)\frac{d\bar y}{ds}(s)=\nabla_y^{\perp}\left(\Gamma_0(\bar y)+\eta_{4\varepsilon}\frac{\varepsilon_j}{2r_j}\,y_1\left(\Gamma_0(y)+\bar A+\Gamma(y)\right)\right)+\varepsilon^2|\log\varepsilon|^{\frac12}\,\eta_{4\varepsilon}\,\mathcal{A}(\bar y,s),\quad \bar y(\tau)=y,$$

where $\mathcal{A}$ satisfies

$$|\partial_s\mathcal{A}(y,s)|+|\mathcal{A}(y,s)|\le M|y|,\quad |\nabla_y\mathcal{A}(y,s)|\le M \tag{7.8}$$ mathb

for some constant M independent of ε. Consider the point $a=(a_1,a_2)$, close to $y=0$, such that

$$\left(\nabla^{\perp}\left(\Gamma_0(y)+\eta_{4\varepsilon}\frac{\varepsilon_j}{2r_j}\,y_1\left(\Gamma_0(y)+\bar A+\Gamma(y)\right)\right)+\varepsilon^2|\log\varepsilon|^{\frac12}\,\eta_{4\varepsilon}\,\mathcal{A}(y,s)\right)_{y=a}=0.$$

A direct inspection gives

$$a_1=-\frac{\varepsilon_j}{8r_j}(\bar A+\Gamma(0))+a_{1\varepsilon}(s),\quad a_2=a_{2\varepsilon}(s),\quad\text{with}\quad |\log\varepsilon|^{\frac12}|\partial_s a_{i\varepsilon}|+|a_{i\varepsilon}|\lesssim\varepsilon^2|\log\varepsilon|^{\frac12},\quad i=1,2.$$

Introducing the change of variable $\bar z=\bar y-a$, we obtain

$$\left(1+\frac{\varepsilon_j}{r_j}\eta_{4\varepsilon}(\bar z_1+a_1)\right)\frac{d\bar z}{ds}(s)=B_a(\bar z)+\varepsilon^2|\log\varepsilon|^{\frac12}\,\eta_{4\varepsilon}\,\mathcal{A}(\bar z+a,s),\quad \bar z(\tau)=y-a,$$

where $\mathcal{A}$ stands for a function satisfying (7.8) and

$$\begin{aligned}B_a(z)&=\nabla_z^{\perp}\left(\Gamma_0(z+a)+\eta_{4\varepsilon}\frac{\varepsilon_j}{2r_j}\,(z_1+a_1)\left(\Gamma_0(z+a)+\bar A+\Gamma(z+a)\right)\right)\\&=\frac{(z+a)^{\perp}}{|z+a|}\left(\Gamma_0'(|z+a|)+\eta_{4\varepsilon}\frac{\varepsilon_j}{2r_j}\,(z_1+a_1)\left(\Gamma_0'(|z+a|)+\Gamma'(|z+a|)\right)\right)\\&\quad+\eta_{4\varepsilon}\frac{\varepsilon_j}{2r_j}\left(\Gamma_0(z+a)+\bar A+\Gamma(z+a)\right)\mathbf{e}_2+\varepsilon^2|\log\varepsilon|^{\frac12}\,\eta_{4\varepsilon}\,\mathcal{A}_1(z+a,s)\end{aligned}$$

where $\mathcal{A}_1$ also satisfies (7.8). The vector field $B_a(z)$ is smooth and the choice of a gives

$$B_a(0)+\varepsilon^2|\log\varepsilon|^{\frac12}\,\eta_{4\varepsilon}\,\mathcal{A}(a,s)=0,\quad\forall s. \tag{7.9}$$ mathb

Let us introduce polar coordinates around a:

$$\bar z=\bar y-a=\bar\rho e^{i\bar\theta}.$$

Hence

$$\frac{d\bar z}{ds}=\bar\rho_s\hat\rho+\bar\rho\bar\theta_s\hat\theta,\quad \hat\rho=e^{i\bar\theta},\quad \hat\theta=ie^{i\bar\theta}.$$

We have that

$$\begin{aligned}B_a(z)\cdot\hat\rho&=\frac{a^{\perp}\cdot\hat\rho}{|z+a|}\left(\Gamma_0'(|z+a|)+\eta_{4\varepsilon}\frac{\varepsilon_j}{2r_j}\,(z_1+a_1)\left(\Gamma_0'(|z+a|)+\Gamma'(|z+a|)\right)\right)\\&\quad+\eta_{4\varepsilon}\frac{\varepsilon_j}{2r_j}\left(\Gamma_0(z+a)+\bar A+\Gamma(z+a)\right)\sin\theta+\varepsilon^2|\log\varepsilon|^{\frac12}\,\eta_{4\varepsilon}\,\mathcal{A}(z+a,s)\cdot\hat\rho\end{aligned}$$

and

$$\begin{aligned}B_a(z)\cdot\hat\theta&=\frac{z^{\perp}\cdot\hat\theta}{|z+a|}\left(\Gamma_0'(|z+a|)+\eta_{4\varepsilon}\frac{\varepsilon_j}{2r_j}\,(z_1+a_1)\left(\Gamma_0'(|z+a|)+\Gamma'(|z+a|)\right)\right)\\&\quad+\frac{a^{\perp}\cdot\hat\theta}{|z+a|}\left(\Gamma_0'(|z+a|)+\eta_{4\varepsilon}\frac{\varepsilon_j}{2r_j}\,(z_1+a_1)\left(\Gamma_0'(|z+a|)+\Gamma'(|z+a|)\right)\right)\\&\quad+\eta_{4\varepsilon}\frac{\varepsilon_j}{2r_j}\left(\Gamma_0(z+a)+\bar A+\Gamma(z+a)\right)\cos\theta+\varepsilon^2|\log\varepsilon|^{\frac12}\,\eta_{4\varepsilon}\,\mathcal{A}(z+a,s)\cdot\hat\theta.\end{aligned}$$

Set

$$\begin{aligned}B_{a1}(z,s)&=\frac{a_1}{|z+a|}\left(\Gamma_0'(|z+a|)+\eta_{4\varepsilon}\frac{\varepsilon_j}{2r_j}\,(z_1+a_1)\left(\Gamma_0'(|z+a|)+\Gamma'(|z+a|)\right)\right)\\&\quad+\eta_{4\varepsilon}\frac{\varepsilon_j}{2r_j}\left(\Gamma_0(z+a)+\bar A+\Gamma(z+a)\right).\end{aligned}$$

In the region we are considering

$$B_{a1}(0,s)=O(\varepsilon^2|\log\varepsilon|^{\frac12}),\quad |B_{a1}(z,s)|=O(\varepsilon|\log\varepsilon|),\quad |\partial_s B_{a1}(z,s)|=|\partial_t B_{a1}|\,\varepsilon^2\,|\log\varepsilon|=O(\varepsilon^3|\log\varepsilon|^{-\frac12}).$$

With the definition of B_{a1}, we can write

$$
\begin{aligned}
B_a(z)\cdot\hat\rho &= B_{a1}\,\sin\theta + \varepsilon^2|\log\varepsilon|^{\frac12}\,\eta_{4\varepsilon}\,\mathcal{A}(z+a,s)\cdot\hat\rho\\
B_a(z)\cdot\hat\theta &= \frac{\rho}{|z+a|}\left(\Gamma_0'(|z+a|) + \eta_{4\varepsilon}\frac{\varepsilon_j}{2r_j}\,(z_1+a_1)\,(\Gamma_0'(|z+a|)+\Gamma'(|z+a|))\right)\\
&\quad + B_{a1}\,\cos\theta + \varepsilon^2|\log\varepsilon|^{\frac12}\,\eta_{4\varepsilon}\,\mathcal{A}(z+a,s)\cdot\hat\theta.
\end{aligned}
$$

Inserting these computations in the ODEs system for $\frac{d\bar z}{ds}$ we get

$$
\begin{aligned}
\left(1+\frac{\varepsilon_j}{r_j}\eta_{4\varepsilon}(\bar z_1+a_1)\right)\,\bar\rho_s &= B_{a1}\,\sin\bar\theta + \varepsilon^2|\log\varepsilon|^{\frac12}\,\eta_{4\varepsilon}\,\mathcal{A}(\bar z+a,s)\cdot\hat\rho,\\
\left(1+\frac{\varepsilon_j}{r_j}\eta_{4\varepsilon}(\bar z_1+a_1)\right)\,\bar\rho\bar\theta_s &= \frac{\bar\rho}{|\bar z+a|}\left(\Gamma_0'(|\bar z+a|) + \eta_{4\varepsilon}\frac{\varepsilon_j}{2r_j}\,(\bar z_1+a_1)\,(\Gamma_0'(|\bar z+a|)+\Gamma'(|\bar z+a|))\right)\\
&\quad + B_{a1}\,\cos\bar\theta + \varepsilon^2|\log\varepsilon|^{\frac12}\,\eta_{4\varepsilon}\,\mathcal{A}(\bar z+a,s)\cdot\hat\theta.
\end{aligned}
\tag{7.10}
$$
mathc

Let us analyse the second equation in (7.10). Using the control on the characteristic curves we obtained in the proof of Lemma 7.1, we have a bound like in (7.6) also for $\bar z(s)$. Thanks to (7.9), we can divide the second equation in (7.10) by $\bar\rho$ and we write it as

$$
\bar\theta_s = -\frac{4}{1+\bar\rho^2}\,(1+g_\varepsilon(s,\bar\rho))\; + \tilde B_{a1}\,\cos\bar\theta\; + \varepsilon^2|\log\varepsilon|^{\frac12}\,\eta_{4\varepsilon}\,\mathcal{A}(z+a,s)\cdot\hat\theta,
$$

where

$$
(1+|z|)\,|\tilde B_{a1}(z,s)| = O(\varepsilon|\log\varepsilon|),\quad (1+|z|)\,|\partial_s\tilde B_{a1}(z,s)| = O(\varepsilon^3|\log\varepsilon|).
$$

Here $|g_\varepsilon(s,\bar\rho)|$ is a function such that $|g_\varepsilon(s,\bar\rho)| = O(|\log\varepsilon|^{-\zeta+1})$, as $\varepsilon\to 0$, where $\zeta>2$ is the number introduced in the definition of $\eta_{4\varepsilon}$ as in (6.24). Let $\bar\theta_0$ be

$$
\bar\theta_0(s) = -\int_\tau^s \frac{4}{1+\bar\rho^2(\eta)}\,d\eta
$$

and let g to solve $\partial_s\left(\bar\theta_0(1+g)\right) = -\frac{4}{1+\bar\rho^2}\,(1+g_\varepsilon(s,\bar\rho))$. This is possible for

$$
\bar\theta_0 g = -\int_\tau^s \frac{4}{1+\bar\rho^2(\eta)}g_\varepsilon(\eta,\bar\rho)d\eta.
$$

Let $\rho=\bar\rho(\tau)$. Following the proof of Lemma 7.1, we have that $\bar\rho(s)^2 = \rho^2(1+o(1))$ as $\varepsilon\to0$, we get

$$
(1+g)\bar\theta_0 = -\frac{4(s-\tau)}{1+\rho^2}\left(1+O\left(\frac{|\log\varepsilon|^{-1}}{1+\rho^2}\right)\right).
$$

Let $\bar\theta_1$ solve at main order

$$
\partial_s\bar\theta_1 = \tilde B_{a1}\,\cos\left(\bar\theta_0\right).
$$

An integration by parts gives, for $z=\bar\theta_0(s)$

$$
\begin{aligned}
\bar\theta_1 &= -\int \tilde B_{a1}\frac{1+\bar\rho^2}{4}\cos z\,dz\\
&= \tilde B_{a1}\frac{1+\bar\rho^2}{4}\sin\left(\bar\theta_0\right) - \int_s^\tau \frac{d}{ds}(\tilde B_{a1})\,\frac{1+\bar\rho^2}{4}\sin\left(\bar\theta_0\right)\\
&= \tilde B_{a1}\frac{1+\bar\rho^2}{4}\sin\left(\bar\theta_0\right)\left(1+O(|\log\varepsilon|^{-\frac12})\right).
\end{aligned}
$$

We write $\bar\theta = \bar\theta_0+\bar\theta_1+\theta_2$. We get $\theta_2 = O(|\log\varepsilon|^{-\frac12})$ and $\partial_s\theta_2 = O(\varepsilon^2|\log\varepsilon|^{\frac12})$ and we conclude that

$$
\begin{aligned}
\bar\theta &= -\frac{4(s-\tau)}{1+\bar\rho^2}\left(1+O\left(\frac{|\log\varepsilon|^{-1}}{1+\bar\rho^2}\right)\right) + \tilde B_{a1}\frac{1+\bar\rho^2}{4}\sin\left(\bar\theta_0\right)\left(1+O(|\log\varepsilon|^{-\frac12})\right) + O(|\log\varepsilon|^{-\frac12})\\
&= -\frac{4(s-\tau)}{1+\bar\rho^2}\left(1+O\left(\frac{|\log\varepsilon|^{-1}}{1+\bar\rho^2} += O(\varepsilon\bar\rho\eta_{4\varepsilon})\right)\right) + O(|\log\varepsilon|^{-\frac12}).
\end{aligned}
\tag{7.11}
$$
perparti0

Inserting this result into the first equation in (7.10), we get

$$
\bar\rho(s) = \rho + \rho O(\varepsilon\rho\eta_{4\varepsilon})\cos(\frac{4(s-\tau)}{1+\rho^2}).
\tag{7.12}
$$
perparti

Since $a = a(s)$, we have that

$$\frac{d\bar y}{dy_\ell} = \frac{d\bar z}{dy_\ell} = \bar\rho_{y_\ell}\hat\rho + \bar\rho\bar\theta_{y_\ell}\hat\theta,$$

and using (7.7) we write

$$\partial_{y_\ell}\phi(y,\tau) = \int_0^\tau \nabla_{\bar y}\mathcal{E}(\bar y, t(s)) \cdot \left(\bar\rho_{y_\ell}\hat\rho + \bar\rho\bar\theta_{y_\ell}\hat\theta\right)\, ds.$$

First we estimate $\int_0^\tau \nabla_{\bar y}\mathcal{E}(\bar y, t(s)) \cdot \bar\rho_{y_\ell}\hat\rho\, ds$. From (7.12) we obtain

$$\bar\rho_{y_\ell} = \left(1 + O(\varepsilon\rho\eta_{4\varepsilon})\cos\frac{4(s-\tau)}{1+\rho^2}\right) + O(\varepsilon\rho\eta_{4\varepsilon})\,\frac{s-\tau}{1+\rho^2}\sin\frac{4(s-\tau)}{1+\rho^2}\,.$$

Set $g(\bar y, s) := \nabla_{\bar y}\mathcal{E}(\bar y(s), t(s)) \cdot \hat\rho$,

$$\begin{aligned}\int_0^\tau \nabla_{\bar y}\mathcal{E}(\bar y, t(s)) \cdot \bar\rho_{y_\ell}\hat\rho\, ds &= \int_0^\tau g\left(1 + O(\varepsilon\rho\eta_{4\varepsilon})\cos\frac{4(s-\tau)}{1+\rho^2}\right)\, ds \\ &\quad + \int_0^\tau g\, O(\varepsilon\rho\eta_{4\varepsilon})\,\frac{s-\tau}{1+\rho^2}\sin\frac{4(s-\tau)}{1+\rho^2}\, ds = A + B.\end{aligned} \tag{7.13}$$

boh

From (7.5) we readily get

$$|A| \lesssim \varepsilon^{-2}|\log\varepsilon|^{-1}(1+|y|)^{\alpha-1}.$$

To estimate B, we integrate by parts, we use that $\int x\sin x\, dx = -x\cos x + \sin x + C$ and obtain

$$\begin{aligned}|B| &\le O(\varepsilon\rho\eta_{4\varepsilon}))\left(|g(\bar y(\tau), t(\tau))|\,\tau\,|\cos\frac{4\tau}{1+\rho^2}| + |g|\,(1+\rho^2)\,|\sin\frac{4\tau}{1+\rho^2}|\right) \\ &\quad + O(\varepsilon\rho\eta_{4\varepsilon}))\,\varepsilon^2|\log\varepsilon|\left(C\,(1+|y|^{\alpha-1})\int_0^\tau s\cos\frac{4s}{1+\rho^2} + \sin\frac{4s}{1+\rho^2}\right) \\ &\lesssim O(\varepsilon\rho\eta_{4\varepsilon}))\,\varepsilon^{-2}|\log\varepsilon|^{-1}\,(1+|y|^{\alpha-1}).\end{aligned}$$

These estimates allow us conclude that

$$\int_0^\tau \nabla_{\bar y}\mathcal{E}(\bar y, t(s)) \cdot \bar\rho_{y_\ell}\hat\rho\, ds \lesssim \varepsilon^{-2}|\log\varepsilon|^{-1}(1+|y|)^{\alpha-1}.$$

Next we treat the term we estimate $\int_0^\tau \nabla_{\bar y}\mathcal{E}(\bar y, t(s)) \cdot \bar\rho\bar\theta_{y_\ell}\hat\theta\, ds$. To this purpose we observe that

$$\frac{d}{ds}\mathcal{E} = \nabla_{\bar y}\mathcal{E}\frac{d\bar y}{ds} + \varepsilon^2|\log\varepsilon|\,\frac{d}{dt}\mathcal{E}.$$

Since $\frac{d\bar y}{ds} = \bar\rho_s\hat\rho + \bar\rho\bar\theta_s\hat\theta$, letting $\gamma = \frac{\bar\theta_{y_\ell}}{\bar\theta_s}$ we get

$$\nabla_{\bar y}\mathcal{E}(\bar y, t(s)) \cdot \bar\rho\bar\theta_{y_\ell}\hat\theta = \gamma\frac{d}{ds}\mathcal{E} - \gamma\nabla_{\bar y}\mathcal{E}\bar\rho_s\hat\rho - \gamma\,\varepsilon^2|\log\varepsilon|\,\frac{d}{dt}\mathcal{E}.$$

We evaluate separately the following integrals

$$I = \int_0^\tau \gamma\frac{d}{ds}\mathcal{E}\, ds, \quad II = \int_0^\tau \gamma\nabla_{\bar y}\mathcal{E}\bar\rho_s\hat\rho\, ds, \quad III = \int_0^\tau \gamma\,\varepsilon^2|\log\varepsilon|\,\frac{d}{dt}\mathcal{E}\, ds.$$

From (7.11) it follows that

$$\begin{aligned}\partial_{y_\ell}\bar\theta &= \frac{8(s-\tau)\bar\rho\bar\rho_{y_\ell}}{(1+\bar\rho^2)^2}\left(1 + O\left(\frac{|\log\varepsilon|^{-1}}{1+\bar\rho^2}\right)\right)\left(1 + O(|\log\varepsilon|^{-\frac12})\right) \\ &\quad + \left((\partial_{y_\ell}\tilde B_{a1}\frac{1+\bar\rho^2}{4} + \tilde B_{a1}\frac{\bar\rho\partial_{y_\ell}\bar\rho}{2})\sin\bar\theta_0 + \tilde B_{a1}\frac{1+\bar\rho^2}{4}\cos\left(\bar\theta_0\right)\partial_{y_\ell}\bar\theta_0\right)\left(1 + O(|\log\varepsilon|^{-\frac12})\right)) \\ &\quad + O(|\log\varepsilon|^{-\frac12}).\end{aligned}$$

Under the assumptions on $\tilde B_{a1}$, we get

$$\left|(\partial_{y_\ell}\tilde B_{a1}\frac{1+\bar\rho^2}{4} + \tilde B_{a1}\frac{\bar\rho\partial_{y_\ell}\bar\rho}{2})\sin\bar\theta_0\right| \lesssim \varepsilon|\sin\bar\theta_0|.$$

Hence

$$(\partial_{y_\ell}\tilde B_{a1}\frac{1+\bar\rho^2}{4} + \tilde B_{a1}\frac{\bar\rho\partial_{y_\ell}\bar\rho}{2})\sin\bar\theta_0 = O(\varepsilon\bar\rho\eta_{4\varepsilon})\frac{(s-\tau)\bar\rho\bar\rho_{y_\ell}}{(1+\bar\rho^2)^2}.$$

We also have

$$\tilde{B}_{a1}\frac{1+\bar{\rho}^2}{4}\cos\bar{\theta}_0\partial_{y_\ell}\bar{\theta}_0 = O(\varepsilon\bar{\rho}\eta_{4\varepsilon})\frac{(s-\tau)\bar{\rho}\bar{\rho}_{y_\ell}}{(1+\bar{\rho}^2)^2}.$$

We thus have

$$\partial_{y_\ell}\bar{\theta} = \frac{8(s-\tau)\bar{\rho}\bar{\rho}_{y_\ell}}{(1+\bar{\rho}^2)^2}\left(1+O\left(\frac{|\log\varepsilon|^{-1}}{1+\bar{\rho}^2}\right)\right)\left(1+O(|\log\varepsilon|^{-\frac{1}{2}})+O(\varepsilon\bar{\rho}\eta_{4\varepsilon})\right)+O(|\log\varepsilon|^{-\frac{1}{2}}).$$

Hence

$$\gamma := \frac{\bar{\theta}_{y_\ell}}{\bar{\theta}_s} = \frac{2(s-\tau)\bar{\rho}\bar{\rho}_{y_\ell}}{(1+\bar{\rho}^2)}\left(1+O\left(\frac{|\log\varepsilon|^{-1}}{1+\bar{\rho}^2}\right)\right)\left(1+O(|\log\varepsilon|^{-\frac{1}{2}})+O(\varepsilon\bar{\rho}\eta_{4\varepsilon})\right)+O(|\log\varepsilon|^{-\frac{1}{2}}),$$

and

$$\frac{d\gamma}{ds} = O\left(\frac{\bar{\rho}\bar{\rho}_{y_\ell}}{(1+\bar{\rho}^2)}\right).$$

Since

$$I = \int_0^\tau \frac{d}{ds}(\gamma E)\,ds - \int_0^\tau \mathcal{E}\frac{d\gamma}{ds}\,ds,$$

we easily get

$$|\int_0^\tau \frac{d}{ds}(\gamma E)\,ds| \lesssim |\mathcal{E}(y,\tau)|\,\frac{\tau}{1+\rho} \lesssim \varepsilon^{-2}\,|\log\varepsilon|^{-1}\,(1+|y|)^{\alpha-1},$$

and

$$\left|\int_0^\tau \mathcal{E}\frac{d\gamma}{ds}\,ds\right| \lesssim \varepsilon^{-2}\,|\log\varepsilon|^{-1}\,(1+|y|)^{\alpha-1},$$

as expected. We now treat integral $II := \int_0^\tau \gamma\nabla_{\bar{y}}\mathcal{E}\bar{\rho}_s\hat{\rho}\,ds$. From the first equation in (7.10) and (7.11), we get

$$II = \int_0^\tau B_{a1}\left(\nabla_{\bar{y}}\mathcal{E}\cdot\hat{\rho}\right)\bar{\rho}\,\bar{\rho}_{y_\ell}\frac{2(s-\tau)}{1+\rho^2}\,\sin\frac{2(s-\tau)}{1+\rho^2}\,(1+o(1))\,ds + O\left(\varepsilon^2|\log\varepsilon|^{\frac{1}{2}}\int_0^\tau \gamma(\nabla_{\bar{y}}\mathcal{E}\cdot\hat{\rho})\right).$$

To estimate the first integral in the above formula we proceed as in the estimate of the term B in formula (7.13). The second integral can be bounded as follows

$$\left|\varepsilon^2|\log\varepsilon|^{\frac{1}{2}}\int_0^\tau \gamma(\nabla_{\bar{y}}\mathcal{E}\cdot\hat{\rho})\right| \lesssim \varepsilon^2|\log\varepsilon|^{\frac{1}{2}}\,(1+|y|)^{\alpha-1}\int_0^\tau (s-\tau)\,ds \lesssim \varepsilon^{-2}|\log\varepsilon|^{-\frac{3}{2}}\,(1+|y|)^{\alpha-1}.$$

We thus get

$$|II| \lesssim \varepsilon^{-2}\,|\log\varepsilon|^{-1}\,(1+|y|)^{\alpha-1}.$$

In an analogous way we can treat the last integral $III = \int_0^\tau \gamma\,\varepsilon^2|\log\varepsilon|\,\frac{d}{dt}\mathcal{E}\,ds$ and we obtain

$$|III| \lesssim \varepsilon^{-2}\,|\log\varepsilon|^{-1}\,(1+|y|)^{\alpha-1}.$$

Collecting all the above estimates, we arrive to the desired bound

$$|\partial_{y_\ell}\phi(y,\tau)| \lesssim \varepsilon^{-2}\,|\log\varepsilon|^{-1}\,(1+|y|)^{\alpha-1}.$$

This concludes the proof of Lemma 6.2.

8. The outer modified transport equation

roofTransportOuter⟩ *Proof.* Let $x\in\Sigma$. We shall prove that $\bar{x}(s;t,x)\in\Sigma$, for all $s\in(0,t)$. By contradiction, assume there exists $s_0\in(0,t)$ such that $\bar{x}(s;t,x)\in\Sigma$, for all $s\in(s_0,t)$ and $\bar{x}(s_0;t,x)\in\partial\Sigma$. Setting $\bar{x}(s_0;t,x) = (\bar{r}(s_0;t,x),\bar{z}(s_0;t,x))$, it holds $\bar{r}(s_0;t,x)=0$. Setting $B(x,t)=(B_1(x,t),B_2(x,t))$, we get from (6.28) that $B_1(\bar{x}(s_0;t,x),s)=0$. See (6.30). Then the ODEs

$$\frac{d}{ds}\bar{x}(s;x,t)\cdot\mathbf{e_1} = B_1(\bar{x}(s;t,x),s),\quad \bar{x}(s_0;x,t)\cdot\mathbf{e_1}=0$$

has the trivial solution $\bar{x}(s;x,t)\cdot\mathbf{e_1}\equiv 0$, which contradicts uniqueness of solutions for ODEs. Formula (6.32) is then well-defined.

In order to get estimates on the solution, we write the solution using Duhamel's representation formula

$$\phi^{out}(x,t) = |\log\varepsilon|^{-1}\int_0^t \tilde{E}_0(\bar{x}(s;x,t),s)\,ds$$

where $\bar{x}(s;x,t)$ are the characteristic curves defined as

$$\frac{d}{ds}\bar{x}(s;x,t) = B(\bar{x}(s;x,t),s) \quad s\in(0,t), \quad \bar{x}(t;x,t)=x.$$

Using the estimates on $B(x,t)$ described in (6.30) we get

$$|\phi(x,t)| \le |\log\varepsilon|^{-1}\tau\|E\|_{L^\infty(\Sigma\times[0,\tau])} \le |\log\varepsilon|^{-1}t\|E\|_{L^\infty(\Sigma\times[0,t])}.$$

We also have that for $1\le p<+\infty$,

$$\int_\Sigma |E(\bar{x}(s;t,x),s)|^p\,dx \le C\int_\Sigma |E(x;t,s)|^p\,dx$$

for some $C>0$, since the differential of area for the change of variable $x\mapsto \bar{x}(s;t,x)$ is a bounded function. From this we readily get

$$\|\phi(\cdot,t)\|_{L^p(\Sigma)} \ \le |\log\varepsilon|^{-1}\ t\sup_{s\in[0,t]}\|E(\cdot,s)\|_{L^p(\Sigma)}.$$

With the change of variables $y=\frac{x-P_j(t)}{\varepsilon}$ an equation of the type (7.1) is satisfied. The result then follows from Lemma 7.2: since E_0 and $\tilde{E}_0$ are zero in the spacial region $\sum_{j=1}^k B(P_j,2|\log\varepsilon|^{-3})$ for all $t\in[0,T)$, there exists $\beta>0$ such that the solution ϕ to (6.27) is zero in $\sum_{j=1}^k B(P_j,\beta|\log\varepsilon|^{-3})$, for all $t\in[0,T)$. An alternative way to see this is to estimate the characteristics for points $x\in B(P_j,\delta|\log\varepsilon|^{-3})$ and use (6.31).

This fact allows us to say that ϕ satisfies an equation of the form

$$\begin{cases} |\log\varepsilon|\partial_t\phi + \nabla_x^\perp H\cdot\nabla\phi = E(x,t) & \text{in } \Sigma\times[0,T], \\ \phi(x,0)=0 & \text{in } \Sigma, \end{cases}$$

where

$$H(x,t) = |\log\varepsilon|\Big(1-\sum_{j=1}^N \eta_j(x,t)\Big)B$$

and η_j is a smooth function with $\eta_j(x,t)=1$ whenever $|x-P_j(t)|<\beta|\log\varepsilon|^{-3}$ and $=0$ if $|x-P_j(t)|>2\beta|\log\varepsilon|^{-3}$. Using the explicit expression of the coefficient B given in (6.28)-(6.31), we see that

$$H(x,t) = \Big(1-\sum_{j=1}^N \eta_j(x,t)\Big)O(|\log\varepsilon|), \quad \text{in } \Sigma\times[0,T],$$

uniformly as $\varepsilon\to 0$. Hence

$$|\bar{x}(s;t,x)| = |x| + O(1), \qquad \text{as} \qquad \varepsilon\to 0,$$

and bounds (6.35) readily follow. □

8.1. **Uniform continuity.** Let us consider an equation of the form

$$\begin{cases} |\log\varepsilon|\, r\,\partial_t\phi + \nabla_x^\perp(r^2(\Psi^0-\alpha_0|\log\varepsilon|+e))\cdot\nabla\phi = E(x,t) & \text{in } \Sigma\times[0,T], \\ \phi(x,0)=0, & \text{in } \Sigma. \end{cases} \tag{8.1}$$

ff

Consider the characteristics $\bar{x}=\bar{x}(s;t,x)=(\bar{r},\bar{z})$ for (8.1)

$$|\log\varepsilon|\frac{d}{ds}\bar{x}(s;t,x) = \bar{r}\nabla^\perp H\left(\bar{x}(s;t,x)\right) + 2H\left(\bar{x}(s;t,x)\right)\mathbf{e}_2,$$
$$\bar{x}(t;t,x)=x$$

where

$$H(x,t) := \Psi^0(x,t) - \alpha_0|\log\varepsilon| + e(x,t), \quad x=(r,z).$$

We know that the characteristics $\bar{x}=\bar{x}(s;t,x)$ are well defined in $\Sigma\times[0,T]$ if $\nabla e,\ r\Delta e\in L^\infty(\Sigma\times[0,T])$. Besides the formula

$$\phi(x,t) = \int_0^t E(\bar{x}(s;t,x),s)\,ds$$

gives the solution of (8.1). We have

⟨modc1⟩ **Lemma 8.1.** *For all $\varrho > 0$ there exists a positive number*

$$\delta = \delta(\varrho, \|\nabla e\|_{L^\infty(\Sigma\times[0,T])}, \|r\Delta e\|_{L^\infty(\Sigma\times[0,T])}, T, \Sigma)$$

such that for all (x_1,t_1), $(x_2,t_2) \in \bar\Sigma \times [0,T]$ we have

$$|t_1 - t_2| + |x_1 - x_2| < \delta \implies |\bar x(s;t_1,x_1) - \bar x(s;t_2,x_2)| < \varrho.$$

If $E(x,t)$ is a bounded function that satisfies

$$\sup_{t\in[0,T],\ |x_1-x_2|<\mu} |E(x_1,t) - E(x_2,t)| \le \Theta(\mu)$$

for a certain function Θ with $\Theta(\mu) \to 0$ as $\mu \to 0$, then the solution $\phi(x,t)$ of (8.1) satisfies that for all (x_1,t_1), $(x_2,t_2) \in \bar\Sigma \times [0,T]$ we have

$$|t_1 - t_2| + |x_1 - x_2| < \delta \implies |\phi(x_1,t_1) - \phi(x_2,t_2)| < \varrho.$$

Proof. By definition

$$|\log\varepsilon| \frac{d}{ds}\bar x(s;t_i,x_i) = \bar r(s;t_i,x_i)\nabla^\perp H\left(\bar x(s;t_i,x_i)\right) + 2H\left(\bar x(s;t_i,x_i)\right)\mathbf{e}_2,$$
$$\bar x(t_i;t_i,x_i) = x_i$$

for $i = 1,2$. Let $h(s) = \bar x(s;t_1,x_1) - \bar x(s;t_2,x_2)$. Then

$$|\frac{d}{ds}h(s)| \le C\,A\,|h(s)|\,|\log(|h(s)|)|,$$

where $A = (\|\nabla e\|_\infty + \|r\Delta e\|_\infty)$. Setting $\beta(s) := |h(s)|^2$, we can assume that $0 < \beta(s) < 1$. Then we get

$$\left|\frac{d}{ds}\left(\log\left(\log\frac{1}{\beta(s)}\right)\right)\right| \le C\,A.$$

Integrating, we obtain

$$e^{-C\,A\,T}\log\frac{1}{\beta(t_1)} \le \log\frac{1}{\beta(s)} \le e^{C\,A\,T}\log\frac{1}{\beta(t_1)}. \tag{8.2}$$ cott

Observe now that

$$|h(t_1)| = |x_1 - x_2| + |x_2 - \bar x(t_1;t_2,x_2)| \le C\,A\,\left(|x_1 - x_2| + |t_1 - t_2|\right). \tag{8.3}$$ cott1

Combining (8.2) and (8.3), we obtain the first statement of the Lemma.

The second statement of the Lemma is a direct consequence of the representation formula

$$\phi(x,t) = \int_0^t E(\bar x(s;t,x),s)\,ds$$

for the solution of (8.1). □

Let us now consider

$$\begin{cases} \phi_\tau + \nabla_x^\perp\left(\Gamma_0(y) + b(y,t)\right)\cdot\nabla\phi = E(y,t) & \text{in } \mathbb{R}^2\times[0,\tau_T], \\ \phi(y,0) = 0, & \text{in } \mathbb{R}^2, \end{cases} \tag{8.4}$$ ff1

with $b(y,t) = 0 = E(y,t)$ for $|y| > R$ and

$$t = |\log\varepsilon|\int_0^\tau \varepsilon_j^2(s)\,ds, \quad T = |\log\varepsilon|\int_0^{\tau_T}\varepsilon_j^2(s)\,ds.$$

This problem has the same form as the one in (7.2), and under our assumption

$$\tau_T \sim \varepsilon^{-2}|\log\varepsilon|^{-1}, \quad R\to\infty \quad \text{as} \quad \varepsilon\to 0.$$

As in the previous problem, the modulus of continuity for the characteristics $\bar y = \bar y(s;\tau,y)$ for (8.4) depends only on $\|\Delta_y b\|_{L^\infty(\mathbb{R}^2\times[0,T])}$, and that of the solution only on a uniform bound for E and for its modulus of continuity. Arguing as in the proof of Lemma 8.1, we can show that

⟨modc2⟩ **Lemma 8.2.** *Assume that*

$$\sup_{\tau\in[0,\tau_T],\ |y_1-y_2|<\mu} |E(y_1,\tau)-E(y_2,\tau)| \le \Theta(\mu)$$

for a certain function Θ *with* $\Theta(\mu)\to 0$ *as* $\mu\to 0$. *Then for each* $\varrho>0$ *there exists a positive number*

$$\delta = \delta(\varrho, \|\Delta b\|_{L^\infty(\mathbb{R}^2\times[0,\tau_T])}, \|E\|_\infty, \Theta, T, \varepsilon)$$

such that the solution $\phi(y,\tau)$ *of (8.4) satisfies that for all* (y_1,τ_1), $(y_2,\tau_2)\in\mathbb{R}^2\times[0,\tau_T]$ *we have*

$$|\tau_1-\tau_2| + |y_1-y_2| < \delta \implies |\phi(y_1,\tau_1)-\phi(y_2,\tau_2)| < \varrho.$$

These results will be useful in Section 11 for the final argument in our construction.

9. The inner-outer gluing procedure

⟨sec8⟩ In the previous sections we proved the existence of points $P_1^1,\dots,P_k^1$ in (4.2) such that, for any choice of points $\mathbf{a}_1,\dots,\mathbf{a}_k$ in (4.2) satisfying (4.5), there exists a good approximate leapfrogging of vortex rings (W^*,Ψ^*) with the form (5.1)-(5.2), as described in Proposition 5.1.

We now set up the inner-outer gluing procedure which will lead us to find an exact solution (Ψ, W) to (4.1) close to the approximation (Ψ^*, W^*).

The solution (Ψ, W) will have the form

$$\begin{aligned}\Psi &= \Psi^*(x,t) + \sum_{j=1}^k \bar\eta_{j2}\frac{1}{r_j}\psi_j\Big(\frac{x-P_j}{\varepsilon_j},t\Big) + \psi^{out}(x,t)\\ W &= W^*(x,t) + \sum_{j=1}^k \bar\eta_{j1}\frac{1}{r_j\varepsilon_j^2}\phi_j\Big(\frac{x-P_j}{\varepsilon_j},t\Big) + \phi^{out}(x,t)\end{aligned} \tag{9.1}$$ f11

where

$$\bar\eta_{jN}(x) = \eta_N\Big(\frac{|\log\varepsilon|^5|x-P_j|}{\delta}\Big).$$

Here η_N is the cut-off function introduced in (2.3). This ansatz has the same form as the one used in (5.1) and (5.2) for the construction of the improved approximate leapfrogging of vortex rings (W^*,Ψ^*). We notice though that here we are taking cut-offs slightly shorter than the ones in (5.3).

Let S_1 and S_2 be the Euler operators introduced in (2.2). Then the operator S_1 evaluated at (W,Ψ) becomes

$$S_1(W,\Psi) = \sum_{j=1}^k \frac{\bar\eta_{1j}}{\varepsilon_j^4} E_j[\phi_j, \psi_j\,\psi^{out}, P] + E^{out}[\phi^{out}, \Psi^{out}, \phi^{in}, \psi^{in}, P] \tag{9.2}$$ fullS1

where

$$\phi^{in}(y,t) = (\phi_1(y,t),\dots,\phi_k(y,t)), \quad \psi^{in}(y,t) = (\psi_1(y,t),\dots,\psi_k(y,t)).$$

In (9.2) E_j, $j=1,\dots,k$, and E^{out} are defined respectively

$$\begin{aligned} E_j^{in}[\phi_j, \psi_j\,\psi^{out}, \mathbf{a}](y,t) &:= |\log\varepsilon|\varepsilon_j^2\Big(1+\frac{\varepsilon_j}{r_j}y_1\Big)\partial_t\phi_j + |\log\varepsilon| B_0(\phi_j)\\ &+ \nabla^\perp\left((1+\frac{\varepsilon_j}{r_j}y_1)^2(\psi_j^0 - r_j\alpha_j|\log\varepsilon_j| + \psi_j^* + \psi_j + r_j\psi^{out}) + \mathcal{R}_j(y,t;P)\right)\cdot\nabla\phi_j\\ &+ \nabla^\perp\left((1+\frac{\varepsilon_j y_1}{r_j})^2(\psi_j + r_j\psi^{out})\right)\nabla(\varepsilon_j^2 r_j W^*)\\ &+ \varepsilon_j^4 S_1(W^*,\Psi^*)(\varepsilon_j y + P_j), \quad |y|<3R, \quad R := \frac{1}{\varepsilon|\log\varepsilon|^5} \end{aligned} \tag{9.3}$$ Ej

with $y = \frac{x-P_j}{\varepsilon_j}$, where B_0 is defined in (4.21), and

$$
\begin{aligned}
E^{out}[\phi^{out}, \Psi^{out}, \phi^{in}, \psi^{in}, \mathbf{a}](x,t) &:= |\log\varepsilon|\, r\, \phi^{out}_t \\
&+ \nabla_x^\perp (r^2(\Psi^* + \sum_{j=1}^k \frac{\bar\eta_{j2}}{r_j}\psi_j(\frac{x-P_j}{\varepsilon_j}) + \psi^{out} - r_0^{-1}|\log\varepsilon|)) \nabla_x \phi^{out} \\
&+ \sum_{j=1}^k \left[r\,|\log\varepsilon|\,\partial_t \bar\eta_{j1} + \nabla_x^\perp (r^2(\Psi^* + \sum_{j=1}^k \frac{\bar\eta_{j2}}{r_j}\psi_j(\frac{x-P_j}{\varepsilon_j}) + \psi^{out} - r_0^{-1}|\log\varepsilon|)) \nabla \bar\eta_{1j} \right] \frac{\phi_j}{\varepsilon_j^2 r_j} \\
&+ \left[\sum_{j=1}^k (\bar\eta_{2j} - \bar\eta_{1j}) \nabla_x^\perp (r^2(\frac{\psi_j}{r_j} + \psi^{out})) + \frac{r^2\psi_j}{r_j}\nabla_x^\perp \bar\eta_{2j} \right] \nabla_x W^* \\
&+ (1 - \sum_{j=1}^k \bar\eta_{2j}) \nabla^\perp (r^2 \psi^{out}) \cdot \nabla W^* + (1 - \sum_{j=1}^k \bar\eta_{j1}) S_1(W^*, \Psi^*) = 0 \quad (x,t) \in \Sigma \times [0,T)
\end{aligned}
$$

It is straightforward to check that a pair (W, Ψ) of the form (9.1) is a solution to (4.1) if $(\phi^{in}, \psi^{in}, \phi^{out}, \psi^{out})$ solve the *inner-outer gluing* system given by the inner problem

$$
\begin{aligned}
E_j^{in}[\phi_j, \psi_j\, \psi^{out}, \mathbf{a}](y,t) = 0, &\quad (y,t) \in B(0;3R) \times [0,T) \\
-\Delta_{5,j}\psi_j = \phi_j, &\quad (y,t) \in B(0;3R) \times [0,T)
\end{aligned} \tag{9.4}
$$

inner

for all $j = 1, \dots, k$, coupled with the outer problem

$$
\begin{aligned}
E^{out}[\phi^{out}, \psi^{out}, \phi^{in}, \psi^{in}, \mathbf{a}](x,t) = 0, &\quad (x,t) \in \Sigma \times [0,T) \\
E_1^{out}[\phi^{out}, \psi^{out}, \phi^{in}, \psi^{in}, \mathbf{a}](x,t) = 0, &\quad (x,t) \in \Sigma \times [0,T)
\end{aligned} \tag{9.5}
$$

out

where

$$
\begin{aligned}
E_1^{out} &:= \Delta_5 \psi^{out} + \phi^{out} + \sum_{j=1}^k (\bar\eta_{j1} - \bar\eta_{j2}) \frac{\phi_j}{r_j \varepsilon_j^2} \\
&+ \sum_{j=1}^k (\frac{\psi_j}{r_j}\Delta_5 \bar\eta_{j2} + 2\nabla_x \bar\eta_{j2} \nabla_x \frac{\psi_j}{r_j}), \quad (x,t) \in \Sigma \times [0,T)
\end{aligned}
$$

with boundary and decay conditions on ψ^{out}

$$
\frac{\partial}{\partial r}\psi^{out}(x,t) = 0, \quad (x,t) \in \partial\Sigma \times [0,T], \quad |\psi^{out}(x,t)| \to 0, \qquad \text{as} \quad |x| \to \infty. \tag{9.6}
$$

boundary

The rest of the paper is devoted to solve the inner-outer gluing system (9.4)-(9.5)-(9.6).

9.1. **Setting the inner problems in the whole $\mathbb{R}^2$.** The inner problems in (9.4) can be extended to the whole space $\mathbb{R}^2$ with the use of cut-off functions. We will use two types of cut-off functions: one with large support much beyond the region $B(0, 3R)$, the other with support bigger that $B(0, 3R)$ but of comparable size.

Using the notation introduce in (2.3), we define

$$
\chi(y_1) = \eta_{10}\left(\frac{|y_1|}{\varepsilon\,|\log\varepsilon|^3}\right), \quad \eta_{4\varepsilon}(y) = \eta_4\left(\frac{|y|}{R}\right), \quad \text{where} \quad R := \frac{1}{\varepsilon|\log\varepsilon|^5}. \tag{9.7}
$$

ccuts

We use χ to extend the operator $\Delta_{5,j}$ to $\mathbb{R}^2$. The functions ϕ_j and ψ_j are related via the operator $\Delta_{5,j}$ as described in (9.4). This operator can be written as

$$
-\operatorname{div}\left((1 + \frac{\varepsilon_j}{r_j}y_1)^3 \nabla\psi_j\right) = (1 + \frac{\varepsilon_j}{r_j}y_1)^3\,\phi_j, \quad (y,t) \in B(0,3R) \times [0,T).
$$

Letting $\mathcal{P}(y_1,t)$ be defined as

$$
\mathcal{P}(y_1,t) = p(y_1,t)^3, \qquad \text{where} \quad p(y_1,t) = \left(1 + \frac{\varepsilon_j}{r_j}y_1\chi\right), \tag{9.8}
$$

defP

it is clear that, if (ψ_j,ϕ_j) are such that

$$-\operatorname{div}(\mathcal{P}\nabla\psi_j)=\mathcal{P}\,\phi_j,\qquad \text{in}\quad \mathbb{R}^2\times[0,T]$$

then their restriction to $B(0,3R)$ satisfy the second relation in (9.4).

Let us introduce the functions

$$\begin{aligned}\psi_{0j}(y,t)&=\Gamma_0(y)+\frac{\varepsilon_j y_1}{2r_j}\,\chi\Big(-3\Gamma_0(y)+A_\varepsilon-4K(P_j,P_j)+\Gamma(y)\Big)\\ \Gamma_{0j}(y,t)&=\Gamma_0(y)+\frac{\varepsilon_j y_1}{2r_j}\,\chi\Big(\Gamma_0(y)+\bar A+\Gamma(y)\Big).\end{aligned}\tag{9.9}$$

For $y\in B(0,3R)$ (and beyond) these functions coincide with part of the expansion of $\psi_j^0+\psi_j^*$ and $(1+\frac{\varepsilon_j}{r_j}y_1)^2(\psi_j^0-r_j\alpha_j|\log\varepsilon_j|+\psi_j^*)$ respectively, as given in (5.9):

$$\begin{aligned}\psi_j^0+\psi_j^*&=\psi_{0j}-4\log\varepsilon_j-\log 8+K(P_j;P_j)+c_j^*(y,t)\\ (1+\frac{\varepsilon_j}{r_j}y_1)^2(\psi_j^0-r_j\alpha_j|\log\varepsilon_j|+\psi_j^*)&=\Gamma_{0j}(y)-(4-\alpha_j r_j)\log\varepsilon_j-\log 8+K(P_j;P_j)+b_j^*(y,t)\end{aligned}\tag{9.10}$$

where

$$|\log\varepsilon|^{\frac12}|\partial_t c_j^*(y,t)|+(1+|y|)|\nabla_y c_j^*(y,t)|+|c_j^*(y,t)|\ \le\ C\varepsilon^2\,(1+|y|^2)\,\log(1+|y|)$$

and

$$|\log\varepsilon|^{\frac12}|\partial_t b_j^*(y,t)|+(1+|y|)|\nabla_y b_j^*(y,t)|+|b_j^*(y,t)|\ \le\ C\varepsilon^2\,(1+|y|^2)\,\log(1+|y|)$$

for $y=\frac{x-P_j}{\varepsilon_j}$, $|y|<|\log\varepsilon|^{-3}$. Besides, from (5.8) we get that

$$\varepsilon_j^2 r_j W^*=f\left((1+\frac{\varepsilon_j}{r_j}y_1)^2(\psi_j^0-r_j\alpha_j|\log\varepsilon_j|+\psi_j^*)\right)(1+O(\varepsilon^2))\sim f_0\left(\Gamma_{0j}(y,t)+b_j^*\right),\quad \text{with}\quad f_0(s)=e^s.$$

Let

$$\begin{aligned}B_0(\phi)=\mathcal{B}_j^0\cdot\nabla\phi,\quad &\mathcal{B}_j^0:=-\varepsilon_j\partial_t\varepsilon_j(1+\frac{\varepsilon_j}{r_j}y_1)y-\frac{\varepsilon_j^2}{r_j}y_1\partial_t P_j\\ (1+\frac{\varepsilon_j}{r_j}y_1\chi)^2(\psi_j^0-r_j\alpha_j|\log\varepsilon_j|+\psi_j^*)+|\log\varepsilon|\,\eta_{4\varepsilon}\,\mathcal{B}_j^0&+\eta_{4\varepsilon}\,\mathcal{R}_j(y,t;P)=\Gamma_{0j}+\eta_{4\varepsilon}\,b_j^*+\eta_{4\varepsilon}b_j^{**},\\ b_j(\psi^{out},\mathbf{a})&=(1+\frac{\varepsilon_j}{r_j}y_1\chi)^2\,\eta_{4\varepsilon}\,(\hat\psi_j+r_j\psi^{out})\\ \varepsilon_j^2 r_j W^*&=f_0\left(\Gamma_{0j}+b_j^*\right)+U^*,\quad f_0(s)=e^s.\end{aligned}\tag{9.11}$$

We explicitly find

$$b_j^{**}=-(4-\alpha_j r_j)\log\varepsilon_j-\log 8+K(P_j;P_j)+|\log\varepsilon|\ \mathcal{B}_j^0+\ \mathcal{R}_j(y,t;P)$$

and, setting $\tilde b_j=b_j^{**}+(4-\alpha_j r_j)\log\varepsilon_j+\log 8-K(P_j;P_j)$,

$$|\log\varepsilon|^{\frac12}|\partial_t\tilde b_j(y,t)|+(1+|y|)|\nabla_y\tilde b_j(y,t)|+|\tilde b_j(y,t)|\ \le\ C\varepsilon^2\,|\log\varepsilon|^{-\frac12}\,|y|.$$

The inner problem (9.3)-(9.4) are the restriction to $B(0,3R)$ of the following equations

$$\begin{aligned}E_j[\phi_j,\psi_j\,\psi^{out},\mathbf{a}](y,t)&:=|\log\varepsilon|\,\varepsilon_j^2\,(1+\frac{\varepsilon_j}{r_j}y_1\chi)\,\partial_t\phi_j\\ &+\nabla^\perp\left(\Gamma_{0j}+\eta_{4\varepsilon}b_j^*+\eta_{4\varepsilon}b_j^{**}+b_j(\psi^{out},\mathbf{a})\right)\cdot\nabla\phi_j\\ &+\nabla^\perp((1+\frac{\varepsilon_j}{r_j}y_1\chi)^2\psi_j)\cdot\nabla\left(f_0\left(\Gamma_{0j}+b_j^*\right)+\eta_{4\varepsilon}U^*\right)\\ &+\nabla^\perp\left(\eta_{4\varepsilon}(1+\frac{\varepsilon_j y_1}{r_1}\chi)^2 r_j\psi^{out}\right)\nabla\left(f_0\left(\Gamma_{0j}+b_j^*\right)+\eta_{4\varepsilon}U^*\right)\\ &+\mathcal{E}_j[\phi_j^*,\psi_j^*\,\psi^{*,out},\mathbf{a}](y,t)\qquad \text{in } \mathbb{R}^2\times[0,T],\\ -\operatorname{div}(\mathcal{P}\nabla\psi_j)&=\mathcal{P}\,\phi_j,\qquad \text{in}\quad \mathbb{R}^2\times[0,T]\end{aligned}\tag{9.12}$$

where $\mathcal{E}_j[\phi_j^*,\psi_j^*\,\psi^{*,out},P]$ is defined and estimated in Proposition 5.1.

9.2. **Decomposition of ϕ_j and ψ_j.** For a function $h:\mathbb{R}^2\to\mathbb{R}$, consider the problem

$$-\operatorname{div}(\mathcal{P}\nabla\psi)=h \qquad \text{in} \qquad \mathbb{R}^2\times[0,T]. \tag{9.13}$$

an

A necessary condition for solvability of (9.13) is that

$$\int_{\mathbb{R}^2} h(y,t)\,dy=0, \quad \forall t\in[0,T). \tag{9.14}$$

an1

We will make use of the following lemma

⟨ii⟩ **Lemma 9.1.** *Assume $h:\mathbb{R}^2\times[0,T]\to\mathbb{R}$ satisfies (9.14) and*

$$\int_{\mathbb{R}^2} U^{-1}(y)h^2(y,t)\,dy<\infty, \quad \forall t\in[0,T], \quad \textit{where} \quad U(y)=\frac{8}{(1+|y|^2)^2}. \tag{9.15}$$

an2

Then there exists a unique solution ψ to (9.13) such that, for all $t\in[0,T]$, $\psi(\cdot,t)\in C^{0,\sigma}(\mathbb{R}^2)$ and

$$|\psi(y,t)|\lesssim\frac{\|U^{-\frac12}h\|_{L^2(\mathbb{R}^2)}}{1+|y|^{1-\sigma}}$$

for some $\sigma\in(0,1)$, and

$$\|U^{-\frac12+\frac1p}\nabla\psi\|_{L^p(\mathbb{R}^2)}\lesssim\|U^{-\frac12}h\|_{L^2(\mathbb{R}^2)}$$

for any $p>2$. If $\|U^{\frac1q-1}h\|_{L^q(\mathbb{R}^2)}<\infty$ for some $q>2$, then

$$|\psi(y,t)|+(1+|y|)|\nabla\psi(y,t)|+(1+|y|)^{1+\sigma}[\nabla\psi]_\sigma\lesssim\frac{\|U^{\frac1q-1}h\|_{L^q(\mathbb{R}^2)}}{1+|y|},$$

where we denote

$$[\nabla\psi]_\sigma(y)=\sup_{y_1,y_2\in B_1(y)}\frac{|\nabla\psi(y_1)-\nabla\psi(y_2)|}{|y_1-y_2|^\sigma}$$

and $\sigma\in(0,1)$.

Proof. It is convenient to pull equation (9.13) into the sphere S^2 by means of the stereographic projection

$$\pi\,:\,S^2-\{(0,0,1)\}:\to\mathbb{R}^2$$

$$\pi(z)=\Big(\frac{z_1}{1-z_3},\frac{z_2}{1-z_3}\Big), \quad z\in S^2-\{(0,0,1)\}, \tag{9.16}$$

stereo

whose inverse is given by $\pi^{-1}:\mathbb{R}^2\to S^2-\{(0,0,1)\}$

$$\pi^{-1}(y)=\Big(\frac{2y_1}{1+|y|^2},\frac{2y_2}{1+|y|^2},\frac{|y|^2-1}{1+|y|^2}\Big), \quad y\in\mathbb{R}^2.$$

For a function $h(y):\mathbb{R}^2\to\mathbb{R}$ we denote by $h(z)$ the function $h(\pi(z))$ defined on S^2. Then the differential of volume is given by $d\pi(z)=\frac{4}{(1+|y|^2)^2}\,dy=\frac{U(y)}{2}\,dy$, where the function U is the one in (1.18). Thus we get

$$\nabla_{\mathbb{R}^2}h(y)=\frac{U^{\frac12}}{\sqrt2}(\nabla_{S^2}h)(z), \quad \Delta_{\mathbb{R}^2}h(y)=\frac{U}{2}(\Delta_{S^2}h)(z), \quad y=\pi(z).$$

and

$$\int_{\mathbb{R}^2}h(y)\,dy \;=\; 2\int_{S^2}h(z)\,U^{-1}(z)\,d\pi(z).$$

For a vector-valued function $F(y):\mathbb{R}^2\to\mathbb{R}^2$ we have

$$\operatorname{div}_{S^2}F(z)=\sqrt2U^{-1}\operatorname{div}_{\mathbb{R}^2}(U^{\frac12}F)(y).$$

Let us denote

$$\tilde h(y,t)=\frac{2\,h(y,t)}{U(y)}$$

Then Equation (9.13) gets transformed into

$$-\operatorname{div}_{S^2}(\mathcal{P}\nabla_{S^2}\psi)=\tilde h \quad \text{in } S^2. \tag{9.17}$$

poisson1

From (9.8) we get that $\mathcal{P}(y_1,t) = 1 + O(\varepsilon R)$, for all $t \in [0,T]$, $y_1 \in \mathbb{R}$. Assumptions (9.14) and (9.15) become

$$\int_{S^2} \tilde h^2(z)\, d\pi(z) \;=\; 2\int_{\mathbb{R}^2} h^2(y) U(y)^{-1}\, dy < \infty, \qquad \int_{S^2} \tilde h(z)\, d\pi(z) \;=\; 0.$$

The latter condition in the above formula implies the existence of a unique solution of (9.17) with mean value zero. This solution is in $H^2(S^2)$, hence it is Hölder continuous of any order. After adding a proper constant, we choose the solution which vanishes at the sounth pole $S = (0,0,1)$. Pulling back this function to a $\psi(y)$ defined in $\mathbb{R}^2$ we see that it satisfies equation (9.13) and it is the only solution that vanishes as $|y| \to \infty$. This and the Hölder condition yields that ψ satisfies

$$|\psi(y)| \;\le\; \frac{C}{1+|y|^{1-\sigma}}\, \|h\, U^{-\frac12}\|_{L^2(\mathbb{R}^2)} \tag{9.18}$$ cota1psi

for an arbitrarily small $\sigma > 0$. Moreover, for all $p > 2$ we have an $L^p(S^2)$-gradient estimate for $\tilde\psi(z)$ of the form

$$\|\nabla_{S^2} \tilde\psi\|_{L^p(S^2)} \;\le\; C\|\tilde\phi\|_{L^2(S^2)}$$

which yields for ψ in (9.13)

$$\|U^{-\frac12+\frac1p}\nabla\psi\|_{L^p(\mathbb{R}^2)} \;\le\; C\|h\, U^{-\frac12}\|_{L^2(\mathbb{R}^2)}. \tag{9.19}$$ gradp

If in addition we have that $\tilde h \in L^q(S^2)$ for some $q > 2$, then a solution of (9.17) satisfies

$$\|\nabla_{S^2} \tilde\psi\|_{C^{0,\sigma}(S^2)} \;\le\; C\|\tilde h\|_{L^q(S^2)}.$$

for some $0 < \sigma < 1$. This estimate translates for ψ into

$$|\psi(y)| \;+\; (1+|y|)\,|\nabla\psi(y)| + (1+|y|)^{1+\sigma}[\nabla\psi]_\sigma(y) \;\le\; \frac{C}{1+|y|}\|U^{\frac1q - 1} h\|_{L^q(\mathbb{R}^2)}.$$

The proof is completed. □

We shall make a decomposition of the functions ϕ_j, ψ_j introduced in (9.1).

Let

$$\mathcal{Z}_{20}(y) = \Gamma_0(y), \quad \mathcal{Z}_{23}(y) = 2\Gamma_0(y) + \nabla_y\Gamma_0(y)\cdot y,$$

where Γ_0 is given in (1.18). We write $\phi_j(y,t)$ in the form

$$\begin{aligned} &\mathcal{P}\phi_j(y,t) = \mathcal{P}\hat\phi_j(y,t) + \sum_{l=0,3} \beta_{jl}(t)\mathcal{P}\mathcal{Z}_{1l}(y,t), \quad \text{where} \\ &\mathcal{Z}_{1\ell} = -\frac{1}{\mathcal{P}}\, \mathrm{div}(\mathcal{P}\nabla\mathcal{Z}_{2\ell}), \quad \ell = 0,3, \end{aligned} \tag{9.20}$$ desco1

for some smooth parameter functions $\beta_{j\ell}$ that we will determine later on. A direct inspection gives

$$\mathcal{Z}_{10} = U(y) - \frac{3\varepsilon_j}{(r_j+\varepsilon_j y_1\chi)}\,(\chi + y_1\chi')\,\partial_{y_1}\mathcal{Z}_{20}(y),$$

$$\mathcal{Z}_{13} = 2U(y) + \nabla_y U(y)\cdot y - \frac{3\varepsilon_j}{(r_j+\varepsilon_j y_1\chi)}\,(\chi + y_1\chi')\,\partial_{y_1}\mathcal{Z}_{23}(y),$$

where χ is defined in (9.8). On $\hat\phi_j$ we impose the following orthogonality conditions

$$\int_{\mathbb{R}^2} \mathcal{P}\hat\phi_j \mathbf{Z}_\ell\, dy \;=\; 0, \quad \ell = 0,1,2,3 \quad \text{for all} \quad t\in[0,T], \tag{9.21}$$ orto1

with

$$\begin{aligned} &\mathbf{Z}_0(y) = 1, \quad p^2\,\mathbf{Z}_3(y,t) = G\,(\Gamma_{0j}) + b_3(y,t) \\ &\mathbf{Z}_1(y,t) = p^{-2}\left(y_1 + \frac{\varepsilon_j}{2r_j}y_1^2\right)\mathbf{1}_{B_{8R}}(y), \quad p^2\,\mathbf{Z}_2(y,t) = (y_2 + \varepsilon y_1 y_2)\,\mathbf{1}_{B_{8R}}(y), \end{aligned} \tag{9.22}$$ bernieb

where $p(y_1,t)$ is defined in (9.8), $\mathbf{1}_{B_{8R}}(y) = 1$ for $y \in B(0,8R)$, $= 0$ otherwise. The function Γ_{0j} is given in (9.9) and G is such that

$$G(\Gamma_0(y)) = \nabla\Gamma_0\cdot y + 2 = \frac{1-|y|^2}{1+|y|^2}.$$

Since $G'(\Gamma_0(y)) \sim |y|^{-2}$ as $|y| \to \infty$, one has

$$G(\Gamma_{0j}) = G(\Gamma_0)(1 + O(\varepsilon R)).$$

Besides, the function $b_3(y,t)$ will be explicitly defined in (10.21); for the moment we think at it as $b_3 = O(\varepsilon^2)$, as $\varepsilon \to 0$. We also observe that $\nabla(p^2 \mathbf{Z}_1) = p\,\mathbf{1}_{B_{8R}}(y)\,\mathbf{e}_1$.

For ϕ_j as in (9.20), we write

$$\hat\phi = \left(\hat\phi_1, \dots, \hat\phi_k\right), \quad \beta = (\beta_1, \dots, \beta_k), \quad \beta_j = (\beta_{j0}, \beta_{j3}). \tag{9.23}$$
nota1

We assume that

$$\int_{\mathbb{R}^2} U^{-1}(y)(\mathcal{P}\hat\phi_j)^2(y,t)\,dy < \infty, \quad \forall t \in [0,T]. \tag{9.24}$$
inte

Let $\hat\psi_j(y,t)$ be the solution of

$$-\operatorname{div}(\mathcal{P}\nabla\hat\psi_j) = \mathcal{P}\hat\phi_j, \qquad \text{in} \quad \mathbb{R}^2, \tag{9.25}$$
?psi1h?

predicted by Lemma 9.1, and define

$$\psi_j(y,t) = \hat\psi_j(y,t) + \sum_{l=0,3} \beta_{jl}(t)\mathcal{Z}_{2l}(y). \tag{9.26}$$
desco2

Then ψ_j satisfies the second condition in (9.4), namely

$$-\Delta_{5,j}\psi_j = \phi_j \quad (y,t) \in B(0,3R) \times [0,T).$$

Recall from (9.3) that

$$R = \frac{1}{\varepsilon|\log\varepsilon|^5}. \tag{9.27}$$
defR

The decomposition for ϕ_j in (9.20) is motivated by the validity of a key estimate for a quadratic form.

⟨pass1⟩ **Lemma 9.2.** *There exists a number $\gamma > 0$ such that for any sufficiently small ε and all ϕ satisfying conditions (9.21)-(9.24), the following holds: let g be given by*

$$g = (f_0'(\Gamma_{0j}))^{-1}\left(\mathcal{P}\phi - \mathcal{P} f_0'(\Gamma_{0j})\, p^2\, \psi\right),$$

where ψ solves $-\operatorname{div}(\mathcal{P}\nabla\psi) = \mathcal{P}\phi$. Then we have

$$\int_{\mathbb{R}^2} \mathcal{P}\phi\, g \geq \frac{\gamma}{|\log R|}\int_{\mathbb{R}^2} (\mathcal{P}\phi)^2 U^{-1}, \tag{9.28}$$
pass

where R is given by (9.27). We recall the definition of $\mathcal{P}(y_1,t)$ and $p(y_1,t)$ in (9.8), and that $f_0(s) = e^s$.

Proof. The proof is divided into two steps: we first prove (9.28) with the test function g replaced by g_0, with

$$g_0 = U^{-1}(\mathcal{P}\phi - U\psi).$$

where U is defined in (1.10). Then we prove (9.28).

Let $\tilde\phi = 2U^{-1}\mathcal{P}\phi$. Using the stereographic projection π introduced in (9.16), we get

$$\int_{\mathbb{R}^2} \mathcal{P}\phi g_0 = 2\int_{\mathbb{R}^2} (\mathcal{P}\phi)^2 U^{-1} - \int_{\mathbb{R}^2} \mathcal{P}\phi\psi = \int_{S^2} \tilde\phi^2 - \int_{S^2} \tilde\phi\psi.$$

Consider the orthonormal basis in $L^2(S^2)$ of spherical harmonics $(e_\ell)_\ell$, where $-\Delta_{S^2} e_\ell = \lambda_\ell e_\ell$. Here $\lambda_0 = 0$ and e_0 is constant, while $\lambda_1 = \lambda_2 = \lambda_3 = 2$, with $e_\ell(z) = z_\ell$, for $\ell = 1,2,3$. We decompose $\tilde\phi$ as

$$\tilde\phi = \sum_{\ell=0}^{\infty} \tilde\phi_\ell e_\ell(z) = \sum_{\ell=0}^{3} \tilde\phi_\ell e_\ell(z) + \tilde\phi^\perp, \quad \tilde\phi_\ell = \int_{S^2} \phi\, e_\ell\, d\pi(z).$$

Since $\int_{\mathbb{R}^2} \mathcal{P}\phi = \int_{S^2} \tilde\phi = 0$, we get $\tilde\phi_0 = 0$. For $\ell \geq 1$, let ψ_ℓ be defined by

$$-\operatorname{div}_{S^2}(\mathcal{P}\nabla_{S^2}\psi_\ell) = e_\ell.$$

From Lemma 9.1 we get that $\psi_\ell \in H^2(S^2)$ and $\|\psi_\ell\|_{H^2(S^2)} \leq C$ for some $C > 0$. Since $-\operatorname{div}_{S^2}(\mathcal{P}\nabla_{S^2}\psi) = \tilde\phi$, we have that $\psi = \sum_{\ell=1}^{\infty} \tilde\phi_\ell\psi_\ell$. The equation satisfied by ψ_ℓ becomes $-\operatorname{div}_{\mathbb{R}^2}(\mathcal{P}\nabla\psi_\ell) = U e_\ell$ in $\mathbb{R}^2$.

It is convenient to write $\psi_\ell = (-\Delta_{S^2})^{-1} e_\ell + \bar\psi_\ell = \frac{2}{\lambda_\ell} e_\ell + \bar\psi_\ell$. Then

$$-\mathrm{div}_{S^2}(\nabla_{S^2}\bar\psi_\ell) = h_\ell, \quad h_\ell = \frac{\nabla_{S^2}\mathcal{P}\nabla_{S^2}\psi_\ell}{\mathcal{P}} + \frac{(1-\mathcal{P})e_\ell}{\mathcal{P}}.$$

A direct computation gives $\int_{S^2} h_\ell = 0$. Besides,

$$\int_{S^2} h_\ell^2 \le \int_{S^2} |\frac{\nabla_{S^2}\mathcal{P}}{\mathcal{P}}\nabla_{S^2}\psi_\ell|^2 + \int_{S^2} |\frac{(1-\mathcal{P})}{\mathcal{P}} e_\ell|^2 \le c|\log\varepsilon|^{-10}\left(\|\psi_\ell\|^2_{H^2(S^2)} + 1\right)$$

for some $c > 0$. We argue as in Lemma 9.1 to get $\bar\psi_\ell \in H^2(S^2)$ and $\|\bar\psi_\ell\|_{H^2(S^2)} = O(\varepsilon R)$ as $\varepsilon \to 0$. Thus

$$\begin{aligned}\int_{\mathbb{R}^2} \mathcal{P}\phi\, g_0 &= \sum_{\ell=4}^{\infty}\left(1 - \frac{2}{\lambda_\ell}\right)\tilde\phi_\ell^2 + O(\varepsilon R)\|\tilde\phi\|^2_{L^2(S^2)} \\ &\ge\ c_1\|\tilde\phi^\perp\|^2_{L^2(S^2)} + O(\varepsilon R)\|\tilde\phi\|^2_{L^2(S^2)}\end{aligned} \tag{9.29}$$

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for some uniform $c_1 > 0$. We also have, for $j = 1, 2$

$$0 = \int_{B(0,5R)} \mathcal{P}\phi\mathbf{Z}_j = c_2\tilde\phi_j + O(\|\tilde\phi^\perp\|_{L^2(S^2)})\,|\log R|^{\frac12}$$

for some uniform $c_2 > 0$, as $\varepsilon \to 0$. On the other hand, we have

$$0 = \int_{\mathbb{R}^2} \mathcal{P}\phi\mathbf{Z}_3 = \tilde\phi_3 + O(\varepsilon R)\|\tilde\phi\|_{L^2(S^2)}.$$

From the above relations we get that for some $c > 0$ independent of ε,

$$\begin{aligned}\|\tilde\phi^\perp\|^2_{L^2(S^2)} &\ge\ \|\tilde\phi\|^2_{L^2(S^2)} - c\sum_{\ell=1}^{3}|\tilde\phi_\ell|^2, \\ &\ge\ (1 - O(\varepsilon R))\,\|\tilde\phi\|^2_{L^2(S^2)}\ -\ c|\log R|\,\|\tilde\phi^\perp\|^2_{L^2(S^2)}.\end{aligned}$$

From here we get

$$(1 + c|\log R|)\,\|\tilde\phi^\perp\|^2_{L^2(S^2)}\ \ge (1 - o(1))\,\|\tilde\phi\|^2_{L^2(S^2)}$$

with $o(1) \to 0$ as $\varepsilon \to 0$, which combines with (9.29) to get

$$\int_{\mathbb{R}^2} \mathcal{P}\phi\, g_0 \ge \frac{\gamma}{|\log R|}\int_{S^2}\tilde\phi^2$$

for some uniform $\gamma > 0$.

We next estimate $\int_{\mathbb{R}^2} \mathcal{P}g$ in terms of $\int_{\mathbb{R}^2} \mathcal{P}g_0$. To this purpose, we write

$$g = g_0 + \left((f_0'(\Gamma_{0j}))^{-1} - U^{-1}\right)(\mathcal{P}\phi - U\psi) + (f_0'(\Gamma_{0j}))^{-1}\left(U - \mathcal{P}f_0'(\Gamma_{0j})p^2\right)\psi$$

Recalling the definition of Γ_{0j} in (9.9), we get

$$f_0'(\Gamma_{0j}) = U(y)\left(1 + O(|\log\varepsilon|^{-2})\right);$$

hence we get

$$\int_{\mathbb{R}^2} \mathcal{P}\phi\ \left((f_0'(\Gamma_{0j}))^{-1} - U^{-1}\right)(\mathcal{P}\phi - U\psi) = O(|\log\varepsilon|^{-2})\int_{\mathbb{R}^2}\mathcal{P}\phi\, g_0.$$

Besides

$$\left|\int_{\mathbb{R}^2} \mathcal{P}\phi\,(f_0'(\Gamma_{0j}))^{-1}\left(U - \mathcal{P}f_0'(\Gamma_{0j})p^2\right)\psi\right| \lesssim |\log\varepsilon|^{-2}\int_{\mathbb{R}^2}|\mathcal{P}\phi\psi| \lesssim |\log\varepsilon|^{-2}\|U^{-\frac12}\phi\|^2_{L^2(\mathbb{R}^2)}.$$

Thus we get

$$\int_{\mathbb{R}^2}\mathcal{P}\phi\, g \ge (1 + O(|\log\varepsilon|^{-2}))\int_{\mathbb{R}^2}\mathcal{P}\phi\, g_0 - O(|\log\varepsilon|^{-2})\int_{\mathbb{R}^2}U^{-1}(\phi)^2 \ge \frac{\tilde\gamma}{|\log R|}\int_{\mathbb{R}^2}U^{-1}(\phi)^2$$

for some new $\tilde\gamma$ which is uniformly positive as $\varepsilon \to 0$. This concludes the proof. □

We will make use of this result to establish a-priori bounds for solutions to a projected version of the inner problem (9.4). In Section 10 we will first establish a-priori bounds in weighted L^2-spaces, which will be used to establish the weighted L^∞ a-priori bounds. Before entering Section 10, we give a sketch of the proof to solve in $(\phi^{in}, \psi^{in}, \phi^{out}, \psi^{out})$ the whole inner-outer gluing system (9.4)-(9.5)-(9.6).

9.3. **Strategy for the rest of the proof.** Let

$$b_j(\beta_j, \psi^{out}, \mathbf{a}) = (1 + \frac{\varepsilon_j}{r_j} y_1 \chi)^2 \, \eta_{4\varepsilon} \, (\hat{\psi}_j + \sum_{l=0,3} \beta_{jl}(t) \mathcal{Z}_{1l} + r_j \psi^{out})$$

where $\eta_{4\varepsilon}$ is defined in (9.7) and introduce the following operators, depending on a homotopy parameter $\lambda \in [0,1]$

$$\begin{aligned} E_{j,\lambda}[\hat{\phi}_j, \beta_j \, \psi^{out}, \mathbf{a}](y,t) &:= |\log \varepsilon| \, \varepsilon_j^2 \, (1 + \frac{\varepsilon_j}{r_j} y_1 \chi) \, \partial_t \hat{\phi}_j \\ &+ \nabla^{\perp} \left(\Gamma_{0j} + \lambda \eta_{4\varepsilon} b_j^* + \lambda \eta_{4\varepsilon} b_j^{**} + \lambda b_j(\beta_j, \psi^{out}, \mathbf{a}) \right) \cdot \nabla \hat{\phi}_j \\ &+ \nabla^{\perp} ((1 + \frac{\varepsilon_j}{r_j} y_1 \chi)^2 \hat{\psi}_j) \cdot \nabla \left(f_0 \left(\Gamma_0 + \lambda b_j^* \right) + \eta_{4\varepsilon} U^* \right) \\ &+ \nabla^{\perp} \left[\varepsilon_j \left(|\log \varepsilon| \partial_t \mathbf{a}_j + D_x \nabla_x \varphi_j(\mathbf{P}_j; \mathbf{P})[\mathbf{a}] + \lambda \eta_{4\varepsilon} (1 + \frac{\varepsilon_j y_1}{r_1})^2 r_j \psi^{out} \right) \cdot y \right] \nabla U \\ &+ \lambda \tilde{\mathcal{E}}_j(\beta_j, \psi^{out}, \mathbf{a}) + |\log \varepsilon| \varepsilon_j^2 (1 + \frac{\varepsilon_j}{r_j} y_1 \chi) \sum_{\ell=0,3} \partial_t(\beta_{j\ell} \mathcal{Z}_{1\ell}) \qquad \text{in } \mathbb{R}^2 \times [0,T] \end{aligned} \tag{9.30}$$ Ejla

where $f_0(s) = e^s$,

$$\begin{aligned} \tilde{\mathcal{E}}_j &= |\log \varepsilon| B_0 (\sum_{\ell=0,3} \beta_{j\ell} \mathcal{Z}_{1\ell}) + \nabla^{\perp} \left(\Gamma_{0j} + \eta_{4\varepsilon} b_j^* + \eta_{4\varepsilon} b_j^{**} + b_j \right) \cdot \nabla (\sum_{\ell=0,3} \beta_{j\ell} \mathcal{Z}_{1\ell}) \\ &+ \nabla^{\perp} \left((1 + \frac{\varepsilon_j y_1}{r_1} \chi)^2 (r_j \psi^{out}) \right) \nabla (\eta_{4\varepsilon} U^*) \\ &+ \nabla^{\perp} \left[(\varepsilon_j |\log \varepsilon| \partial_t \mathbf{a}_j + \varepsilon_j \nabla_x \varphi_j(P_j; P) - \nabla_x \varphi_j(\mathbf{P}; \mathbf{P}) - D_x \nabla_x \varphi_j(\mathbf{P}_j; \mathbf{P})[\mathbf{a}]) \cdot y \right] \nabla U \\ &+ \mathcal{E}_j[\phi_j^*, \psi_j^* \, \psi^{*,out}, \mathbf{a}](y,t) \end{aligned}$$

where $\mathcal{E}_j[\phi_j^*, \psi_j^* \, \psi^{*,out}, P]$ is defined and estimated in Proposition 5.1.

If $\lambda = 1$ and we restrict the problem to $B(0,3R)$, we get the operator E_j^{in} defined in (9.3), or equivalently (9.12).

Recalling the notation introduced in (9.23), we define

$$\begin{aligned} E_{1,\lambda}^{out}(\phi^{out}, \psi^{out}, \hat{\phi}, \beta, \mathbf{a}) \; &= \; |\log \varepsilon| \, r \, \partial_t \phi^{out} \\ &+ \; \nabla^{\perp} \big(r^2 (\Psi^* - r_0^{-1} |\log \varepsilon| + \lambda \sum_{j=1}^{k} \frac{\bar{\eta}_{j2}}{r_j} \psi_j (\frac{x - P_j}{\varepsilon_j}) + \psi^{out}) \big) \cdot \nabla \phi^{out} \\ &+ \tilde{\mathcal{E}}_1^{out}(\hat{\phi}, \beta, \psi^{out}, \mathbf{a}) \quad (x,t) \in \Sigma \times [0,T) \end{aligned}$$

where

$$\begin{aligned} &\tilde{\mathcal{E}}_{\lambda}^{out}(\hat{\phi}, \beta, \psi^{out}, \mathbf{a}) \; := \\ &\sum_{j=1}^{k} \left[r \, |\log \varepsilon| \, \partial_t \bar{\eta}_{j1} + \nabla_x^{\perp} (r^2 (\Psi^* + \sum_{j=1}^{k} \frac{\bar{\eta}_{j2}}{r_j} \psi_j (\frac{x - P_j}{\varepsilon_j}) + \psi^{out} - r_0^{-1} |\log \varepsilon|)) \nabla \bar{\eta}_{1j} \right] \frac{\phi_j}{\varepsilon_j^2 r_j} \\ &+ \left[\sum_{j=1}^{k} (\bar{\eta}_{2j} - \bar{\eta}_{1j}) \nabla_x^{\perp} (r^2 (\frac{\psi_j}{r_j} + \psi^{out})) + \frac{r^2 \psi_j}{r_j} \nabla_x^{\perp} \bar{\eta}_{2j} \right] \nabla_x W^* \\ &+ (1 - \sum_{j=1}^{k} \bar{\eta}_{2j}) \nabla^{\perp} (r^2 \psi^{out}) \cdot \nabla W^* \\ &+ (1 - \sum_{j=1}^{k} \bar{\eta}_{j1}) S_1(W^*, \Psi^*) = 0 \quad (x,t) \in \Sigma \times [0,T), \end{aligned} \tag{9.31}$$ EE1

We also define

$$E^{out}_{1,\lambda}(\psi^{out},\phi^{out},\hat\phi,\beta,\mathbf{a}) \;=\; \Delta_5\psi^{out}+\lambda\phi^{out}+\lambda\sum_{j=1}^{k}(\bar\eta_{j1}-\bar\eta_{j2})\frac{\phi_j}{r_j\varepsilon_j^2}$$
$$+\lambda\sum_{j=1}^{k}(\frac{\psi_j}{r_j}\Delta_5\bar\eta_{j2}+2\nabla_x\bar\eta_{j2}\nabla_x\frac{\psi_j}{r_j}),\quad (x,t)\in\Sigma\times[0,T)$$

The key observation is that for ϕ_j, ψ_j given by (9.20)-(9.26) we have the identities, when $\lambda=1$

$$\begin{aligned}E_{j,1}(\hat\phi,\beta,\phi^{out},\psi^{out},\mathbf{a}) &= E^{in}_j(\phi_j,\psi_j,\psi^{out};\beta,\mathbf{a})\quad\text{in } B(0,3R)\times[0,T],\\ E^{out}_1(\phi^{out},\psi^{out},\hat\phi,\beta,\mathbf{a}) &= E^{out}(\phi^{out},\psi^{out},\phi^{in},\psi^{in};\beta,\mathbf{a})\quad\text{in } \Sigma\times[0,T),\\ E^{out}_{1,1}(\psi^{out},\phi^{out},\hat\phi,\beta,\mathbf{a}) &= E^{out}_1(\psi^{out},\psi^{in},\phi^{out};\beta,\mathbf{a})\quad\text{in } \Sigma\times[0,T),\end{aligned}$$

with boundary and decay conditions

$$\frac{\partial}{\partial r}\psi^{out}(x,t)=0,\quad (x,t)\in\partial\Sigma\times[0,T],\quad |\psi^{out}(x,t)|\to 0,\qquad\text{as}\quad |x|\to\infty.$$

Here E^{in}_j, E^{out} and E^{out}_1 are defined respectively in (9.3), (9.4) and (9.5). In other words, solving the *inner-outer gluing* system (9.4)-(9.5) coupled with the boundary and decay conditions (9.6) amounts to make the three quantities above equal to zero (keeping the boundary and decay conditions). We will do this by a continuation argument that involves finding uniform a priori estimates for the corresponding equations along the deformation parameter λ imposing in addition initial condition 0 for all the parameter functions.

We use the following strategy. We consider the functions $\beta_j,\psi^{out},\mathbf{a}$ as given and require that $\hat\phi_j$ satisfies an initial value problem of the form

$$\begin{cases} E_{j,\lambda}(\hat\phi_j,\beta_j,\phi^{out},\psi^{out},\mathbf{a})=\sum_{l=0}^{3}c_{lj}(t)z_{1l}(y)\quad\text{in }\mathbb{R}^2\times[0,T],\\ \hat\phi_j(y,0)=0\quad\text{in }\mathbb{R}^2\end{cases}\tag{9.32}$$

equinner

where

$$\begin{aligned}z_{10}(y)=U_0(y),&\quad z_{11}(y)=\partial_{y_1}U_0(y),\\ z_{12}(y)=\partial_{y_2}U_0(y),&\quad z_{13}(y)=2U_0(y)+\nabla_yU_0(y)\cdot y.\end{aligned}\tag{9.33}$$

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for some explicit functions $c_{lj}(t)$. We prove that $c_{lj}(t)$ are linearly dependent on $\hat\phi_j$ and can be computed after integrating the equation against $\mathbf{Z}_\ell$ in space variable. See (9.22) for the definition of $\mathbf{Z}_\ell$.

Solving $E_{\lambda,j}\equiv 0$ is equivalent to solving the initial value problems

$$\begin{aligned}c_{lj}[\hat\phi_j,\beta_j,\phi^{out},\psi^{out},\mathbf{a},\lambda](t)=0\quad\text{for all}\quad t\in[0,T],\ \ell=0,1,2,3\\ \mathbf{a}_j(0)=\beta_j(0)=0\end{aligned}\tag{9.34}$$

equc

for all $j=1,\dots,k$. We require

$$\begin{aligned}E^{out}_\lambda(\phi^{out},\psi^{out},\hat\phi_j,\beta_j,\mathbf{a})=0\quad\text{in }\Sigma\times[0,T],\\ \phi^{out}(\cdot,0)=0\quad\text{in }\Sigma\end{aligned}\tag{9.35}$$

equout1

and

$$\begin{aligned}E^{out}_{1,\lambda}(\psi^{out},\phi^{out},\hat\phi_j,\beta_j,\mathbf{a})=0\quad\text{in}\quad\Sigma\times[0,T],\\ \frac{\partial}{\partial r}\psi^{out}=0\quad\text{on }\partial\Sigma\times[0,T],\quad |\psi^{out}(x,t)|\to0,\qquad\text{as}\quad|x|\to\infty.\end{aligned}\tag{9.36}$$

equout2

We recall the form of ϕ_j and ψ_j, as in (9.20)-(9.26)

$$\begin{aligned}\phi_j(y,t)&=\hat\phi_j(y,t)+\sum_{l=0,3}\beta_{jl}(t)Z_{1l}(y,t),\\ \psi_j(y,t)&=\hat\psi_j(y,t)+\sum_{l=0,3}\beta_{jl}(t)\mathcal{Z}_{2l}(y).\end{aligned}$$

We can write the system of equations (9.32), (9.34), (9.35), (9.36) in the form of a fixed point problem for the variable

$$\vec{p} = (\hat{\phi}, \beta, \phi^{out}, \psi^{out}, \mathbf{a}).$$

The fix point problem has the form

$$\vec{p} = \mathcal{F}(\vec{p}, \lambda), \quad \vec{p} \in \mathcal{O}. \tag{9.37}$$ equp

Here $\mathcal{O}$ designates a bounded open set in an appropriate Banach space with $\vec{p} = \vec{0} \in \mathcal{O}$ and $\mathcal{F}(\cdot, \lambda)$ is a homotopy of nonlinear compact operators on $\bar{\mathcal{O}}$ with $\mathcal{F}(\cdot, 0)$ linear.

We shall prove that a suitable choice of a small $\mathcal{O}$ yields that for all $\lambda \in [0,1]$ no solution of (9.37) with $\vec{p} \in \partial\mathcal{O}$ exists. Existence of a solution of (9.37) for $\lambda = 1$ thus follows from standard degree theory. But this precisely corresponds to a solution of the original problem. The definition of the norm and the set $\mathcal{O}$ will yield the desired properties of the solution of Euler equation thus obtained.

In order to find the desired a priori estimates we need several preliminary considerations which we make in the next section.

10. Some a-priori estimates

⟨sec9⟩ Let us consider functions $\phi(y,t)$, $y \in \mathbb{R}^2$, $t \in [0,T]$ that satisfy

$$\|\mathcal{P}\phi\, U^{-\frac{1}{2}}\|_{L^2(\mathbb{R}^2)} < \infty, \quad \text{where} \quad \mathcal{P}(y_1,t) = p^3(y_1,t), \quad p(y_1,t) = (1 + \frac{\varepsilon_j}{r_j} y_1 \chi).$$

We recall that the function U is defined in (1.10), $\mathcal{P}$, p were introduced in (9.8) and χ is in (9.7). We also assume the orthogonality conditions on $\mathcal{P}\phi$

$$\int_{\mathbb{R}^2} \mathcal{P}\phi(y,t)\mathbf{Z}_\ell(y)\, dy \;=\; 0, \quad \ell = 0,1,2,3, \quad \forall t \in [0,T] \tag{10.1}$$ orto40

where the function $\mathbf{Z}_\ell$, $\ell = 0,1,2,3$ are defined in (9.22). Let $\psi(y,t)$ be the solution, predicted by Lemma 9.1 of

$$-\operatorname{div}(\mathcal{P}\nabla\psi) = \mathcal{P}\phi, \qquad \text{in} \qquad \mathbb{R}^2. \tag{10.2}$$ psi1

We let $f_0(v) = e^v$. This section is devoted to establish a series of a-priori estimates for solutions to a linear transport equation of the form

$$\begin{aligned} |\log\varepsilon|\, \varepsilon_j^2\, p\, \partial_t \phi + \nabla_y^\perp \left(\Gamma_{0j} + a_* + a\right) \cdot \nabla_y \left(\phi - f_0'(\Gamma_{0j} + a_*)\, p^2\, \psi\right) + E(y,t) \;&=\; 0 \quad \text{in } \mathbb{R}^2 \times (0,T), \\ \phi(\cdot, 0) \;&=\; 0 \quad \text{in } \mathbb{R}^2 \end{aligned} \tag{10.3}$$ innerL

These estimates will be used to treat (9.32).

On the functions $a_*(y,t)$, and $a(y,t)$ that appear in (10.3) we assume

$$a_*(y,t),\, a(y,t) \;= 0 \quad \text{for } |y| \geq 8R, \quad \Delta_y(a + a_*) \in L^\infty(\mathbb{R}^2 \times (0,T)) \tag{10.4}$$ nu0

and for some numbers $C > 0$, $\nu > 0$,

$$\begin{aligned} |\log\varepsilon|^{\frac{1}{2}} |\partial_t a_*(y,t)| + (1+|y|)|\nabla_y a_*(y,t)| + |a_*(y,t)| \;&\leq C\varepsilon^2\,(1+|y|^2)\,\log(1+|y|) \\ |\nabla_y a(y,t)| \;&\leq \varepsilon^{2+\nu}. \end{aligned} \tag{10.5}$$ nu

These assumptions are consistent with the description of Problem (9.32), in the version contained in (9.30): we will take

$$a_* = \lambda \eta_{4\varepsilon}\, b_j^*, \quad a = \lambda\left(\eta_{4\varepsilon} b_j^{**} + b_j\right), \tag{10.6}$$ aastar

where b_j^*, b_j^{**} and b_j are defined as in (9.11).

10.1. **An L^2-weighted a priori estimate.** Our first result is an L^2-weighted a-priori estimate on a solution to (10.3). We have the following

⟨lin⟩ **Lemma 10.1.** *There exists a constant $C>0$ such that for any a, a_* satisfying (10.4)-(10.5), R given by (9.27), all sufficiently small ε and any solution ϕ of (10.3)-(10.1) with*

$$\sup_{t\in[0,T]} \|U^{-\frac12}\,\mathcal{P}\phi(\cdot,t)\|_{L^2(\mathbb{R}^2)} \ < \ +\infty \tag{10.7}$$ aaa

we have

$$\sup_{t\in[0,T]} \|U^{-\frac12}\,\mathcal{P}\phi(\cdot,t)\|_{L^2(\mathbb{R}^2)} \ \le \ C\,\varepsilon^{-2}\,|\log\varepsilon|^{-\frac12} \sup_{t\in[0,T]} \|E(\cdot,t)\,U^{-1/2}\|_{L^2(\mathbb{R}^2)}. \tag{10.8}$$ pass1

Proof. Let us assume that

$$\sup_{t\in[0,T]} \|E(\cdot,t)\,U^{-1/2}\|_{L^2(\mathbb{R}^2)} \ < \ +\infty$$

and define the functions

$$U_1 = f_0'(\Gamma_{0j}+a_*), \quad U_1 g_1 = \phi - f_0'(\Gamma_{0j}+a_*)(1+\frac{\varepsilon_j}{r_j}y_1\chi)^2\psi,$$

where $f_0(s)=e^s$. We multiply equation (10.3) against g_1 and integrate in $\mathbb{R}^2$. One term is

$$\begin{aligned}\int_{\mathbb{R}^2}(1+\frac{\varepsilon_j}{r_j}y_1\chi)\,\partial_t\phi\,g_1\,dy &= \int_{\mathbb{R}^2}(1+\frac{\varepsilon_j}{r_j}y_1\chi)\partial_t\phi\frac{\phi}{U_1}\,dy - \int_{\mathbb{R}^2}\partial_t\phi\,\mathcal{P}\psi dy\\ &= \frac12\partial_t\int_{\mathbb{R}^2}(1+\frac{\varepsilon_j}{r_j}y_1\chi)U_1^{-1}\phi^2\,dy - \frac12\int_{\mathbb{R}^2}\partial_t\left((1+\frac{\varepsilon_j}{r_j}y_1\chi)U_1^{-1}\right)\phi^2\\ &\quad -\int_{\mathbb{R}^2}\partial_t\,(\mathcal{P}\phi\,)\,\psi dy + \int_{\mathbb{R}^2}\phi\,\partial_t\mathcal{P}\psi dy.\end{aligned}$$

We recall that the weight $\mathcal{P}(y_1,t)=\left(1+\frac{\varepsilon_j}{r_j}y_1\chi\right)^3$ as in (9.8). Since $-\operatorname{div}(\mathcal{P}\nabla\psi)=\mathcal{P}\phi$,

$$-\int_{\mathbb{R}^2}\partial_t\,(\mathcal{P}\phi\,)\,\psi dy = \int_{\mathbb{R}^2}\partial_t\,(\operatorname{div}(\mathcal{P}\nabla\psi\,)\,)\,\psi dy = \int_{\mathbb{R}^2}\operatorname{div}(\partial_t\mathcal{P}\nabla\psi)\,\psi dy + \int_{\mathbb{R}^2}\operatorname{div}(\mathcal{P}\nabla\partial_t\psi)\,\psi dy$$

On the other hand, using the symmetry of the form $\int_{\mathbb{R}^2}\operatorname{div}(\mathcal{P}\nabla\psi_1)\psi_2 dy$ we have that

$$\begin{aligned}\partial_t\int_{\mathbb{R}^2}\operatorname{div}(\mathcal{P}\nabla\psi)\psi\,dy &= \int_{\mathbb{R}^2}\operatorname{div}(\partial_t\mathcal{P}\nabla\psi)\psi\,dy + \int_{\mathbb{R}^2}\operatorname{div}(\mathcal{P}\nabla\partial_t\psi)\psi\,dy + \int_{\mathbb{R}^2}\operatorname{div}(\mathcal{P}\nabla\psi)\partial_t\psi\,dy\\ &= \int_{\mathbb{R}^2}\operatorname{div}(\partial_t\mathcal{P}\nabla\psi)\psi\,dy + 2\int_{\mathbb{R}^2}\operatorname{div}(\mathcal{P}\nabla\partial_t\psi)\psi\,dy\end{aligned}$$

from which we get that

$$-\int_{\mathbb{R}^2}\partial_t\,(\mathcal{P}\phi)\,\psi dy = \frac12\partial_t\int_{\mathbb{R}^2}\operatorname{div}(\mathcal{P}\nabla\psi)\psi\,dy + \frac12\int_{\mathbb{R}^2}\operatorname{div}(\partial_t\mathcal{P}\nabla\psi)\psi\,dy.$$

Thus we conclude that

$$\begin{aligned}\int_{\mathbb{R}^2}(1+\frac{\varepsilon_j}{r_j}y_1\chi)\,\partial_t\phi\,g_1\,dy &= \frac12\partial_t\int_{\mathbb{R}^2}(1+\frac{\varepsilon_j}{r_j}y_1\chi)\phi g_1\,dy - \frac12\int_{\mathbb{R}^2}\partial_t\left((1+\frac{\varepsilon_j}{r_j}y_1\chi)U_1^{-1}\right)\phi^2\\ &\quad + \int_{\mathbb{R}^2}\phi\,\partial_t\mathcal{P}\psi dy + \frac12\int_{\mathbb{R}^2}\operatorname{div}(\partial_t\mathcal{P}\nabla\psi)\psi\,dy.\end{aligned}$$

Moreover

$$\begin{aligned}\int_{\mathbb{R}^2}\nabla_y^\perp(\Gamma_{0j}+a_*+a)\cdot\nabla_y(U_1g_1)g_1 &= \int_{\mathbb{R}^2}U_1^{-1}\nabla_y^\perp(\Gamma_{0j}+a_*+a)\cdot\nabla_y(\frac{U_1^2g_1^2}{2})\,dy\\ = -\int_{\mathbb{R}^2}\nabla\cdot\left(U_1^{-1}\nabla_y^\perp(\Gamma_{0j}+a_*+a)\right)\frac{U_1^2g_1^2}{2}\,dy &= \int_{\mathbb{R}^2}\nabla_y^\perp(\Gamma_{0j}+a_*+a)\frac{\nabla U_1}{U_1}\frac{U_1g_1^2}{2}\,dy,\end{aligned}$$

and we conclude that

$$\begin{aligned}&\frac{|\log\varepsilon|\varepsilon_j^2}{2}\partial_t\int_{\mathbb{R}^2}(1+\frac{\varepsilon_j}{r_j}y_1\chi)\phi g_1\,dy-\frac{|\log\varepsilon|\varepsilon_j^2}{2}\int_{\mathbb{R}^2}\partial_t\left((1+\frac{\varepsilon_j}{r_j}y_1\chi)U_1^{-1}\right)\phi^2\\&+|\log\varepsilon|\varepsilon_j^2\int_{\mathbb{R}^2}\phi\,\partial_t\mathcal{P}\psi dy+\frac{|\log\varepsilon|\varepsilon_j^2}{2}\int_{\mathbb{R}^2}\operatorname{div}(\partial_t\mathcal{P}\nabla\psi)\psi\,dy\\&+\int_{\mathbb{R}^2}\nabla_y^{\perp}(\Gamma_{0j}+a_*+a)\frac{\nabla U_1}{U_1}\frac{U_1g_1^2}{2}\,dy+\int_{\mathbb{R}^2}Eg_1\,dy=0.\end{aligned}\tag{10.9}$$

Next, we estimate the last four terms in the above expression. Using estimates on U_1, the bounds on ψ as in Lemma 9.1 and the explicit definition of $\mathcal{P}$ in (9.8), we get

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}\phi^2\partial_t(U_1^{-1}(1+\frac{\varepsilon_j}{r_j}y_1\chi))\,dy\right|\lesssim|\log\varepsilon|^{-\frac{9}{2}}\int_{\mathbb{R}^2}\phi^2U^{-1}\\&\left|\int_{\mathbb{R}^2}\phi\psi\partial_t\mathcal{P}\,dy\right|\lesssim|\log\varepsilon|^{-\frac{9}{2}}(\int_{\mathbb{R}^2}\phi^2U^{-1})^{\frac12}(\int_{\mathbb{R}^2}\psi^2U)^{\frac12}\lesssim|\log\varepsilon|^{-\frac{9}{2}}\int_{\mathbb{R}^2}\phi^2U^{-1}\\&\left|\int_{\mathbb{R}^2}\operatorname{div}(\partial_t\mathcal{P}\nabla\psi)\psi\,dy\right|\lesssim|\log\varepsilon|^{-\frac{9}{2}}\int_{\mathbb{R}^2}\phi^2U^{-1},\end{aligned}$$

To estimate $\int_{\mathbb{R}^2}\nabla_y^{\perp}(\Gamma_{0j}+a_*+a)\frac{\nabla U_1}{U_1}\frac{U_1g_1^2}{2}\,dy$, we observe that

$$\int_{\mathbb{R}^2}\nabla_y^{\perp}(\Gamma_{0j}+a_*+a)\frac{\nabla U_1}{U_1}\frac{U_1g_1^2}{2}\,dy=\int_{\mathbb{R}^2}\nabla_y^{\perp}a\frac{\nabla U_1}{U_1}\frac{U_1g_1^2}{2}\,dy.$$

From (10.5) we get

$$\left|\int_{\mathbb{R}^2}\nabla_y^{\perp}a\frac{\nabla U_1}{U_1}\frac{U_1g_1^2}{2}\,dy\right|\lesssim\varepsilon^{2+\nu}\int_{\mathbb{R}^2}\frac{U_1g_1^2}{2}\,dy.$$

Finally

$$\left|\int_{\mathbb{R}^2}Eg_1\,dy\right|\le\|EU^{-\frac12}\|_{L^2(\mathbb{R}^2)}\,\|U^{\frac12}g_1\|_{L^2(\mathbb{R}^2)}.$$

Since $\mathcal{P}(y_1,t)=1+O(\varepsilon R)$ uniformly as $\varepsilon\to0$ for $y_1\in\mathbb{R}$, $t\in[0,T]$, from Lemma 9.2 and (9.28) we obtain

$$\int_{\mathbb{R}^2}U_1g_1^2\le C\int_{\mathbb{R}^2}\phi^2U^{-1}\le C|\log\varepsilon|\int_{\mathbb{R}^2}(1+\frac{\varepsilon_j}{r_j}y_1\chi)\phi g_1\,dy,$$

for some $C>0$. Define

$$f(t)=\left(\int_{\mathbb{R}^2}(1+\frac{\varepsilon_j}{r_j}y_1\chi)\phi g_1\,dy\right)^{\frac12}.$$

We conclude that

$$\begin{aligned}&\varepsilon_j^2|\log\varepsilon|\left|\int_{\mathbb{R}^2}\phi^2\partial_t(U_1^{-1}(1+\frac{\varepsilon_j}{r_j}y_1\chi))\,dy\right|\lesssim\varepsilon_j^2|\log\varepsilon||\log\varepsilon|^{-\frac12}f^2(t),\\&\varepsilon_j^2|\log\varepsilon|\left|\int_{\mathbb{R}^2}\phi\psi\partial_t\mathcal{P}\,dy\right|\lesssim\varepsilon_j^2|\log\varepsilon||\log\varepsilon|^{-\frac32}f^2(t),\\&\varepsilon_j^2|\log\varepsilon|\left|\int_{\mathbb{R}^2}\operatorname{div}(\partial_t\mathcal{P}\nabla\psi)\psi\,dy\right|\lesssim\varepsilon_j^2|\log\varepsilon||\log\varepsilon|^{-\frac32}f^2(t),\\&\left|\int_{\mathbb{R}^2}Eg_1\,dy\right|\lesssim\|EU^{-\frac12}\|_{L^2(\mathbb{R}^2)}f(t).\end{aligned}$$

Inserting these estimates in (10.9), we get

$$\varepsilon_j^2|\log\varepsilon|\frac{d}{dt}f^2(t)\le\varepsilon_j^2|\log\varepsilon|A_\varepsilon f^2(t)+C\|EU^{-\frac12}\|_{L^2(\mathbb{R}^2)}f(t),$$

with $A_\varepsilon\to0$ as $\varepsilon\to0$. Then we find

$$\varepsilon_j^2|\log\varepsilon|\frac{df}{dt}(t)\;\le\;\varepsilon_j^2|\log\varepsilon|A_\varepsilon f(t)+C\|E(\cdot,t)\,U^{-1/2}\|_{L^2}.$$

Performing the change of variable $\varepsilon_j^2|\log\varepsilon|\frac{d}{dt} = \frac{d}{d\tau}$, Gronwall's inequality yields

$$f(t) \,\le\, C\,\varepsilon^{-2}|\log\varepsilon|^{-1} \sup_{0\le t\le T} \|E(\cdot,t)\,U^{-1/2}\|_{L^2}.$$

This inequality and Lemma 9.2 yield (10.8). ☐

10.2. **An L^∞-weighted a priori estimate.** We consider now a class of functions $E(y,t)$ such that for a number $0<\beta<1$ we have

$$\|E\|_{3+\beta} := \sup_{(y,t)\in\mathbb{R}^2\times[0,T]} |(1+|y|)^{3+\beta}E(y,t)\| < +\infty. \tag{10.10}$$ Enu

We observe that there exists $C_*>0$ such that

$$\sup_{t\in[0,T]} \|E(\cdot,t)\,U^{-1/2}\|_{L^2(\mathbb{R}^2)} \,\le\, C_*\,\|E\|_{3+\beta}.$$

Hence Lemma 10.1 is applicable.

⟨lin1⟩ **Lemma 10.2.** *Under the assumptions of Lemma 10.1, there exists a small $\sigma>0$ such that, for some $0<\alpha<1$ and all small ε, we have that*

$$\begin{aligned} &|\psi(y,t)| \,+\, (1+|y|)|\nabla_y\psi(y,t)| + (1+|y|)^{1+\alpha}[\nabla\psi(\cdot,t)]_\alpha(y) \\ &\le \frac{\varepsilon^{-2-\sigma}}{1+|y|}\,\|E\|_{3+\beta} \quad \text{for all} \quad (y,t)\in\mathbb{R}^2\times[0,T]. \end{aligned} \tag{10.11}$$ pass2

where $\psi(y,t)$ is given by (10.2) where $\phi(y,t)$ is a solution of (10.3)-(10.1) satisfying (10.7).

Proof. From (10.3) we get that $\phi(y,t)$ satisfies the transport equation

$$\begin{aligned} \varepsilon_j^2|\log\varepsilon|(1+\frac{\varepsilon_j}{r_j}y_1\chi)\phi_t + \nabla_y^\perp(\Gamma_{0j}+a_*+a)\cdot\nabla_y\phi + \tilde E(y,t) \,&=\, 0 \quad \text{in } \mathbb{R}^2\times(0,T), \\ \phi(\cdot,0) \,&=\, 0 \quad \text{in } \mathbb{R}^2 \end{aligned} \tag{10.12}$$ innerL1

where

$$\tilde E(y,t) \,=\, E(y,t) - \nabla_y^\perp\,(\Gamma_{0j}+a_*+a)\cdot\nabla_y\left(f_0'(\Gamma_{0j}+a_*)(1+\frac{\varepsilon_j}{r_j}y_1\chi)^2\psi\right).$$

The result of Lemma 7.1 is still valid for a transport equation of the form (10.12). Let us fix a number p with $2<p<\frac{2}{1-\beta}$. Then we have that

$$\sup_{t\in[0,T]} \|U^{\frac1p-1}E(\cdot,t)\|_{L^p(\mathbb{R}^2)} \,\le\, C\,\|E\|_{3+\beta}.$$

Since ϕ solves equation (10.12) and Lemma 7.1 apply to yield

$$\|U^{\frac1p-1}\phi\|_{L^p(\mathbb{R}^2)} \le C\varepsilon^{-2}|\log\varepsilon|^{-1}\|U^{\frac1p-1}\tilde E\|_{L^p(\mathbb{R}^2)}.$$

Let us estimate this weighted L^p norm for the second term in $\tilde E$. We have

$$\left|\nabla_y^\perp(\Gamma_{0j}+a_*+a)\cdot\nabla_y(f_0'(\Gamma_{0j}+a_*)(1+\frac{\varepsilon_j}{r_j}y_1\chi)^2\psi)\right| \,\le\, C\,[\frac{1}{1+|y|^5}|\nabla\psi| + \frac{1}{1+|y|^6}|\psi|\,].$$

Since

$$\left\|U^{\frac1p-1}\frac{|\nabla\psi|}{1+|y|^5}\right\|_{L^p(\mathbb{R}^2)} \le C\left\|U^{-\frac12+\frac1p}|\nabla\psi|\right\|_{L^p(\mathbb{R}^2)} \le C\,\left\|U^{-\frac12}\phi\right\|_{L^2(\mathbb{R}^2)},$$

where the last inequality follows from (9.19). From (9.18) we get

$$\left\|U^{\frac1p-1}\frac{|\psi|}{1+|y|^6}\right\|_{L^p(\mathbb{R}^2)} \le C\left\|U^{-\frac12}\phi\right\|_{L^2(\mathbb{R}^2)}.$$

Combining the above estimates, we conclude

$$\sup_{t\in[0,T]}\|U^{\frac1p-1}\tilde E(\cdot,t)\|_{L^p(\mathbb{R}^2)} \,\le C\left(\|E\|_{3+\beta} + \sup_{t\in[0,T]}\|U^{-\frac12}\phi\|\|_{L^2(\mathbb{R}^2)}\right).$$

We now apply estimate (10.8) and we conclude that

$$\sup_{t\in[0,T]} \|U^{\frac{1}{p}-1}\phi(\cdot,t)\|_{L^p(\mathbb{R}^2)} \le C\varepsilon^{-4}\,|\log\varepsilon|^{-\frac{3}{2}} \sup_{t\in[0,T]} \|U^{-\frac{1}{2}}E(\cdot,t)\|_{L^2(\mathbb{R}^2)}.$$

Next we use an interpolation argument to control $\|U^{\frac{1}{q}-1}\phi\|_{L^q(\mathbb{R}^2)}$ for some $q\in(2,p)$. Write q in the form in the form

$$q = 2(1-\lambda)+\lambda p, \quad \lambda\in(0,1).$$

Using Hölder's inequality we check that

$$\begin{aligned}\|U^{\frac{1}{q}-1}\phi\|^q_{L^q(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} U^{1-q}|\phi|^q\,dy \le \left(\int_{\mathbb{R}^2} U^{-1}|\phi|^2\,dy\right)^{(1-\lambda)} \left(\int_{\mathbb{R}^2} U^{1-p}|\phi|^p\,dy\right)^{\lambda}\\ &= \|U^{-\frac{1}{2}}\phi\|^{2(1-\lambda)}_{L^2(\mathbb{R}^2)}\,\|U^{\frac{1}{p}-1}\phi\|^{p\lambda}_{L^p(\mathbb{R}^2)},\end{aligned}$$

We thus get that

$$\|U^{\frac{1}{q}-1}\phi\|_{L^q(\mathbb{R}^2)} \le C\,\varepsilon^{-2-2\frac{p\lambda}{q}}\,|\log\varepsilon|^{-\frac{1-\lambda}{q}-\frac{3p\lambda}{2q}}\,\|E\|_{3+\beta}.$$

Inserting this information in the above estimate we get

$$\|U^{\frac{1}{q}-1}\phi\|_{L^q(\mathbb{R}^2)} \le C\,\varepsilon^{-2-\sigma}\,\|E\|_{3+\beta},$$

for some $\sigma>0$. Estimate (10.11) follows from Lemma 9.1. The number $\sigma>0$ can be taken arbitrarily small (asking λ to be close to 0), in particular satisfying $\sigma<\beta$. The proof is concluded. □

As a consequence of the above result we can also get an L^∞-weighted estimate for ϕ.

⟨co71⟩ **Corollary 10.1.** *Under the assumptions of Lemma 10.1, we also have the estimate*

$$|\phi(y,t)| \;\le\; C\left[\frac{\varepsilon^{-2-\sigma}}{1+|y|^{3+\beta}}+\frac{\varepsilon^{-4-\sigma}}{1+|y|^{7}}\right]\|E\|_{3+\beta}. \tag{10.13}$$ pass3

Proof. Recall that $\phi(y,t)$ solves (10.12). From Lemma 10.2, we get

$$|\tilde E(y,t)| \le C\Big[\frac{1}{1+|y|^{3+\beta}}+\frac{\varepsilon^{-2-\sigma}}{1+|y|^{7}}\Big]\,\|E\|_{3+\beta}.$$

Estimate (10.13) then follows as a direct application of Lemma 7.1 for $p=+\infty$. □

We next revisit the estimates for ψ given in Lemma (9.1) and in Lemma 10.2, in view of Corollary 10.1. This will be useful in the sequel.

⟨iii⟩ **Lemma 10.3.** *Assume the validity of the assumptions of Lemma 10.1, and let $\psi(y,t)$ be given by (10.2) where $\phi(y,t)$ is a solution of (10.3)-(10.1) satisfying (10.7). Then*

$$\psi = \psi_1+\psi_2$$

where, for $y = r\,e^{i\theta}$,

$$\psi_1(y,t) = A_1(r,t)\cos\theta + A_2(r,t)\sin\theta,$$

$$(1+|y|)|\partial_r A_j(r,t)| + |A_j(r,t)| \lesssim \frac{1}{1+|y|^{1+\sigma}}\|U^{\frac{1}{q}-1}\phi\|_{L^q(\mathbb{R}^2)} + \frac{R^{\frac{2}{q}-1}}{1+|y|}\,\|U^{\frac{1}{q}-1}\phi\|_{L^q(\mathbb{R}^2)}, \quad j=1,2 \tag{10.14}$$ es111

and

$$(1+|y|)|\nabla_y\psi_2(y,t)| + |\psi_2(y,t)| \lesssim \left[\frac{\varepsilon^{-2-\sigma}}{1+|y|^{1+\beta}}+\frac{\varepsilon^{-4-\sigma}}{1+|y|^{5}}\right]\|E\|_{3+\beta}, \tag{10.15}$$ es1

for some $\sigma>0$.

Proof. Under the orthogonality conditions (10.1) on $\mathcal{P}\phi$, the result in Lemma 9.1 gives the existence and uniqueness of a solution ψ to (10.2) such that $|\psi(y,t)|\to 0$ as $|y|\to\infty$, for all $t\in[0,T]$.

Decompose $h := \mathcal{P}\phi$ in Fourier series

$$h(r,\theta,t) = H_1+H_2, \quad , \quad H_1 = \sum_{n=\pm1} h_n(r,t)e^{in\theta}, \quad H_2 = \sum_{n\neq 0,\pm1} h_n(r,t)e^{in\theta}.$$

We let ψ_1 to be the solution to $-\Delta\psi_1 = H_1$ with decay to 0 at infinity. It has the form in (10.14). Next we show the validity of the bounds in (10.14).

Using the orthogonality conditions (9.21) for $\ell = 0, 1, 2$, we write

$$\psi_1(y,t) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\left(\log\frac{1}{|y-z|} - \log\frac{1}{|y|} - \frac{y\cdot z}{|y|^2}\right) H_1(z,t)\,dz - \frac{1}{2\pi}\int_{\mathbb{R}^2\setminus B(0,5R)}\frac{y\cdot z}{|y|^2} H_1(z,t)\,dz.$$

The second integral can be easily estimated

$$\left|\int_{\mathbb{R}^2\setminus B(0,5R)}\frac{y\cdot z}{|y|^2} H_1(z,t)\,dz\right| \le \frac{C}{1+|y|}\|U^{\frac{1}{q}-1}\phi\|_{L^q(\mathbb{R}^2)}\left(\int_{\rho>5R}\frac{\rho^{\frac{q}{q-1}}}{1+\rho^{\frac{4q-4}{q-1}}}\rho\,d\rho\right)^{\frac{q-1}{q}} \le R^{\frac{2}{q}-1}\frac{C}{1+|y|}\|U^{\frac{1}{q}-1}\phi\|_{L^q(\mathbb{R}^2)}$$

To estimate the first integral, we split the region of integration in $|z| < \frac{|y|}{2}$ and its complement. For $|z| < \frac{|y|}{2}$, we Taylor expand the quantity inside the bracket and get

$$\Big|\int_{|z|<\frac{|y|}{2}}\left(\log\frac{1}{|y-z|} - \log\frac{1}{|y|} - \frac{y\cdot z}{|y|^2}\right)\mathcal{P}(z,t)\phi(z,t)\,dz\Big| \le \frac{C}{1+|y|^2}\int_{|z|<\frac{|y|}{2}}|z|^2\phi\,dz$$
$$\le \frac{C}{1+|y|^2}\|U^{\frac{1}{q}-1}\phi\|_{L^q(\mathbb{R}^2)}\left(\int_{|z|<\frac{|y|}{2}}\frac{\rho^{\frac{2q}{q-1}}}{1+\rho^{\frac{4q-4}{q-1}}}\rho\,d\rho\right)^{\frac{q-1}{q}} \le \frac{C}{1+|y|^{2-\frac{2}{q}}}\|U^{\frac{1}{q}-1}\phi\|_{L^q(\mathbb{R}^2)}.$$

In the complementary region, we get

$$\Big|\int_{|z|>\frac{|y|}{2}}\left(\log\frac{1}{|y-z|} - \log\frac{1}{|y|} - \frac{y\cdot z}{|y|^2}\right)\mathcal{P}(z,t)\phi(z,t)\,dz\Big| \le \Big|\int_{|z|>\frac{|y|}{2}}\left(\log\frac{1}{|y-z|} - \frac{y\cdot z}{|y|^2}\right)\mathcal{P}(z,t)\phi(z,t)\,dz\Big|$$
$$\le \frac{C}{1+|y|^{2-\frac{2}{q}}}\log(1+|y|)\|U^{\frac{1}{q}-1}\phi\|_{L^q(\mathbb{R}^2)}.$$

Thus we conclude that

$$|\psi_1(y,t)| \lesssim \frac{1}{1+|y|^{2-\frac{2}{q}-\sigma'}}\|U^{\frac{1}{q}-1}\phi\|_{L^q(\mathbb{R}^2)} + \frac{R^{\frac{2}{q}-1}}{1+|y|}\|U^{\frac{1}{q}-1}\phi\|_{L^q(\mathbb{R}^2)},$$

for any $\sigma' > 0$. Proceeding in a similar way, we get the estimate for $\nabla\psi_1$ and the validity of (10.14).

Consider the function

$$H = H_2 + (\mathcal{P}-1)H_1 + \nabla\mathcal{P}\nabla\psi_1,$$

and let ψ_2 solve $-\operatorname{div}(\mathcal{P}\nabla\psi_2) = H$. A direct computation gives

$$\int_{\mathbb{R}^2}[(\mathcal{P}-1)H_1 + \nabla\mathcal{P}\cdot\nabla\psi_1]\,dy = -\int_{\mathbb{R}^2}H_1 + \int_{\mathbb{R}^2}\mathcal{P}H_1 - \int_{\mathbb{R}^2}\mathcal{P}\Delta\psi_1 = 0.$$

A direct inspection gives that H has no mode 1 in its Fourier decomposition and

$$|H| \lesssim \left[\frac{\varepsilon^{-2-\sigma}}{1+|y|^{3+\beta}} + \frac{\varepsilon^{-4-\sigma}}{1+|y|^7}\right]\|E\|_{3+\beta}.$$

Define $\bar\psi = (-\Delta)^{-1}\left(\frac{\varepsilon^{-2-\sigma}}{1+|y|^{3+\beta}} + \frac{\varepsilon^{-4-\sigma}}{1+|y|^7}\right)$ given by the Newtonnian potential in $\mathbb{R}^2$:

$$\bar\psi(y,t) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\log\frac{1}{|y-z|}\left(\frac{\varepsilon^{-2-\sigma}}{1+|z|^{3+\beta}} + \frac{\varepsilon^{-4-\sigma}}{1+|z|^7}\right)dz.$$

One can show that

$$(1+|y|)|\nabla\bar\psi| + |\bar\psi| \lesssim \left[\frac{\varepsilon^{-2-\sigma}}{1+|y|^{1+\beta}} + \frac{\varepsilon^{-4-\sigma}}{1+|y|^5}\right]\|E\|_{3+\beta}.$$

Then $\hat\psi =: M\,\|E\|_{3+\beta}\,\bar\psi$ satisfies

$$\mathrm{div}(\mathcal{P}\nabla\hat\psi) + H \le 0,$$

provided the constant $M>0$ is taken large enough. This gives the bound (10.15) on ψ_2. □

10.3. **Estimates for a projected problem.** Here we consider the "projected version" of Problem (10.3),

$$\begin{aligned}\varepsilon_j^2|\log\varepsilon|\,p\,\partial_t\phi + \nabla_y^\perp\left(\Gamma_{0j}+a_*+a\right)\cdot\nabla_y\left(\phi - f_0'(\Gamma_{0j}+a_*)p^2\psi\right) \,+\, E(y,t) &= \sum_{l=0}^{3} c_l(t) z_{1l}(y) \quad \text{in } \mathbb{R}^2\times(0,T)\\ \phi(\cdot,0) \,=\, 0 \quad \text{in } \mathbb{R}^2 &\end{aligned} \tag{10.16}$$

innerL2

under the same assumptions on a and a_* as in (10.3), and

$$\|E\|_{3+\beta} \;<\; +\infty, \tag{10.17}$$

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for some $0<\beta<1$ (see (10.10) for the definition of this norm). We recall that

$$p(y_1,t) = \left(1+\frac{\varepsilon_j y_1}{r_j}\chi\right) \quad \text{and} \quad \mathcal{P}(y_1,t) = \left(1+\frac{\varepsilon_j y_1}{r_j}\chi\right)^3,$$

as defined in (9.8). Besides ψ is given by (10.2) and ϕ satisfies the orthogonality conditions (10.1). The functions $z_{1\ell}$ are given in (9.33), we recall them here

$$z_{10}(y) = U_0(y),\quad z_{11}(y) = \partial_{y_1}U_0(y),\quad z_{12}(y) = \partial_{y_2}U_0(y),\quad z_{13}(y) = 2U_0(y) + \nabla_y U_0(y)\cdot y.$$

Our next goal is to get a useful expression for the functions $c_\ell(t)$ in (10.16). We can get it if we use the explicit form of $\Gamma_{0j}+a_*$.

In accordance with (9.10) we write

$$p^2(\psi_j^0 - r_j\alpha_j|\log\varepsilon_j| + \psi_j^*) = \Gamma_{0j}(y) - (4-\alpha_j r_j)\log\varepsilon_j - \log 8 + K(P_j;P_j) + \eta_{4\varepsilon}b_j^*$$

and we also write

$$p^2(\psi_j^0 - r_j\alpha_j|\log\varepsilon_j| + \psi_j^*) = p^2\left(\psi_{0j}+g_*\right).$$

where ψ_{0j} is defined in (9.9) up to constant, and g_* is

$$g_* = -r_j\alpha_j|\log\varepsilon_j| + g_{**}$$

with g_{**}

$$|\log\varepsilon|^{\frac12}|\partial_t g_{**}(y,t)| + (1+|y|)|\nabla_y g_{**}(y,t)| + |g_{**}(y,t)| \;\le\; C\varepsilon^2\,(1+|y|^2)\,\log(1+|y|).$$

Taking $a_* = \eta_{4\varepsilon}b_j^*$ (see (10.6) with $\lambda=1$) we get

$$\Gamma_{0j} + a_* - (4-\alpha_j r_j)\log\varepsilon_j - \log 8 + K(P_j;P_j) = p^2(\psi_{0j}+g_*). \tag{10.18}$$

newe

Setting again $f_0(s)=e^s$, we have

$$f_0\left(\Gamma_{0j}+a_*\right) = \tilde f\left(p^2(\psi_{0j}+g_*)\right) := 8e^{-K(P_j;P_j)}\varepsilon_j^{4-\alpha_j r_j}e^{p^2(\psi_{0j}+g_*)}$$

Let

$$\begin{aligned}\mathbf{B}(y,t) &:= \mathrm{div}\left(\mathcal{P}\nabla(\psi_{0j}+g_*)\right) + \mathcal{P}\tilde f\left(p^2\left(\psi_{0j}+g_*\right)\right)\\ &= \mathrm{div}\left(\mathcal{P}\nabla(\psi_{0j}+g_*)\right) + \mathcal{P}\,f_0\left(\Gamma_{0j}+a_*\right), \quad \text{and}\\ \mathbf{L}(\psi) &:= \mathrm{div}\left(\mathcal{P}\nabla\psi\right) + \mathcal{P}\,f_0'\left(\Gamma_{0j}+a_*\right)\,p^2\psi.\end{aligned} \tag{10.19}$$

defBL

The construction in Proposition 5.1 gives that for some $\nu>0$, for all $(y,t)\in\mathbb{R}^2\times[0,T]$

$$|\mathbf{B}(y,t)| \le \frac{\varepsilon^{2+\nu}}{(1+|y|)^{1+\nu}}.$$

We also have, after differentiating first with respect to y_2, then with respect to y_1,

$$
\begin{aligned}
&|\mathbf{L}(\partial_2(\psi_{0j}+g_*))| = \left|\mathrm{div}(\mathcal{P}\nabla\partial_2(\psi_{0j}+g_*)) + \mathcal{P}f_0'(\Gamma_{0j}+a_*)p^2\partial_2(\psi_{0j}+g_*)\right| \le \frac{\varepsilon^{2+\nu}}{(1+|y|)^{2+\nu}}\\
&\Delta_{5,j}\partial_1(\psi_{0j}+g_*) + f_0'(\Gamma_{0j}+a_*)\,\partial_1(p^2(\psi_{0j}+g_*)) - \frac{3\varepsilon^2}{p^2}\partial_1(\psi_{0j}+g_*) = R_1, \quad \text{with} \qquad (10.20)\ \texttt{uuu}\\
&|R_1(y,t)| \le \frac{\varepsilon^{2+\nu}}{(1+|y|)^{2+\nu}}
\end{aligned}
$$

Besides, we choose the function $b_3(y,t)$ in the definition of $\mathbf{Z}_3$ in (9.22): we choose

$$
b_3(y,t) := G'(\Gamma_{0j})\, a_*(y,t) \tag{10.21}
$$
`defb3`

where G has been introduced in (9.22). We will also assume conditions (10.5) on a_* and a.

Under these further assumptions, we prove the following

⟨prop71⟩ **Proposition 10.1.** *For all sufficiently small $\varepsilon > 0$ and any functions $E(y,t)$ and $\phi(y,t)$ that satisfy (10.1), (10.16) and (10.17), we have that the numbers $c_\ell(t)$ define linear functionals of E which satisfy the estimate*

$$
\gamma_\ell c_\ell(t) = \int_{\mathbb{R}^2} E(\cdot,t)\mathbf{Z}_\ell\, dy + o(1)\|E\|_{3+\beta}, \quad \textit{with} \quad o(1)\to 0, \qquad \textit{as} \quad \varepsilon\to 0.
$$

Besides ϕ and ψ satisfy the estimates (10.8), (10.11), (10.13).

Proof. We define

$$
\tilde E(y,t) = E(y,t) - \sum_{l=0}^{3} c_l(t) z_{1l}(y),
$$

so that $\|\tilde E\|_{3+\beta} < \infty$. Lemma 10.1 gives that

$$
\|U^{-\frac12}\phi\|_{L^2(\mathbb{R}^2)} \lesssim \varepsilon^{-2}|\log\varepsilon|^{-\frac12}\mathcal{M},
$$

where

$$
\mathcal{M} := \|E\|_{3+\beta} + \sum_{\ell=0}^{3}\|c_\ell\|_{L^\infty(0,T)}\,.
$$

Arguing as in the proof of Lemma 10.2 we have that for any $\sigma > 0$ small, we can find $q > 2$ such that ϕ solution to (10.16) satisfies

$$
\|U^{\frac1q-1}\phi\|_{L^q(\mathbb{R}^2)} \lesssim \varepsilon^{-2-\sigma}\mathcal{M}.
$$

Thanks to Corollary 10.1 we also have the validity of the pointwise estimate

$$
|\phi(y,t)| \le C\left[\frac{\varepsilon^{-2-\sigma}}{1+|y|^{3+\beta}} + \frac{\varepsilon^{-4-\sigma}}{1+|y|^7}\right]\mathcal{M}.
$$

Recalling the relations (10.1)

$$
\int_{\mathbb{R}^2}\mathcal{P}\phi\,\mathbf{Z}_\ell = 0, \quad \ell = 0,1,2,3,
$$

where $\mathbf{Z}_\ell$ are defined in (9.22), we multiply (10.16) against $p^2\mathbf{Z}_\ell$ and integrate on $\mathbb{R}^2$ to get

$$
\begin{aligned}
\gamma_l c_l(t) &= -\varepsilon_j^2|\log\varepsilon|\int_{\mathbb{R}^2}\phi\,\partial_t(\mathcal{P}\mathbf{Z}_\ell)\,dy - \sum_{m\ne\ell} c_m(t)\int_{\mathbb{R}^2} p^2 z_{1m}\mathbf{Z}_\ell\,dy + \int_{\mathbb{R}^2} E(\cdot,t)\,p^2\mathbf{Z}_\ell\,dy\\
&+ \int_{\mathbb{R}^2}\nabla_y^\perp(\Gamma_{0j}+a_*+a)\cdot\nabla_y\left(\phi - f_0'(\Gamma_{0j}+a_*)\,p^2\,\psi\right)\,p^2\mathbf{Z}_\ell\,dy.
\end{aligned}
$$

where

$$
\gamma_\ell = \int_{\mathbb{R}^2} p^2\, z_{1\ell}\mathbf{Z}_\ell\,dy, \quad l = 0,1,2,3.
$$

A direct computation gives

$$
\gamma_\ell = (1+O(\varepsilon R))\int_{\mathbb{R}^2} z_{1\ell}\mathbf{Z}_\ell\,dy,
$$

as $\varepsilon \to 0$, for all ℓ. Besides, for $m \neq \ell$, $\int_{\mathbb{R}^2} p^2 z_{1m} \mathbf{Z}_\ell \, dy = O(\varepsilon R)$.

Take $\ell = 3$ and recall that $p^2\, \mathbf{Z}_3 = G * \Gamma_{0j}) + b_3$. It is straighforward to see that

$$\varepsilon_j^2 |\log \varepsilon| \int_{\mathbb{R}^2} \phi\, \partial_t(\mathcal{P}\mathbf{Z}_3)\, dy = O(\varepsilon^3 |\log \varepsilon|^{\frac12}) \|\phi(\cdot,t) U^{-\frac12}\|_{L^2(\mathbb{R}^2)}.$$

Integrating by parts we get

$$\begin{aligned}
&\int_{\mathbb{R}^2} \nabla_y^{\perp} \left(\Gamma_{0j} + a_* + a\right) \cdot \nabla_y \left(\phi - f_0'(\Gamma_{0j} + a_*)\, p^2\, \psi\right)(\cdot, t)\, p^2 \mathbf{Z}_3\, dy\,. \\
&= \; - \int_{\mathbb{R}^2} \left(\phi - f_0'(\Gamma_{0j} + a_*)\, p^2\, \psi\right) \nabla_y (p^2 \mathbf{Z}_3) \cdot \nabla_y^{\perp} (\Gamma_{0j} + a_* + a).
\end{aligned}$$

Recalling that $p^2\, \mathbf{Z}_3 = G(\Gamma_{0j}(y)) + G'(\Gamma_{0j})\, a_*(y,t)$, we write

$$p^2 \mathbf{Z}_3(y,t) = G(\Gamma_{0j} + a_*\,) + \tilde{\mathbf{Z}}_3(y,t),$$

and we get

$$\nabla_y (p^2 \mathbf{Z}_3 \cdot \nabla_y^{\perp} (\Gamma_{0j} + a_* + a)) = \nabla_y \mathbf{Z}_3 \cdot \nabla_y^{\perp} a + \nabla_y \tilde{\mathbf{Z}}_3 \cdot \nabla_y^{\perp} (\Gamma_{0j} + a_*)$$

and, using (10.21),

$$\int_{\mathbb{R}^2} \left(\phi - f_0'(\Gamma_{0j} + a_*)\, p^2\, \psi\right)\; \nabla_y (p^2 \mathbf{Z}_3) \cdot \nabla_y^{\perp} (\Gamma_{0j} + a_* + a) = O(\varepsilon^2) \|\phi(\cdot,t) U^{-\frac12}\|_{L^2(\mathbb{R}^2)}.$$

We conclude that

$$\begin{aligned}
\gamma_3 c_3(t) &= \int_{\mathbb{R}^2} E(\cdot,t)\, \mathbf{Z}_3\, dy \, + O(\varepsilon^2) \|\phi(\cdot,t) U^{-\frac12}\|_{L^2(\mathbb{R}^2)} + O(\varepsilon R) \sum_{\ell=0}^{2} |c_\ell(t)| \\
&= \int_{\mathbb{R}^2} E(\cdot,t)\, \mathbf{Z}_3\, dy \, + o(1)\mathcal{M} + O(\varepsilon R) \sum_{\ell=0}^{2} |c_\ell(t)|, \quad o(1) \to 0, \qquad \text{as} \quad \varepsilon \to 0.
\end{aligned}$$

Take now $\ell = 0$. We have

$$\varepsilon_j^2 |\log \varepsilon| \int_{\mathbb{R}^2} \phi\, \partial_t(\mathcal{P}\mathbf{Z}_0)\, dy = \varepsilon_j^2 |\log \varepsilon| \int_{\mathbb{R}^2} \phi\, \partial_t(\mathcal{P})\, \mathbf{Z}_0\, dy = O(\varepsilon^3 |\log \varepsilon|^{\frac12}) \|\phi(\cdot,t) U^{-\frac12}\|_{L^2(\mathbb{R}^2)}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^2} \nabla_y^{\perp} \left(\Gamma_{0j} + a_* + a\right) \cdot \nabla_y \left(\phi - f_0'(\Gamma_{0j} + a_*)\, p^2\, \psi\right)\; p^2\, \mathbf{Z}_0\, dy \\
&= \; - \int_{\mathbb{R}^2} \left(\phi - f_0'(\Gamma_{0j} + a_*)\, p^2\, \psi\right)\; \nabla_y^{\perp} (\Gamma_{0j} + a_* + a) \cdot \nabla_y p^2 \\
&= \; -2 \frac{\varepsilon_j}{r_j} \int_{\mathbb{R}^2} \left(\phi - f_0'(\Gamma_{0j} + a_*)\, p^2\, \psi\right)\; (1 + \frac{\varepsilon_j}{r_j} y_1 \chi)\, (\chi + y_1 \chi') \partial_2 \Gamma_0 + O(\varepsilon^2) \|\phi(\cdot,t) U^{-\frac12}\|_{L^2(\mathbb{R}^2)}
\end{aligned}$$

as

$$\nabla p^2 = 2 \frac{\varepsilon_j}{r_j} (1 + \frac{\varepsilon_j}{r_j}\, y_1\, \chi)\, (\chi + y_1 \chi') \mathbf{e}_1$$

with χ given in (9.8). We recall now that

$$-\phi = \Delta \psi + \frac{3\varepsilon_j}{(r_j + \varepsilon_j y_1 \chi)} \partial_1 \psi.$$

Integrating by parts we get

$$\begin{aligned}
- \int_{\mathbb{R}^2} \phi (1 + \frac{\varepsilon_j}{r_j} y_1 \chi)\, (\chi + y_1 \chi') \partial_2 \Gamma_0 &= \int_{\mathbb{R}^2} \Delta(\partial_2 \Gamma_0)(1 + \frac{\varepsilon_j}{r_j} y_1 \chi)\, (\chi + y_1 \chi') \psi \\
&+ \int_{\mathbb{R}^2} \Delta \left((1 + \frac{\varepsilon_j}{r_j} y_1 \chi)\, (\chi + y_1 \chi') \right) \partial_2 \Gamma_0 \psi + \frac{3\varepsilon_j}{r_j} \int_{\mathbb{R}^2} (\chi + y_1 \chi') \partial_2 \Gamma_0 \partial_1 \psi \\
&= \int_{\mathbb{R}^2} \Delta(\partial_2 \Gamma_0) \psi + O(\varepsilon) \|\phi(\cdot,t) U^{-\frac12}\|_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

Since $\Delta(\partial_2\Gamma_0)+f_0'(\Gamma_0)\partial_2\Gamma_0=0$, we get

$$\int_{\mathbb{R}^2}\left(\phi-f_0'(\Gamma_{0j}+a_*)\,p^2\,\psi\right)\,(1+\frac{\varepsilon_j}{r_j}y_1\chi)\,(\chi+y_1\chi')\partial_2\Gamma_0=O(\varepsilon)\|\phi(\cdot,t)U^{-\frac12}\|_{L^2(\mathbb{R}^2)}$$

and we conclude that

$$\gamma_0c_0(t)=\int_{\mathbb{R}^2}E(\cdot,t)\,\mathbf{Z}_3\,dy\;+o(1)\mathcal{M}+O(\varepsilon R)\sum_{\ell\neq0}|c_\ell(t)|,\quad o(1)\to0,\qquad\text{as}\quad\varepsilon\to0.$$

Consider now $l=1$. Recalling that $\Gamma_{0j}+a_*=p^2(\psi_{0j}+g_*)$ in (10.18) and that $\nabla\left(p^2\,\mathbf{Z}_1\right)=p\,\mathbf{1}_{B_{8R}}\,\mathbf{e}_1$ we get

$$\begin{aligned}&\int_{\mathbb{R}^2}\nabla_y^\perp(\Gamma_{0j}+a_*+a)\cdot\nabla_y\left(\phi-f_0'(\Gamma_{0j}+a_*)\,p^2\,\psi\right)\,p^2\,\mathbf{Z}_1\,dy\\&=-\int_{B_{8R}}\left(\phi-f_0'(\Gamma_{0j}+a_*)\,p^2\,\psi\right)\,\nabla_y^\perp(\Gamma_{0j}+a_*+a)\cdot\,p\,\mathbf{e}_1\,dy\\&+\int_{\partial B_{8R}}(\phi-f_0'(\Gamma_{0j})\psi)\,\nabla_y^\perp\Gamma_{0j}\cdot\nu(y)\,p^2\,\mathbf{Z}_1\,d\sigma\\&=\int_{B_{8R}}\left(\phi-f_0'(\Gamma_{0j}+a_*)\,p^2\,\psi\right)\mathcal{P}\partial_2(\Gamma_{0j}+a_*)\,dy+\,O(\varepsilon^2)\|\phi(\cdot,t)U^{-\frac12}\|_{L^2(\mathbb{R}^2)}.\end{aligned}$$

Since $-\frac{1}{\mathcal{P}}\operatorname{div}(\mathcal{P}\nabla\psi)=\phi$, we have, for an arbitrary h

$$\int_{B_{8R}}\mathcal{P}\phi h=-\int_{B_{8R}}\operatorname{div}(\mathcal{P}\nabla\psi)h=-\int_{B_{8R}}\frac{1}{\mathcal{P}}\operatorname{div}(\mathcal{P}\nabla h)\mathcal{P}\psi+\int_{\partial B_{8R}}\mathcal{P}(\nabla\psi\cdot\nu h-\psi\nabla h\cdot\nu).$$

We now take $h=\partial_2(\Gamma_{0j}+a_*)$. Using the estimates in Lemma 10.3, (10.14) and (10.15), we obtain

$$\int_{\partial B_{8R}}\mathcal{P}(\nabla\psi\cdot\nu h-\psi\nabla h\cdot\nu)=o(1)\,\mathcal{M}.$$

Moreover we get

$$\begin{aligned}&\int_{B_{8R}}\left(\phi-f_0'(\Gamma_{0j}+a_*)\,p^2\,\psi\right)\,\mathcal{P}\,\partial_2(\Gamma_{0j}+a_*)\,dy\\&\qquad=-\int_{B_{8R}}\mathbf{L}\left(\partial_2(\Gamma_{0j}+a_*)\right)\,\psi\,dy+o(1)\mathcal{M},\end{aligned}$$

where $\mathbf{L}$ is defined in (10.19). From (10.18) we observe that $\Gamma_{0j}+a_*$ and $p^2(\psi_{0j}+g_*)$ differ by a constant term, so $\partial_2(\Gamma_{0j}+a_*)=\partial_2\left(p^2(\psi_{0j}+g_*)\right)$. Hence we conclude that

$$\begin{aligned}&\int_{B_{8R}}\left(\phi-f_0'(\Gamma_{0j}+a_*)\,p^2\,\psi\right)\,\mathcal{P}\,\partial_2(\Gamma_{0j}+a_*)\,dy\\&\qquad=O(\varepsilon^{2+\nu})\|U^{-\frac12}\phi\|_{L^2(\mathbb{R}^2)}+O(\varepsilon^2)\|\phi(\cdot,t)U^{-\frac12}\|_{L^2(\mathbb{R}^2)}+o(1)\mathcal{M}\end{aligned}$$

thanks to (10.20). Besides

$$\varepsilon_j^2|\log\varepsilon|\int_{\mathbb{R}^2}\phi\,\partial_t(\mathcal{P}\mathbf{Z}_1)\,dy=O(\varepsilon^3|\log\varepsilon|^{\frac12})\|\phi(\cdot,t)U^{-\frac12}\|_{L^2(\mathbb{R}^2)},$$

so that

$$\gamma_1c_1(t)\;=\int_{\mathbb{R}^2}E(\cdot,t)\,p^2\mathbf{Z}_1\,dy+o(1)\mathcal{M}+O(\varepsilon R)\sum_{\ell\neq1}|c_\ell(t)|,\quad o(1)\to0,\qquad\text{as}\quad\varepsilon\to0.$$

The last case to consider is $l = 2$. Recall that $p^2\,\mathbf{Z}_2 = (y_2 + \varepsilon y_1 y_2)\,\mathbf{1}_{B_{8R}}$, so

$$\begin{aligned}
&\int_{\mathbb{R}^2} \nabla_y^\perp(\Gamma_{0j} + a_* + a)\cdot\nabla_y\left(\phi - f_0'(\Gamma_{0j} + a_*)\,p^2\,\psi\right)\,p^2\,\mathbf{Z}_2\,dy\\
&= -\int_{B_{8R}} \left(\phi - f_0'(\Gamma_{0j} + a_*)\,p^2\,\psi\right)\,\nabla_y^\perp(\Gamma_{0j} + a_* + a)\cdot\nabla(p^2\mathbf{Z}_2)\,dy\\
&+\int_{\partial B_{8R}} (\phi - f_0'(\Gamma_{0j})\psi)\,\nabla_y^\perp\Gamma_{0j}\cdot\nu(y)\,p^2\mathbf{Z}_2\,d\sigma\\
&= \int_{B_{8R}} \left(\phi - f_0'(\Gamma_{0j} + a_*)\,p^2\,\psi\right)\cdot\nabla_y^\perp(\Gamma_{0j} + a_*)\cdot\nabla(p^2\mathbf{Z}_2)\,dy + O(\varepsilon^2)\|\phi(\cdot,t)U^{-\frac{1}{2}}\|_{L^2(\mathbb{R}^2)}\\
&= \int_{B_{8R}} \left(-\Delta_{5,j}\psi - f_0'(\Gamma_{0j} + a_*)\,p^2\,\psi\right)\,\nabla_y^\perp(\Gamma_{0j} + a_*)\cdot\nabla(p^2\mathbf{Z}_2)\,dy + O(\varepsilon^2)\|\phi(\cdot,t)U^{-\frac{1}{2}}\|_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

Use the fact that $\Gamma + a_* = p^2(\psi_{0j} + g_*)$ in (10.18) to compute

$$\begin{aligned}
h_0 &:= \nabla_y^\perp(\Gamma_{0j} + a_*)\cdot\nabla(p^2\mathbf{Z}_2) = h_1 + h_2\\
h_1 &= \partial_1(p^2(\psi_{0j} + g_*))\\
h_2 &= \varepsilon\,\left(y_1\partial_1(p^2(\psi_{0j} + g_*)) - y_2\partial_2(p^2(\psi_{0j} + g_*))\right)
\end{aligned}$$

Integrating by parts and using estimates (10.3), (10.14) and (10.15) we get

$$\begin{aligned}
\int_{B_{8R}} (-\Delta_{5,j}\psi - f_0'(\Gamma_{0j} + a_*)\,p^2\,\psi\,\nabla_y^\perp(\Gamma_{0j} + a_*)\cdot\nabla(p^2\mathbf{Z}_2)\,dy &= \int_{B_{8R}} \left(-\Delta_{5,j}\psi - f_0'(\Gamma_{0j} + a_*)\,p^2\,\psi\right)\,h_0\,dy\\
&= \int_{B_{8R}} \left(-\Delta_{5,j}h_0 + \frac{6\varepsilon}{p}\partial_1 h_0 - \frac{3\varepsilon^2}{h^2}h_0 - f_0'(\Gamma_{0j} + a_*)\,p^2\,h_0\right)\,\psi\,dy\\
&= \int_{B_{8R}} \left(-\Delta_{5,j}h_0 + 6\varepsilon\partial_1 h_0 - f_0'(\Gamma_{0j} + a_*)\,p^2\,h_0\right)\,\psi\,dy + O(\varepsilon^2)\|\phi(\cdot,t)U^{-\frac{1}{2}}\|_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

We have

$$\begin{aligned}
&\int_{B_{8R}} \left(-\Delta_{5,j}h_1 + 6\varepsilon\partial_1 h_1 - f_0'(\Gamma_{0j} + a_*)\,p^2\,h_1\right)\,\psi\,dy\\
&\quad= \int_{B_{8R}} p^2\left(-\Delta_{5,j}\partial_1(\psi_{0j} + g_*) - f_0'(\Gamma_{0j} + a_*)\,\partial_1(p^2(\psi_{0j} + g_*))\right)\,\psi\,dy\\
&\quad+ \int_{B_{8R}} p^2\left(4\varepsilon\partial_1^2(\psi_{0j} + g_*) - 2\varepsilon\Delta(\psi_{0j} + g_*)\right)\,\psi\,dy + O(\varepsilon^2)\|\phi(\cdot,t)U^{-\frac{1}{2}}\|_{L^2(\mathbb{R}^2)}\\
&\quad= \int_{B_{8R}} p^2\left(-\Delta_{5,j}\partial_1(\psi_{0j} + g_*) - f_0'(\Gamma_{0j} + a_*)\,\partial_1(p^2(\psi_{0j} + g_*))\right)\,\psi\,dy\\
&\quad+ \int_{B_{8R}} \left(2\varepsilon\partial_1^2\Gamma_0 - 2\varepsilon\partial_2^2\Gamma_0\right)\,\psi\,dy + O(\varepsilon^2)\|\phi(\cdot,t)U^{-\frac{1}{2}}\|_{L^2(\mathbb{R}^2)}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{B_{8R}} \left(-\Delta_{5,j}h_2 + 6\varepsilon\partial_1 h_2 - f_0'(\Gamma_{0j} + a_*)\,p^2\,h_2\right)\,\psi\,dy\\
&\quad= \varepsilon\int_{B_{8R}} (-\Delta(y_1\partial_1\Gamma_0) - U\;y_1\partial_1\Gamma_0 + \Delta(y_2\partial_2\Gamma_0) - U\;y_2\partial_2\Gamma_0)\,\psi\,dy + O(\varepsilon^2)\|\phi(\cdot,t)U^{-\frac{1}{2}}\|_{L^2(\mathbb{R}^2)}\\
&\quad= \varepsilon\int_{B_{8R}} \left(-2\partial_1^2\Gamma_0 + 2\partial_2^2\Gamma_0\right)\,\psi\,dy + O(\varepsilon^2)\|\phi(\cdot,t)U^{-\frac{1}{2}}\|_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

In the latter computation we use that

$$\Delta(y_j\partial_j\Gamma_0) + U\;y_j\partial_j\Gamma_0 = y_j\left(\Delta(\partial_j\Gamma_0) + U\;\partial_j\Gamma_0\right) + 2\partial_j^2\Gamma_0 = 2\partial_j^2\Gamma_0, \quad j = 1, 2.$$

We conclude that

$$\int_{B_{8R}} (-\Delta_{5,j}\psi - f_0'(\Gamma_{0j}+a_*)\, p^2\, \psi\, \nabla_y^{\perp}(\Gamma_{0j}+a_*)\cdot\, \nabla(p^2\mathbf{Z}_2)\, dy$$
$$= \int_{B_{8R}} p^2 \left(-\Delta_{5,j}\partial_1(\psi_{0j}+g_*) - f_0'(\Gamma_{0j}+a_*)\, \partial_1(p^2(\psi_{0j}+g_*))\right)\, \psi\, dy$$
$$+\, O(\varepsilon^2)\|\phi(\cdot,t)U^{-\frac12}\|_{L^2(\mathbb{R}^2)}.$$

We now use the second estimate in (10.20) to conclude that

$$\int_{B_{8R}} \left(-\Delta_{5,j}\psi - f_0'(\Gamma_{0j}+a_*)\, p^2\, \psi\right)\, \nabla_y^{\perp}(\Gamma_{0j}+a_*)\cdot\, \nabla(p^2\mathbf{Z}_2)\, dy = O(\varepsilon^2)\|\phi(\cdot,t)U^{-\frac12}\|_{L^2(\mathbb{R}^2)},$$

and hence

$$\int_{\mathbb{R}^2} \nabla_y^{\perp}(\Gamma_{0j}+a_*+a)\cdot\nabla_y \left(\phi - f_0'(\Gamma_{0j}+a_*)\, p^2\, \psi\right)\, p^2\, \mathbf{Z}_2\, dy = O(\varepsilon^2)\|\phi(\cdot,t)U^{-\frac12}\|_{L^2(\mathbb{R}^2)}.$$

Thus we get

$$\gamma_2 c_2(t) \;= \int_{\mathbb{R}^2} E(\cdot,t)\, p^2\mathbf{Z}_2\, dy + o(1)\mathcal{M} + O(\varepsilon R)\sum_{\ell\neq 1}|c_\ell(t)|, \quad o(1)\to 0, \qquad \text{as} \quad \varepsilon\to 0.$$

Combining the above estimates, we obtain the expected result. □

We want to apply Proposition 10.1 to obtain a priori estimates for the projected non-linear problem (9.32). We take

$$a_* = \lambda\eta_{4\varepsilon}\, b_j^*, \quad a = \lambda\left(\eta_{4\varepsilon} b_j^{**} + b_j\right)$$

in the inner operator defined in (9.30), that we write in the form

$$\varepsilon_j^2|\log\varepsilon|\, p\, \partial_t\phi + \nabla_y^{\perp}(\Gamma_0+a_*+a)\cdot\nabla_y(\phi - f'(\Gamma_0+a_*)p^2\psi) +\, \Theta_{j,\lambda}$$
$$+\, \nabla^{\perp}((1+\frac{\varepsilon_j}{r_j}y_1\chi)^2\hat\psi_j)\cdot\nabla\left(\lambda\,\eta_{4\varepsilon}U^*\right) = \sum_{l=0}^{3} c_l(t)\mathcal{Z}_{1l}(y) \quad \text{in } \mathbb{R}^2\times(0,T)$$
$$\phi(\cdot,0) \;=\; 0 \quad \text{in } \mathbb{R}^2$$

where

$$\Theta_{j,\lambda} = \nabla^{\perp}\left[\varepsilon_j\left(|\log\varepsilon|\partial_t\mathbf{a}_j + D_x\nabla_x\varphi_j(\mathbf{P}_j;\mathbf{P})[\mathbf{a}] + \lambda\eta_{4\varepsilon}(1+\frac{\varepsilon_j y_1}{r_1})^2 r_j\psi^{out}\right)\cdot y\right]\nabla U$$
$$+\, \lambda\tilde{\mathcal{E}}_j(\beta_j,\psi^{out},\mathbf{a}) + |\log\varepsilon|\varepsilon_j^2(1+\frac{\varepsilon_j}{r_j}y_1\chi)\sum_{\ell=0,3}\partial_t(\beta_{j\ell}\mathcal{Z}_{1\ell}) \qquad \text{in } \mathbb{R}^2\times[0,T]$$

Besides we have a priori bounds of the form

$$\gamma_\ell c_l(t) = \int_{\mathbb{R}^2} \Theta_{j,\lambda}(\cdot,t)\mathbf{Z}_\ell\, dy \;+\; o(1)\|\Theta_{j,\lambda}\|_{3+\beta}. \tag{10.22}$$

roger1

for some $o(1)\to 0$ as $\varepsilon\to 0$ readily follow, of course provided that $a(y,t)$ satisfies the required smallness assumptions. This is guaranteed by the choice of spaces for the parameter functions $\beta_j, \psi^{out}, \mathbf{a}$. This is what we shall specify in the next section.

11. Fixed point formulation and conclusion of the proof

⟨sec10⟩

In this section we set up the system (9.32), (9.34), (9.35), (9.36) as a fixed point problem in the form (9.37) in an appropriate Banach space for the parameter functions

$$\vec p = (\hat\phi, \beta, \phi^{out}, \psi^{out}, \mathbf{a}).$$

11.1. **The space for the parameter functions.** We begin by defining an appropriate norm for the functions $\hat\phi_j(y,t)$, $\phi^{out}(x,t)$, $\psi^{out}(x,t)$, $\beta(t)$ and $\mathbf{a}(t)$.

Let us fix a small number $0<\beta<1$. For and arbitrary functions $\phi(y,t)$ we define the inner norm

$$\begin{aligned}\|\phi\|_i \;:=\; &\sup_{t\in[0,T]}\|\phi(\cdot,t)U^{-\frac12}\|_{L^2(\mathbb{R}^2)}\\ &+\sup_{(y,t)\in\mathbb{R}^2\times[0,T]}|\,(1+|y|)^{3+\beta}\min\{1,\varepsilon^{2+\frac{\beta}{4}}(1+|y|)^{4-\beta}\}\phi(y,t)\,|\end{aligned}$$

For the outer functions ψ^{out}, ϕ^{out} we consider the following norms for functions $\phi(x,t)$, $\psi(x,t)$ defined in $\Sigma\times[0,T]$.

$$\begin{aligned}\|\phi\|_{o1} &:= \|(1+|x|)^{2+\nu}\phi\|_{L^\infty(\Sigma\times[0,T])},\\ \|\psi\|_{o2} &:= \|(1+|x|)^{\nu}\psi\|_{L^\infty(\Sigma\times[0,T])}+\|(1+|x|)^{1+\nu}\nabla_x\psi\|_{L^\infty(\Sigma\times[0,T])}\,.\end{aligned}$$

We consider the space X of all continuous functions $\vec p=(\hat\phi,\beta,\phi^{out},\psi^{out},\mathbf{a})$ such that

$$\nabla_y\hat\psi(y,t),\quad \nabla_x\psi^{out}(x,t),\quad \frac{d}{dt}\beta(t),\quad \frac{d}{dt}\mathbf{a}(t)$$

exist and are continuous and such that

$$\|\vec p\|_X \;:=\; \|\phi^{out}\|_{o1}+\|\psi^{out}\|_{o2}+\sum_{j=1}^k\left(\|\hat\phi_j\|_i+\|\beta_j\|_{C^1[0,T]}+\|\mathbf{a}\|_{C^1[0,T]}\right)\;<\;+\infty.$$

We define the set $\mathcal{O}$ as a "deformed ball" centered at $\vec p=\vec 0$. We fix an arbitrarily small number $\sigma>0$ and let $\mathcal{O}$ be the set of all functions $\vec p=(\tilde\phi,\alpha,\tilde\xi,\phi^{out},\psi^{out})\in X$ such that

$$\begin{cases}\displaystyle\sum_{j=1}^k\|\hat\phi_j\|_i\;<\;\varepsilon^{3-3\sigma},\\ \displaystyle\sum_{j=1}^k\|\beta_j\|_{C^1[0,T]}\;<\;\varepsilon^{3-3\sigma},\quad \sum_{j=1}^k\|\mathbf{a}\|_{C^1[0,T]}\;<\;\varepsilon^{4-3\sigma},\\ \|\phi^{out}\|_{o1}\;<\;\varepsilon^{4-3\sigma},\quad \|\psi^{out}\|_{o2}\;<\;\varepsilon^{4-3\sigma}.\end{cases}\tag{11.1}$$

OO

11.2. **Fixed point formulation.** Let us express System (9.32), (9.34), (9.35), (9.36) in the fixed point form (9.37) for a suitable operator $\mathcal{F}(\cdot,\lambda)$, in a region of the form (11.1).

We start with (9.32). For a given function $a(y,t)$ let us consider the transport operator

$$\mathcal{T}_j(a)[\phi]:=|\log\varepsilon|\,\varepsilon_j^2\,(1+\frac{\varepsilon_j}{r_j}y_1\chi)\,\partial_t\phi+\nabla_y^\perp(\Gamma_{0j}+a)\cdot\nabla_y\phi,$$

and for a bounded function $E(y,t)$ the linear equation

$$\begin{aligned}\mathcal{T}_j(a)[\phi]+E=0\quad&\text{in }\mathbb{R}^2\times[0,T]\\ \phi(\cdot,0)=0\quad&\text{in }\mathbb{R}^2.\end{aligned}$$

The result of Lemma 7.1 is still valid for a transport equation of the form

$$\begin{aligned}|\log\varepsilon|\varepsilon_j^2(1+\frac{\varepsilon_j}{r_j}y_1\chi)\partial_t\phi+\nabla_y^\perp(\Gamma_{0j}+a)\cdot\nabla_y\phi+E(y,t)\;&=\;0\quad\text{in }\mathbb{R}^2\times(0,T),\\ \phi(\cdot,0)\;&=\;0\quad\text{in }\mathbb{R}^2\end{aligned}$$

Here the function $\chi(y_1)$ is defined in (9.7) and the function $a(y,t)$ satisfy

$$a(y,t)\;=0\quad\text{for }|y|\ge 8R,\quad \Delta_y(a)\in L^\infty(\mathbb{R}^2\times(0,T))$$

and for some numbers $C>0$, $\nu>0$,

$$(1+|y|)^{-1}|\log\varepsilon|^{\frac12}|\partial_t a(y,t)|+|\,\nabla_y a(y,t)|\;\le C\varepsilon^2(1+|y|)\,\log(1+|y|).$$

We call $\phi = \mathcal{T}_j^{-1}(a)[E]$ the unique solution of this problem, through the representation formula (7.4), which defines a linear operator of E. Let us write the operator $E_{j,\lambda}$ in (9.30) in the form

$$E_{j,\lambda}(\hat\phi_j, \beta_j, \psi^{out}; \mathbf{a}) = \mathcal{T}_j(\lambda\eta_{4\varepsilon}b_j^*(\mathbf{a}) + \lambda\eta_{4\varepsilon}b_j^{**}(\mathbf{a}) + \lambda b_j(\hat\psi_j, \beta_j, \psi^{out}, \mathbf{a}))[\hat\phi_j] + \nabla^\perp(p^2\hat\psi_j)\cdot\nabla(\lambda\,\eta_{4\varepsilon}U^*) + \Theta_{j\lambda}(\hat\psi_j, \beta_j, \psi^{out}; \mathbf{a}).$$

and we reformulate equations (9.32) as

$$\hat\phi_j = \mathcal{F}_\lambda^{in}(\hat\phi_j, \beta_j, \psi^{out}, \mathbf{a}),$$

where

$$\mathcal{F}_\lambda^{in}(\hat\phi_j, \beta_j, \psi^{out}, \mathbf{a}) = \mathcal{T}_j^{-1}(\lambda\eta_{4\varepsilon}b_j^*(\mathbf{a}) + \lambda\eta_{4\varepsilon}b_j^{**}(\mathbf{a}) + \lambda b_j(\hat\psi_j, \beta_j, \psi^{out}, \mathbf{a}))\circ \left[\nabla^\perp(p^2\hat\psi_j)\cdot\nabla(\lambda\,\eta_{4\varepsilon}U^*) + \Theta_{j\lambda}(\hat\psi_j, \beta_j, \psi^{out}; \mathbf{a}) - \sum_{l=0}^{3} c_{lj}z_{1l}\right] \tag{11.2}$$

fixed1

We reformulate the outer equations (9.35)-(9.36) in a similar way. For a given function $e(x,t)$ with $r\Delta_x e$, $\nabla e \in L^\infty(\Sigma\times[0,T])$ let us consider the transport operator

$$\mathcal{T}_o(e)[\phi] := |\log\varepsilon|\, r\,\partial_t\phi + \nabla_y^\perp(r^2(\Psi_* - r_0^{-1}|\log\varepsilon| + e))\cdot\nabla_x\phi,$$

and for a bounded function $E(x,t)$, the linear equation

$$\mathcal{T}_o(e)[\phi] + E = 0 \quad \text{in } \Sigma\times[0,T]$$
$$\phi(\cdot,0) = 0 \quad \text{in } \Sigma.$$

We call $\phi = \mathcal{T}_o^{-1}(e)[E]$ the unique solution of this problem, through the representation formula (6.32). We write (9.35)-(9.36) in the form

$$\begin{aligned} \phi^{out} &= \mathcal{F}_{1\lambda}^{out}(\hat\phi, \beta, \psi^{out}, \mathbf{a}) \\ \psi^{out} &= \mathcal{F}_{2\lambda}^{out}(\hat\phi, \beta, \psi^{out}, \mathbf{a}) \end{aligned} \tag{11.3}$$

fixed2

where

$$\mathcal{F}_{1\lambda}^{out}(\hat\phi, \beta, \psi^{out}, \mathbf{a}) := \mathcal{T}_o^{-1}(\lambda\sum_{j=1}^{k}\frac{\bar\eta_{j2}}{r_j}\psi_j(\frac{x-P_j}{\varepsilon_j}) + \lambda\psi^{out})\left[\lambda\tilde{\mathcal{E}}_1^{out}(\hat\phi, \beta, \psi^{out}; \mathbf{a})\right]$$

with $\tilde{\mathcal{E}}_1^{out}$ given by (9.31) and

$$\mathcal{F}_{2\lambda}^{out}(\hat\phi, \beta, \psi^{out}, \mathbf{a}) := \mathcal{T}^{-1}\left[\lambda\,\phi^{out} + \lambda\sum_{j=1}^{k}(\bar\eta_{j1} - \bar\eta_{j2})\frac{\phi_j}{r_j\varepsilon_j^2} + \lambda\sum_{j=1}^{k}(\frac{\psi_j}{r_j}\Delta_5\bar\eta_{j2} + 2\nabla_x\bar\eta_{j2}\nabla_x\frac{\psi_j}{r_j})\right]$$

where $\psi = \mathcal{T}^{-1}h$ is the unique solution of Problem (4.14)

$$\Delta_5\psi + h = 0, \quad \text{in } \Sigma, \quad \frac{\partial\psi}{\partial r} = 0 \quad \text{on } \partial\Sigma, \quad \psi(x)\to 0, \quad \text{as} \quad |x|\to\infty,$$

for a smooth function h satisfying (4.15). Recall that $\Delta_5 = \frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$, for $x=(r,z)\in\Sigma$.

Using (10.22), we write equations (9.34) $c_{\ell j} = 0$ as

$$\begin{cases} |\log\varepsilon|\partial_t\mathbf{a}_j = -D_x\nabla_x\varphi_j(\mathbf{P}_j;\mathbf{P})[\mathbf{a}] + \lambda\varepsilon_j^{-1}\mathcal{G}_j(\hat\phi, \beta, \psi^{out}; \mathbf{a}), \\ |\log\varepsilon|\partial_t\beta_j = \lambda\varepsilon_j^{-2}\mathcal{H}_j(\hat\phi, \beta, \psi^{out}; \mathbf{a}), \quad j = 1,\dots,k, \\ \qquad \mathbf{a}(0) = 0, \quad \beta(0) = 0, \end{cases} \tag{11.4}$$

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where

$$\beta_j = (\alpha_{0j}, \alpha_{3j}), \quad \mathbf{a}_j = (a_{j1}, a_{j2}), \quad \beta = (\beta_1,\dots,\beta_k), \quad \mathbf{a} = (\mathbf{a}_1,\dots,\mathbf{a}_k)$$

and, for

$$\mathcal{G}_j = (\mathcal{G}_{1j}, \mathcal{G}_{2j}), \quad \mathcal{H}_j = (\mathcal{H}_{0j}, \mathcal{H}_{3j}),$$

$$
\begin{aligned}
\mathcal{G}_{\ell j}(\hat\phi,\beta,\psi^{out},\mathbf{a})(t) \ &:= \gamma_\ell\lambda\int_{\mathbb{R}^2}\big[\,\nabla^\perp[\eta_{4\varepsilon}p^2 r_j\psi^{out}\cdot y]\nabla U+\tilde{\mathcal{E}}_j(\beta_j,\psi^{out},\mathbf{a})\,\big]\mathbf{Z}_\ell\,dy\\
&\quad \text{for } \ell=1,2,\\
\mathcal{H}_{mj}(\hat\phi,\beta,\psi^{out},\mathbf{a})(t) \ &:=\gamma_m\lambda\int_{\mathbb{R}^2}\big[\,\nabla^\perp[\eta_{4\varepsilon}p^2 r_j\psi^{out}\cdot y]\nabla U+\tilde{\mathcal{E}}_j(\beta_j,\psi^{out},\mathbf{a})\,\big]\mathbf{Z}_m\,dy\\
&\quad +\gamma_m\int_{\mathbb{R}^2}p\sum_{i=0,3}\beta_{ji}\partial_t\mathcal{Z}_{1i})\mathbf{Z}_m\\
&\quad \text{for } m=0,3.
\end{aligned}
$$

Recall that $p=1+\frac{\varepsilon_j}{r_j}\,y_1\,\chi$. Equations (11.4) can be written in fixed point form as

$$
\begin{aligned}
\mathbf{a}(t)&=\mathcal{F}_{1\lambda}(\hat\phi,\beta,\psi^{out},\mathbf{a})=|\log\varepsilon|^{-1}\int_0^t\big(B(s)[\mathbf{a}]+\lambda\varepsilon_j^{-1}\mathcal{G}_j(\hat\phi,\beta,\psi^{out};\mathbf{a})\big)\,ds\\
\beta(t)&=\mathcal{F}_{0\lambda}(\hat\phi,\beta,\psi^{out},\mathbf{a})=|\log\varepsilon|^{-1}\int_0^t\lambda\varepsilon_j^{-2}\mathcal{H}_j(\hat\phi,\beta,\psi^{out};\mathbf{a})\,ds
\end{aligned}
\tag{11.5}
$$

sisalfa

where

$$
\big(B(t)[\mathbf{a}]\big)_j=-D_x\nabla_x\varphi_j(\mathbf{P}_j;\mathbf{P})[\mathbf{a}],\quad \mathcal{G}=(\mathcal{G}_1,\dots,\mathcal{G}_k),\quad \mathcal{H}=(\mathcal{H}_1,\dots,\mathcal{H}_k).
$$

System (9.32), (9.34), (9.35), (9.36) can be written as a fixed point problem (11.2)-(11.3)-(11.5), which we write as

$$
\begin{aligned}
\hat\phi &=\mathcal{F}^{in}_\lambda(\hat\phi,\beta_j,\psi^{out},\mathbf{a}),\quad \phi^{out}=\tilde{\mathcal{F}}^{out}_{1\lambda}(\hat\phi,\beta,\psi^{out},\mathbf{a}),\quad \psi^{out}=\tilde{\mathcal{F}}^{out}_{2\lambda}(\hat\phi,\beta,\psi^{out},\mathbf{a})\\
\mathbf{a}(t)&=\tilde{\mathcal{F}}_{1\lambda}(\hat\phi,\beta,\psi^{out},\mathbf{a}),\quad \beta(t)=\tilde{\mathcal{F}}_{0\lambda}(\hat\phi,\beta,\psi^{out},\mathbf{a})
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\mathcal{F}}^{out}_{1\lambda}(\hat\phi,\beta,\psi^{out},\mathbf{a})&=F^{out}_{1\lambda}(\mathcal{F}^{in}_\lambda(\hat\phi,\beta_j,\psi^{out},\mathbf{a}),\beta,\psi^{out},\mathbf{a})\\
\tilde{\mathcal{F}}^{out}_{2\lambda}(\hat\phi,\beta,\psi^{out},\mathbf{a})&=\mathcal{F}^{out}_{2\lambda}(\mathcal{F}^{in}_\lambda(\hat\phi,\beta_j,\psi^{out},\mathbf{a}),\beta,\tilde{\mathcal{F}}^{out}_{1\lambda}(\hat\phi,\beta,\psi^{out},\mathbf{a}),\mathbf{a})\\
\tilde{\mathcal{F}}_{1\lambda}(\hat\phi,\beta,\psi^{out},\mathbf{a})&=\mathcal{F}_{1\lambda}(\mathcal{F}^{in}_\lambda(\hat\phi,\beta_j,\psi^{out},\mathbf{a}),\beta,\tilde{\mathcal{F}}^{out}_{1\lambda}(\hat\phi,\beta,\psi^{out},\mathbf{a}),\mathbf{a})\\
\tilde{\mathcal{F}}_{0\lambda}(\hat\phi,\beta,\psi^{out},\mathbf{a})&=\mathcal{F}_{0\lambda}(\mathcal{F}^{in}_\lambda(\hat\phi,\beta_j,\psi^{out},\mathbf{a}),\beta,\tilde{\mathcal{F}}^{out}_{1\lambda}(\hat\phi,\beta,\psi^{out},\mathbf{a}),\mathbf{a}).
\end{aligned}
$$

In a more compact form, the above system can be expressed as

$$
\vec p \ = \ \tilde{\mathcal{F}}_\lambda(\vec p),\quad \vec p\in\bar{\mathcal{O}}
\tag{11.6}
$$

ecc

where $\mathcal{O}$ is defined in (11.1) and

$$
\begin{cases}
\tilde{\mathcal{F}}_\lambda(\vec p) \ :=(\mathcal{F}^{in}_\lambda(\vec p),\tilde{\mathcal{F}}_{0\lambda}(\vec p),\tilde{\mathcal{F}}_{1\lambda}(\vec p),\tilde{\mathcal{F}}^{out}_{1\lambda}(\vec p),\tilde{\mathcal{F}}^{out}_{2\lambda}(\vec p)),\\
\qquad\vec p=(\hat\phi,\beta,\phi^{out},\psi^{out},\mathbf{a}).
\end{cases}
\tag{11.7}
$$

opera

Lemma 11.1. *The operator* $\tilde{\mathcal{F}}:\mathcal{O}\times[0,1]\to X$ *given by* $\tilde{\mathcal{F}}(\cdot,\lambda)=\tilde{\mathcal{F}}_\lambda$ *in* (11.7) *is compact.*

Proof. We check that each of the five operators defining $\tilde{\mathcal{F}}_\lambda(\vec p)$ is compact in $\mathcal{O}$ (uniformly in λ). This operator $\mathcal{F}^{in}_\lambda(\vec p)$ is defined through $g=\mathcal{T}_j^{-1}(b)[h]$ which has the property in Lemma 8.2. This gives that a uniform bound in $\Delta_y b$ and a control of the modulus of continuity in y of $h(y,t)$ uniformly in t yields a uniform control of the modulus of continuity of g in both variables (y,t). We see in (11.2) that for a certain $C_\varepsilon>0$ we have

$$
\|\Delta_y(b_j^*+b_j)\|_{L^\infty(\mathbb{R}^2\times[0,T])}\le C_\varepsilon\quad\text{for all}\quad \vec p\in\bar{\mathcal{O}}.
$$

and it vanishes outside a compact set. Moreover, we have a uniform Hölder control in space variables on the corresponding arguments h for $\vec p\in\bar{\mathcal{O}}$ as it follows from the Hölder estimates for the gradients of $\hat\psi_j$ and ψ^{out} inherited from the uniform bounds holding for $\tilde\psi$ and ϕ^{out} in the definition of $\mathcal{O}$ (see the argument in the proof of (10.11)). Also, the numbers $c_{lj}(t)$ have a uniform bound, thanks to (10.22). Uniform Lipschitz bounds hold for the remaining errors, as it follows in particular from the control of the terms involving $\nabla_y\phi_j^*$. Lemma 8.2 then implies that $\tilde{\mathcal{F}}^{in}_\lambda(\bar{\mathcal{O}})$ is a set of continuous functions $g:\mathbb{R}^2\times[0,T]\to\mathbb{R}^N$ whose restrictions to any compact set defines a uniformly bounded, equicontinuous set. Hence, any sequence $\phi_n\in\tilde{\mathcal{F}}^{in}_\lambda(\mathcal{O})$ has a subsequence $\phi_{n'}$ which is uniformly convergent on each compact set. Finally, we observe that $\|\phi_n\|_4 \le C_\varepsilon$ since the argument of the transport operator has this property. This implies that $\phi_{n'}$ is actually convergent in the space of continuous functions with finite $\|\cdot\|_{3+\beta}$-norm, since $0 < \beta < 1$. Hence $\tilde{\mathcal{F}}^{in}_\lambda(\mathcal{O})$ is precompact in this space. The compactness of the operator $\tilde{\mathcal{F}}^{out}_{1\lambda}$ into $C(\bar\Omega \times [0,T])$ follows directly from Arzela-Ascoli's theorem, again from the corresponding control for the transport equation and the uniform controls on space and time variables valid for the operator $\tilde{\mathcal{F}}^{in}_{1\lambda}$. From here the compactness for $\tilde{\mathcal{F}}^{out}_{2\lambda}$ follows in similar manner. Finally, the compactness of the operators $\tilde{\mathcal{F}}_{0\lambda}(\vec p), \tilde{\mathcal{F}}_{1\lambda}(\vec p)$ into $C^1([0,T])$ follows again from the equicontinuity in t inherited for the different terms involved in their definition. The proof is concluded. □

11.3. **Conclusion of the proof of Theorem 1.** The original problem has been so far reduced to finding a solution of the fixed point problem (11.6) for $\lambda = 1$. To do this, we will prove that for all $\lambda \in [0,1]$ this equation has no solution $\vec p \in \partial\mathcal{O}$, at least whenever ε is chosen sufficiently small. Let us assume that $\vec p \in \bar{\mathcal{O}}$ satisfies (11.6) for some λ. We claim that actually $\vec p \in \mathcal{O}$. Examining the function $\Theta_{j,\lambda}$ in (**??**) we see that if β is chosen sufficiently small then

$$\|\Theta_j(\beta, \psi^{out}, \mathbf{a})\|_{3+\beta} \le \varepsilon^{5-\frac{3}{2}\beta}$$

Corollary 10.1 and Lemma 10.1 then yield, by definition of the inner norm,

$$\|\hat\phi\|_i \le \varepsilon^{3-2\beta} \ll \varepsilon^{3-3\beta},$$

the latter number being that involved in the definition of $\mathcal{O}$ in (11.1). Let us consider the outer equations. Examining expression (9.31) that determines the size of ϕ^{out}, we see that its magnitude does not exceed the order $O(\varepsilon^{4-\beta})$. Here we have used the remote size of $\hat\phi$ implicit in the norm $\|\hat\phi_j\|_i$. Indeed using the size induced in $\hat\psi$, we find that

$$\|\phi^{out}\|_{o1} + \|\psi^{out}\|_{o2} \le \varepsilon^{4-2\beta} \ll \varepsilon^{4-3\beta}.$$

Finally from the size of Θ_j we readily see that

$$\|\mathbf{a}_j\|_{C^1[0,T]} + \varepsilon\|\beta_j\|_{C^1[0,T]} \le \varepsilon^{4-2\beta} \ll \varepsilon^{4-3\beta}.$$

As a conclusion, we get that $\vec p \in \mathcal{O}$ and the claim has been proven.

Standard degree theory applies then to yield that the degree $\deg(I - \tilde{\mathcal{F}}(\cdot,\lambda), \mathcal{O}, 0)$ is well-defined and it is constant in $\lambda \in [0,1]$. Since $\tilde{\mathcal{F}}(\cdot, 0)$ is a linear compact operator, this constant is actually non-zero. Existence of a solution in $\mathcal{O}$ for $\lambda = 1$ then follows. The proof is concluded. □

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