Global minimizers of the Allen-Cahn equation in dimension $n \geq 8$

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Abstract

We prove the existence of nontrivial global minimizers of the Allen-Cahn equation in dimension 8 and above. More precisely, given any strict area-minimizing Lawson's cone, there is a family of global minimizers whose nodal sets are asymptotic to this cone. As a consequence of Jerison-Monneau's program we then establish the existence of many new counter-examples to the De Giorgi conjecture whose nodal sets are different from the Bombieri-De Giorgi-Giusti minimal graph.

1 Introduction and main results

Bounded entire solutions of the Allen-Cahn equation

$$-\Delta u = u - u^3 \quad \text{in } \mathbb{R}^n, \ \ |u| < 1, \tag{1}$$

has attracted a lot of attention in recent years, partly due to its intricate connection with minimal surfaces. For $n = 1$, (1) has a heteroclinic solution $H(x) = \tanh\left(\frac{x}{\sqrt{2}}\right)$. Up to a translation, this is the unique monotone increasing solution in $\mathbb{R}$. De Giorgi [8] conjectured that for $n \leq 8$, if a solution to (1) is monotone in one direction, then up to translation and rotation it equals H.

De Giorgi conjecture is parallel to the Bernstein conjecture in minimal surface theory, which states that if $F:\mathbb{R}^n\rightarrow\mathbb{R}$ is a solution to the minimal surface equation

$$\operatorname{div}\frac{\nabla F}{\sqrt{1+|\nabla F|^2}}=0,$$

then F must be a linear function in its variables. The Bernstein conjecture has been shown to be true for $n\leq 7$. The famous Bombieri-De Giorgi-Giusti minimal graph (see [4]) gives a counter-example for $n=8$, which also disproves the Bernstein conjecture for all $n\geq 8$. As for the De Giorgi conjecture, it has been proven to be true for $n=2$ (Ghoussoub-Gui [17]), $n=3$ (Ambrosio-Cabré [3]), and for $4\leq n\leq 8$ (Savin [25]), under an additional limiting condition

$$\lim_{x_n\rightarrow\pm\infty}u\left(x',x_n\right)=\pm 1. \tag{2}$$

This condition together with the monotone property implies that u is a global minimizer in the sense that, for any smooth bounded domain $\Omega\subset\mathbb{R}^n$,

$$\mathcal{J}\left(u\right)\leq\mathcal{J}\left(u+\phi\right),\ \text{ for all }\phi\in C_0^\infty\left(\Omega\right),$$

where

$$\mathcal{J}\left(u\right):=\int_\Omega\left[\frac{1}{2}\left|\nabla u\right|^2+\frac{\left(u^2-1\right)^2}{4}\right],$$

see Alberti-Ambrosio-Cabré [1] and Savin [25]. In Farina-Valdinoci [15], these De Giorgi type classification results have been obtained for more general quasi-linear equations and under less restrictive condition as (2). For example, it is proven there that certain symmetry assumptions on the asymptotic profile $\lim_{x_n\rightarrow+\infty}u\left(x',x_n\right)$ will be sufficient to guarantee the 1D symmetry for the original solutions. We refer to [15] for details and more complete history on this subject. Let us also mention that in a recent paper of Farina and Valdinoci [16], by imposing energy bound and certain geometric information on the interface, they prove 1D symmetry for monotone solutions without minimality assumption.

On the other hand, it turns out that for $n\geq 9$, there do exist monotone solutions which are not one dimensional. These nontrivial examples have been constructed in [12] using the machinery of infinite dimensional Lyapunov-Schmidt reduction. The nodal sets of these solutions are actually close to the Bombieri-De Giorgi-Giusti minimal graphs. The construction has been successfully extended to other settings. For example, it is proved in [13] that for any nondegenerate minimal surfaces with finite total curvature in $\mathbb{R}^3$, one could construct family of solutions for the Allen-Cahn equation whose nodal sets "follow" these minimal surfaces. These results indicate that the minimal surface theory is deeply related to the Allen-Chan equation.

Regarding solutions which are not necessary monotone, Savin [25] also proved that if u is a global minimizer and $n\leq 7$, then u is one dimensional. While the monotone solutions of Del Pino-Kowalczyk-Wei provides examples of nontrivial global minimizers in dimension $n \geq 9$, it is still not known whether there are nontrivial global minimizers for $n = 8$. A result in Farina-Valdinoci [15](Theorem 1.7 there), which generalizes the previously mentioned result of Savin, already tells us that *monotone global minimizers* in dimension 8 must be 1D. However, due to the connection with minimal surface theory, one expects nontrivial global minimizers should exist in dimension 8 and higher. This was pointed out in [19, Section 1.3] and posed as an open question in [7]. The existence of global minimizers will be our main focus in this paper.

Let us now recall some preliminary facts from the minimal surface theory. It is known that in $\mathbb{R}^8$, there is a minimal (meaning that the mean curvature is equal to zero) cone with one singularity at the origin which minimizes the area, called Simons cone. It is given explicitly by:

$$\left\{x_1^2 + ... + x_4^2 = x_5^2 + ... + x_8^2\right\}.$$

The minimality (area-minimizing property) of this cone is proved in [4]. More generally, if we consider the so-called Lawson's cone $(2 \leq i \leq j)$

$$\mathcal{C}_{i,j} := \left\{(x, y) \in \mathbb{R}^i \oplus \mathbb{R}^j : |x|^2 = \frac{i-1}{j-1} |y|^2\right\},$$

then it has mean curvature zero except at the origin and hence is a minimal hypersurface with one singular point. For $i+j \leq 7$, the cone is unstable (Simons [28]). Indeed, it is now known that for $i + j \geq 8$ with $(i, j) \neq (2, 6)$, $\mathcal{C}_{i,j}$ are area-minimizing, and $\mathcal{C}_{2,6}$ is not area-minimizing but it is one-sided minimizer. (See [2], [9], [21], [23]...). Note that $\mathcal{C}_{i,j}$ has the $O(i) \times O(j)$ symmetry, that is, it is invariant under the natural group actions of $O(i)$ on the first i variables and $O(j)$ on the last j variables. We refer to [22] and references therein for more complete history and details on related subjects. It is worth mentioning that the class of cones $\mathcal{C}_{i,j}$ does not exhaust the list of minimal cones in Lawson's original paper [23]. But here we will only focus on $\mathcal{C}_{i,j}$, although one expects that the results in this paper could be extended to more general strict area-minimizing cones. Note that in [27], $\mathcal{C}_{i,j}$ is also called quadratic cone. We refer to [27] for more geometric properties of these cones.

It turns out there are analogous objects as $\mathcal{C}_{i,i}$ in the theory of Allen-Cahn equation. They are the so-called saddle-shaped solutions, which are solutions in $\mathbb{R}^{2i}$ of (1) vanishes exactly on the cone $\mathcal{C}_{i,i}$ (Cabré-Terra [5, 6] and Cabré [7]). We denote them by D_i. It has been proved in [5] that these solutions are unique in the class of symmetric functions. Furthermore in [5, 6] it is proved that for $2 \leq i \leq 3$, the saddle-shaped solution is unstable, while for $i \geq 7$, they are stable ([7]). It is also conjectured in [7] that for $i \geq 4$, D_i should be a global minimizer. This turns out to be a difficult problem.

To state our results, we need to introduce some notations. Denote

$$r = \sqrt{x_1^2 + ... + x_i^2}, \quad s = \sqrt{x_{i+1}^2 + ... + x_{i+j}^2}.$$

Note that actually r, s depend on the indices i and j. Suppose that $2 \leq i \leq j$ satisfy one of the following two conditions:
(C1) $i + j \geq 9$.
(C2) $i + j = 8$, $|i - j| < 4$.

Set $n = i+j$. We have mentioned that under condition (C1) or (C2), the cone $\mathcal{C}_{i,j}$ is area-minimizing. Indeed, it is strict area-minimizing. (We refer to [18] for the definition of strict area-minimizing property). Results of [18] (see also [2]) then tell us that $\mathbb{R}^n$ is foliated by a family of minimal hypersurfaces which are invariant under $O(i) \times O(j)$. Each minimal hypersurface in this family is asymptotic to $\mathcal{C}_{i,j}$ at the rate Cr^{α} as r tends to infinity, where

$$\alpha = \frac{-(n-3) + \sqrt{(n-3)^2 - 4(n-2)}}{2}.$$

One could check that $-2 \leq \alpha < -1$, if $n \geq 8$.

Theorem 1 *Suppose $2 \leq i \leq j$ satisfy condition (C1) or (C2). There exists a constant $\beta_{i,j}$ such that for each $a \in \mathbb{R}$, there exists a global minimizer U_a of the Allen-Cahn equation* (1) *which is invariant under $O(i) \times O(j)$ and has the following property: If the nodal set of U_a is represented by $s = F_{U_a}(r)$, then*

$$F_{U_a}(r) = \sqrt{\frac{j-1}{i-1}}r + \beta_{i,j}r^{-1} + ar^{\alpha} + o(r^{\alpha}), \text{ as } r \to +\infty.$$

Moreover, if $i = j$, $\beta_{i,j} = 0$.

Remark 2 *In the case of $i = j \geq 4$, if one could show that this family of solutions U_a depends continuously on a and is ordered: $U_{k_1} < U_{k_2}$ for $k_1 < k_2$, then the saddle-shaped solution D_i will be a global minimizer. However, our variational construction only yields the existence and gives no information on the ordering and continuity in a.*

As we will see later on, the existence of at least one global minimizer is an easy consequence of the existence of certain family of ordered solutions constructed in [24]. (It should be known in the literature but we can not locate exact reference.) Recall that a result of Jerison and Monneau [19] tells us that the existence of a nontrivial global minimizer in $\mathbb{R}^8$ which is even in all of its variables implies the existence of a family of counter-examples for the De Giorgi conjecture in $\mathbb{R}^9$. Hence an immediate corollary of Theorem 1 is the following

Corollary 3 *Suppose $2 \leq i \leq j$ satisfy condition (C1) or (C2). There is a family of monotone solutions to the Allen-Cahn equation* (1) *in $\mathbb{R}^{i+j+1}$, which is not one-dimensional and having $O(i) \times O(j)$ symmetry in the first $i + j$ variables.*

This corollary could be regarded as a parallel result to Simon's existence result on entire minimal graphs [26].

Now let us sketch the main idea of the proof of Theorem 1. We shall firstly construct minimizers on bounded domains, with suitable boundary conditions. As we enlarge the domain, we will see that a subsequence of solutions on these bounded domains will converge to a global minimizer, as one expects. To ensure that these solutions converge, we will use the family of solutions constructed by Pacard-Wei [24] as barriers. The condition that the cone we start with is strict area-minimizing is used to show that the solutions of Pacard-Wei are ordered. To show the compactness and precise asymptotic behavior we use the convenient tool of Fermi coordinate. The rest of the paper is devoted to the proof of Theorem 1.

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2 Solutions on bounded domains and their asymptotic behavior

In this section, we deal with the case that the cone is the Simons cone in $\mathbb{R}^8$. The starting point of our construction of global minimizers will be the solutions of Pacard-Wei[24] which we shall describe below.

Let $\nu(\cdot)$ be the unit normal of the Simons cone $\mathcal{C}_{4,4}$ pointing towards the region $\{(r,s): r<s\}$. Since we are interested in solutions with $O(4)\times O(4)$ symmetry, let us introduce

$$r=\sqrt{x_1^2+...+x_4^2},\quad s=\sqrt{x_5^2+...+x_8^2}. \tag{3}$$

There is a smooth minimal surface Γ^+ lying on one side of the Simons cone, which is asymptotic to this cone and has the following properties (see [18] or [24]). Γ^+ is invariant under the group of action of $O(4)\times O(4)$. Outside of a large ball, Γ^+ is a graph over $C_{4,4}$ and

$$\Gamma^+=\left\{X+\left[\frac{1}{\sqrt{2}r^2}+o\left(r^{-2}\right)\right]\nu(X): X\in C_{4,4}\right\},\ \text{as } r\to+\infty.$$

Similarly, there is a smooth minimal hypersurface Γ^- on the other side of the cone. For $\lambda>0$, let $\Gamma_\lambda^\pm=\lambda\Gamma^\pm$ be the family of homotheties of $\Gamma^\pm$. Then it is known that $\Gamma_\lambda^\pm$ together with $\mathcal{C}_{4,4}$ forms a foliation of $\mathbb{R}^8$. We also write Γ_λ^+ as the graph of the function $s=f(\lambda,r)$.

We use N_u to denote the nodal set of a function u. The following existence result for a family of ordered solutions is proved by Pacard-Wei in [24].

Theorem 4 (Pacard-Wei) *For all λ large enough(say $\lambda > \lambda_0$), there exist solutions $U_\lambda^{\pm}$ to the Allen-Cahn equation in $\mathbb{R}^8$ which are $O(4) \times O(4)$ invariant. These solutions depend continuously on the parameter λ and are ordered. That is,*

$$U_{\lambda_1}^{+}(X) < U_{\lambda_2}^{+}(X), \lambda_1 < \lambda_2.$$
$$U_{\lambda_1}^{-}(X) < U_{\lambda_2}^{-}(X), \lambda_1 > \lambda_2,$$
$$U_{\lambda_0}^{-}(X) < U_{\lambda_0}^{+}(X).$$

Suppose that in the (r, s) coordinate,

$$N_{U_\lambda^{\pm}} = \left\{(r, s) : s = F_\lambda^{\pm}(r)\right\}.$$

Then they have the following asymptotic behavior:

$$F_\lambda^{\pm}(r) = r \pm \lambda^3 r^{-2} + o\left(r^{-2}\right), \text{ as } r \to +\infty.$$

Moreover, for $|X|$ large,

$$U_\lambda^{+}(X) = H(X_1) + 3\eta(X_1) l^{-2} + o\left(l^{-2}\right). \tag{4}$$

Here X_1 is the signed distance from X to $N_{U_\lambda^+}$, positive in the region $\{r > s\}$. Let $P(X)$ be the point on $N_{U_\lambda^+}$ which realizes this distance, then l is the r-coordinate of $P(X)$. η is the function defined in (5). Similar result holds for U_λ^{-}.

Remark 5 *The order property of $U_\lambda^{\pm}(\cdot)$ is proved in [24, Proposition 12.1], which essentially follows from (4). The asymptotic behavior (4) will also be used later on. This will be discussed in the next section in terms of Fermi coordinate. We also refer to Lemma 16 for the analysis of the solutions in the case corresponding to general Lawson's cones. It should be emphasized that the construction in [24] only gives us these solutions when λ is sufficiently large.*

Proposition 6 *As $\lambda \to +\infty$, $U_\lambda^{\pm} \to \pm 1$ uniformly on any compact set of $\mathbb{R}^8$.*

Proof. Set $\varepsilon = \lambda^{-1}$ and $w_\varepsilon^{\pm}(X) := U_\lambda^{\pm}(\lambda X)$. Then $w_\varepsilon^{\pm}$ are solutions to the singularly perturbed Allen-Cahn equation

$$-\varepsilon \Delta w_\varepsilon^{\pm} = \frac{1}{\varepsilon}\left(w_\varepsilon^{\pm} - w_\varepsilon^{\pm 3}\right).$$

Moreover, the construction of [24] implies that $\{w_\varepsilon^{\pm} = 0\}$ lies in an $O(\varepsilon)$ neighborhood of $\Gamma^{\pm}$. Because the distance from the origin to $\Gamma^{\pm}$ is positive, by the equation, we see $w_\varepsilon^{\pm}$ is close to ± 1 in a fixed ball around the origin. Rescaling back we finish the proof. ∎

2.1 Minimizing arguments and solutions with $O(4)\times O(4)$ symmetry

For each $a\in\mathbb{R}$, we would like to construct a solution whose nodal set in the r-s coordinate is asymptotic to the curve

$$s=r+ar^{-2}$$

at infinity. Without loss of generality, let us assume $a\geq 0$.

Consider the first quadrant of the r-s plane. Let l designate a choice of a local coordinate on Γ_a^+. For each point on Γ_a^+, l could just be defined to be its r-coordinate. Now let (l,t) be the Fermi coordinate around the minimal surface Γ_a^+, More precisely, for each P, we denote by $\pi(P)$ to be the point on Γ_a^+ which realizes the distance between P and $\pi(P)$. Then the Fermi coordinate of P is defined to be (l,t), where t is the signed distance from P to Γ_a^+, positive in the region $\{(r,s):r>s\}$, and l is understood to be the r-coordinate of $\pi(P)$. Keep in mind that the Fermi coordinate is not smoothly defined on the whole space $\mathbb{R}^8$, because it is smooth at the r,s axes.

For each d large, let L_d be the line orthogonal to Γ_a^+ at the point $(d,f(a,d))$. Denote the domain enclosed by L_d and the r,s axes by Ω_d. (This domain actually should be considered as a domain in $\mathbb{R}^8$. We still denote it by Ω_d for notational simplicity.) We shall impose Neumann boundary condition on r,s axes and suitable Dirichlet boundary condition on L_d, to get a minimizer for the energy functional $\mathcal{J}$.

Recall that H is the one dimensional heteroclinic solution. Since $\int_{\mathbb{R}} tH'(t)^2\,dt=0$, there exists a unique bounded solution η of the problem

$$\begin{cases} -\eta''+\left(3H^2-1\right)\eta=-tH', \\ \int_{\mathbb{R}}\eta H'dt=0. \end{cases} \tag{5}$$

There is actually an explicit form for η, see [12], but we do not need this fact.

Let us now fix a number $\lambda^*>2\max\left\{a^{\frac{1}{3}},\lambda_0\right\}$. Let $\varepsilon>0$ be a small constant. Let ρ be a cut-off function defined outside the unit ball. We require that ρ equals 1 in the region $\left\{\varepsilon s<r<\varepsilon^{-1}s\right\}$ and equals 0 near the r,s axes. It is worth pointing out that the Fermi coordinate is smoothly defined in the region $\left\{\varepsilon s<r<\varepsilon^{-1}s\right\}\backslash B_R(0)$, for R sufficiently large. We seek a minimizer of the functional $\mathcal{J}$ within the class of functions

$$S_d:=\left\{\phi\in H^{1,2}(\Omega_d):\phi=H_d^*+\rho\eta(t)A_2 \text{ on } L_d\right\}.$$

Here $A_2=\sum k_i^2$ is the squared norm of the second fundamental form of the minimal surface Γ_a^+, with k_i being the principle curvatures of Γ_a^+. Hence A_2 decays like $O\left(r^{-2}\right)$ as r tends to infinity. Since the functions $U_\lambda^\pm$ have the asymptotic behavior (4), we could choose H_d^* such that it equals $H(t)$ in the support of ρ and satisfies

$$U_{\lambda^*}^-<H_d^*+\rho\eta(t)A_2<U_{\lambda^*}^+ \text{ in } L_d. \tag{6}$$

Let $u = u_d$ be a minimizer of the functional $\mathcal{J}$ over S_d. The existence of u follows immediately from standard arguments in Calculus of Variations. But in principle, we may not have uniqueness. Intuitively speaking, the uniqueness of minimizer should be an issue related to minimizing property of the saddle-shaped solutions.

Proposition 7 *u_d is invariant under the natural action of $O(4) \times O(4)$.*

Proof. Let $e \in O(4) \times O(4)$. Then due to the invariance of the energy functional and the boundary condition, $u(e\cdot)$ is still a minimizer. By elliptic regularity, u is smooth.

Suppose the action of e is given by: $(x_1, x_2, ..., x_8) \to (-x_1, x_2, ..., x_8)$. We first show that $u(x) = u(ex)$ for any $x \in \Omega_d$. Assume to the contrary that this is not true. Let us consider the functions

$$w_1(x) := \min\{u(x), u(ex)\},$$
$$w_2(x) := \max\{u(x), u(ex)\}.$$

Since $u(\cdot)$ and $u(e\cdot)$ have the same boundary data, w_1 and w_2 are also minimizers of the functional $\mathcal{J}$. Hence they are solutions of the Allen-Cahn equation. Since $w_1 \leq w_2$ and $w_1(0, x_2, ..., x_8) = w_2(0, x_2, ..., x_8)$, by the strong maximum principle, $w_1 = w_2$. It follows that $u(x) = u(ex)$.

Let us use x to denote the first four coordinates $(x_1, ..., x_4)$ and y denote the last four coordinates $(x_5, ..., x_8)$. Let $\bar{e}$ be any reflection across a three dimensional hyperplane L in $\mathbb{R}^4$ which passes through origin. This gives us a corresponding element in $O(4) \times O(4)$, still denoted by $\bar{e}$,

$$\bar{e}(x, y) := (\bar{e}(x), y).$$

Similar arguments as above tell us that $u(x) = u(\bar{e}x)$ for any $x \in \Omega_d$. This in turn would imply that u is invariant under the group of action of $O(4) \times O(4)$. ∎

Lemma 8 *For any d large, there holds*

$$U^-_{\lambda^*} < u_d < U^+_{\lambda^*}. \tag{7}$$

Proof. By Proposition 6, for λ sufficiently large, $u_d < U^+_\lambda$ on Ω_d. Now let us continuously decrease the value of λ. Since we have a continuous family of solutions U^+_λ, and for $\lambda > \lambda^*$, each of them is greater than u_d on the boundary of Ω_d (recalling (6) and the ordering properties of $U^\pm_\lambda$), hence by the strong maximum principle, $u_d < U^+_{\lambda^*}$. Similarly, one could show that $U^-_{\lambda^*} < u_d$. This finishes the proof. ∎

2.2 Asymptotic analysis of the solutions

The minimizers u_d are invariant under $O(4) \times O(4)$ action. Using the uniform estimate (7), we could show that as d goes to infinity, up to a subsequence,

u_d converges in $C^2_{loc}\left(\mathbb{R}^8\right)$ to a nontrivial entire solution U of the Allen-Cahn equation.

We then claim

Lemma 9 *U is a global minimizer.*

Proof. Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be any fixed function. Then for $d > 2R_0$, where $supp(\phi) \subset B_{R_0}(0)$, we have

$$\mathcal{J}(u_d) \leq \mathcal{J}(u_d + \phi).$$

Letting $d \to +\infty$ we arrive at the conclusion. ■

By Lemma 8, the nodal set N_U of U must lie between $N_{U^+_{\lambda^*}}$ and $N_{U^-_{\lambda^*}}$. We use $s = F(r)$ to represent N_U.

The main result of this section is the following proposition, which proves Theorem 1 in the case of Simons cone.

Proposition 10 *The nodal set of U has the following asymptotic behavior:*

$$F(r) = r + ar^{-2} + o\left(r^{-2}\right), \text{ as } r \to +\infty. \tag{8}$$

The proof of this proposition relies on detailed analysis of the asymptotic behavior of the sequence of solutions u_d.

To begin with, let us define an approximate solution as

$$\bar{H}(l,t) = \rho H(t - h_d(l)) + (1-\rho)\frac{H(t - h_d(l))}{|H(t - h_d(l))|},$$

where ρ is the cutoff function introduced in Section 2.1, and h_d is a small function defined on Γ_a^+ to be determined in the sequel. We shall write the solution u_d as

$$u_d = \bar{H} + \phi_d, \tag{9}$$

Introduce the notation $\bar{H}' := \rho H'(t-h)$. The reason that we put ρ in this definition is that the Fermi coordinate is not defined on the whole space. We require the following orthogonality condition to be satisfied by ϕ_d :

$$\int_{\mathbb{R}} \phi_d \bar{H}' dt = 0. \tag{10}$$

Since u_d is close to $H(t)$, we could find a unique small function h_d satisfying (9), using implicit function theorem in the same spirit as Lemma 5.1 of [20].

Our starting point for the asymptotic analysis is the following (non-optimal) estimate:

Lemma 11 *The function h_d and ϕ_d satisfy*

$$|h_d| + |\phi_d| + |h_d'| + |h_d''| \leq Cl^{-2},$$

where C does not depend on d.

Proof. We first prove $|h_d| \leq Cl^{-2}$. By the orthogonal condition,

$$\int_{\mathbb{R}} \left(u_d - \bar{H}\right) \bar{H}' dt = 0. \tag{11}$$

Hence

$$\int_{\mathbb{R}} [u_d - H(t)] \bar{H}' dt = \int_{\mathbb{R}} \left[\bar{H} - H(t)\right] \bar{H}' dt = h_d \int_{\mathbb{R}} \bar{H}'^2 dt + o(h_d). \tag{12}$$

Recall that

$$U_{\lambda^*}^- \leq u_d \leq U_{\lambda^*}^+,$$

and by the asymptotic behavior (4),

$$\left|U_{\lambda^*}^\pm - H(t)\right| \leq Cl^{-2}.$$

From these two estimates and (12), we infer that

$$|h_d| \leq C \frac{\left|\int_{\mathbb{R}} [u_d - H(t)] \bar{H}' dt\right|}{\int_{\mathbb{R}} \bar{H}'^2 dt} \leq Cl^{-2}.$$

As an consequence,

$$|\phi_d| = \left|u_d - \bar{H}\right| \leq Cl^{-2}.$$

Next we show $|h_d'| \leq Cl^{-2}$. To see this, we differentiate equation (11) with respect to l. This yields

$$\int_{\mathbb{R}} \left(\partial_l u_d - \partial_l \bar{H}\right) \bar{H}' dt = -\int_{\mathbb{R}} \left(u_d - \bar{H}\right) \partial_l \bar{H}' dt.$$

As a consequence,

$$\left|h_d' \int_{\mathbb{R}} \bar{H}'^2 dt\right| \leq C \left|\int_{\mathbb{R}} \partial_l u_d \bar{H}' dt\right| + Cl^{-2}. \tag{13}$$

Observe that u_d is a solution trapped between $U_{\lambda^*}^+$ and $U_{\lambda^*}^-$. Hence elliptic regularity tells us

$$\partial_l u_d = O\left(l^{-2}\right). \tag{14}$$

Here we use $O\left(l^{-2}\right)$ to denote a term bounded by Cl^{-2}, where C is a universal constant independent of d. (14) together with (13) yields $|h_d'| \leq Cl^{-2}$. Similarly, we can prove that $|h_d''| \leq Cl^{-2}$. ∎

Let ν be the unit normal of Γ_a^+, pointing to the region $\{(r,s) : r > s\}$. The Laplacian operator Δ has the following expansion(see [14]) in the Fermi coordinate (l,t):

$$\Delta = \Delta_{\gamma^t} + \partial_t^2 - M_t \partial_t.$$

Here M_t is the mean curvature of the surface

$$\gamma^t := \left\{X + t\nu(X), X \in \Gamma_a^+\right\}.$$

Δ_{γ^t} is the Laplacian-Beltrami operator on γ^t. We use $g^t = \left(g^t_{i,j}\right)$ to denote the induced metric on the surface γ^t. Then

$$\Delta_{\gamma^t}\varphi = \frac{\partial_i\left(g^{i,j,t}\sqrt{|g^t|}\partial_j\varphi\right)}{\sqrt{|g^t|}}. \tag{15}$$

Here we have used $|g^t|$ to denote the determinant of the matrix of the metric tensor on γ^t. For $t = 0$, $\gamma^0 = \Gamma^+_a$. The metric tensor on Γ^+_a has the form

$$\left(1 + f'^2\right) dr^2 + \left(r^2 + f^2\right) d\sigma^2,$$

where $d\sigma^2$ is the metric tensor on $S^3\left(\frac{1}{\sqrt{2}}\right) \times S^3\left(\frac{1}{\sqrt{2}}\right)$. In general, when $t \neq 0$, the metric on γ^t and γ^0 in (l,t)-coordinate is related by (see [14, Section 2.1]):

$$g^t = g^0\left(I - tA\right)^2. \tag{16}$$

Here A is the matrix represents the second fundamental form of Γ^+_a. In particular,

$$g^t_{i,j} = g^0_{i,j} + O\left(l^{-1}\right).$$

Lemma 12 *For a function φ invariant under $O(4)\times O(4)$ action, the Laplacian-Beltrami operator Δ_{γ^t} has the form*

$$\Delta_{\gamma^t}\varphi = \Delta_{\gamma^0}\varphi + P_1(l,t)\, D\varphi + P_2(l,t)\, D^2\varphi,$$

where

$$|P_1(l,t)| \leq Cl^{-2}, |P_2(l,t)| \leq Cl^{-1}.$$

Proof. Let $g^t = \left(g^{i,j,t}\right)$ be the inverse matrix of $\left(g^t_{i,j}\right)$. Using (15), we get

$$\Delta_{\gamma^t}\varphi = \frac{\partial_i\left(g^{i,j,t}\sqrt{|g^t|}\partial_j\varphi\right)}{\sqrt{|g^t|}} = \partial_i g^{i,j,t}\partial_j\varphi + g^{i,j,t}\partial_{ij}\varphi + g^{i,j,t}\partial_j\varphi\partial_i \ln\sqrt{|g^t|}.$$

We compute

$$\begin{aligned}\Delta_{\gamma^t}\varphi - \Delta_{\gamma^0}\varphi &= \partial_i\left(g^{i,j,t} - g^{i,j,0}\right)\partial_j\varphi + \left(g^{i,j,t} - g^{i,j,0}\right)\partial_{ij}\varphi \\ &\quad + g^{i,j,t}\partial_j\varphi\partial_i \ln\frac{\sqrt{|g^t|}}{\sqrt{|g^0|}} + \left(g^{i,j,t} - g^{i,j,0}\right)\partial_j\varphi\partial_i \ln\sqrt{|g^0|}. \end{aligned} \tag{17}$$

The estimate follows from this formula, identity (16), and the asymptotic behavior of the principle curvatures of Γ^+_a. ■

One main step of our analysis will be the estimate of the approximate solution. Recall that $k_i, i = 1, ..., 7,$ are the principle curvatures of Γ^+_a.

Lemma 13 *The error of the approximate solution $\bar{H}$ has the following estimate:*

$$\Delta \bar{H} + \bar{H} - \bar{H}^3 = -\frac{1}{2}\left(h_d'' + \frac{6}{l}h_d'\right)\bar{H}' + O\left(h_d'^2\right) + O\left(l^{-2}h_d'\right) + O\left(l^{-1}h_d''\right) - \left(tA_2 + t^3\sum_{i=1}^{7}k_i^4\right)\bar{H}' + O\left(l^{-5}\right).$$

Proof. Computing the Laplacian in the Fermi coordinate, we obtain, up to an exponential decaying term introduced by the cutoff function ρ,

$$\begin{aligned}\Delta \bar{H} + \bar{H} - \bar{H}^3 &= \Delta_{\gamma^t}\bar{H} + \partial_t^2\bar{H} - M_t\partial_t\bar{H} + \bar{H} - \bar{H}^3 \\ &= \Delta_{\gamma^t}\bar{H} - M_t\partial_t\bar{H}.\end{aligned}$$

Since Γ_a^+ is a minimal surface,

$$\begin{aligned}M_t &= \sum_{i=1}^{7}\frac{k_i}{1-tk_i} \\ &= tA_2 + t^2\sum_{i=1}^{7}k_i^3 + t^3\sum_{i=1}^{7}k_i^4 + O\left(k_i^5\right).\end{aligned}$$

Observe that k_i^3 decays like $O\left(r^{-3}\right)$ at infinity. However, we would like to show that $\sum_{i=1}^{7}k_i^3$ actually has a faster decay. Indeed, recall that(see [2]), in an arc length parametrization of the curve Γ_a^+,

$$\begin{aligned}k_1 &= \frac{-r''s' + r's''}{\left(r'^2 + s'^2\right)^{\frac{3}{2}}}, \\ k_2 = k_3 = k_4 &= \frac{s'}{r\sqrt{r'^2 + s'^2}}, \\ k_5 = k_6 = k_7 &= \frac{-r'}{s\sqrt{r'^2 + s'^2}}.\end{aligned}$$

In particular, using the fact that along Γ_a^+, $s = r + ar^{-2} + o\left(r^{-2}\right)$, we estimate

$$k_1 = O\left(r^{-4}\right), \text{ and } |k_2| = ... = |k_7| = \frac{1}{\sqrt{2}r} + O\left(r^{-4}\right).$$

Therefore we obtain

$$\sum_{i=1}^{7}k_i^3 = O\left(r^{-5}\right).$$

It follows that,

$$M_t = tA_2 + t^3\sum_{i=1}^{7}k_i^4 + O\left(r^{-5}\right).$$

Next we compute $\Delta_{\gamma^t}\bar{H}$. By Lemma 12, in the Fermi coordinate,

$$\begin{aligned}\Delta_{\gamma^t}\bar{H} &= \frac{\partial_i\left(g^{i,j,t}\sqrt{|g^t|}\partial_j\left[H\left(t-h_d\left(l\right)\right)\right]\right)}{\sqrt{|g^t|}} \\ &= \Delta_{\gamma^0}\bar{H}+O\left(l^{-2}\right)h_d'+O\left(l^{-1}\right)h_d''. \\ &= -\frac{1}{2}\left(h_d''+\frac{6}{l}h_d'\right)\bar{H}'+O\left(h_d'^2\right)+O\left(l^{-2}\right)h_d'+O\left(l^{-1}\right)h_d''+O\left(l^{-5}\right).\end{aligned}$$

Here in the last identity, we have used the asymptotic expansion

$$g_{1,1}^0 = 2+O\left(l^{-3}\right).$$

Note that the fact $\frac{-1}{2}$ appears before $h_d''+\frac{6}{l}h_d'$ because l represents the r-coordinate of the projected point. We now obtain

$$\begin{aligned}\Delta\bar{H}+\bar{H}-\bar{H}^3 &= -\frac{1}{2}\left(h_d''+\frac{6}{l}h_d'\right)\bar{H}'+O\left(h_d'^2\right)+O\left(l^{-2}\right)h_d'+O\left(l^{-1}\right)h_d'' \\ &\quad-\left(tA_2+t^3\sum_{i=1}^{7}k_i^4\right)\bar{H}'+O\left(l^{-5}\right).\end{aligned}$$

■

Let us set

$$J\left(h_d\right)=\frac{1}{2}\left[h_d''+\frac{6}{l}h_d'+\frac{6}{l^2}h_d\right].$$

Note that this is essentially the Jacobi operator of the Simons cone, in the r-coordinate. With all these understood, we are ready to prove the following

Proposition 14 *The function h_d satisfies*

$$\left|J\left(h_d\right)\right|\leq Cl^{-5},$$

where C is a constant independent of d.

Proof. Frequently, we drop the subscript d for notational simplicity if there is no confusion.

Since $\phi+\bar{H}$ solves the Allen-Cahn equation, ϕ satisfies

$$-\Delta\phi+\left(3\bar{H}^2-1\right)\phi=\Delta\bar{H}+\bar{H}-\bar{H}^3-3\bar{H}\phi^2-\phi^3.$$

By Lemma 13,

$$\begin{aligned}&-\Delta\phi+\left(3\bar{H}^2-1\right)\phi \\ &= -J\left(h\right)\bar{H}'+\left(t-h\right)\bar{H}'A_2+\left(t-h\right)^3\bar{H}'\sum_{i=1}^{7}k_i^4 \\ &+O\left(h'^2\right)+O\left(l^{-2}h'\right)+O\left(l^{-1}h''\right) \\ &-3\phi^2\bar{H}-\phi^3+O\left(l^{-5}\right).\end{aligned}$$

The function $(t-h)\,A_2\bar{H}'$ is essentially orthogonal to $\bar{H}'$ and decays like $O\left(l^{-2}\right).$ This is a slow decaying term. Recall that we defined a function η satisfying

$$-\eta''+\left(3H^2-1\right)\eta=-tH'\left(t\right).$$

We introduce $\bar{\eta}=\eta\left(t-h\right).$ Straightforward computation yields

$$\begin{aligned}
&-\Delta\left(\bar{\eta}A_2\right)+\left(3\bar{H}^2-1\right)\bar{\eta}A_2\\
&=-\Delta_{\gamma^t}\left(\bar{\eta}A_2\right)-\partial_t^2\left(\bar{\eta}A_2\right)+M_t\partial_t\left(\bar{\eta}A_2\right)+\left(3\bar{H}^2-1\right)\bar{\eta}A_2\\
&=-\Delta_{\gamma^t}\left(\bar{\eta}A_2\right)+M_t\partial_t\left(\bar{\eta}A_2\right)-\left(t-h\right)\bar{H}'A_2\\
&=-\bar{\eta}\Delta_{\gamma^0}A_2+\left(t-h\right)\partial_t\bar{\eta}A_2^2-\left(t-h\right)\bar{H}'A_2+\partial_t\bar{\eta}\Delta_{\gamma^0}hA_2+O\left(l^{-5}\right).
\end{aligned}$$

Although $-\bar{\eta}\Delta_{\gamma^0}A_2+(t-h)\,\partial_t\bar{\eta}A_2^2$ decays only like $O\left(l^{-4}\right)$, due to the fact that η is odd, it is orthogonal to $\bar{H}'$.

Let $\phi=\rho\bar{\eta}\left|A\right|^2+\bar{\phi}$. Then the new function $\bar{\phi}$ satisfies

$$\begin{aligned}
-\Delta\bar{\phi}+\left(3\bar{H}^2-1\right)\bar{\phi}=&-J\left(h\right)\bar{H}'+\left(t-h\right)^3H'\sum_{i=1}^{7}k_i^4\\
&+\bar{\eta}\Delta_{\gamma^0}A_2-\left(t-h\right)\partial_t\bar{\eta}A_2^2+\partial_t\bar{\eta}\Delta_{\gamma^0}hA_2\\
&+O\left(h'^2\right)+O\left(l^{-2}\right)h'+O\left(l^{-1}\right)h''\\
&+3\left(\bar{\eta}A_2+\bar{\phi}\right)^2\bar{H}+\left(\bar{\eta}A_2+\bar{\phi}\right)^3+O\left(l^{-5}\right).
\end{aligned}\tag{18}$$

Denote the right hand side by E. We would like to estimate $\int_{\mathbb{R}}E\bar{H}'dt$, which is the projection of E onto $\bar{H}'$.

Multiplying both sides with $\bar{H}'$ and integrating in t, using Lemma 12, we get

$$\begin{aligned}
\int_{\mathbb{R}}E\bar{H}'dt&=\int_{\mathbb{R}}\left[-\Delta\bar{\phi}+\left(3\bar{H}^2-1\right)\bar{\phi}\right]\bar{H}'dt\\
&=\int_{\mathbb{R}}\left[-\Delta_{\gamma^t}\bar{\phi}-\partial_t^2\bar{\phi}+M_t\partial_t\bar{\phi}+\left(3\bar{H}^2-1\right)\bar{\phi}\right]\bar{H}'dt\\
&=\int_{\mathbb{R}}\left[-\Delta_{\gamma^0}\bar{\phi}-\partial_t^2\bar{\phi}+\left(3\bar{H}^2-1\right)\bar{\phi}\right]\bar{H}'dt+l^{-1}O\left(\bar{\phi}\right).
\end{aligned}\tag{19}$$

Here $O\left(\bar{\phi}\right)$ denotes a term satisfying

$$\left|O\left(\bar{\phi}\right)\right|\leq C\left(\left|\bar{\phi}\right|+\left|D\bar{\phi}\right|+\left|D^2\bar{\phi}\right|\right),\ \text{as } l\to+\infty.$$

Since $\int_{\mathbb{R}}\phi\bar{H}'dt=0$ and η is orthogonal to H', we have

$$\int_{\mathbb{R}}\bar{\phi}\bar{H}'dt=-\int_{\mathbb{R}}\rho\bar{\eta}\left|A\right|^2\bar{H}'dt=O\left(e^{-l}\right).\tag{20}$$

We then deduce from integrating by parts that

$$\int_{\mathbb{R}}\left[-\partial_t^2\bar{\phi}+\left(3\bar{H}^2-1\right)\bar{\phi}\right]\bar{H}'dt=l^{-1}O\left(\bar{\phi}\right)+O\left(e^{-l}\right).\tag{21}$$

We also calculate

$$\left|\int_{\mathbb{R}} \Delta_{\gamma^0}\bar{\phi}\bar{H}'dt\right| \leq C\left|\frac{d^2\left(\int_{\mathbb{R}}\bar{\phi}\bar{H}'dt\right)}{dl^2}\right| + C\left|\frac{d\left(\int_{\mathbb{R}}\bar{\phi}\bar{H}'dt\right)}{dl}\right| + C\int_{\mathbb{R}}\left|\phi\frac{d\bar{H}'}{dl}\right|dt + C\int_{\mathbb{R}}\left|\phi\frac{d^2\bar{H}'}{dl^2}\right|dt.$$

It then follows from the estimate of h' and h'' that,

$$\left|\int_{\mathbb{R}} \Delta_{\gamma^0}\bar{\phi}\bar{H}'dt\right| \leq Ce^{-l} + Cl^{-2}\int_{\mathbb{R}}|\phi|\,dt. \tag{22}$$

Inserting (21) and (22) into (19) , we find that

$$\int_{\mathbb{R}}\left[-\Delta\bar{\phi} + \left(3\bar{H}^2 - 1\right)\bar{\phi}\right]\bar{H}'dt = \int_{\mathbb{R}} E\bar{H}'dt = l^{-1}O\left(\bar{\phi}\right) + O\left(e^{-l}\right). \tag{23}$$

As a consequence of this fact, we get

$$\begin{aligned} -\Delta\bar{\phi} + \left(3\bar{H}^2 - 1\right)\bar{\phi} &= E - \frac{\int_{\mathbb{R}} E\bar{H}'dt}{\int_{\mathbb{R}}\bar{H}'^2dt}\bar{H}' + \frac{\int_{\mathbb{R}} E\bar{H}'dt}{\int_{\mathbb{R}}\bar{H}'^2dt}\bar{H}' \\ &= E - \frac{\int_{\mathbb{R}} E\bar{H}'dt}{\int_{\mathbb{R}}\bar{H}'^2dt}\bar{H}' + l^{-1}O\left(\bar{\phi}\right) + O\left(e^{-l}\right). \end{aligned} \tag{24}$$

Note that

$$\begin{aligned} E - \frac{\int_{\mathbb{R}} E\bar{H}'dt}{\int_{\mathbb{R}}\bar{H}'^2dt}\bar{H}' &= (t-h)^3\,\bar{H}'\sum_{i=1}^{7}k_i^4 \\ &\quad + \bar{\eta}\Delta_{\Gamma_a^+}A_2 - (t-h)\,\partial_t\bar{\eta}A_2^2 \\ &\quad + O\left(h'^2\right) + O\left(l^{-2}\right)h' + O\left(l^{-1}\right)h'' \\ &\quad + 3\bar{\eta}^2A_2^2\bar{H} + o\left(\bar{\phi}\right) + O\left(l^{-5}\right). \end{aligned}$$

By the estimate of h, h', h'', we get the following (non-optimal) estimate:

$$E - \frac{\int_{\mathbb{R}} E\bar{H}'dt}{\int_{\mathbb{R}}\bar{H}'^2dt}H' = O\left(l^{-3}\right) + o\left(\bar{\phi}\right).$$

Now by (24) , $\bar{\phi}$ satisfies an equation of the form

$$-\Delta\bar{\phi} + \left(3\bar{H}^2 - 1\right)\bar{\phi} = O\left(l^{-3}\right) + o\left(\bar{\phi}\right).$$

We claim that $\bar{\phi}$ satisfies

$$\left|\bar{\phi}\right| + \left|D\bar{\phi}\right| + \left|D^2\bar{\phi}\right| \leq Cl^{-3}, \tag{25}$$

with C independent of d.

To prove the claim, we shall use the linear theory of the operator $-\Delta + 3H^2 - 1$, in the same spirit as that of Proposition 8.1 in [24]. Here we sketch the proof. Assume to the contrary that (25) is not true. Then we could find a sequence $d_n \to +\infty$, a sequence of functions $\{\bar{\phi}_{d_n}\}$, and a sequence of points $\{l_n, t_n\}$ with $l_n \to +\infty$, such that

$$A_n := \left|\bar{\phi}_{d_n}(l_n, t_n)\right| l_n^3 = \max_{(l,t)} \left|\bar{\phi}(l,t)\, l^3\right| \to +\infty.$$

Note that $\bar{\phi}$ is only defined for $l \le d$. Define a sequence of new functions

$$\Phi_n(l,t) := \frac{\bar{\phi}_n(l + l_n, t + t_n)(l + l_n)^3}{A_n}.$$

Then Φ_n satisfies an equation of the form

$$-\partial_l^2 \Phi_n - \partial_t^2 \Phi_n + \left(3H^2 - 1\right)\Phi_n = o(1) + o(\Phi_n), \text{ as } n \to +\infty.$$

Φ_n also satisfies $\Phi_n(0,0) = 1$ and $|\Phi_n| \le 1$. We consider four cases.

Case 1. $d_n - l_n \to +\infty$ and $|t_n| \le C$.

In this case, Φ_n converges in $C^2_{loc}(\mathbb{R}^2)$ to a bounded entire solution Φ of the equation

$$-\partial_l^2 \Phi - \partial_t^2 \Phi + \left(3H^2 - 1\right)\Phi = 0, \text{ in } \mathbb{R}^2.$$

This implies that $\Phi = cH'$, with $c \ne 0$. This contradicts with the fact that $\int_{\mathbb{R}} \Phi H' dt = 0$.

Case 2. $d_n - l_n \to +\infty$ and $|t_n| \to +\infty$.

In this case, Φ_n converges in $C^2_{loc}(\mathbb{R}^2)$ to a bounded entire solution Φ of the equation

$$-\partial_l^2 \Phi - \partial_t^2 \Phi + 2\Phi = 0, \text{ in } \mathbb{R}^2.$$

This is not possible, since $\Phi(0,0) = 1$ and $|\Phi| \le 1$.

Case 3. $d_n - l_n \to C_0 < +\infty$ and $|t_n| \le C$.

Note that $\bar{\phi}_n = O\left(e^{-d}\right)$ on the boundary L_d. Hence Φ_n converges to a bounded solution Φ of the equation

$$-\partial_l^2 \Phi - \partial_t^2 \Phi + \left(3H^2 - 1\right)\Phi = 0, \text{ in } (-\infty, C_0] \times \mathbb{R},$$

with $\Phi(C_0, t) = 0$. Reflecting Φ across the line $l = C_0$, we again obtain an entire solution on $\mathbb{R}^2$ satisfies the orthogonality condition $\int_{\mathbb{R}} \Phi H' dt = 0$. This is a contradiction.

Case 4. $d_n - l_n \to C_0 < +\infty$ and $|t_n| \to +\infty$.

Then Φ_n converges to a bounded solution Φ of the equation

$$-\partial_l^2 \Phi - \partial_t^2 \Phi + 2\Phi = 0, \text{ in } (-\infty, C_0] \times \mathbb{R}.$$

Reflecting across the line $l = C_0$, we also get a contradiction. The proof of the claim is completed.

Now multiplying both sides of (18) by $\bar{H}'$ again and integrating in t, using the estimate (25) of $\bar{\phi}$, we get

$$J(h)+O\left(h'^2\right)+O\left(l^{-2}\right)h'+O\left(l^{-1}h''\right)=\int_{\mathbb{R}}\left[-\Delta\bar{\phi}+\left(3\bar{H}^2-1\right)\bar{\phi}\right]\bar{H}'dt+O\left(l^{-5}\right). \tag{26}$$

On the other hand, by (23) and (25),

$$\int_{\mathbb{R}}\left[-\Delta\bar{\phi}+\left(3\bar{H}^2-1\right)\bar{\phi}\right]\bar{H}'dt=O\left(l^{-4}\right).$$

This together with (26) implies

$$J(h)+O\left(h'^2\right)+O\left(l^{-2}\right)h'+O\left(l^{-1}\right)h''=O\left(l^{-4}\right). \tag{27}$$

Using the fact that $|h'|,|h''|\leq Cl^{-2}$, we deduce from (27) that

$$|h'|\leq Cl^{-3},|h''|\leq Cl^{-4}. \tag{28}$$

Inserting this estimate back into (24), we get an improved estimate for $\bar{\phi}$:

$$\left|\bar{\phi}\right|\leq Cl^{-4}. \tag{29}$$

Then by (23) again,

$$\int_{\mathbb{R}}\left[-\Delta\bar{\phi}+\left(3\bar{H}^2-1\right)\bar{\phi}\right]\bar{H}'dt=O\left(l^{-5}\right).$$

This combined with with (26) and (28) leads to

$$J(h)=O\left(l^{-5}\right). \tag{30}$$

This is the desired estimate. ■

Proof of Proposition 10. We shall use (30) to get uniform estimates for h_d. Let $r_0>0$ be a fixed large constant. We now know

$$\begin{cases} J(h_d)=O\left(l^{-5}\right), \text{ in } (r_0,d), \\ h_d(d)=0. \end{cases}$$

Variation of parameters formula tells us that there are constants $c_{1,d},c_{2,d}$ and function ζ_d, such that

$$h_d(l)=c_{1,d}l^{-2}+c_{2,d}l^{-3}+\zeta_d. \tag{31}$$

Here $\zeta_d(d)=0$ and

$$|\zeta_d|\leq Cl^{-\frac{5}{2}}. \tag{32}$$

From the boundary condition $h_d(d)=0$, we infer

$$c_{1,d}+c_{2,d}d^{-1}=0. \tag{33}$$

On the other hand, (31) and the estimate $|h_d(l)| \leq Cl^{-2}$ yield

$$\left|c_{1,d} + c_{2,d} l^{-1}\right| \leq C. \tag{34}$$

Equations (33) and (34) lead to

$$|c_{1,d}| \leq Cd^{-1}, \ \ |c_{2,d}| \leq C. \tag{35}$$

Proposition 10 then follows from the uniform estimate (32) of ζ_d, uniform estimate (29) for $\phi_d = \rho\bar{\eta} |A|^2 + \bar{\phi}$ and the bound (35). We point out that the main order term of ϕ_d is $\rho\bar{\eta} |A|^2$ which decays like Cl^{-2}, but $\bar{\eta}$ satisfies $\bar{\eta}(0) = 0$. ■

3 Global minimizers from Lawson's area-minimizing cones

We have obtained global minimizers from the minimal surfaces asymptotic to the Simons cone. Now we consider the area-minimizing cone $\mathcal{C}_{i,j}$. We assume throughout this section that $j \geq i \geq 2$, and either

$$i + j \geq 9,$$

or

$$i + j = 8, \ \ |i - j| < 4.$$

Let

$$r = \sqrt{x_1^2 + ... + x_i^2}, \quad s = \sqrt{x_{i+1}^2 + ... + x_{i+j}^2}.$$

Put $i + j = n$. The Jacobi operator of $\mathcal{C}_{i,j}$, acting on function $h(r)$ defined on $\mathcal{C}_{i,j}$ which depends only on r, has the form

$$J(h) = \frac{i-1}{n-2}\left[h'' + \frac{n-2}{r}h' + \frac{n-2}{r^2}h\right].$$

Solutions of the equation $J(h) = 0$ are given by

$$c_1 r^{\alpha^+} + c_2 r^{\alpha^-},$$

where

$$\alpha^{\pm} = \frac{-(n-3) \pm \sqrt{(n-3)^2 - 4(n-2)}}{2}.$$

One could check that for $n \geq 8$, we always have

$$-2 \leq \alpha^+ < -1.$$

Note that when $n = 8$, $\alpha^+ = -2$ and $\alpha^- = -3$.

In this section, we prove the following proposition, from which Theorem 1 easily follows.

Proposition 15 *There exists a constant $\beta_{i,j}$, with $\beta_{i,i}=0$, such that for each $a\in\mathbb{R}$, we could construct a solution U_a to the Allen-Cahn equation. Use $s=F_{U_k}(r)$ to denote the nodal set of U_a. Then F_{U_k} has the asymptotic behavior:*

$$F_{U_k}(r)=\sqrt{\frac{j-1}{i-1}}r+\beta_{i,j}r^{-1}+ar^{\alpha^+}+o\left(r^{\alpha^+}\right).$$

For notational convenience, let us simply consider the cone $\mathcal{C}_{3,5}$ over the product of spheres $S^2\left(\sqrt{\frac{2}{6}}\right)\times S^4\left(\sqrt{\frac{4}{6}}\right)$. The proof for other cases are similar. Under a choice of the unit normal, the principle curvatures of $\mathcal{C}_{3,5}$ are given by

$$k_1=0,\quad k_2=k_3=\frac{\sqrt{6}}{3r},\quad k_4=k_5=k_6=k_7=-\frac{\sqrt{6}}{6r}.$$

It is well known that $\mathcal{C}_{3,5}$ is an area-minimizing cone. Actually, it is strict area-minimizing (see [21]). Then it follows from results of [18] that there is a foliation of $\mathbb{R}^8$ by minimal hypersurfaces. Each minimal hypersurface is asymptotic to $\mathcal{C}_{3,5}$ at the slower rate Cr^{-2} predicted by the Jacobi operator. By slightly abusing the notation, we still use $\Gamma_\lambda^{\pm}$ to denote this foliation. For λ sufficiently large (say $\lambda\geq\lambda_0$), the construction of Pacard-Wei again gives us a family of solutions $U_\lambda^{\pm}$ whose nodal set is close to $\Gamma_\lambda^{\pm}$. We want to show that this family of solutions is ordered. This is the content of the following

Lemma 16 *The family of solutions $U_\lambda^{\pm}$ is ordered. That is,*

$$\begin{aligned}&U_{\lambda_1}^{+}(X)<U_{\lambda_2}^{+}(X),\lambda_1<\lambda_2.\\&U_{\lambda_1}^{-}(X)<U_{\lambda_2}^{-}(X),\lambda_1>\lambda_2,\\&U_{\lambda_0}^{-}(X)<U_{\lambda_0}^{+}(X).\end{aligned}$$

Proof. Since this has not been proven in the paper [24](see Proposition 12.1 there), we give a sketch of the proof.

We only consider the family of solutions U_λ^+. The case of U_λ^- is similar. U_λ^+ is obtained from Lyapunov-Schmidt reduction. Adopting the notations of the previous sections, let (l,t) be the Fermi coordinate with respect to Γ_λ^+. Let ν be the unit normal of Γ_λ^+ pointing to $\{(r,s):r>s\}$. Let us still set

$$\gamma^t:=\left\{X+t\nu(X):X\in\Gamma_\lambda^+\right\}.$$

Then in the Fermi coordinate, U_λ^+ has the form

$$U_\lambda^+=H(t-h(l))+\phi:=\bar{H}+\phi.$$

where h is chosen such that ϕ is orthogonal to $\bar{H}'$ in the sense of (10).

As before, ϕ satisfies

$$-\Delta\phi+\left(3\bar{H}^2-1\right)\phi=\Delta_{\gamma^t}\bar{H}-M_t\bar{H}'+o(\phi).\tag{36}$$

To simplify the notation, we set $\bar{t} = t - h(l)$. Let us assume for this moment that $|h| \leq Cl^{-1}$(which actually is true). Let $k_{i,\lambda}$ be the principle curvatures of Γ_λ^+. We set

$$A_m := \sum_i k_{i,\lambda}^m.$$

Although we do not write this explicitly, keep in mind that A_m actually depends on λ. Recall that

$$\begin{aligned} M_t \bar{H}' &= \left(tA_2 + t^2 A_3 + t^3 A_4\right) \bar{H}' + O\left(l^{-5}\right) \\ &= \left(\bar{t}A_2 + \bar{t}^2 A_3 + \bar{t}^3 A_4\right) \bar{H}' + hA_2 \bar{H}' + 2\bar{t}hA_3\bar{H}' + O\left(l^{-5}\right). \end{aligned}$$

Inspecting the projection of these terms onto $\bar{H}'$, we find that the main order term of the projection should be

$$\bar{t}^2 A_3 \bar{H}' + hA_2 \bar{H}'.$$

Inspecting each term in (17) carefully and using (16), we get

$$\Delta_{\gamma^t} \bar{H} = -J_\lambda(h) \bar{H}' + O\left(h'^2\right) H''(\bar{t}) + \left[O\left(h'' l^{-1}\right) + O\left(h' l^{-2}\right)\right] \bar{t}\bar{H}'.$$

Let J_λ be the Jacobi operator on Γ_λ^+ :

$$J_\lambda(h) = \Delta_{\Gamma_\lambda^+} h + A_2 h.$$

Inserting the expansion of $M_t \bar{H}'$ into (36), we then find that the function ϕ should satisfy

$$\begin{aligned} -\Delta\phi + \left(3\bar{H}^2 - 1\right)\phi &= \Delta_{\gamma^t}\bar{H} - M_t\bar{H}' + o(\phi) \\ &= -J_\lambda(h)\bar{H}' + O\left(h'^2\right) H''(\bar{t}) + \left[O\left(h'' l^{-1}\right) + O\left(h' l^{-2}\right)\right]\bar{t}\bar{H}' \\ &\quad - \left(\bar{t}A_2 + \bar{t}^2 A_3 + \bar{t}^3 A_4\right)\bar{H}' - 2\bar{t}hA_3\bar{H}' + O\left(l^{-5}\right) + o(\phi). \end{aligned}$$

Concerning the projection onto $\bar{H}'$, the main order term at the right hand side should be

$$-J_\lambda(h)\bar{H}' - \bar{t}^2 A_3 \bar{H}'.$$

Hence we find that the main order term h_0 of h should satisfy the equation

$$J_\lambda(h_0) = c^* A_3,$$

where

$$c^* = -\frac{\int_{\mathbb{R}} t^2 H'^2 dt}{\int_{\mathbb{R}} H'^2 dt}.$$

Let $\bar{h}_0(l) = h_0(\lambda l)$. We find that $\bar{h}_0$ should satisfy

$$J_1\left(\bar{h}_0\right) = c^* \lambda^2 A_3(\lambda l).$$

Recall that J_1 is the Jacobi operator on the rescaled minimal surface Γ_1^+. Since the family of minimal foliation is asymptotic to the cone at the slower rate Cl^{-2}, we find that J_1 has a kernel decays like Cl^{-2}. Then using the invertibility theory of the Jacobi operator J_1(see Lemma 10.1 of [24]), we could find a solution ξ of

$$J_1(\xi) = c^* \lambda^2 A_3(\lambda l), \tag{37}$$

satisfying

$$\xi(l) = c_0 l^{-1} + o\left(l^{-2}\right) \text{ and } \xi'(l) = -c_0 l^{-2} + o\left(l^{-3}\right). \tag{38}$$

Here c_0 is a constant independent of λ. Then the main order of U_λ^+ will be

$$H\left(t - \frac{1}{\lambda}\xi\left(\frac{l}{\lambda}\right)\right).$$

We want to prove

$$\partial_\lambda U_\lambda^+ > 0, \tag{39}$$

provided that λ is large enough. To emphasize the dependence on λ, we shall use (l_λ, t_λ) to denote the Fermi coordinate with respect to Γ_λ^+.

We will first of all prove (39) in a radius K tubular neighbourhood Ξ of Γ_λ where K is large enough but fixed. Recall that we use $s = f(\lambda, r)$ to represent the minimal surface Γ_λ^+. Note that

$$f(\lambda, r) = \lambda f\left(1, \frac{r}{\lambda}\right).$$

Consider the system

$$\begin{cases} l_\lambda + t_\lambda \frac{f'}{\sqrt{1+f'^2}} = r^*, \\ f - t_\lambda \frac{1}{\sqrt{1+f'^2}} = s^*. \end{cases}$$

where (r^*, s^*) is the (r,s)-coordinate of P, and $f' = \partial_r f(\lambda, l_\lambda)$. Differentiate this system with respect to λ, we find that

$$\partial_\lambda l_\lambda = -\frac{t_\lambda \partial_\lambda f' + f' \partial_\lambda f \sqrt{1+f'^2}}{(1+f'^2)^{\frac{3}{2}} + t_\lambda f''}, \tag{40}$$

and

$$\partial_\lambda t_\lambda = \frac{\partial_\lambda f}{\sqrt{1+f'^2}}.$$

Hence, using the fact that

$$f(1, r) = \sqrt{2} r + r^{-2} + o\left(r^{-2}\right),$$

we get

$$\partial_\lambda t_\lambda \geq C\left[f\left(1, \frac{l_\lambda}{\lambda}\right) - \frac{l_\lambda}{\lambda} f'\left(1, \frac{l_\lambda}{\lambda}\right)\right] \geq \frac{C\lambda^2}{l_\lambda^2}. \tag{41}$$

Here we have also used the fact that $f(1,r) - r f'(1,r) \geq c_\delta > 0$ in $[0, \delta]$, which follows from the area-minimizing property of Γ^+.

Next we compute

$$\partial_\lambda\left(\frac{1}{\lambda}\xi\left(\frac{l_\lambda}{\lambda}\right)\right)=-\frac{1}{\lambda^2}\xi\left(\frac{l_\lambda}{\lambda}\right)-\frac{l_\lambda}{\lambda^3}\xi'\left(\frac{l_\lambda}{\lambda}\right)+\frac{1}{\lambda^2}\xi'\left(\frac{l_\lambda}{\lambda}\right)\partial_\lambda l_\lambda.$$

By formula (40) , we have the estimate

$$|\partial_\lambda l_\lambda|\leq C.$$

Taking into account of the estimate (38), we then obtain

$$\left|\partial_\lambda\left(\frac{1}{\lambda}\xi\left(\frac{l_\lambda}{\lambda}\right)\right)\right|\leq\frac{C}{l_\lambda^2}. \tag{42}$$

Combining (41) and (42) , we find that

$$\partial_\lambda\left\{H\left[t_\lambda-\frac{1}{\lambda}\xi\left(\frac{l_\lambda}{\lambda}\right)\right]\right\}\geq\frac{C\lambda^2}{l_\lambda^2}, \tag{43}$$

provided λ is large enough.

On the other hand, estimating the error $U_\lambda^+-H\left[t_\lambda-\frac{1}{\lambda}\xi\left(\frac{l_\lambda}{\lambda}\right)\right]$ similarly as the last section of [24], one could show

$$\left|\partial_\lambda\left\{U_\lambda^+-H\left[t_\lambda-\frac{1}{\lambda}\xi\left(\frac{l_\lambda}{\lambda}\right)\right]\right\}\right|\leq\frac{C\lambda}{l_\lambda^2}.$$

This together with (43) clearly yields $\partial_\lambda U_\lambda^+>0$ in Ξ. Then we could use the maximum principle to conclude that $\partial_\lambda U_\lambda^+>0$ in the whole space.

Finally, we need to compare U_λ^+ with U_λ^-. Observe that for the main order term $H\left(t-\frac{1}{\lambda}\xi\left(\frac{l}{\lambda}\right)\right)$ of U_λ^+ and U_λ^-, the order l^{-1} terms in ξ are actually same, which essentially follows from (38) . On the other hand, the lower order terms in $U_\lambda^\pm$ could be well controlled. Hence $U_\lambda^-<U_\lambda^+$. Here we emphasize that the Fermi coordinate (t,l) are with respect to different minimal hypersurfaces for U_λ^+ and U_λ^-. ■

We would like to remark that for a general area-minimizing cone (but not strict area-minimizing), although the family of Pacard-Wei solutions still exists, we do not know if it is ordered, because the arguments of Lemma 16 do not apply in this case.

By Lemma 16, the family of solutions $U_\lambda^\pm$ forms a foliation. We could use them as sub and super solutions to obtain solutions between them and we have similar results as in the Simons cone case. However, in the current situation, we will show that the nodal set of each solution will be asymptotic to the curve $s=\sqrt{2}r+\frac{\beta_{i,j}}{r}$. Here the constant $\beta_{i,j}=-\sqrt{3}c_0$, where c_0 is the constant in (38).

Now we are ready to prove Proposition 15. We still focus on the case $(i,j)=(3,5)$. Since the main steps are the same as the case of Simons cone, we shall only sketch the proof and point out the main difference.

Proof of Proposition 15. Let $a > 0$ be a fixed real number, the case of $a \leq 0$ being similar. Let (l, t) be the Fermi coordinate with respect to the minimal hypersurface asymptotic to the cone $\mathcal{C}_{3,5}$ with the asymptotic behavior:

$$s = f(a, r) = \sqrt{2}r + ar^{-2} + o\left(r^{-2}\right).$$

We could construct minimizers on a sequence of bounded domains Ω_d. Let L_d be the line orthogonal to the minimal surface at $(d, f(a, d))$. On L_d we impose the Dirichlet boundary condition that

$$u_d = H\left(t - \frac{1}{a}\xi\left(\frac{l}{a}\right)\right) + \eta(t) A_2 \text{ on } L_d.$$

in the region where the Fermi coordinate is well defined. Recall that we have ordered solutions $U_\lambda^\pm$ of Pacard-Wei. We could assume that the boundary function is trapped between two solutions $U_{\lambda*}^+$ and $U_{\lambda^*}^-$ for some fixed λ^*. Then the minimizer u_d on Ω_d will be between $U_{\lambda^*}^+$ and $U_{\lambda^*}^-$. We could take the limit for a subsequence of solutions $\{u_d\}$ obtained in this way, as $d \to +\infty$.

We need to get uniform bounds on $\{u_d\}$.

Define the approximate solution $\bar{H}(t-h)$ as before and write $u_d = \bar{H} + \phi$. Then we get

$$\begin{aligned}
-\Delta\phi + \left(3\bar{H}^2 - 1\right)\phi &= \Delta_{\gamma^t}\bar{H} - M_t\bar{H}' + o(\phi) \\
&= -J(h)\bar{H}' + H''(\bar{t})\, O\left(h'^2\right) + \left[O\left(h''l^{-1}\right) + O\left(h'l^{-2}\right)\right]\bar{t}\bar{H}' \\
&\quad - \left(\bar{t}A_2 + \bar{t}^2A_3 + \bar{t}^3A_4\right)\bar{H}' - 2\bar{t}\bar{H}'hA_3 + O\left(l^{-5}\right) + o(\phi).
\end{aligned}$$

Write $h(l)$ as $h^*(l) + \frac{1}{a}\xi\left(\frac{l}{a}\right)$, where ξ is the function appearing in (37). Then

$$\begin{aligned}
E &:= \Delta\bar{H} + \bar{H} - \bar{H}^3 \\
&= -J(h)\bar{H}' + H''(\bar{t})\, O\left(h'^2\right) + \left[O\left(h''l^{-1}\right) + O\left(h'l^{-2}\right)\right]\bar{t}\bar{H}' \\
&\quad - \left(\bar{t}A_2 + \bar{t}^2A_3 + \bar{t}^3A_4\right)\bar{H}' - 2\bar{t}\bar{H}'hA_3 + O\left(l^{-5}\right).
\end{aligned}$$

Inspecting each term in this error, it turns out that the main order terms contributing to the projection of E onto $\bar{H}'$ should be

$$-J(h^*)\bar{H}' + O\left(l^{-5}\right).$$

On the other hand, since $u = \bar{H} + \phi$ and ϕ satisfies

$$-\Delta\phi + \left(3\bar{H}^2 - 1\right)\phi = E - \frac{\int_{\mathbb{R}} E\bar{H}'dt}{\int_{\mathbb{R}} \bar{H}'^2 dt}\bar{H}' + o(\phi),$$

arguing in the same spirit as Proposition 14, we find that

$$\phi = \eta(\bar{t}) A_2 + \phi^*,$$

with

$$|\phi^*| \leq Cl^{-3}.$$

Note that in the proof of Proposition 14, we have shown that the function $\bar{\phi}$ there is bounded by Cl^{-4}. But here ϕ^* only decays like Cl^{-3}. A refined analysis shows that actually the $O\left(r^{-3}\right)$ term in ϕ^* could be written as $\eta_2\left(\bar{t}\right)A_3$, where η_2 satisfies

$$-\eta_2''\left(t\right)+\left(3H^2-1\right)\eta_2\left(t\right)=-t^2H'+\frac{\int_{\mathbb{R}}t^2H'^2dt}{\int_{\mathbb{R}}H'^2dt}H'.$$

Then the term $\int_{\mathbb{R}}\Delta_{\gamma^t}\phi^*\bar{H}'dt$ could be bounded by Cl^{-5}. Now multiplying the equation

$$-\Delta\phi+\left(3\bar{H}^2-1\right)\phi=\Delta\bar{H}+\bar{H}-\bar{H}^3-3\bar{H}\phi^2-\phi^3$$

by $\bar{H}'$ and integrating in t, one could show that

$$J\left(h^*\right)=O\left(l^{-5}\right). \tag{44}$$

With (44) at hand, we could proceed similarly as in Section 2.2 and conclude the proof. We remark that the constant $\beta_{i,j}$ comes from the function ξ, which decays like Cr^{-1}. We know that the appearance of ξ is related to the fact that when $i\neq j$, the term A_3 decays like Cr^{-3}. In the case that $i=j$, A_3 is of the order $O\left(r^{-5}\right)$, hence $\beta_{i,i}=0$. ■

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